Opuscula Math. 37, no. 5 (2017), 647–664
http://dx.doi.org/10.7494/OpMath.2017.37.5.647 Opuscula Mathematica

BLOCK COLOURINGS OF 6-CYCLE SYSTEMS

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Communicated by Mariusz Meszka

Abstract. Let $\Sigma = (X, \mathcal{B})$ be a 6-cycle system of order v, so $v \equiv 1, 9 \mod 12$. A c-colouring of type s is a map $\phi\colon \mathcal{B} \to \mathcal{C}$, with C set of colours, such that exactly c colours are used and for every vertex x all the blocks containing x are coloured exactly with s colours. Let $\frac{v-1}{2} = qs + r$, with $q, r \geq 0$. ϕ is *equitable* if for every vertex x the set of the $\frac{v-1}{2}$ blocks containing x is partitioned in r colour classes of cardinality $q+1$ and $s-r$ colour classes of cardinality q. In this paper we study bicolourings and tricolourings, for which, respectively, $s = 2$ and $s = 3$, distinguishing the cases $v = 12k+1$ and $v = 12k+9$. In particular, we settle completely the case of $s = 2$, while for $s = 3$ we determine upper and lower bounds for c.

Keywords: 6-cycles, block-colourings, G-decompositions.

Mathematics Subject Classification: 05C15, 05B05.

1. INTRODUCTION

Block colourings of 4-cycle systems have been introduced and studied in [3, 4, 9, 11]. In this paper we study block colourings of 6-cycle systems, in what follows just "colourings".

Let K_v be the complete simple graph on v vertices. The graph having vertices $a_1, a_2, \dots, a_k$, with $k \geq 3$, and having edges $\{a_k, a_1\}$ and $\{a_i, a_{i+1}\}$ for $i = 1, \dots, k-1$ is a k-cycle and it will be denoted by $(a_1, a_2, \dots, a_k)$. A n-cycle system of order v, briefly $nCS(v)$, is a pair $\Sigma = (X, \mathcal{B})$, where X is the set of vertices and $\mathcal{B}$ is a set of n-cycles, called *blocks*, that partitions the edges of K_v.

A colouring of a $nCS(v)$ $\Sigma = (X, \mathcal{B})$ is a mapping $\phi\colon \mathcal{B} \to \mathcal{C}$, where $\mathcal{C}$ is a set of colours. A c-colouring is a colouring in which exactly c colours are used. The set of blocks coloured with a colour of $\mathcal{C}$ is a *colour class*. A c-colouring of type s is a colouring in which, for every vertex x, all the blocks containing x are coloured with exactly s colours.

Let $\Sigma = (X, \mathcal{B})$ be an $nCS(v)$, let $\phi\colon \mathcal{B} \to \mathcal{C}$ be a c-colouring of type s and let $\frac{v-1}{2} = qs + r$ with $q, r \geq 0$. Note that each vertex of an $nCS(v)$ is contained in exactly

$\frac{v-1}{2}$ blocks. ϕ is *equitable* if for every vertex x the set of the $\frac{v-1}{2}$ blocks containing x is partitioned in r colour classes of cardinality $q+1$ and $s-r$ colour classes of cardinality q. A bicolouring, tricolouring or quadricolouring is an equitable colouring with $s=2$, $s=3$ or $s=4$.

The colour spectrum of an $nCS(v)$ $\Sigma=(X,\mathcal{B})$ is the set:

$$\Omega_s^{(n)}(\Sigma)=\{c \mid \text{there exists an equitable } c\text{-colouring of type } s \text{ of } \Sigma\}.$$

We also consider the set $\Omega_s^{(n)}(v)=\bigcup \Omega_s^{(n)}(\Sigma)$, where Σ varies in the set of all the $nCS(v)$.

We will consider the *lower s-chromatic index* $\chi_s^{(n)}(\Sigma)=\min \Omega_s^{(n)}(\Sigma)$ and the *upper s-chromatic index* $\overline{\chi}_s^{(n)}(\Sigma)=\max \Omega_s^{(n)}(\Sigma)$. If $\Omega_s^{(n)}(\Sigma)=\emptyset$, then we say that Σ is uncolourable.

In the same way we consider $\chi_s^{(n)}(v)=\min \Omega_s^{(n)}(v)$ and $\overline{\chi}_s^{(n)}(v)=\max \Omega_s^{(n)}(v)$.

Block colourings for $s=2$, $s=3$ and $s=4$ of $4CS$ have been studied in [3,9,11]. The problem arose as a consequence of colourings of Steiner systems studied in [7,10,12,18]. For further references on such topics see [2,5,14,19].

The case $n=5$, which the authors have been studying, appears to be definitely more complex than those studied previously. In this paper we will consider the case $n=6$. It is known (see [15]) that a $6CS(v)$ exists if and only if $v\equiv 1,9 \mod 12$. We will study block colourings for $6CS$ in the cases $s=2$ and $s=3$, distinguishing the cases $v=12k+1$ and $v=12k+9$.

In what follows, to construct 6-cycle systems we will use sometimes the difference method. This means that we fix as a vertex set $X=\mathbb{Z}_v$ and, defined a base-block $B=(a_1,a_2,a_3,a_4,a_5,a_6)$, its translates will be all the blocks of type

$$B+i=(a_1+i,a_2+i,a_3+i,a_4+i,a_5+i,a_6+i)$$

for every $i\in\mathbb{Z}$. Then, given $x,y\in X$, $x\neq y$, the edge $\{x,y\}$ will belong to one of the blocks $B+i$ for some i if and only if $|x-y|\in\{|a_i-a_{i+1}|\colon i=1,\dots,6\}$, where the indices are taken mod 6.

2. BICOLOURINGS FOR $v=12k+1$

In this section we will consider bicolourings in the case $v=12k+1$. We will deal with the case $v=12k+9$ in the next section. First, we determine a bound for the number c of colours of bicolourings.

Lemma 2.1. *Let $\Sigma=(V,\mathcal{B})$ be a $6CS(v)$, where $v=12k+1$, and let $\phi\colon\mathcal{B}\to C$ be a c-bicolouring of Σ. Then $c\leq 3$.*

Proof. Let $|C|=c$ and let $\gamma\in C$. Any element $v\in V$ incident with blocks coloured with γ must be incident with precisely $3k$ blocks coloured with γ. This means that there are at least $6k+1$ vertices incident with blocks coloured with γ. This means that

$$c(1+6k)\leq 2(1+12k),$$

so that $c\leq 3$. □

In the following theorems we determine the sets $\Omega_2^{(6)}(12k+1)$, but we find two different results, depending on the parity of k.

Theorem 2.2. *If k is odd, then $\Omega_2^{(6)}(12k+1) = \emptyset$.*

Proof. Let $\Sigma = (V, \mathcal{B})$ be a $6CS(v)$, where $v = 12k+1$, and let $\phi\colon \mathcal{B} \to C$ be a 2-bicolouring of Σ. Let $\gamma \in C$ and let $\mathcal{B}_\gamma$ the set of blocks of $\mathcal{B}$ coloured with γ. Then it must be:

$$|\mathcal{B}_\gamma| = \frac{v \cdot 3k}{6}.$$

Since k is odd, we get a contradiction.

Now, let $\Sigma = (V, \mathcal{B})$ be a $6CS(v)$, where $v = 12k+1$, and let $\phi\colon \mathcal{B} \to C$ be a 3-bicolouring of Σ. In this case we proceed as in [9, Lemma 2.1]. We can suppose that $C = \{1,2,3\}$ and we denote by X the set of vertices incident with blocks of colour 1 and 2, by Y the set of vertices incident with blocks of colour 1 and 3 and by Z the set of vertices incident with blocks of colour 2 and 3. Let $x = |X|$, $y = |Y|$ and $z = |Z|$.

We can note that these sets are pairwise disjoint and that in each block we can have vertices at most of two types. Moreover, it is easy to see that a block can not contain an odd number of edges having vertices of different types.

This implies that the products xy, xz, yz are all even and so among x, y and z at most one is odd. However, since $x+y+z = v$, one of them is odd, while the others are even. Since

$$|B_1| = \frac{3k \cdot (x+y)}{6},$$
$$|B_2| = \frac{3k \cdot (x+z)}{6},$$
$$|B_3| = \frac{3k \cdot (y+z)}{6},$$

then we get a contradiction, because k is odd. This shows that there is no $3 \notin \Omega_2^{(6)}(12k+1)$. By Lemma 2.1, we get the statement. □

Theorem 2.3. *If k is even, then $\Omega_2^{(6)}(12k+1) = \{2,3\}$.*

Proof. Let $V = \mathbb{Z}_{12k+1}$. Consider on $\mathbb{Z}_{12k+1}$ the following base blocks:

$$A_i = (0, 6k+1-i, 5k, 9k+i, 11k+1, 2k+i),$$

for $i \in \{1,\dots,k\}$. If $k = 2h$, assign the colour 1 to the blocks A_i and all their translated forms, for $i \in \{1,\dots,h\}$ and the colour 2 to the blocks A_i and all their translated forms, for $i \in \{h+1,\dots,2h\}$. If $\mathcal{B}$ is the set of all these blocks, $\Sigma = (\mathbb{Z}_{12k+1}, \mathcal{B})$ is a $6CS(12k+1)$ and the previous assignment determines a 2-bicolouring of Σ.

Now we prove that $3 \in \Omega_2^{(6)}(12k+1)$. Let $k = 2h$ and consider two disjoint sets A and B, with $|A| = |B| = 12h$, and en element $\infty \notin A \cup B$. By [15] we can consider two $6CS(12h+1)$, $\Sigma_1 = (A \cup \{\infty\}, \mathcal{B}_1)$ and $\Sigma_2 = (B \cup \{\infty\}, \mathcal{B}_2)$. By [17] we can take a $6CS$ $\Sigma_3 = (K_{A,B}, \mathcal{B}_3)$ on the bipartite graph $K_{A,B}$. Then $\Sigma = (A \cup B \cup \{\infty\}, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$

is a $6CS(12k+1)$. Assigning the colour i to the blocks of $\mathcal{B}_i$, for $i = 1, 2, 3$, we get a 3-bicolouring of the Σ.

This proves that $3 \in \Omega_2^{(6)}(12k+1)$ and by Lemma 2.1 we get the statement. □

3. BICOLOURINGS FOR $v = 12k + 9$

In this section we study bicolouring for $6CS$ of order $v = 12k + 9$. First, we determine a bound for the number c of colours.

Lemma 3.1. *Let $\Sigma = (V, \mathcal{B})$ be a $6CS(v)$, where $v = 12k + 9$, and let $\phi\colon \mathcal{B} \to C$ be a c-bicolouring of Σ. Then $c \leq 3$.*

Proof. Let $|C| = c$ and let $\gamma \in C$. Any element $v \in V$ incident with blocks coloured with γ must be incident with precisely $3k+2$ blocks coloured with γ. This means that there are at least $6k + 5$ vertices incident with blocks coloured with γ. This means that

$$c(5 + 6k) \leq 2(9 + 12k),$$

so that $c \leq 3$. □

As done in the case $v = 12k + 1$, also in the case $v = 12k + 9$ we are going to get two distinct results, based on the parity of k. Indeed, the following result can be proved as Theorem 2.2.

Theorem 3.2. *If k is odd, then $\Omega_2^{(6)}(12k+9) = \emptyset$.*

Proof. The proof proceeds as in Theorem 2.2, because, in a bicolouring of a $6CS$ of order $12k + 9$ on a vertex set V, any element $v \in V$ is incident with $3k + 2$ blocks coloured with one colour and $3k + 2$ blocks coloured with another one. So, if k is odd, $3k+2$ is odd too and, proceeding as in Theorem 2.2, we show that $2, 3 \notin \Omega_2^{(6)}(12k+9)$ for any k odd. By Lemma 3.1 the statement follows. □

Now we are going to deal with the case $v = 12k + 9$ when k is even. Let us first prove, using the difference method, the following result.

Theorem 3.3. *If k is even, then $\chi_2^{(6)}(12k+9) = 2$ for any $k \geq 0$ and $\Omega_2^{(6)}(9) = \{2\}$.*

Proof. 1) Let $v = 12k + 9$ and let $k = 2h$. Consider on $\mathbb{Z}_{24h+9}$ the following base blocks:

$$A_i = (0, 12h + 5 - i, 20h + 9, 18h + 4 + i, 22h + 9, 4h + 4 + i)$$

for $i \in \{1, \ldots, 2h\}$, in the case $h \geq 1$. Consider on $\mathbb{Z}_{24h+9}$ the family $\mathcal{A}$ of blocks of all the translated forms of the blocks A_i, for $i \in \{1, \ldots, 2h\}$. Consider also the following blocks:

$$B_j = (3j, 3j + 1, 3j + 4, 3j + 5, 3j + 6, 3j + 2),$$
$$C_j = (3j, 3j + 3, 3j + 1, 3j + 5, 3j + 2, 3j + 4)$$

for $j \in \{0, \ldots, 8h + 2\}$. Then $\Sigma = (\mathbb{Z}_{24h+9}, \mathcal{A} \cup \bigcup B_j \cup \bigcup C_j)$ (if $h = 0$ take $\mathcal{A} = \emptyset$) is a $6CS(24h + 9)$.

Let us assign the colour 1 to the blocks A_i and all their translated forms for $i \in \{1, \ldots, h\}$ and all the blocks B_j and the colour 2 to the blocks A_i and all their translated forms for $i \in \{h+1, \ldots, 2h\}$ and all the blocks C_j. In this way we get a 2-bicolouring of Σ.

2) Let $v = 9$, let $\Sigma = (V, \mathcal{B})$ be a $6CS(9)$ and let $\phi\colon \mathcal{B} \to C$ be a 3-bicolouring of Σ. We can suppose that $C = \{1, 2, 3\}$ and let us denote by $\mathcal{B}_i$ the set of blocks coloured with i and by X_i the set of vertices incident with these blocks. Any vertex $x \in X$ incident with blocks coloured with the colour i must be incident with precisely 2 blocks coloured with i. So, since $|\mathcal{B}| = 6$, then $|\mathcal{B}_i| = 2$ for any $i = 1, 2, 3$ and by

$$|\mathcal{B}_i| = \frac{2|X_i|}{6}$$

we see that it must be $|X_i| = 6$ for any i. Let $X = \{a_1, \ldots, a_9\}$ and suppose that $X_1 = \{a_1, \ldots, a_6\}$. We can suppose that the edge $\{a_1, a_2\}$ is not incident with the blocks of $\mathcal{B}_1$. This implies that we can suppose that $\{a_1, a_2\}$ will be incident with one of the blocks of $\mathcal{B}_2$. So $a_7, a_8, a_9 \in X_2$, but $|X_2| = 6$. This means that we can suppose that $a_3 \in X_2$, but a_3 is adjacent with a_1 and a_2 in the blocks of $\mathcal{B}_1$. So in the blocks of $\mathcal{B}_2$ a_3 can be adjacent only with the a_7, a_8, a_9. This is is not possible and so by Lemma 3.1 we have that $\Omega_2^{(6)}(9) = \{2\}$. □

Now we need to prove that $3 \in \Omega_2(12k+9)$ for k even, $k \geq 2$. In order to do this, we will need some technical lemmas. First, let us recall that the union $G_1 \cup G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph having $V_1 \cup V_2$ as vertex set and edges those of $E_1 \cup E_2$.

Definition 3.4. A 1-factorization $\{F_1, \ldots, F_{2n-1}\}$ of the complete graph K_{2n} is called *uniform* if the graphs $F_i \cup F_j$ are all isomorphic for $i \neq j$.

Since $F_i \cup F_j$ is a 2-regular graph, it is isomorphic to a disjoint union of even cycles. If these cycles have length $k_1, \ldots, k_r$, then we say that the uniform 1-factorization is of type $(k_1, \ldots, k_r)$.

Lemma 3.5 ([6, 8])**.** *There exists a uniform* 1*-factorization of* K_{12} *of type* $(6, 6)$ *and it is unique up to isomorphisms.*

The previous lemma, together with the following ones, provides us the decomposition technique that will be required later.

Lemma 3.6. *Let* $h \geq 1$ *and let* X *and* Y *be disjont sets such that* $|X| = 12h$ *and* $|Y| = 3$*. Then:*

1. *the graph* $K_{X,Y} \cup K_X$ *can be decomposed into* 6*-cycles;*
2. *for any* r *such that* $1 \leq r \leq 5$ *there exist pairwise disjoint factors* $F_1, \ldots, F_{2r}$ *of* K_X *such that the graph* $K_{X,Y} \cup (K_X - F_1 - \ldots - F_{2r})$ *can be decomposed into* 6*-cycles and for any* $j = 0, \ldots, r-1$ *the graph* $F_{2j+1} \cup F_{2j+2}$ *can be decomposed into* 6*-cycles.*

Proof. The first part of the statement is a direct consequence of the existence of maximum packings of K_n with 6-cycles when $n \equiv 3 \mod 12$ (see [13]). We will prove the second part of the statement by induction. Let $h = 1$. By Lemma 3.5, we can consider a uniform factorization $\mathcal{F} = \{F_1, \ldots, F_{11}\}$ of K_X, with $X = \{0, 1, \ldots, 11\}$. Let $F_{11} = \{\{i, i+6\} \mid i = 0, \ldots, 5\}$ and let $Y = \{a, b, c\}$. Then the following cycles:

$$(a, i+8, b, i, c, i+4) \quad \text{for } i = 0, 1, 2, 3,$$
$$(a, 0, 6, b, 7, 1),\ (a, 2, 8, c, 9, 3),\ (b, 4, 10, c, 11, 5)$$

determine a 6-cycles decomposition of the graph $K_{X,Y} \cup F_{11}$. Then Lemma 3.5 easily leads us to the statement in the case $h = 1$. Indeed, $K_X - F_{11} = F_1 \cup \ldots \cup F_{10}$. This proves the base case $h = 1$, because the factorization $\mathcal{F}$ is uniform.

Now we prove the inductive step. Let $h > 1$ and let $Y = \{a, b, c\}$. Let $X = \bigcup_{i=1}^{h} X_i$, where $X_i \cap X_j = \emptyset$ for $i \neq j$ and $|X_i| = 12$ for any i. Note that

$$K_X = K_{X_1} \cup \ldots \cup K_{X_h} \cup \bigcup_{i<j} K_{X_i, X_j} \tag{3.1}$$

and also that

$$K_{X,Y} = K_{X_1,Y} \cup \ldots \cup K_{X_h,Y}. \tag{3.2}$$

By induction, for any i and r, with $1 \leq r \leq 5$, we can find $F_1^{(i)}, \ldots, F_{2r}^{(i)}$ such that $K_{X_i,Y} \cup (K_{X_i} - F_1^{(i)} - \ldots - F_{2r}^{(i)})$ can be decomposed into 6-cycles and for any $j = 0, \ldots, r-1$ $F_{2j+1}^{(i)} \cup F_{2j+2}^{(i)}$ can be decomposed in 6-cycles.

Let $F_j = \bigcup_{i=1}^{h} F_j^{(i)}$ for any j, so that each F_j is a factor of X and $F_1, \ldots, F_{2r}$ are pairwise disjoint. So by (3.1) and (3.2) and by the fact that K_{X_i,X_j} can be decomposed into 6-cycles, for any $i \neq j$, $F_1, \ldots, F_{2r}$ are such that $K_{X,Y} \cup (K_X - F_1 - \ldots - F_{2r})$ can be decomposed into 6-cycles. Moreover, obviously for any $j = 0, \ldots, r-1$ $F_{2j+1} \cup F_{2j+2}$ can be decomposed into 6-cycles. □

The last technical lemma needed is the following.

Lemma 3.7. *Let $h \geq 1$ and let X and Y be disjont sets such that $|X| = 12h$ and $|Y| = 3$. Then, given a 1-factor F of K_X, the graph $K_{X,Y} \cup F$ can be decomposed into 6-cycles.*

Proof. In Lemma 3.6 the statement has been proved in the case $h = 1$. Now let $h > 1$. We know that $|F| = 6h$. So we can decompose F in h disjoint subsets $F_1, \ldots, F_h$ and we can call X_i the vertex set of F_i. So $X = \bigcup_{i=1}^{h} X_i$, where $X_i \cap X_j = \emptyset$ for $i \neq j$, $|X_i| = 12$ and F_i is a factor of X_i.

We can apply the statement to each X_i and F_i, so that $K_{X_i,Y} \cup F_i$ can be decomposed into 6-cycles. Now note that

$$K_{X,Y} \cup F = K_{X_1,Y} \cup \ldots \cup K_{X_h,Y} \cup F_1 \cup \ldots \cup F_h.$$

This clearly proves the statement. □

Now we are ready to prove the following result.

Theorem 3.8. *If k is even, $k \geq 2$, then $\Omega_2^{(6)}(12k+9) = \{2,3\}$.*

Proof. 1) Let $v = 33$. Let us consider four pairwise disjoint sets X, Y, Z and T, with $|X| = 6$, $|Y| = 12$, $|Z| = 3$, $|T| = 12$ and $X = \{x_1, \dots, x_6\}$, $Y = \{y_1, \dots, y_{12}\}$, $Z = \{z_1, z_2, z_3\}$ and $T = \{t_1, \dots, t_{12}\}$. We will determine a 3-bicolouring for a $6CS$ on $X' = X \cup Y \cup Z \cup T$.

Let us consider the factor $F_1 = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ on K_X. By [1, Theorem 1.1], we can decompose the graph $K_X - F_1$ into 6-cycles, obtaining the blocks A_1 and A_2. Similarly, we can consider the factor:

$$F_2 = \{\{y_1, y_2\}, \{y_3, y_4\}, \{y_5, y_6\}, \{y_7, y_8\}, \{y_9, y_{10}\}, \{y_{11}, y_{12}\}\}$$

on K_Y. As before, by [1, Theorem 1.1] we can decompose the graph $K_Y - F_2$ into 6-cycles, obtaining the blocks $B_1,\dots,B_{10}$. Moreover, by [17] we can decompose the complete bipartite graph $K_{X,Y}$ into 6-cycles, obtaining the blocks $C_1,\dots, C_{12}$.

Let us consider, also, the blocks

$$D_1 = (x_1, x_2, z_1, x_3, z_3, z_2), \quad D_2 = (x_3, x_4, z_3, x_1, z_1, z_2),$$
$$D_3 = (x_5, x_6, z_2, x_4, z_1, z_3), \quad D_4 = (x_2, z_3, x_6, z_1, x_5, z_2).$$

These blocks represent a decomposition of the graph $K_Z \cup F_1 \cup K_{X,Z}$. We will also consider the blocks $E_1,\dots,E_{12}$, that we obtain by decomposing $K_{X,T}$ into 6-cycles (again by [17]). Moreover, consider the following blocks:

$$G_i = (z_1, t_{i+4}, z_3, t_i, z_2, t_{i+8})$$

for $i = 1, 2, 3, 4$. These blocks represent a decomposition of $K_{Z,T} - \mathcal{G}$, where

$$\mathcal{G} = \{\{z_i, t_j\} \mid i = 1, 2, 3,\ j = 4i-3, 4i-2, 4i-1, 4i\}.$$

By Lemma 3.5, we can find pairwise disjoint factors F_3, F_4, F_5 of K_T in such a way that the graph $K_T - F_3 - F_4 - F_5$ can be decomposed into 6-cycles that we call $H_1,\dots,H_8$.

Consider the graph $K_{Y,Z} \cup F_2$. By Lemma 3.7, we can decompose this graph into 6-cycles $I_1,\dots, I_7$. Similarly, by Lemma 3.7, we can get:

- a decomposition in 6-cycles of the graph $K_{T,\{y_4,y_5,y_6\}} \cup F_3$, obtaining the blocks $J_1,\dots,J_7$,
- a decomposition in 6-cycles of the graph $K_{T,\{y_7,y_8,y_9\}} \cup F_4$, obtaining the blocks $K_1,\dots,K_7$,
- a decomposition in 6-cycles of the graph $K_{T,\{y_{10},y_{11},y_{12}\}} \cup F_5$, obtaining the blocks $L_1,\dots,L_7$.

At last, decompose $\mathcal{G} \cup K_{T,\{y_1,y_2,y_3\}}$ in the following blocks:

$$\begin{aligned}
M_1 &= (z_1, t_2, y_2, t_4, y_1, t_1),\\
M_2 &= (z_1, t_4, y_3, t_2, y_1, t_3),\\
M_3 &= (z_2, t_6, y_1, t_8, y_2, t_5),\\
M_4 &= (z_2, t_8, y_3, t_6, y_2, t_7),\\
M_5 &= (z_3, t_{10}, y_1, t_{12}, y_3, t_9),\\
M_6 &= (z_3, t_{12}, y_2, t_{10}, y_3, t_{11}),\\
M_7 &= (y_1, t_5, y_3, t_1, y_2, t_9),\\
M_8 &= (y_1, t_7, y_3, t_3, y_2, t_{11}).
\end{aligned}$$

Let us call $\mathcal{B}$ the set of all these blocks. Then clearly that the system $\Sigma = (X', \mathcal{B})$ is a $6CS$ of order 33.

Now let us consider the colouring $\phi\colon \mathcal{B} \to \{1, 2, 3\}$ such that:

- the blocks A_i B_i and C_i are coloured with the colour 1,
- the blocks D_i, E_i, G_i and H_i are coloured with the colour 2,
- the remaining blocks I_i, J_i, K_i, L_i and M_i are coloured with the colour 3.

This is a 3-bicolouring of Σ. Indeed, in the blocks coloured with 1 we have only the vertices of X and Y and each of them belongs to 8 of these blocks; in the blocks coloured with 2 we have only the vertices of X, Z and T and each of them belongs to 8 of these blocks; in the blocks coloured with 3 we have only the vertices of Y, Z and T and each of them belongs to 8 of these blocks. This proves that $3 \in \Omega_2^{(6)}(33)$ and by Lemma 3.1 we get that $\Omega_2^{(6)}(33) = \{2, 3\}$.

2) Let $v = 24h + 9$, with $h \geq 2$. Let us consider the $6CS$ $\Sigma = (X', \mathcal{B})$ of order 33 constructed previously with the given 3-bicolouring. Let $\mathcal{B}_1$ be the set of blocks coloured with 1, $\mathcal{B}_2$ the set of blocks coloured with 2 and $\mathcal{B}_3$ the set of blocks coloured with the colour 3.

We have $X' = X \cup Y \cup Z \cup T$, where $|X| = 6$, $|Y| = 12$, $|Z| = 3$ and $|T| = 12$ and X, Y, Z and T are pairwise disjoint. Let us consider two other sets Y' and T', disjoint from X', such that $|Y'| = |T'| = 12h - 12$ and $Y' \cap T' = \emptyset$. We will determine a 3-bicolouring for a $6CS$ on $X'' = X' \cup Y' \cup T'$, where $|X''| = 24h + 9$.

Let I_1 be a factor of $K_{Y'}$, so that by [1] we can decompose $K_{Y'} - I_1$ into 6-cycles A_i for $i = 1, \ldots, (h-1)(12h-14)$. By [17], we can also decompose $K_{X \cup Y, Y'}$ into 6-cycles $B_1, \ldots, B_{36h-36}$.

By Lemma 3.6, we can find pairwise disjoint factors I_2, I_3, I_4 and I_5 of $K_{T'}$ such that $K_{Z,T'} \cup (K_{T'} - I_2 - I_3 - I_4 - I_5)$ can be decomposed into 6-cycles C_i for $i = 1, \ldots, (h-1)(12h-11)$ and $I_2 \cup I_3$ and $I_4 \cup I_5$ can also decomposed into 6-cyles.

By [17], we can also decompose $K_{X \cup T, T'}$ into 6-cycles $D_1, \ldots, D_{36h-36}$.

By Lemma 3.7, we can decompose $K_{Y',Z} \cup I_1$ into 6-cycles $E_1, \ldots, E_{7h-7}$. By [17], we can decompose $K_{Y \cup Y', T'}$ into 6-cycles $F_1, \ldots, F_{2h(12h-12)}$ and $K_{Y',T}$ into 6-cycles $G_1, \ldots, G_{24h-24}$. At last we can decompose $I_2 \cup I_3$ and $I_4 \cup I_5$ into 6-cycles $H_1, \ldots, H_{4h-4}$.

Let us call $\mathcal{B}$ the set of these blocks. Then it is easily seen that the system $\Sigma = (X'', \mathcal{B})$ is a $6CS$ of order $24h + 9$.

Now let us consider the colouring $\phi\colon \mathcal{B} \to \{1,2,3\}$ such that:

- the blocks of $\mathcal{B}_1$ and A_i and B_i are coloured with the colour 1,
- the blocks of $\mathcal{B}_2$ and C_i and D_i are coloured with the colour 2,
- the remaining blocks of $\mathcal{B}_3$ and the remaining blocks E_i, F_i, G_i and H_i are coloured with the colour 3.

This is a 3-bicolouring of Σ. Indeed, in the blocks coloured with 1 we have only the vertices of X, Y and Y' and each of them belongs to $6h+2$ of these blocks; in the blocks coloured with 2 we have only the vertices of X, Z, T and T' and each of them belongs to $6h+2$ of these blocks; in the blocks coloured with 3 we have only the vertices of Y, Y', Z, T and T' and each of them belongs to $6h+2$ of these blocks. This proves that $3 \in \Omega_2^{(6)}(24h+9)$ and, by Lemma 3.1, we get that $\Omega_2^{(6)}(24h+9) = \{2,3\}$ for any $h \geq 1$. □

4. LOWER 3-CHROMATIC INDEX

In this section we study tricolourings, so that $s = 3$, analizing the lower 3-chromatic index. First, we determine an upper bound for the number of colours required.

Lemma 4.1. *Let $\Sigma = (V, \mathcal{B})$ be a $6CS(v)$ and let $\phi\colon \mathcal{B} \to C$ be a c-tricolouring of Σ. Then:*

1. *if $v = 13$, $c \leq 7$,*
2. *if $v \equiv 1 \mod 12$ and $v > 13$, $c \leq 8$,*
3. *if $v \equiv 9 \mod 12$, $c \leq 9$.*

Proof. Let $v = 12k+1$, for some $k \geq 1$ and let $|C| = c$ and let $\gamma \in C$. Any element $v \in V$ incident with blocks coloured with γ must be incident with precisely $2k$ blocks coloured with γ. This means that there are at least $4k+1$ vertices incident with blocks coloured with γ. This means that

$$c(1+4k) \leq 3(1+12k),$$

so that $c \leq 8$, if $k \geq 2$, otherwise we get $c \leq 7$ if $k = 1$.

Let $v = 12k+9$, for some $k \geq 0$ and let $|C| = c$ e let $\gamma \in C$. Any element $v \in V$ incident with blocks coloured with γ must be incident with either $2k+2$ or $2k+1$ blocks coloured with γ. This means that there are at least $4k+3$ vertices incident with blocks coloured with γ. This means that

$$c(3+4k) \leq 3(9+12k),$$

so that $c \leq 9$. □

Since $v \equiv 1, 9 \mod 12$, we are going to distinguish the two cases, being this time the case $v \equiv 1 \mod 12$ more difficult to deal with. Indeed, we will determine the exact value of $\chi_3^{(6)}(12k+1)$ only for $k = 1$, $k = 2$ and $k \equiv 0 \mod 3$, while we will determine the exact value of $\chi_3^{(6)}(12k+9)$ for any $k \geq 0$.

Theorem 4.2. *If $k \equiv 1,2 \mod 3$, $\chi_3^{(6)}(12k+1) \geq 4$. If $k \equiv 0 \mod 3$, $\chi_3^{(6)}(12k+1) = 3$.*

Proof. Let $\Sigma = (V, \mathcal{B})$ be a $6CS(v)$ and let $\phi\colon \mathcal{B} \to C$ be a 3-tricolouring of Σ. Any element $v \in V$ incident with blocks coloured with γ must be incident with precisely $2k$ blocks coloured with γ. So, if $\mathcal{B}_\gamma$ is the set of blocks coloured with γ, it must be

$$|B_\gamma| = \frac{2kv}{6} = \frac{kv}{3}.$$

However, if $k \equiv 1,2 \mod 3$, this number is not an integer. This shows that, if $k \equiv 1,2 \mod 3$, $\chi_3^{(6)}(12k+1) \geq 4$.

Now, let $v = 36h+1$, for some $h \geq 1$. Let us consider three sets A, B, C such that $|A| = |B| = |C| = 12h$ and $A \cap B = A \cap C = B \cap C = \emptyset$ and let us consider also an element $\infty \notin A \cup B \cup C$.

By [15], we can decompose the complete graphs $K_{A\cup\{\infty\}}$, $K_{B\cup\{\infty\}}$ and $K_{C\cup\{\infty\}}$ into 6-cycles, that we call, respectively, D_i, E_i and F_i for $i = 1,\ldots,12h^2+h$. Moreover, by [17] we can decompose the complete bipartite graphs $K_{A,B}$, $K_{A,C}$ and $K_{B,C}$ into 6-cycles that we call, respectively, G_i, H_i and I_i for $i = 1,\ldots,24h^2$. Called $\mathcal{B}$ the set of all these blocks, it is easy to see that the system $\Sigma = (A \cup B \cup C \cup \{\infty\}, \mathcal{B})$ is a $6CS$ of order $36h+1$.

Consider, now, the colouring $\phi\colon \mathcal{B} \to \{1,2,3\}$ obtained by assigning the colour 1 to the blocks D_i and I_i, the colour 2 to the blocks E_i and H_i and the colour 3 to the blocks F_i and G_i. Then it is easy to see that this is a 3-tricolouring of Σ. □

In the following result we see that the lower 3-chromatic index in the cases $v = 13$ and $v = 25$ is 4. It is reasonable to conjecture that, in general, if $k \equiv 1,2 \mod 3$, then $\chi_3^{(6)}(12k+1) = 4$.

Theorem 4.3. *$\chi_3^{(6)}(13) = 4$ and $\chi_3^{(6)}(25) = 4$.*

Proof. 1) Let $v = 13$. Let us consider three sets $A = \{a_1,a_2,a_3,a_4\}$, $B = \{b_1,b_2,b_3,b_4\}$, $C = \{c_1,c_2,c_3,c_4\}$, pairwise disjoint, and an element $\infty \notin A \cup B \cup C$. On $X = A \cup B \cup C \cup \{\infty\}$ let us consider the following blocks:

$D_1 = (\infty,a_1,b_2,a_3,b_3,a_2)$, $D_2 = (b_1,b_2,b_4,a_4,\infty,a_3)$, $D_3 = (b_3,b_4,a_1,a_2,b_1,a_4)$,
$D_4 = (\infty,c_1,a_1,a_3,a_2,c_2)$, $D_5 = (c_1,c_3,c_2,a_4,a_3,c_4)$, $D_6 = (c_3,\infty,c_4,a_2,a_4,a_1)$,
$D_7 = (\infty,b_1,c_2,b_2,c_3,b_3)$, $D_8 = (c_1,c_2,c_4,b_4,\infty,b_2)$, $D_9 = (c_3,c_4,b_1,b_3,c_1,b_4)$,
$D_{10} = (a_1,b_3,b_2,a_2,b_4,c_2)$, $D_{11} = (a_1,b_1,c_3,a_4,b_2,c_4)$, $D_{12} = (a_2,c_1,b_1,b_4,a_3,c_3)$,
$D_{13} = (a_3,c_1,a_4,c_4,b_3,c_2)$.

Then $\Sigma = (X, \bigcup_{i=1}^{13} D_i)$ is $6CS$ of order 13. Let us consider, now, the colouring $\phi\colon \bigcup_{i=1}^{13} D_i \to \{1,2,3,4\}$ obtained in the following way:

- assign the colour 1 to the blocks D_1, D_2 and D_3,
- assign the colour 2 to the blocks D_4, D_5 and D_6,
- assign the colour 3 to the blocks D_7, D_8 and D_9,
- assign the colour 4 to the remaning blocks D_{10}, D_{11}, D_{12} and D_{13}.

Then ϕ is a 4-tricolouring of Σ, so that $4 \in \Omega_3^{(6)}(13)$. By Theorem 4.2, we get that $\chi_3^{(6)}(13) = 4$.

2) Let $v = 25$. Let $X = \mathbb{Z}_4 \times \{1,2,3,4,5,6\} \cup \{\infty\}$, with $\infty \notin \mathbb{Z}_4 \times \{1,2,3,4,5,6\}$. Let us consider on X the following blocks:

$$\begin{array}{lll}
A_1 = (0_5, 1_5, 1_4, 3_5, 2_5, 0_4), & A_2 = (0_5, 2_5, 3_4, 1_5, 3_5, 2_4), & A_3 = (0_5, 3_5, 0_4, 1_5, 2_5, 1_4), \\
A_4 = (0_6, 1_6, 1_3, 3_6, 2_6, 0_3), & A_5 = (0_6, 2_6, 3_3, 1_6, 3_6, 2_3), & A_6 = (0_6, 3_6, 0_3, 1_6, 2_6, 1_3), \\
A_7 = (\infty, 0_5, 3_4, 0_2, 3_3, 0_6), & A_8 = (\infty, 1_5, 2_4, 0_2, 2_3, 1_6), & A_9 = (\infty, 2_5, 2_4, 2_2, 2_3, 2_6), \\
A_{10} = (\infty, 3_5, 3_4, 2_2, 3_3, 3_6), & A_{11} = (0_2, 0_4, 3_2, 2_4, 1_2, 1_4), & A_{12} = (0_2, 0_3, 3_2, 2_3, 1_2, 1_3), \\
A_{13} = (2_2, 0_4, 1_2, 3_4, 3_2, 1_4), & A_{14} = (2_2, 0_3, 1_2, 3_3, 3_2, 1_3), &
\end{array}$$

which represent a decomposition in 6-cycles of the graph:

$$K_{\{0_5,1_5,2_5,3_5\}} \cup K_{\{0_6,1_6,2_6,3_6\}} \cup K_{\{0_2,1_2,2_2,3_2\} \cup \{0_5,1_5,2_5,3_5\},\{0_4,1_4,2_4,3_4\}}$$
$$\cup K_{\{0_2,1_2,2_2,3_2\} \cup \{0_6,1_6,2_6,3_6\},\{0_3,1_3,2_3,3_3\}} \cup K_{\{\infty\},\{0_5,1_5,2_5,3_5\} \cup \{0_6,1_6,2_6,3_6\}}.$$

Also, by [15], we can decompose:

- the complete graph on $\{0_1, 1_1, 2_1, 3_1\} \cup \{0_2, 1_2, 2_2, 3_2\} \cup \{\infty\}$ into 6-cycles $B_1, \ldots, B_6$,
- the complete graph on $\{0_3, 1_3, 2_3, 3_3\} \cup \{0_4, 1_4, 2_4, 3_4\} \cup \{\infty\}$ into 6-cycles $C_1, \ldots, C_6$.

By [16, Theorem 2.2], given $K_{\{0_1,1_1,2_1,3_1\},\{0_5,1_5,2_5,3_5\},\{0_6,1_6,2_6,3_6\}}$, we can decompose this equipartite graph into 6-cycles $D_1, \ldots, D_8$. Moreover, let us consider the blocks $E_{ij} = (i_1, j_3, i_5, j_2, i_6, j_4)$ for any $i, j \in \{0,1,2,3\}$. Let $\mathcal{B}$ the set of all these blocks. Then $\Sigma = (X, \mathcal{B})$ is a $6CS$ of order 25.

Consider, now, the colouring $\phi\colon \mathcal{B} \to \{1,2,3,4\}$ obtained in the following way:

- assign the colour 1 to the blocks A_i,
- assign the colour 2 to the blocks B_i,
- assign the colour 3 to the blocks C_i and D_i,
- assign the colour 4 to the blocks E_{ij}.

Then ϕ is a 4-tricolouring of Σ, so that $4 \in \Omega_3^{(6)}(25)$ and by Theorem 4.2 we get that $\chi_3^{(6)}(25) = 4$. □

In the following theorem we will see that $3 \in \Omega_3^{(6)}(12k+9)$ for any $k \geq 0$, using the difference method technique.

Theorem 4.4. *For any $k \geq 0$, $\chi_3^{(6)}(12k+9) = 3$.*

Proof. 1) Let $k = 0$. Let us consider the following 6-cycles on $X = \mathbb{Z}_9$:

$$\begin{array}{lll}
A_1 = (1,2,3,4,5,7), & A_2 = (1,3,0,6,2,8), & A_3 = (1,6,3,5,2,4), \\
A_4 = (6,7,4,8,0,5), & A_5 = (1,5,8,7,2,0), & A_6 = (3,7,0,4,6,8).
\end{array}$$

Given $\mathcal{B} = \bigcup_{i=1}^{6} A_i$, the system $\Sigma = (X, \mathcal{B})$ is a $6CS$ on X. Consider, now, the colouring $\phi\colon \mathcal{B} \to \{1,2,3\}$ obtained by assigning the colour 1 to the blocks A_1 and A_2, the colour 2 to the blocks A_3 and A_4 and the colour 3 to the blocks A_5 and A_6.

Then it is easy to see that this is a 3-tricolouring of Σ.

2) Let $k \geq 1$ and let $v = 12k + 9$. Consider $X = \mathbb{Z}_{4k+3} \times \{1, 2, 3\}$. We will construct a $6CS$ Σ on X and a 3-tricolouring of Σ. Consider the following blocks on X:

- $A_j = (0_1, j_1, 0_3, (4k + 3 - j)_3, 0_2, (4k + 3 - j)_2)$ for $j \in \{1, \ldots, k\}$,
- $B_j = (0_1, j_1, (2k+1)_3, (j+2k+1)_3, (3k+2)_2, (j+3k+2)_2)$ for $j \in \{k+1, \ldots, 2k+1\}$,
- $C_j = (0_1, j_2, 0_3, j_1, 0_2, j_3)$ for $j \in \{k + 1, \ldots, 2k + 1\}$.

By using the difference method on X it is easy to see that, if $\mathcal{B}$ is the collection of all these blocks and their translates, the system $\Sigma = (X, \mathcal{B})$ is a $6CS$ on X.

Suppose now that $k = 1$. Consider the colouring $\phi \colon \mathcal{B} \to \{1, 2, 3\}$ on Σ obtained in the following way:

1. assign the colour 1 to the block A_1 and all its translates and to the blocks $C_2 + i$ for $i \in \{0, \ldots, 4\}$,
2. assign the colour 2 to the blocks B_2 and all its translates and to the blocks $C_3 + i$ for $i \in \{0, 1, 5, 6\}$,
3. assign the colour 3 to the block B_3 and all its translates, to the blocks $C_2 + i$ for $i = 5, 6$ and to the blocks $C_3 + i$ for $i = 2, 3, 4$.

This is a 3-tricolouring of Σ. Any element in X belongs to 10 blocks of Σ and in a 3-tricolouring of Σ these blocks must be divided into three sets of cardinality 4, 3 and 3, each a subset of a colour class. With the assigned colouring we see that:

- the elements $2_i, 3_i, 4_i$, for $i = 1, 2, 3$, belong to 4 blocks coloured with 1, while the remaining ones belong to 3 blocks coloured with 1,
- the elements 1_i, for $i = 1, 2, 3$, belong to 4 blocks coloured with 2, while the remaining ones belong to 3 blocks coloured with 2,
- the elements $0_i, 5_i, 6_i$, for $i = 1, 2, 3$, belong to 4 blocks coloured with 3, while the remaining ones belong to 3 blocks coloured with 3.

Suppose now that $k \geq 2$ and consider the colouring $\phi \colon \mathcal{B} \to \{1, 2, 3\}$ obtained in the following way:

1. assign the colour 1 to the blocks A_j, for $j \in \{1, \ldots, k\}$, and all their translates and to the blocks $C_{2k} + i$ for $i \in \{0, \ldots, 3k + 1\}$,
2. assign the colour 2 to the blocks B_j, for $j \in \{k+1, \ldots, 2k\}$, and all their translates and to the blocks $C_{2k+1} + i$ for $i \in \{0, \ldots, 2k - 1\} \cup \{3k + 2, \ldots, 4k + 2\}$,
3. assign the colour 3 to the block B_{2k+1} and all its translates, to the blocks C_j, for $j \in \{k + 1, \ldots, 2k - 1\}$, and all their translates, to the blocks $C_{2k} + i$ for $i \in \{3k + 2, \ldots, 4k + 2\}$ and to the blocks $C_{2k+1} + i$ for $i \in \{2k, \ldots, 3k + 1\}$.

This is a 3-tricolouring of Σ. Any elements in X belongs to $6k + 4$ blocks of Σ and in a 3-tricolouring of Σ these blocks must be divided into three sets of cardinality $2k + 2$, $2k + 1$ and $2k + 1$, each a subset of a colour class. With the assigned colouring we see that:

- the elements $\{0_i, \ldots, (k - 2)_i\} \cup \{(2k)_i, \ldots, (3k + 1)_i\}$, for $i = 1, 2, 3$, belong to $2k + 2$ blocks coloured with 1, while the remaining elements belong to $2k + 1$ blocks coloured with 1,

- the elements $\{k_i, \ldots, (2k-1)_i\}$, for $i = 1, 2, 3$, and $\{(3k+2)_i, \ldots, (4k)_i\}$ for $i = 1, 2, 3$, belong to $2k+2$ blocks coloured with 2, while the remaining elements belong to $2k+1$ blocks coloured with 2,
- the elements $(k-1)_i, (4k+1)_i, (4k+2)_i$, for $i = 1, 2, 3$, belong to $2k+2$ blocks coloured with 3, while the remaining elements belong to $2k+1$ blocks coloured with 3.

This shows that ϕ is a 3-tricolouring of Σ. □

5. UPPER 3-CHROMATIC INDEX

In this last section we study the upper 3-chromatic index, finding, in general, an upper bound and in just some cases its exact value. Again, we will study separately the cases $v = 12k+1$ and $v = 12k+9$.

Theorem 5.1. $\overline{\chi}_3^{(6)}(12k+1) = 7$ *for* $k \equiv 0, 2 \mod 3$ *and* $\overline{\chi}_3^{(6)}(12k+1) \leq 7$ *for* $k \equiv 1 \mod 3$.

Proof. By Lemma 4.1, we know that $\overline{\chi}_3^{(6)}(12k+1) \leq 8$ for $k \geq 2$, while $\overline{\chi}_3^{(6)}(13) \leq 7$. So we can suppose that $k \geq 2$. Suppose that there exists an 8-tricolouring of a $6CS$ $\Sigma = (X, \mathcal{B})$ of order $12k+1$. Let $\mathcal{B}_i$ be the family of blocks coloured with the colour i and let X_i be the set of vertices incident with the blocks of $\mathcal{B}_i$. Then any $x \in X_i$ belongs to $2k$ blocks of $\mathcal{B}_i$, so that $|X_i| \geq 4k+1$ for any i. So we have that $|X_i| = 4k+1+k_i$ for any i. However, we know that

$$\sum_{i=1}^{8} |X_i| = 3(12k+1) \Rightarrow \sum_{i=1}^{8} k_i = 4k-5.$$

Note now that, if $x, y \in X_i \cap X_j$, with $x \neq y$ and $i \neq j$, then the edge $\{x, y\}$ may belong to just one block either in $\mathcal{B}_i$ or in $\mathcal{B}_j$. So y is either one of the elements of X_i not adjacent to x in the blocks of $\mathcal{B}_i$ (of which there are at most k_i) or one of the elements of X_j not adjacent to x in the blocks of $\mathcal{B}_j$ (of which there are at most k_j). This means that

$$|X_i \cap X_j| \leq k_i + k_j + 1.$$

So we have

$$2|X_i| = \sum_{j \in \{1,\ldots,8\} \setminus \{i\}} |X_i \cap X_j| \Rightarrow 2(4k+1+k_i) \leq \sum_{j \in \{1,\ldots,8\} \setminus \{i\}} (k_i + k_j + 1)$$
$$\Rightarrow 8k+2+2k_i \leq 6k_i + 4k + 2 \Rightarrow k_i \geq k.$$

Since $\sum_{i=1}^{8} k_i = 4k-3$, we get $4k-3 \geq 8k$, so that $4k \leq -3$, which is a contradiction. So $\overline{\chi}_3^{(6)}(12k+1) \leq 7$ for any $k \geq 1$.

Now, let $k \equiv 0, 2 \mod 3$ and let $v = 12k+1$. Let us consider $A_1, \ldots, A_6$ pairwise disjoint sets such that $|A_i| = 2k$ for any i and take an element $\infty \notin A_i$ for any i. Let

$X = \bigcup_{i=1}^{6} A_i \cup \{\infty\}$. By [15], we can decompose the complete graph $K_{A_{2i+1} \cup A_{2i+2} \cup \{\infty\}}$ for $i = 0, 1, 2$ into 6-cycles determining the system $\Sigma_i = (A_{2i+1} \cup A_{2i+2} \cup \{\infty\}, \mathcal{B}_i)$ for $i = 0, 1, 2$. By [16], we can decompose the complete equipartite graphs K_{A_1,A_3,A_5}, K_{A_1,A_4,A_6}, K_{A_2,A_3,A_6} and K_{A_2,A_4,A_5} into 6-cycles, determining, respectively, the family of blocks $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$.

It is easy to see that $\Sigma = (X, \bigcup_{i=1}^{3} \mathcal{B}_i \cup \bigcup_{i=1}^{4} \mathcal{C}_i)$ is a $6CS$ of order v. Let $\phi\colon \bigcup_{i=1}^{3} \mathcal{B}_i \cup \bigcup_{i=1}^{4} \mathcal{C}_i \to \{1, \ldots, 7\}$ be a colouring which assigns the colour i to the blocks of $\mathcal{B}_i$, for $i = 1, 2, 3$ and the colour j to the blocks of C_{j-3} for $j = 4, 5, 6, 7$. It is easy to see that ϕ is a 7-tricolouring of Σ and this proves that $\overline{\chi}_3^{(6)}(12k + 1) = 7$ for $k \equiv 0, 2 \mod 3$. □

It is possible to determine the spectrum of tricolourings for $6CS$ of order 13.

Theorem 5.2. $\Omega_3^{(6)}(13) = \{4, 5\}$.

Proof. Let $\Sigma = (X, \mathcal{B})$ be a $6CS(13)$. We need to show that, given a tricolouring $\phi\colon \mathcal{B} \to \{1, \ldots, c\}$, then $c \le 5$. By Lemma 4.1, we know that $c \le 7$. Let $\mathcal{B}_i$ the set of blocks coloured with i and X_i the set of vertices incident with the blocks of $\mathcal{B}_i$.

Let $c = 7$. It must be $|B_i| \ge 2$ for any i, while however

$$13 = |\mathcal{B}| = \sum_{i=1}^{7} |\mathcal{B}_i|.$$

This is not possible and so $c \le 6$.

Let $c = 6$. Since $|B_i| \ge 2$ for any i and $13 = |\mathcal{B}| = \sum_{i=1}^{6} |\mathcal{B}_i|$, then we can say that $|\mathcal{B}_i| = 2$ for $i = 1, \ldots, 5$ and $|B_6| = 3$. Note that $|\mathcal{B}_i| = \frac{2|X_i|}{6}$ and so $|X_i| = 6$ for $i = 1, \ldots, 5$ and $|X_6| = 9$. Since, for any $i = 1, \ldots, 5$, any $x \in X_i$ is incident to both blocks of $\mathcal{B}_i$, we see that for any $x \in X_i$ there exists just one $y \in X_i$ such that the edge $\{x, y\}$ does not belong to the blocks of $\mathcal{B}_i$. This implies that $|X_i \cap X_j| \le 2$ for any $i, j = 1, \ldots, 5$, $i \ne j$. However,

$$39 = 3|X| = \sum_{1 \le i < j \le 6} |X_i \cap X_j| \Rightarrow 2|X_6| = \sum_{i=1}^{5} |X_i \cap X_6| \ge 19.$$

Since $|X_6| = 9$, we have a contradiction, and so $c \le 5$.

Now, by Theorem 4.3, to get the statement we need to show that there exists a 5-tricolouring of a $6CS$ of order 13. On $\mathbb{Z}_{13}$ consider the following blocks:

- A_1 and A_2, obtained by decomposing $K_{\{0,1,2,3,4,5\}} - \{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$ (see [1, Theorem 1.1]) in 6-cycles,
- A_3 and A_4, obtained by decomposing $K_{\{0,1,6,7,8,9\}} - \{\{0, 6\}, \{1, 7\}, \{8, 9\}\}$ in 6-cycles,
- A_5 and A_6, obtained by decomposing $K_{\{0,2,6,10,11,12\}} - \{\{0, 2\}, \{6, 10\}, \{11, 12\}\}$ in 6-cycles,
- $A_7 = (3, 8, 4, 7, 5, 9)$, $A_8 = (3, 11, 4, 10, 5, 12)$, $A_9 = (7, 11, 8, 10, 9, 12)$, $A_{10} = (1, 7, 3, 6, 5, 11)$, $A_{11} = (1, 10, 3, 2, 8, 12)$, $A_{12} = (2, 7, 10, 6, 4, 9)$ and $A_{13} = (4, 5, 8, 9, 11, 12)$.

It is easy to see that the system $\Sigma = (\mathbb{Z}_{13}, \bigcup_{i=1}^{13} A_i)$ is a $6CS(13)$. Let us consider now a colouring $\phi\colon \bigcup_{i=1}^{13} A_i \to \{1,\dots,5\}$ defined in the following way:

- assign the colour 1 to the blocks A_1, A_2,
- assign the colour 2 to the blocks A_3, A_4,
- assign the colour 3 to the blocks A_5, A_6,
- assign the colour 4 to the blocks A_7, A_8, A_9,
- assign the colour 5 to the blocks $A_{10}, A_{11}, A_{12}, A_{13}$.

It is easy to see that this is a 5-tricolouring of Σ. □

Now we determine an upper bound for $\overline{\chi}_3^{(6)}(12k+9)$.

Theorem 5.3. $\overline{\chi}_3^{(6)}(12k+9) \le 7$ *for* $k \ge 1$.

Proof. By Lemma 4.1, we know that $\overline{\chi}_3^{(6)}(12k+9) \le 9$.

Suppose that there exists a 9-tricolouring of a $6CS$ $\Sigma = (X, \mathcal{B})$ of order $12k+9$. Let $\mathcal{B}_i$ be the family of blocks coloured with the colour i and let X_i be the set of vertices incident with the blocks of $\mathcal{B}_i$. Then any $x \in X_i$ belongs to either $2k+1$ or $2k+2$ blocks of $\mathcal{B}_i$, so that $|X_i| \ge 4k+3$ for any i. So we have that $|X_i| = 4k+3+k_i$ for any i, with $k_i \ge 0$. However we know that

$$\sum_{i=1}^{9} |X_i| = 3(12k+9) \Rightarrow \sum_{i=1}^{9} k_i = 0.$$

So $k_i = 0$ for any i. However, this is not possible, because in such a way no element of X belongs to $2k+2$ blocks of $\mathcal{B}_i$ for some i. So we have a contradiction and $\overline{\chi}_3^{(6)}(12k+9) \le 8$.

As before, suppose that there exists an 8-tricolouring of a $6CS$ $\Sigma = (X, \mathcal{B})$ of order $12k+9$. Let $\mathcal{B}_i$ be the family of blocks coloured with the colour i and let X_i be the set of vertices incident with the blocks of $\mathcal{B}_i$. Then any $x \in X_i$ belongs to either $2k+1$ or $2k+2$ blocks of $\mathcal{B}_i$, so that $|X_i| \ge 4k+3$ for any i. So we have that $|X_i| = 4k+3+k_i$ for any i, with $k_i \ge 0$. However,

$$\sum_{i=1}^{8} |X_i| = 3(12k+9) \Rightarrow \sum_{i=1}^{8} k_i = 4k+3.$$

Note now that, if $x, y \in X_i \cap X_j$, with $x \ne y$ and $i \ne j$, then the edge $\{x, y\}$ may belong to just one block either in $\mathcal{B}_i$ or in $\mathcal{B}_j$. So y is either one of the elements of X_i not adjacent to x in the blocks of $\mathcal{B}_i$ (of which there are at most k_i) or one of the elements of X_j not adjacent to x in the blocks of $\mathcal{B}_j$ (of which there are at most k_j). This means that

$$|X_i \cap X_j| \le k_i + k_j + 1.$$

So we have

$$2|X_i| = \sum_{j \in \{1,\dots,8\}\setminus\{i\}} |X_i \cap X_j| \Rightarrow 2(4k+3+k_i) \le \sum_{j \in \{1,\dots,8\}\setminus\{i\}} (k_i + k_j + 1)$$
$$\Rightarrow 8k+6+2k_i \le 6k_i + 4k + 10 \Rightarrow k_i \ge k-1.$$

Since $\sum_{i=1}^{8} k_i = 4k+3$, we get $4k+3 \geq 8k-8$, so that $4k \leq 11$. This means that the only possibilities are $k=2$ and $k=1$.

Let $k=2$, so that $v=33$ and any vertex $x \in X_i$ belongs to either 6 or 5 blocks of $\mathcal{B}_i$. Since $k_i \geq k-1$, we have that $k_i \geq 1$ for any i. Moreover, $\sum_{i=1}^{8} k_i = 4k+3 = 11$. So we can suppose that $k_i = 1$ and $|X_i| = 12$ for any $i = 1, \ldots, 5$. This means that any element in X_i, for $i = 1, \ldots, 5$, belongs to exactly 5 blocks of $\mathcal{B}_i$ and that for any $x \in X_i$ there exists just one $y \in X_i$ such that $\{x, y\}$ is not incident with some block of $\mathcal{B}_i$. In particular, we get that $X_i \cap X_j \cap X_k = \emptyset$ for any pairwise distinct $i, j, k = 1, \ldots, 5$. Let us recall also that $|X_i \cap X_j| \leq k_i + k_j + 1 = 3$ for any $i, j =, 1, \ldots, 5$. Since

$$33 \geq |X_1 \cup \ldots \cup X_5| = \sum_{i=1}^{5} |X_i| - \sum_{1 \leq i < j \leq 5} |X_i \cap X_j| \Rightarrow \sum_{1 \leq i < j \leq 5} |X_i \cap X_j| \geq 27,$$

we see that there exists $i, j = 1, \ldots, 5$, with $i \neq j$, such that $|X_i \cap X_j| = 3$. Let $X_i \cap X_j = \{x, y, z\}$. By what remarked previously, we can suppose that $\{x, y\}$ is incident with some block in $\mathcal{B}_i$ and similarly either $\{x, z\}$ or $\{y, z\}$ to some block in $\mathcal{B}_i$. In both cases we get a contradiction and so we see that $k=2$ is impossible.

So let $k=1$. In this case, $|X_i| = 7 + k_i$ for any i and $\sum_{i=1}^{8} k_i = 7$. So we can say that $k_1 = 0$ and $|X_1| = 7$. Since in this case $v = 21$ and any $x \in X_i$ belongs to either 4 or 3 blocks of $\mathcal{B}_i$, we can say that the blocks of $\mathcal{B}_1$ are a decomposition of the complete graph on X_1. By [15], this is impossible because $7 \not\equiv 1, 9 \mod 12$. □

At last we determine the spectrum of $\Omega_3^{(6)}(9)$.

Theorem 5.4. $\Omega_3^{(6)}(9) = \{3, 4\}$.

Proof. By Lemma 4.1, we know that $\overline{\chi}_3^{(6)}(9) \leq 9$. Let $\Sigma = (X, \mathcal{B})$ be a $6CS$ and let $\phi\colon \mathcal{B} \to \{1, 2, \ldots, c\}$ be c-tricolouring of Σ. Since $|\mathcal{B}| = 6$, it follows that $c \leq 6$.

Since ϕ is a tricolouring, we see that any vertex belongs to 4 blocks, 2 of them coloured with the same colour and the other two with other two different colours. So, if $c = 6$, then any two blocks are coloured with different colours, which is clearly impossible in a tricolouring. If $c = 5$, then only 2 of 6 blocks are coloured with the same colour. So at most only 6 of the 9 vertices belongs to two blocks coloured with same colour. So $c \leq 4$.

Now we will prove that $\overline{\chi}_3^{(6)}(9) = 4$. On $X = \mathbb{Z}_9$ consider the following blocks:

$$B_j = (3j, 3j+1, 3j+4, 3j+5, 3j+6, 3j+2),$$
$$C_j = (3j, 3j+3, 3j+1, 3j+5, 3j+2, 3j+4)$$

for $j = 0, 1, 2$. Then $\Sigma = (X, \bigcup_{j=0}^{2} B_j \cup C_j)$ is a $6CS$ on X. Consider the following colouring $\phi\colon \bigcup_{j=0}^{2} B_j \cup C_j \to \{1, 2, 3, 4\}$:

- assign the colour 1 to the blocks B_j for $j = 0, 1, 2$,
- assign the colour j, for $j = 2, 3, 4$, to the block C_{j-2}.

Then it is easy to see that ϕ is a 4-tricolouring of Σ, so that $\overline{\chi}_3^{(6)}(9) = 4$. By Theorem 4.4, we get that $\Omega_3^{(6)}(9) = \{3, 4\}$. □

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Received: September 19, 2016.
Revised: December 10, 2016.
Accepted: December 21, 2016.