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EXISTENCE OF MINIMAL AND MAXIMAL SOLUTIONS TO RL FRACTIONAL INTEGRO-DIFFERENTIAL INITIAL VALUE PROBLEMS

Z. Denton and J.D. Ramírez

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Abstract. In this work we investigate integro–differential initial value problems with Riemann Liouville fractional derivatives where the forcing function is a sum of an increasing function and a decreasing function. We will apply the method of lower and upper solutions and develop two monotone iterative techniques by constructing two sequences that converge uniformly and monotonically to minimal and maximal solutions. In the first theorem we will construct two natural sequences and in the second theorem we will construct two intertwined sequences. Finally, we illustrate our results with an example.

Keywords: Riemann Liouville derivative, integro-differential equation, monotone method.

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1. INTRODUCTION

Fractional differential equations arise in science and engineering as a more useful tool than their integer counterpart for the modeling of natural phenomena, see the books [1, 6, 9, 11, 22, 23] and the article [14] for more information. On the other hand, monotone iterative methods are well established for nonlinear ordinary differential equations, as it can be found in [10]. In recent years these methods have been applied to differential equations with fractional derivatives, see the book [11] and the papers [3–5, 7, 8, 13–19, 24–28] and [29].

Additionally, the basic theory and properties of integro–differential equations are provided in the book [12]. This book introduces a monotone method for first order ordinary integro–differential equations with periodic boundary conditions.

In this work we first establish a comparison theorem equivalent to the result developed in [11] for a RL fractional integro–differential equation of order q, $0 < q < 1$, with initial condition. Next, we will use the method of lower and upper solutions combined with a generalized monotone iterative technique to prove the existence of coupled minimal and maximal solutions. Finally we will prove that there exist either natural or intertwined sequences that converge uniformly and monotonically to coupled minimal and maximal solutions of the integro–differential initial value problem.

2. PRELIMINARY RESULTS

In this section we state the definition of the Riemann Liouville derivative and mention several results that will be necessary to prove the main result of this work.

We begin by giving the definition of the Mittag–Leffler function.

Definition 2.1. The two parameter Mittag–Leffler function is defined as

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}.$$

In particular $E_{1,1}(t) = e^t$, and $E_{\alpha,\beta}(t)$ is also called the generalized exponential function.

Let $J = [a, b]$ be a finite real interval. The definition of Riemann Liouville fractional derivative is given in [6, 9, 11, 23] as follows.

Definition 2.2. The Riemann Liouville fractional derivative of order α, where $n-1 \leq \alpha < n$ and $n \in \mathbb{N}$, is denoted by D^α and defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s) ds.$$

Consider the nonlinear initial value problem of the form

$$\begin{aligned} D^q u(t) &= f(t, u(t)), \\ u(t)(t-a)^{1-q}|_{t=a} &= u^0. \end{aligned} \tag{2.1}$$

Throughout this paper we will consider the Riemann Liouville derivative of order q, where $0 < q < 1$.

We recall the following definition.

Definition 2.3. Let $0 < q < 1$ and $p = 1 - q$. If G is an open set in $\mathbb{R}$, then we denote by $C_p([a, b], G)$ the function space

$$C_p([a, b], G) = \left\{ u \in C((a, b], G) \;\middle|\; (t-a)^p u(t) \in C([a, b], G) \right\}.$$

If $u \in C_p([a, b], G)$, then u is said to be C_p continuous in $[a, b]$.

Remark 2.4. In [9] and [11] it was proven that if $0 < q < 1$, $G \subset \mathbb{R}$ is an open set, and $f : (a, b] \times G \to \mathbb{R}$ is such that for any $u \in G$, $f \in C_p([a, b], G)$, then u satisfies (2.1) if and only if it satisfies the Volterra fractional integral equation

$$u(t) = u^0(t) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, u(s)) ds, \tag{2.2}$$

where $u^0(t) = \frac{u^0(t-a)^{q-1}}{\Gamma(q)}$

This relationship is especially true if $f : [a, b] \times G \to \mathbb{R}$ is continuous.

If (2.1) is a non-homogeneous linear fractional differential equation, that is if $f(t, u(t)) = Mu(t) + f(t)$, where M is a real number and $f \in C_p([a, b], \mathbb{R})$, then the solution is given by

$$u(t) = u^0(t-a)^{q-1} E_{q,q}\left(M(t-a)^q\right) + \int_a^t (t-s)^{q-1} E_{q,q}\left(M(t-s)^q\right) f(s) ds \tag{2.3}$$

where $t \in (a, b]$, and $E_{q,q}(t)$ is the two parameter Mittag–Leffler function. See [11] for details.

Now assume that $u \in C_p([a, b], \mathbb{R})$, $Tu(t) = \int_a^t K(t, s) u(s) ds$, and $K \in C([a, b] \times [a, b], \mathbb{R})$ is a positive function. Since K is continuous and the integral of a C_p continuous function is also C_p continuous, then Tu is C_p continuous and Remark 2.4 can be generalized as follows:

Remark 2.5. The nonlinear integro–differential initial value problem

$$\begin{aligned} D^q u &= f(t, u(t), Tu(t)), \\ u(t)(t-a)^p|_{t=a} &= u^0, \end{aligned} \tag{2.4}$$

is equivalent to the Volterra fractional integral equation

$$u(t) = u^0(t) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} f(s, u(s), Tu(s)) ds. \tag{2.5}$$

That is, every solution of (2.4) is a solution of (2.5) and viceversa.

In the rest of this section we state several comparison results relative to initial value problems with the Riemann Liouville derivative.

Lemma 2.6. *Let $m \in C_p([a, b], \mathbb{R})$ and suppose that for any $t_1 \in (a, b]$ we have that on (a, t_1), $m(t) \leq 0$, $m(t_1) = 0$ and $m(t)(t-a)^p|_{t=a} \leq 0$. Then $D^q m(t_1) \geq 0$.*

Remark 2.7. The previous lemma was proven in [11] when $m(t)$ is Hölder continuous of order $\lambda > q$. However, in iterative methods it is not possible to prove that each of the iterates are Hölder continuous of order $\lambda > q$. In [5] the authors proved this result without assuming that m is Hölder continuous.

We conclude this section with a comparison theorem and some important consequences.

Theorem 2.8. *Let $J = [a, b]$, and suppose that there exist two functions $v^0(t), w^0(t) \in C[J, \mathbb{R}]$ with $v^0(t) < w^0(t)$ such that the following conditions hold*

(a) $f, g \in C\left(J \times \left[v^0(t), w^0(t)\right] \times \left[Tv^0(t), Tw^0(t)\right]\right)$,
(b) *f is increasing in u and Tu, g is decreasing in u and Tu, and*
(c) *For $v(t), w(t) \in C_p[J, \mathbb{R}]$ such that $v^0(t) \le (t-a)^p v(t), (t-a)^p w(t) \le w^0(t)$ the following inequalities are true for $t \in (a, b]$,*

$$\begin{aligned} D^q v(t) &\le f(t, v(t), Tv(t)) + g(t, w(t), Tw(t)), \\ v(t)(t-a)^p|_{t=a} &\le u^0, \ \text{and} \\ D^q w(t) &\ge f(t, w(t), Tw(t)) + g(t, v(t), Tv(t)), \\ w(t)(t-a)^p|_{t=a} &\ge u^0. \end{aligned} \tag{2.6}$$

Suppose further that $f(t, u, Tu)$ and $g(t, u, Tu)$ satisfy the following Lipschitz condition for $L_1, L_2 > 0$, $M_1, M_2 \ge 0$, and $x \ge y$,

$$\begin{aligned} f(t, x, Tx) - f(t, y, Ty) &\le L_1(x-y) + M_1 T(x-y), \\ g(t, x, Tx) - g(t, y, Ty) &\ge -L_2(x-y) - M_2 T(x-y), \end{aligned} \tag{2.7}$$

then $v(t)(t-a)^p|_{t=a} \le w(t)(t-a)^p|_{t=a}$ implies that

$$v(t) \le w(t), \text{ for } a < t \le b.$$

Proof. Assume first without loss of generality that one of the inequalities in (2.6) is strict, say $D^q v(t) < f(t, v(t), Tv(t)) + g(t, w(t), Tw(t))$, and $v_0 < w_0$, where $(t-a)^p v(t)|_{t=a} = v_0$ and $(t-a)^p w(t)|_{t=a} = w_0$. We will show that $v(t) < w(t)$ for $t \in [a, b]$.

Suppose, to the contrary, that there exists t_1 such that $a < t_1 \le b$ for which

$$v(t_1) = w(t_1), \text{ and } v(t) < w(t) \text{ for } t < t_1.$$

Setting $m(t) = v(t) - w(t)$ it follows that $m(t_1) = 0$ and $m(t) < 0$ for $a < t < t_1$. Also, if $a < s \le t_1$ then $v(s) \le w(s)$ and

$$Tv(t_1) = \int_a^{t_1} K(t_1, s) v(s) ds \le \int_a^{t_1} K(t_1, s) w(s) ds = Tw(t_1).$$

Then by Lemma 2.6 we have that $D^q m(t_1) \ge 0$. Thus

$$\begin{aligned} &f(t_1, v(t_1), Tv(t_1)) + g(t_1, w(t_1), Tw(t_1)) \\ &> Dv(t_1) \ge Dw(t_1) \\ &\ge f(t_1, w(t_1), Tw(t_1)) + g(t_1, v(t_1), Tv(t_1)), \end{aligned}$$

which is a contradiction to the assumption $v(t_1) = w(t_1)$. Therefore $v(t) < w(t)$ for $t > a$.

Now assume that the inequalities in (2.6) are non strict. We will show that $v(t) \leq w(t)$.

Set

$$v_\varepsilon(t) = v(t) - \varepsilon(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right), \text{ and}$$
$$w_\varepsilon(t) = w(t) + \varepsilon(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right),$$

where $\varepsilon > 0$, and $\lambda > 1$ is a constant that will be determined later.

This implies that

$$v_\varepsilon(t)(t-a)^p|_{t=a} = v_\varepsilon^0 = v(t)(t-a)^p|_{t=a} - \varepsilon E_{q,q}(0) < v^0,$$
$$w_\varepsilon(t)(t-a)^p|_{t=a} = w_\varepsilon^0 = w(t)(t-a)^p|_{t=a} + \varepsilon E_{q,q}(0) > w^0,$$

$v_\varepsilon(t) < v(t)$, and $w_\varepsilon(t) > w(t)$ for $a < t \leq b$.

Hence,

$$Tv_\varepsilon(t) = \int_a^t K(t,s)v_\varepsilon(s)ds \leq \int_a^t K(t,s)v(s)ds = Tv(t),$$

and

$$Tw_\varepsilon(t) = \int_a^t K(t,s)w_\varepsilon(s)ds \geq \int_a^t K(t,s)w(s)ds = Tw(t),$$

for $t > a$.

Using (2.6) and the Lipschitz condition (2.7), we find for $t > a$ that

$$\begin{aligned}
&D^q v_\varepsilon(t) \\
&= D^q v(t) - \varepsilon\lambda(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right) \\
&\leq f\left(t, v(t), Tv(t)\right) + g\left(t, w(t), Tw(t)\right) - \varepsilon\lambda(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right) \\
&= f\left(t, v(t), Tv(t)\right) + g\left(t, w(t), Tw(t)\right) - f\left(t, v_\varepsilon(t), Tv_\varepsilon(t)\right) \\
&\quad - g\left(t, w_\varepsilon(t), Tw_\varepsilon(t)\right) + f\left(t, v_\varepsilon(t), Tv_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), Tw_\varepsilon(t)\right) \\
&\quad - \varepsilon\lambda(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right) \\
&\leq L_1\left(v(t) - v_\varepsilon(t)\right) + M_1 T\left(v(t) - v_\varepsilon(t)\right) + L_2\left(w_\varepsilon(t) - w(t)\right) \\
&\quad + M_2 T\left(w_\varepsilon(t) - w(t)\right) + f\left(t, v_\varepsilon(t), Tv_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), Tw_\varepsilon(t)\right) \\
&\quad - \varepsilon\lambda(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right) \\
&= \varepsilon L_1(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right) + \varepsilon M_1 T\left((t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)\right) \\
&\quad + \varepsilon L_2(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right) + \varepsilon M_2 T\left((t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)\right) \\
&\quad + f\left(t, v_\varepsilon(t), Tv_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), Tw_\varepsilon(t)\right) - \varepsilon\lambda(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right) \\
&= \varepsilon\left(L_1 + L_2\right)(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right) \\
&\quad + \varepsilon\left(M_1 + M_2\right) T\left((t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)\right) \\
&\quad + f\left(t, v_\varepsilon(t), Tv_\varepsilon(t)\right) + g\left(t, w_\varepsilon(t), Tw_\varepsilon(t)\right) - \varepsilon\lambda(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right).
\end{aligned}$$

Now consider the expression

$$T\left((t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)\right)=\int\limits_a^t K(t,s)(t-a)^{q-1}E_{q,q}\left(\lambda(s-a)^q\right)ds,$$

and let $K_0=\max\limits_{a\le s\le t\le b}\{\Gamma(q)K(t,s)(t-s)^p\}$. Clearly $K_0>0$.

Then,

$$\begin{aligned}
&T\left((t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)\right)\\
&=\int\limits_a^t K(t,s)(s-a)^{q-1}E_{q,q}\left(\lambda(s-a)^q\right)\left(\frac{\Gamma(q)(t-s)^{q-1}}{\Gamma(q)(t-s)^{q-1}}\right)ds\\
&\le\frac{K_0}{\Gamma(q)}\int\limits_a^t(t-s)^{q-1}(s-a)^{q-1}E_{q,q}\left(\lambda(s-a)^q\right)ds\\
&=\lim_{r\to a^+}\frac{K_0}{\Gamma(q)}\int\limits_r^t(t-s)^{q-1}(s-a)^{q-1}E_{q,q}\left(\lambda(s-a)^q\right)ds\\
&=\lim_{r\to a^+}\frac{K_0}{\lambda}(s-a)^{q-1}E_{q,q}\left(\lambda(s-a)^q\right)\Big|_r^t\\
&=\frac{K_0}{\lambda}(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)-\lim_{r\to a^+}\frac{K_0}{\lambda}(r-a)^{q-1}E_{q,q}\left(\lambda(r-a)^q\right)\\
&\le\frac{K_0}{\lambda}(t-a)^{q-1}E_{q,q}(\lambda(t-a)^q).
\end{aligned}$$

We have now obtained that

$$\begin{aligned}
{}^cD^qv_\varepsilon(t)\le&\ \varepsilon\left(L_1+L_2\right)(t-a)^{q-1}\left(E_{q,q}\left(\lambda(t-a)^q\right)\right)\\
&+\varepsilon\left\{\frac{K_0\left(M_1+M_2\right)}{\lambda}\right\}(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)\\
&+f\left(t,v_\varepsilon(t),Tv_\varepsilon(t)\right)+g\left(t,w_\varepsilon(t),Tw_\varepsilon(t)\right)\\
&-\varepsilon\lambda(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)\\
=&\ \varepsilon\left(L_1+L_2+\frac{K_0\left(M_1+M_2\right)}{\lambda}-\lambda\right)(t-a)^{q-1}E_{q,q}\left(\lambda(t-a)^q\right)\\
&+f\left(t,v_\varepsilon(t),Tv_\varepsilon(t)\right)+g\left(t,w_\varepsilon(t),Tw_\varepsilon(t)\right).
\end{aligned}$$

Choose $\lambda=2\left[(L_1+L_2)+K_0(M_1+M_2)\right]+1$, then

$$L_1+L_2+\frac{K_0\left(M_1+M_2\right)}{\lambda}-\lambda<0,$$

and

$${}^cD^qv_\varepsilon(t)<f\left(t,v_\varepsilon(t),Tv_\varepsilon(t)\right)+g\left(t,w_\varepsilon(t),Tw_\varepsilon(t)\right).$$

By a similar argument, we can show that

$$^{c}D^{q}w_{\varepsilon}(t) > f\left(t, w_{\varepsilon}(t), Tw_{\varepsilon}(t)\right) + g\left(t, v_{\varepsilon}(t), Tv_{\varepsilon}(t)\right).$$

Now applying the result for strict inequalities to $v_{\varepsilon}(t), w_{\varepsilon}(t)$, we get that $v_{\varepsilon}(t) < w_{\varepsilon}(t)$ for $t \in J$ and for every $\varepsilon > 0$. That is

$$v(t) - \varepsilon E_{q}\left(\lambda(t-a)^{q}\right) < w(t) + \varepsilon E_{q}\left(\lambda(t-a)^{q}\right),$$

or

$$v(t) < w(t) + 2\varepsilon E_{q}\left(\lambda(t-a)^{q}\right).$$

Consequently, making $\varepsilon \to 0$, we get that $v(t) \leq w(t)$ for $t \in J$. □

We now state the following corollary that will be needed to obtain our main results.

Corollary 2.9. *Let $m \in C_{p}[J, \mathbb{R}]$ be such that*

$$\begin{aligned} D^{q}m(t) &\leq Lm(t) + MTm(t), \quad t \in (a, b], \\ m(t)(t-a)^{p}|_{t=a} &\leq 0, \end{aligned}$$

where $L > 0, M \geq 0$. Then we have from the previous theorem that

$$m(t) \leq 0,$$

for $a < t \leq b$. Similarly, if $m \in C_{p}[J, \mathbb{R}]$ is such that

$$\begin{aligned} D^{q}m(t) &\geq -Lm(t) - MTm(t), \quad t \in (a, b], \\ m(t)(t-a)^{p}|_{t=a} &\geq 0, \end{aligned}$$

for $L > 0, M \geq 0$, then we have from the previous theorem that

$$m(t) \geq 0,$$

for $a < t \leq b$.

The result of Corollary 2.9 is still true even if $L = M = 0$, which we state separately.

Corollary 2.10. *Let $D^{q}m(t) \leq 0$ on $(a, b]$. Then $m(t) \leq 0$ for $t \in (a, b]$, if $m(t)(t-a)^{p}|_{t=a} \leq 0$.*

3. MAIN RESULTS

In this section we will initially give the definition of coupled lower and upper solutions and then we will develop two generalized monotone iterative techniques for the nonlinear integro–differential initial value problem (3.1), given below.

Consider the problem

$$\begin{aligned} D^{q}u(t) &= f\left(t, u(t), Tu(t)\right) + g\left(t, u(t), Tu(t)\right), \\ u(t)(t-a)^{p}|_{t=a} &= u^{0}, \end{aligned} \tag{3.1}$$

where $J = [a, b]$, $f, g \in C[J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$, $u \in C_p[J \times \mathbb{R}]$, $K \in C(J \times J, \mathbb{R})$ is a positive function, and $Tu(t) = \int_a^t K(t,s)u(s)ds$.

If $u \in C_p[a, b]$ satisfies the fractional differential equation

$$D^q u(t) = f(t, u(t), Tu(t)) + g(t, u(t), Tu(t)),$$

for $t \in (a, b]$ and u is such that $u(t)(t-a)^p|_{t=a} = u^0$, then u is said to be a solution of (3.1).

Throughout the rest of this paper, we will assume that f is increasing in u and Tu, and g is decreasing in u and Tu for $t \in (a, b]$.

Next we provide the definition of coupled lower and upper solutions of (3.1).

Definition 3.1. Let $v_0, w_0 \in C_p[J, \mathbb{R}]$. Then v_0 and w_0 are said to be

(a) natural lower and upper solutions of (3.1) if

$$\begin{aligned} D^q v_0(t) &\le f(t, v_0(t), Tv_0(t)) + g(t, v_0(t), Tv_0(t)) \\ v_0(t)(t-a)^p|_{t=a} &\le u^0, \\ D^q w_0(t) &\ge f(t, w_0(t), Tw_0(t)) + g(t, w_0(t), Tw_0(t)) \\ w_0(t)(t-a)^p|_{t=a} &\ge u^0; \end{aligned} \tag{3.2}$$

(b) coupled lower and upper solutions of Type I of (3.1) if

$$\begin{aligned} D^q v_0(t) &\le f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t)) \\ v_0(t)(t-a)^p|_{t=a} &\le u^0, \\ D^q w_0(t) &\ge f(t, w_0(t), Tw_0(t)) + g(t, v_0(t), Tv_0(t)) \\ w_0(t)(t-a)^p|_{t=a} &\ge u^0; \end{aligned} \tag{3.3}$$

(c) coupled lower and upper solutions of Type II of (3.1) if

$$\begin{aligned} D^q v_0(t) &\le f(t, w_0(t), Tw_0(t)) + g(t, v_0(t), Tv_0(t)) \\ v_0(t)(t-a)^p|_{t=a} &\le u^0, \\ D^q w_0(t) &\ge f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t)) \\ w_0(t)(t-a)^p|_{t=a} &\ge u^0; \end{aligned} \tag{3.4}$$

(d) coupled lower and upper solutions of Type III of (3.1) if

$$\begin{aligned} D^q v_0(t) &\le f(t, w_0(t), Tw_0(t)) + g(t, w_0(t), Tw_0(t)) \\ v_0(t)(t-a)^p|_{t=a} &\le u^0, \\ D^q w_0(t) &\ge f(t, v_0(t), Tv_0(t)) + g(t, v_0(t), Tv_0(t)) \\ w_0(t)(t-a)^p|_{t=a} &\ge u^0. \end{aligned} \tag{3.5}$$

If we have upper and lower solutions of (3.1) of any Type then we can guarantee that a solution u of (3.1) exists between them on $(a, b]$ provided the correct requirements are met. These requirements are detailed in the following lemma. We will only consider equations of Type I and II since they are the specific ones we utilize in our main result.

Lemma 3.2. *Assume that v_0, w_0 are coupled lower and upper solutions of type I or II for (3.1) such that $v_0(t) \le w_0(t)$ for $t \in (a, b]$. Further suppose*

$$f, g \in C\left(J \times [v_0(t), w_0(t)] \times [Tv_0(t), Tw_0(t)], \mathbb{R}\right).$$

Then there exists a solution $u \in C_p(J, \mathbb{R})$ of (3.1) such that $v_0(t) \le u(t) \le w_0(t)$ on $(a, b]$.

This lemma is proved in the same way as seen in [5], with only minor additions to account for the integral transformation. The following theorem concerns coupled lower and upper solutions of the form (3.3). Next, we develop a generalized monotone iterative technique for the integro-differential initial value problem. Finally, we obtain natural sequences which converge uniformly and monotonically to coupled minimal and maximal solutions of (3.1).

Theorem 3.3. *Assume that*

(A1) *v_0, w_0 are coupled lower and upper solutions of type I for (3.1) with $v_0(t) \le w_0(t)$ in $(a, b]$; and*
(A2) *$f, g \in C\left(J \times [v_0(t), w_0(t)] \times [Tv_0(t), Tw_0(t)], \mathbb{R}\right)$, where $f\left(t, u(t), Tu(t)\right)$ is increasing in u and Tu and $g\left(t, u(t), Tu(t)\right)$ is decreasing in u and in Tu.*

If $u(t)$ is a solution of (3.1) such that $v_0(t) \le u(t) \le w_0(t)$ for all $t \in (a, b]$, then the sequences defined by

$$\begin{aligned} D^q v_{n+1}(t) &= f\left(t, v_n(t), Tv_n(t)\right) + g\left(t, w_n(t), Tw_n(t)\right), \\ v_{n+1}(t)(t-a)^p|_{t=a} &= u^0, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} D^q w_{n+1}(t) &= f\big(t, w_n(t), Tw_n(t)\big) + g\big(t, v_n(t), Tv_n(t)\big), \\ w_{n+1}(t)(t-a)^p|_{t=a} &= u^0, \end{aligned} \tag{3.7}$$

are such that

$$v_0 \le v_1 \le \ldots \le v_n \le v_{n+1} \le u \le w_{n+1} \le w_n \le \ldots \le w_1 \le w_0,$$

in $(a, b]$, where the weighted sequences $\{(t-a)^p v_n(t)\}$ and $\{(t-a)^p w_n(t)\}$ are such that $(t-a)^p v_n(t) \to (t-a)^p \rho(t)$ and $(t-a)^p w_n(t) \to (t-a)^p r(t)$ uniformly and monotonically in $C[J, \mathbb{R}]$, and ρ, r are coupled minimal and maximal solutions of (3.1); i.e., ρ and r satisfy the coupled system

$$\begin{aligned} D^q \rho(t) &= f\left(t, \rho(t), T\rho(t)\right) + g\left(t, r(t), Tr(t)\right), \quad t \in (a, b], \\ \rho(t)(t-a)^p|_{t=a} &= u^0, \end{aligned}$$

and

$$\begin{aligned} D^q r(t) &= f\left(t, r(t), Tr(t)\right) + g\left(t, \rho(t), T\rho(t)\right), \quad t \in (a,b], \\ r(t)(t-a)^p|_{t=a} &= u^0, \end{aligned}$$

with $\rho \le u \le r$, $t \in (a,b]$.

Proof. By Lemma 3.2, we know a solution u of (3.1) exists such that $v_0 \le u \le w_0$ as described in our hypothesis. We will show that $v_0 \le v_1 \le u \le w_1 \le w_0$.

It follows from (3.3) that

$$\begin{aligned} D^q v_0(t) &\le f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t)), \\ v_0(t)(t-a)^p|_{t=a} &\le u_0, \\ D^q w_0(t) &\ge f(t, w_0(t), Tw_0(t)) + g(t, v_0(t), Tv_0(t)), \\ w_0(t)(t-a)^p|_{t=a} &\ge u_0, \end{aligned}$$

and by (3.6), we get that

$$\begin{aligned} D^q v_1 &= f\left(t, v_0(t), Tv_0(t)\right) + g\left(t, w_0(t), Tw_0(t)\right), \\ v_1(t)(t-a)^p &= u^0. \end{aligned}$$

Therefore, $v_0(t)(t-a)^p|_{t=a} \le u^0 = v_1(t)(t-a)^p|_{t=a}$. If we let $m = v_0 - v_1$, then $m(t)(t-a)^p|_{t=a} \le 0$ and

$$\begin{aligned} D^q m &= D^q v_0 - D^q v_1 \\ &\le f\left(t, v_0, Tv_0\right) + g\left(t, w_0, Tw_0\right) - f\left(t, v_0, Tv_0\right) - g\left(t, w_0, Tw_0\right) = 0. \end{aligned}$$

Since $D^q m \le 0$ and $m(t)(t-a)^p_{t=a} \le 0$, by an application of Corollary 2.10 we have that $m(t) \le 0$ and, consequently, $v_0(t) \le v_1(t)$ in $(a,b]$.

Suppose that u is a solution of (3.1) such that $v_0(t) \le u(t) \le w_0(t)$. In order to prove that $v_1(t) \le u(t)$, observe that since $v_0(t) \le u(t)$ for each t in $(a,b]$ and $K > 0$, then

$$Tv_0(t) = \int_a^t K(t,s) v_0(s) ds \le \int_a^t K(t,s) u(s) ds = Tu(t)$$

for each $t \in (a,b]$. Similarly we have that $Tu(t) \le Tw_0(t)$ for each $t \in (a,b]$.

Letting $m(t) = v_1(t) - u(t)$, we have that

$$m(t)(t-a)^p|_{t=a} = (v_1 - u)(t)(t-a)^p|_{t=a} = u^0 - u^0 = 0.$$

Moreover, by the increasing nature of f and the decreasing nature of g we have that

$$\begin{aligned} D^q m &= D^q v_1 - D^q u \\ &= f\left(t, v_0(t), Tv_0(t)\right) + g\left(t, w_0(t), Tw_0(t)\right) \\ &\quad - f\left(t, u(t), Tu(t)\right) - g\left(t, u(t), Tu(t)\right) \le 0, \end{aligned}$$

and, by Corollary 2.10, we have that $v_1(t) \leq u(t)$. By a similar argument, we can show that $u(t) \leq w_1(t)$ and $w_1(t) \leq w_0(t)$. Thus, $v_0(t) \leq v_1(t) \leq u(t) \leq w_1(t) \leq w_0(t)$.

Now we will show that $v_k \leq v_{k+1}$ for $k \geq 1$.

Assume that

$$v_{k-1}(t) \leq v_k(t) \leq u(t) \leq w_k(t) \leq w_{k-1}(t),$$

for $k > 1$ and $t \in (a, b]$.

If $a < s \leq t \leq b$, we have that $x_1(s) \leq x_2(s)$ implies that

$$Tx_1(t) = \int_a^t K(t,s)x_1(s)ds \leq \int_a^t K(t,s)x_2(s)ds = Tx_2(t).$$

Thus

$$Tv_{k-1}(t) \leq Tv_k(t) \leq Tu(t) \leq Tw_k(t) \leq Tw_{k-1}(t).$$

Let $m = v_k - v_{k+1}$. Then

$$v_k(t)(t-a)^p|_{t=a} = u^0 = v_{k+1}(t)(t-a)^p|_{t=a},$$

so $m(t)(t-a)^{1-a}|_{t=a} = 0$. By the increasing nature of f and the decreasing nature of g, it follows that

$$\begin{aligned} D^q m &= D^q v_k - D^q v_{k+1} \\ &= f(t, v_{k-1}, Tv_{k-1}) + g(t, w_{k-1}, Tw_{k-1}) - f(t, v_k, Tv_k) - g(t, w_k, Tw_k) \leq 0. \end{aligned}$$

Similarly, by Corollary 2.10, we have that $m(t) \leq 0$ and consequently $v_k(t) \leq v_{k+1}(t)$.

Using the hypothesis that $v_0(t) \leq u(t) \leq w_0(t)$ in $(a, b]$, the above argument and induction we can also show that $w_{k+1} \leq w_k$, $v_{k+1} \leq u$, and $u \leq w_{k+1}$. Therefore, for $n > 0$,

$$v_0 \leq v_1 \leq v_2 \leq \ldots \leq v_n \leq u \leq w_n \leq \ldots \leq w_2 \leq w_1 \leq w_0.$$

Now we have to show that the weighted sequences converge uniformly. We will use the Arzela-Ascoli Theorem by showing that the sequences are uniformly bounded and equicontinuous.

First we show uniform boundedness. By hypothesis both $v_0(t)(t-a)^p$ and $w_0(t)(t-a)^p$ are bounded on $[a, b]$, then there exists $M > 0$ such that for any $t \in [a, b]$, $|v_0(t)(t-a)^p| \leq M$ and $|w_0(t)(t-a)^p| \leq M$. Since

$$v_0(t)(t-a)^p \leq v_n(t)(t-a)^p \leq w_n(t)(t-a)^p \leq w_0(t)(t-a)^p$$

for each $n > 0$, it follows that

$$0 \leq (v_n(t) - v_0(t))(t-a)^p \leq (w_n(t) - v_0(t))(t-a)^p \leq (w_0(t) - v_0(t))(t-a)^p,$$

and consequently $\{v_n(t)(t-a)^p\}$ and $\{w_n(t)(t-a)^p\}$ are uniformly bounded.

We will now prove that the weighted sequences are equicontinuous. Doing so will require a few preliminaries. First, we can also show that the sequences $\{Tv_n\}, \{Tw_n\}$ are uniformly bounded using that the weighted sequences are also uniformly bounded. To do so let V be the uniform bound of $\{(t-a)^p v_n\}$, and let κ be the upper bound of the function $K(s,t)$ on $J \times J$. Then we have

$$\begin{aligned}|Tv_n| &\leq \int_a^t \big|K(t,s)(s-a)^p v_n(s)\big|(s-a)^{q-1}ds \\ &\leq \kappa V \int_a^t (s-a)^{q-1}ds \leq \frac{\kappa V(b-a)^q}{q}.\end{aligned}$$

We can show that $\{Tw_n\}$ is uniformly bounded similarly.

Now, for simplicity, let F_n be the function defined as

$$F_n(t) = f(t, v_n(t), Tv_n(t)) + g(t, w_n(t), Tw_n(t)).$$

Since f, g are continuous on J, and since each v_n, w_n are C_p continuous then there exist continuous functions $\tilde{f}, \tilde{g}$ such that

$$f(t, v_n, Tv_n) + g(t, w_n, Tw_n) = \tilde{f}(t, (t-a)^p v_n, Tv_n) + \tilde{g}(t, (t-a)^p w_n, Tw_n).$$

Given this, and that the weighted sequences and the transformed sequences are uniformly bounded we can choose an $N \geq 0$ such that $F_n(t) \leq N$ for all $t \in J$.

The last property we need to show that $\{v_n(t)(t-a)^p\}$ is equicontinuous is that the function

$$\phi(t) = (t-a)^p(t-s)^{-p}$$

is decreasing in t for $a < s \leq t$. To prove this note that

$$\begin{aligned}\tfrac{d}{dt}\phi(t) &= p(t-a)^{p-1}(t-s)^{-p} - p(t-a)^p(t-s)^{-p-1} \\ &= -p(t-a)^{p-1}(t-s)^{-p-1}(s+a) \leq 0.\end{aligned}$$

Now, let $t, \tau \in (a,b]$ and without loss of generality suppose $t \geq \tau$, implying $\phi(t) \leq \phi(\tau)$. Now we can show that

$$\begin{aligned}&|(t-a)^p v_n(t) - (\tau-a)^p v_n(\tau)| \\ &= \Big|\frac{(t-a)^p}{\Gamma(q)} \int_a^t (t-s)^{q-1}F_n(s)ds - \frac{(\tau-a)^p}{\Gamma(q)} \int_a^\tau (\tau-s)^{q-1}F_n(s)ds\Big|\end{aligned}$$

$$\leq \frac{(t-a)^p}{\Gamma(q)} \int\limits_{\tau}^{t} (t-s)^{q-1}|F_n(s)|ds + \frac{1}{\Gamma(q)} \int\limits_{a}^{\tau} |\phi(t)-\phi(\tau)||F_n(s)|ds$$

$$\leq \frac{N(t-a)^p}{\Gamma(q)} \int\limits_{\tau}^{t} (t-s)^{q-1}ds + \frac{N}{\Gamma(q)} \int\limits_{a}^{\tau} \big(\phi(\tau)-\phi(t)\big)ds$$

$$= \frac{N}{\Gamma(q)}\Big((t-a)^p \frac{(t-\tau)^q}{q} + (\tau-a)^p \int\limits_{a}^{\tau} (\tau-s)^{q-1}ds - (t-a)^p \int\limits_{a}^{\tau} (t-s)^{q-1}ds\Big)$$

$$= \frac{N}{q\Gamma(q)}\Big(2(t-a)^p(t-\tau)^q + (\tau-a) - (t-a)\Big) \leq \frac{2N(b-a)^p}{\Gamma(q+1)}(t-\tau)^q.$$

In the case that $\tau = a$ note

$$|(t-a)^p v_n(t) - u^0| \leq \frac{(b-a)^p}{\Gamma(q)} \int\limits_{a}^{t} (t-s)^{q-1}|F_n(s)|ds \leq \frac{N(b-a)^p}{\Gamma(q+1)}(t-a)^q.$$

So, letting $\omega = \frac{2N(b-a)^p}{\Gamma(q+1)}$ we have that

$$|(t-a)^p v_n(t) - (\tau-a)^p v_n(\tau)| \leq \omega|t-\tau|^q$$

for all $t, \tau \in J$, implying that $\{(t-a)^p v_n(t)\}$ is equicontinuous. Similarly, we can show that $\{(t-a)^p w_n(t)\}$ is equicontinuous. Since the weighted sequences are uniformly bounded and equicontinuous, by the Ascoli-Arzelà Theorem, both have uniformly convergent subsequences. Further, since both weighted sequences are monotone then they must both converge uniformly. So, let $(t-a)^p\rho$ and $(t-a)^p r$ be the uniform limits of $\{(t-a)^p v_n\}$ and $\{(t-a)^p w_n\}$ respectively. Given this we also obtain that $v_n \to \rho$ and $w_n \to r$ pointwise on $(a, b]$.

Now we wish to show that ρ, r are coupled solutions of (3.1). To begin, we note that every iterate of $\{K(s,t)v_n(s)\}, \{K(s,t)w_n(s)\}$ are dominated by an integrable function. To show this note

$$|K(s,t)v_n(s)| \leq \kappa V(s-a)^{q-1},$$

where κ and V are defined as above, and

$$\int\limits_{a}^{t} \kappa V(s-a)^{q-1}ds \leq \frac{\kappa V}{q}(b-a)^q < \infty,$$

for all $t \in (a, b]$. We can show a similar result for $\{K(s,t)w_n(s)\}$. Therefore, by applying the Lebesgue Dominated Convergence Theorem we can show that $Tv_n(t) \to T\rho(t)$

and $Tw_n(t) \to Tr(t)$ pointwise for any t in $(a,b]$. Now due to all of the convergence results we have shown and the continuity of f and g we now have that

$$(t-a)^p v_{n+1} = \frac{u^0}{\Gamma(q)} + \frac{(t-a)^p}{\Gamma(q)} \int_a^t (t-s)^{q-1}\big(f(t,v_n,Tv_n) + g(t,w_n,Tw_n)\big)ds$$

converges uniformly to

$$(t-a)^p \rho = \frac{u^0}{\Gamma(q)} + \frac{(t-a)^p}{\Gamma(q)} \int_a^t (t-s)^{q-1}\big(f(t,\rho,T\rho) + g(t,r,Tr)\big)ds$$

on J. Now dividing by $(t-a)^p$ we get

$$\rho(t) = u^0(t) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1}\big(f(t,\rho,T\rho) + g(t,r,Tr)\big)ds,$$

on $(a,b]$. Further we can also show that

$$r(t) = u^0(t) + \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1}\big(f(t,r,Tr) + g(t,\rho,T\rho)\big)ds.$$

Remark 2.5 yields that ρ, r are coupled solutions of (3.1).

To prove that ρ, r are minimal and maximal comes from previous work. We have already proven that $v_n \le u \le w_n$ for all n. Since u was a solution of (3.1) with $v_0 \le u \le w_0$, if we let x be any solution of (3.1) with $(t-a)^p x(t)|_{t=a} = u^0$ and $v_0 \le x \le w_0$ on $(a,b]$, which we know exists thanks to Lemma 3.2, we can use the same inductive arguments to prove that $v_n \le x \le w_n$ for all $n \ge 0$ and for $t \in (a,b]$. Therefore, $\rho \le x \le r$ on $(a,b]$. This implies that ρ, r are minimal and maximal coupled solutions of (3.1), and completes the proof. □

Finding coupled lower and upper solutions of Type I as in (3.3) is more challenging than finding solutions of Type II, see the recent papers [2, 20, 21] for methods to construct lower and upper solutions of the form (3.3) for different types of initial value problems. With an additional assumption on the first iterate of each sequence, we can construct intertwined sequences that converge uniformly and monotonically to minimal and maximal solutions by using coupled lower and upper solutions of Type II (3.4). Moreover, these sequences converge to a unique solution. The proof is similar to the one in Theorem 3.3, so we state the result without a proof. We state the conditions for uniqueness separately.

Theorem 3.4. *Assume that*

(B1) v_0, w_0 *are coupled lower and upper solutions of type II for* (3.1) *with* $v_0(t) \leq w_0(t)$ *in* $(a,b]$*; and*

(B2) $f, g \in C(J \times [v_0(t), w_0(t)] \times [Tv_0(t), Tw_0(t)], \mathbb{R})$*, where* $f(t, u(t), Tu(t))$ *is increasing in* u *and* Tu *and* $g(t, u(t), Tu(t))$ *is decreasing in* u *and* Tu*.*

Define the following sequences,

$$\begin{aligned} D^q v_{n+1}(t) &= f(t, w_n(t), Tw_n(t)) + g(t, v_n(t), Tv_n(t)), \\ v_{n+1}(t)(t-a)^p|_{t=a} &= u^0, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} D^q w_{n+1}(t) &= f\big(t, v_n(t), Tv_n(t)\big) + g\big(t, w_n(t), Tw_n(t)\big), \\ w_{n+1}(t)(t-a)^p|_{t=a} &= u^0. \end{aligned} \tag{3.9}$$

If $u(t)$ *is a solution of* (3.1) *such that* $v_0(t) \leq w_1(t) \leq u(t) \leq v_1(t) \leq w_0(t)$*,* $t \in (a,b]$*, then* (3.8) *and* (3.9) *provide intertwined sequences of the form*

$$\begin{aligned} v_0 &\leq w_1 \leq v_2 \leq \ldots \leq v_{2n} \leq w_{2n+1} \leq u \\ &\leq v_{2n+1} \leq w_{2n} \leq \ldots \leq w_2 \leq v_1 \leq w_0, \end{aligned}$$

where

$$\{(t-a)^p v_{2n}(t), (t-a)^p w_{2n+1}(t)\} \to (t-a)^p \rho(t)$$

and

$$\{(t-a)^p w_{2n}(t), (t-a)^p v_{2n+1}(t)\} \to (t-a)^p r(t)$$

uniformly and monotonically in $C[J, \mathbb{R}]$*, and* ρ, r *are coupled minimal and maximal solutions of* (3.1)*, respectively; i.e.,* ρ *and* r *satisfy the coupled system*

$$\begin{aligned} D^q \rho(t) &= f(t, \rho(t), T\rho(t)) + g(t, r(t), Tr(t)), \quad t \in (a,b], \\ \rho(t)(t-a)^{1-q} &= u^0, \end{aligned}$$

and

$$\begin{aligned} D^q r(t) &= f(t, r(t), Tr(t)) + g(t, \rho(t), T\rho(t)), \quad t \in (a,b], \\ r(t)(t-a)^p|_{t=a} &= u^0, \end{aligned}$$

with $\rho \leq u \leq r$ *in* $(a,b]$*.*

Remark 3.5. In addition to conditions (A1)–(A2) of Theorem 3.3 or (B1)–(B2) of Theorem 3.4, suppose that there exist positive constants M_1, M_2, and non negative constants N_1, N_2 such that f and g satisfy the following one–sided Lipschitz conditions for $x \geq y$,

$$\begin{aligned} f(t, x, Tx) - f(t, y, Ty) &\leq M_1(x-y) + N_1 T(x-y), \\ g(t, x, Tx) - g(t, y, Ty) &\geq -M_2(x-y) - N_2 T(x-y), \end{aligned} \tag{3.10}$$

then $\rho = r = u$; i.e., the sequences converge to a unique solution.

We already proved that $\rho \leq r$. In order to show that $r \leq \rho$, let $p(t) = r(t) - \rho(t)$. Clearly,

$$p(t)(t-a)^p|_{t=a} = (r-\rho)(t)(t-a)^p|_{t=a} = u_0 - u_0 = 0.$$

Since $\rho \leq r$ we have from the conclusion of Theorem 3.3 and (3.10) that

$$\begin{aligned} D^q p &= D^q r - D^q \rho \\ &= f(t, r, Tr) + g(t, \rho, T\rho) - f(t, \rho, T\rho) - g(t, r, Tr) \\ &\leq M_1(r-\rho) + N_1 T(r-\rho) + M_2(r-\rho) + N_2 T(r-\rho) \\ &= (M_1 + M_2)(r-\rho) + (N_1 + N_2)T(r-\rho) \\ &= (M_1 + M_2)p + (N_1 + N_2)Tp. \end{aligned}$$

We obtain from Corollary 2.9 that $p(t) \leq 0$ for $t \in (a, b]$ and, consequently, $r(t) \leq \rho(t)$. Therefore $\rho(t) = r(t) = u(t)$, and the sequences converge to the same solution.

4. NUMERICAL RESULTS

In this section we present one example that illustrates the result from Theorem 3.4.

Example 4.1. Consider the following integro–differential initial value problem of order $q = \frac{1}{2}$ on $J = [0, 1]$,

$$\begin{aligned} D^{1/2}u &= 2 - \frac{t}{8} + \frac{1}{8}u(t) + \frac{1}{8}\int_0^t (1+s)u(s)ds \\ &\qquad - \frac{1}{16}u^2(t) - \frac{1}{16}\left[\int_0^t (1+s)u(s)ds\right]^2, \\ u(0) &= 0. \end{aligned} \tag{4.1}$$

Here

$$Tu(t) = \int_0^t (1+s)u(s)ds.$$

Then the function

$$f(t, u(t), Tu(t)) = 2 - \frac{t}{8} + \frac{1}{8}u(t) + \frac{1}{8}\int_0^t (1+s)u(s)ds$$

is increasing in u and Tu, and

$$g(t, u(t), Tu(t)) = -\frac{1}{16}u^2(t) - \frac{1}{16}\left[\int_0^t (1+s)u(s)ds\right]^2$$

is decreasing in u and Tu for all $t \in (0,1]$. We will show graphically that $v_0 = 0$ and $w_0 = 3 + \sqrt{t}$ are coupled lower and upper solutions of type II that satisfy (3.4) on the interval $J = [0,1]$. Clearly $v_0(t)t^{1/2}|_{t=0} = w_0(t)t^{1/2}|_{t=0} = 0$. Also,

$$D^{1/2}v_0(t) = 0,$$
$$D^{1/2}w_0(t) = \frac{3}{\sqrt{\pi t}} + \frac{2}{\sqrt{\pi}}.$$

In Figure 1 we show the graph of $f(t, w_0(t), Tw_0(t)) + g(t, v_0(t), Tv_0(t))$ and in Figure 2 we show the graph of $f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t))$.

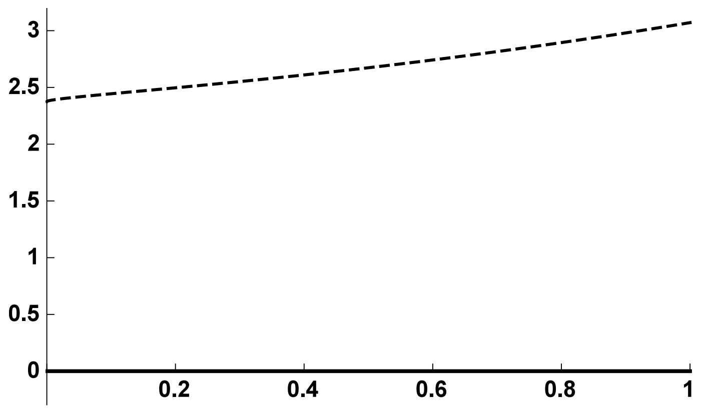

Fig. 1. $0 = D^{1/2}v_0(t) \le f(t, w_0(t), Tw_0(t)) + g(t, v_0(t), Tv_0(t))$

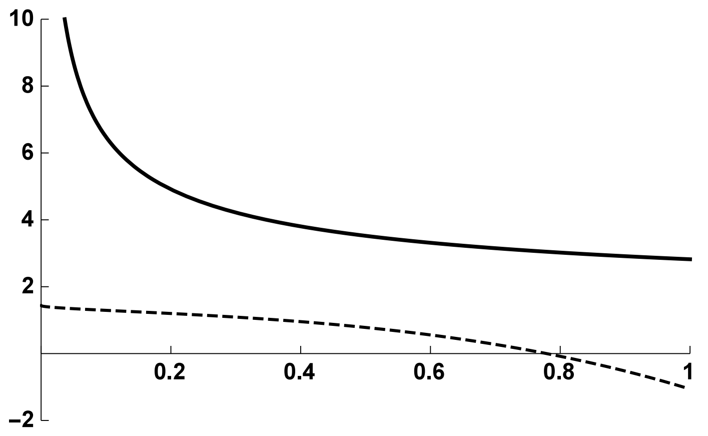

Fig. 2. ${}^cD^{1/2}w_0(t) \ge f(t, v_0(t), Tv_0(t)) + g(t, w_0(t), Tw_0(t))$

We construct the sequences according to Theorem 3.4, in Figure 3 we show four iterates of $\{t^{1/2}v_n(t)\}$ and four iterates of $\{t^{1/2}w_n(t)\}$ on $[0,1]$.

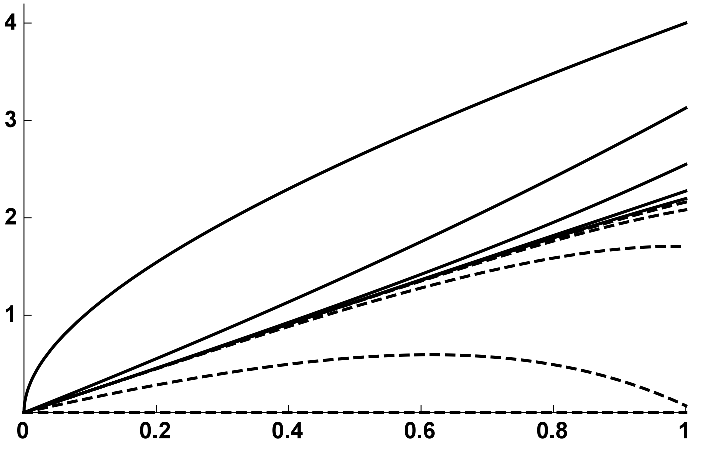

Fig. 3. Dashed: $t^{1/2}v_0 \le t^{1/2}w_1 \le t^{1/2}v_2 \le t^{1/2}w_3 \le t^{1/2}v_4$;
Solid: $t^{1/2}w_4 \le t^{1/2}v_3 \le t^{1/2}w_2 \le t^{1/2}v_1 \le t^{1/2}w_0$

Table 1. Table of ten points in $[0,1]$ of $v_4(t)$ and $w_4(t)$ for equation (4.1)

t	$v_4(t)$	$w_4(t)$
0.0	0.000000	0.000000
0.1	0.226289	0.226289
0.2	0.453429	0.453431
0.3	0.680861	0.680881
0.4	0.907949	0.908047
0.5	1.133821	1.134178
0.6	1.357143	1.358235
0.7	1.575812	1.578753
0.8	1.786538	1.793770
0.9	1.984275	2.000883
1.0	2.161438	2.197557

We have used Mathematica to compute the iterates, the graphs and the tables.

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Z. Denton
zdenton@ncat.edu

North Carolina A&T State University
Department of Mathematics
Greensboro, NC, 27411 USA

J.D. Ramírez
ramirezd@savannahstate.edu

Savannah State University
Department of Mathematics
Savannah, GA 31404, USA

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