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ON 3-TOTAL EDGE PRODUCT CORDIAL CONNECTED GRAPHS

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Abstract. A k-total edge product cordial labeling is a variant of the well-known cordial labeling. In this paper we characterize connected graphs of order at least 15 admitting a 3-total edge product cordial labeling.

Keywords: 3-total edge product cordial labelings, 3-TEPC graphs.

Mathematics Subject Classification: 05C78.

1. INTRODUCTION

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G. The sum of order and size of G is denoted by $\tau(G)$, i.e., $\tau(G) = |V(G)| + |E(G)|$. The subgraph of a graph G induced by $A \subseteq E(G)$ is denoted by $G[A]$. The set of vertices of G adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$. The cardinality of this set is called the degree of v. As usual $\delta(G)$ stands for the minimum degree among vertices of G. For integers p, q we denote by $[p, q]$ the set of all integers z satisfying $p \leq z \leq q$.

Let $k \geq 2$ be an integer. For a graph G, a mapping $\varphi : E(G) \to [0, k-1]$ induces a vertex mapping $\varphi^* : V(G) \to [0, k-1]$ defined by

$$\varphi^*(v) \equiv \prod_{u \in N_G(v)} \varphi(vu) \pmod{k}.$$

Set

$$\mu_\varphi(i) := |\{v \in V(G) : \varphi^*(v) = i\}| + |\{e \in E(G) : \varphi(e) = i\}|$$

for each $i \in [0, k-1]$. A mapping $\varphi : E(G) \to [0, k-1]$ is called a *k-total edge product cordial* (for short k-TEPC) *labeling* of G if

$$|\mu_\varphi(i) - \mu_\varphi(j)| \leq 1 \quad \text{for all} \quad i, j \in [0, k-1].$$

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A graph that admits a k-TEPC labeling is called a *k-total edge product cordial* (k-TEPC) graph.

A k-total edge product cordial labeling is a version of the well-known cordial labeling defined by Cahit [2]. Vaidya and Barasara [12] introduced the concept of a 2-TEPC labeling as the edge analogue of a total product cordial labeling defined by Sundaram *et al.* [10]. They called this labeling the total edge product cordial labeling. In [12,13] they proved that cycles C_n for $n \neq 4$, complete graphs K_n for $n > 2$, wheels, fans, double fans and some cycle related graphs are 2-TEPC. In [14] they proved that any graph can be embedded as an induced subgraph of a 2-TEPC graph. An extension of the total product cordial labeling is a k-total product cordial labeling introduced by Ponraj *et al.* [6]. In [6–8] they presented some classes of 3-total product cordial graphs. Tenguria and Verma [11] also deal with 3-total product cordial labelings. The 4-total cordial labelings are studied in [9]. Azaizeh *et al.* [1] introduced the concept of k-TEPC graphs as the edge analogue of k-total product cordial graphs. They proved that paths P_n for $n \geq 4$, cycles C_n for $3 < n \neq 6$, some trees and some unicyclic graphs are 3-TEPC graphs. In [5] there is shown that dense graphs admit k-TEPC labelings. We refer the reader to [3] for comprehensive references.

Let us recall two results from [1], which we shall use hereinafter.

Proposition 1.1. *The star $K_{1,n}$, $n \geq 3$, is 3-total edge product cordial if and only if $n \not\equiv 1 \pmod 3$.*

Proposition 1.2. *The cycle C_n, $n \geq 3$, is 3-total edge product cordial if and only if $n \notin \{3, 6\}$.*

In this paper we will deal with 3-TEPC graphs.

2. AUXILIARY RESULTS

The following claim is evident.

Observation 2.1. *A mapping $\varphi : E(G) \to [0, 2]$ is a 3-TEPC labeling of a graph G if and only if*

$$\left\lfloor \frac{\tau(G)}{3} \right\rfloor \leq \mu_\varphi(i) \leq \left\lceil \frac{\tau(G)}{3} \right\rceil \quad \textit{for each } \ i \in [0, 2].$$

Lemma 2.2. *Let G be a graph without isolated vertices and let t be an integer belonging to $[0, \tau(G)]$. There exists a mapping $\varphi : E(G) \to [0, 2]$ satisfying $\mu_\varphi(0) = t$ if and only if there is a subset A of $E(G)$ such that $\tau(G[A]) = t$.*

Proof. Suppose that there is a mapping $\varphi : E(G) \to [0, 2]$ such that $\mu_\varphi(0) = t$. Set $A = \{e \in E(G) : \varphi(e) = 0\}$. Since $\varphi^*(v) = 0$ whenever v is incident with an edge of A, $\mu_\varphi(0) = \tau(G[A])$.

On the other hand, let A be a subset of $E(G)$. Consider the mapping $\psi : E(G) \to [0, 2]$ defined by

$$\psi(e) = \begin{cases} 0 & \text{when } \ e \in A, \\ 1 & \text{when } \ e \notin A. \end{cases}$$

Clearly, $\mu_\psi(0) = \tau(G[A])$. □

Evidently, $3 \le \tau(H) \ne 4$ for any graph H without isolated vertices. Observation 2.1 and Lemma 2.2 imply the following observation.

Observation 2.3. *Any 3-total edge product cordial graph G satisfies*

$$7 \le \tau(G) \ne 12.$$

Given a mapping $\varphi : E(G) \to [0,2]$. Clearly, $\varphi^*(v) = 2$, $v \in V(G)$, if and only if v is incident with the odd number of edges having label 2 and no edge having label 0. Therefore, we immediately have the following claim.

Observation 2.4. *Let $\varphi : E(G) \to [0,2]$ be a mapping. Let $e' = uv$ be an edge of G such that $\varphi(e') = 1$. The mapping $\psi : E(G) \to [0,2]$ defined by*

$$\psi(e) = \begin{cases} \varphi(e) & \text{when } e \ne e', \\ 2 & \text{when } e = e', \end{cases}$$

satisfies $\mu_\psi(0) = \mu_\varphi(0)$ *and*

$$\mu_\psi(2) = \begin{cases} \mu_\varphi(2) - 1 & \text{when } \varphi^*(u) = \varphi^*(v) = 2, \\ \mu_\varphi(2) & \text{when } \{\varphi^*(u), \varphi^*(v)\} = \{0,2\}, \\ \mu_\varphi(2) + 1 & \text{when } \varphi^*(u) = \varphi^*(v) = 0, \\ \mu_\varphi(2) + 1 & \text{when } \{\varphi^*(u), \varphi^*(v)\} = \{1,2\}, \\ \mu_\varphi(2) + 2 & \text{when } \{\varphi^*(u), \varphi^*(v)\} = \{0,1\}, \\ \mu_\varphi(2) + 3 & \text{when } \varphi^*(u) = \varphi^*(v) = 1. \end{cases}$$

Given a graph G. Let A be a subset of $E(G)$. An edge $e \in E(G) - A$ is called *AA-edge* if its both end vertices belong to $V(G[A])$. A pendant edge $e \in E(G) - A$ is called *AP-edge* if its end vertex belongs to $V(G[A])$ (clearly, it is the end vertex of degree greater than 1).

Lemma 2.5. *Let G be a connected graph and let A be a subset of $E(G)$ such that*

$$\tau(G[A]) \in \left\{ \left\lfloor \frac{\tau(G)}{3} \right\rfloor, \left\lceil \frac{\tau(G)}{3} \right\rceil \right\}.$$

If G contains either an AA-edge and an AP-edge or two distinct AA-edges, then it is a 3-TEPC graph.

Proof. As $\tau(G[A]) \in \{\lfloor \tau(G)/3 \rfloor, \lceil \tau(G)/3 \rceil\}$, there are integers t_1 and t_2 such that $\lceil \tau(G)/3 \rceil \ge t_1 \ge t_2 \ge \lfloor \tau(G)/3 \rfloor$ and $\tau(G[A]) + t_1 + t_2 = \tau(G)$.

Let T be a spanning tree of G such that $A_T := E(T) \cap A \ne \emptyset$. Then

$$|E(G) - A| \ge |E(T) - A_T| = |E(T)| - |A_T| = (|V(T)| - 1) - |A_T|.$$

Since $|A_T| + 1 \le |V(T[A_T])|$,

$$|E(G) - A| \ge |V(T)| - |V(T[A_T])| = |V(T) - V(T[A_T])| \ge |V(G) - V(G[A])|.$$

As
$$t_1 + t_2 = \tau(G) - \tau(G[A]) = |E(G) - A| + |V(G) - V(G[A])|,$$
we have $|E(G) - A| \geq t_1 \geq t_2$.

Suppose that e_A and e'_A are assumed edges of G (i.e., e_A is an AA-edge and e'_A is either an AP-edge or an AA-edge). Denote by e_1, e_2, ..., e_q the edges of $E(G) - (A \cup \{e_A, e'_A\})$ (clearly, $q \geq t_2 - 2$). For every $i \in [0, q]$ define a set B_i by $B_0 = \emptyset$ and $B_i = B_{i-1} \cup \{e_i\}$. Let φ_i, for $i \in [0, q]$, be a mapping from $E(G)$ to $[0, 2]$ given by
$$\varphi_i(e) = \begin{cases} 0 & \text{when } \ e \in A, \\ 2 & \text{when } \ e \in B_i, \\ 1 & \text{otherwise.} \end{cases}$$
Clearly, $\mu_{\varphi_i}(0) = \tau(G[A])$, for every $i \in [0, q]$.

Denote by p the largest integer of $[0, q]$ such that $\mu_{\varphi_i}(2) \leq t_2$ for each $i \leq p$. If $p < q$, then by Observation 2.4, $\mu_{\varphi_p}(2) + 3 \geq \mu_{\varphi_{p+1}}(2) > t_2$. Therefore, $t_2 - 2 \leq \mu_{\varphi_p}(2) \leq t_2$. If $p = q$, then
$$\mu_{\varphi_p}(2) \geq |\{e : \varphi_p(e) = 2\}| = |B_p| = p = q \geq t_2 - 2.$$
So, again $t_2 - 2 \leq \mu_{\varphi_p}(2) \leq t_2$. Now define a set $B \subset E(G)$ by
$$B = \begin{cases} B_p & \text{when } \ \mu_{\varphi_p}(2) = t_2, \\ B_p \cup \{e_A\} & \text{when } \ \mu_{\varphi_p}(2) = t_2 - 1, \\ B_p \cup \{e'_A\} & \text{when } \ \mu_{\varphi_p}(2) = t_2 - 2 \text{ and } e'_A \text{ is an } AP\text{-edge}, \\ B_p \cup \{e_A, e'_A\} & \text{when } \ \mu_{\varphi_p}(2) = t_2 - 2 \text{ and } e'_A \text{ is an } AA\text{-edge}. \end{cases}$$
It is easy to see that a mapping $\psi : E(G) \to [0, 2]$ defined by
$$\psi(e) = \begin{cases} 0 & \text{when } \ e \in A, \\ 2 & \text{when } \ e \in B, \\ 1 & \text{otherwise} \end{cases}$$
satisfies $\mu_\psi(0) = \tau(G[A])$, $\mu_\psi(2) = t_2$ and $\mu_\psi(1) = t_1$. Thus, ψ is a desired 3-TEPC labeling of G. □

Lemma 2.6. *Let G be a connected graph of size at least $5\big(|V(G)| - 1\big)$. Then G is a 3-TEPC graph.*

Proof. Since $|E(G)| \geq 5\big(|V(G)| - 1\big)$, G is a graph of order at least 10 and $\tau(G) \geq 55$. As G is a connected graph, there is a spanning tree T of G. Moreover, for G we have
$$\tau(G) = |V(G)| + |E(G)| \geq 3\big(2|V(G)| - 1\big) - 2.$$
Therefore, $\lceil \tau(G)/3 \rceil \geq 2|V(G)| - 1$. Thus, there exists a set $A \subset E(G)$ such that $E(T) \subseteq A$ and $|A| = \lceil \tau(G)/3 \rceil - |V(G)|$. Then $\tau(G[A]) = \lceil \tau(G)/3 \rceil$, every edge of $E(G) - A$ is an AA-edge and
$$|E(G) - A| = \tau(G) - \big(|V(G)| + |A|\big) = \tau(G) - \lceil \tau(G)/3 \rceil \geq 36 > 2.$$
According to Lemma 2.5, G is a 3-TEPC graph. □

A *matching* in a graph is a set of pairwise nonadjacent edges. A *maximum matching* is a matching that contains the largest possible number of edges. The number of edges in a maximum matching of a graph G is denoted by $\alpha(G)$.

Lemma 2.7. *Let G be a connected graph such that $16 \leq \tau(G) \not\equiv 3 \pmod 6$, $\delta(G) = 1$ and $\alpha(G) \geq 2$. Then G is a 3-TEPC graph.*

Proof. As $16 \leq \tau(G) \not\equiv 3 \pmod 6$, G is a graph of order at least 6 and there is an even integer $t_0 \geq 6$ such that $t_0 \in \{\lfloor \tau(G)/3 \rfloor, \lceil \tau(G)/3 \rceil\}$. Moreover, according to Lemma 2.6, it is enough to consider $|E(G)| < 5(|V(G)|-1)$. Then, $\tau(G) < 6|V(G)|-5$ and $t_0 \leq 2|V(G)| - 2$. As t_0 is even, there is a positive integer s such that $t_0 = 2s+2$. Clearly, $2 \leq s \leq |V(G)| - 2$.

Since $\delta(G) = 1$, there is a pendant vertex in G. Suppose that w is a pendant vertex of G such that $\alpha(G-w)$ is the largest possible. If $\alpha(G-w) = 1$, then $G-w$ is a star of order at least 5 and w is adjacent to a pendant vertex of the star. Clearly, for any pendant vertex $x \neq w$ in G, we have $\alpha(G-x) = 2 > \alpha(G-w)$, a contradiction. Thus, $\alpha(H) \geq 2$, for $H := G-w$. So, there are two nonadjacent edges in H. Any minimal connected subgraph of H containing these edges is a path of length at least 3. Let P be a path of length 3 in H such that the distance between w and P (a vertex of P) in the graph G is the smallest possible. If w is adjacent to no vertex of P, then there is a path of length at least 2 between w and P and a continuing path of length at least 2 in P. So, there is a path of length 3 in H such that w is adjacent to a vertex of this path, a contradiction. Therefore, w is adjacent to a vertex of P.

Denote by e_1, e_2, e_3 the edges of P in such a way that e_1 and e_3 are independent edges of P. Clearly, e_1 and e_3 are also independent edges of G. Moreover, there is a spanning tree T of H which contains P. Set $p = |V(G)| - 2$ and denote by $e_4, \ldots, e_p$ the edges of $E(T) - \{e_1, e_2, e_3\}$ in such a way that the subgraph of H induced by $\{e_1, \ldots, e_j\}$ is a connected graph for each $j \in [1, p]$. The edge of G incident with w denote by e_0. Clearly, the subgraph of G induced by $\{e_i : i \in [0, p]\}$ is its spanning tree. Set

$$A = \begin{cases} \{e_1, e_3, e_4, \ldots, e_{s+1}\} & \text{when } s < p, \\ \{e_0, e_1, e_3, e_4, \ldots, e_p\} & \text{when } s = p. \end{cases}$$

The graph which we obtain from $G[A]$ by adding the edge e_2 is a tree. Therefore, $G[A]$ is a forest with two connected components and so $|E(G[A])| = s$, $|V(G[A])| = s+2$, i.e., $\tau(G[A]) = t_0$. Moreover, e_0 is an AP-edge and e_2 is an AA-edge when $s < p$, and every edge of $E(G) - A$ is an AA-edge when $s = p$. According to Lemma 2.5, G is a 3-TEPC graph. □

Lemma 2.8. *Let G be a connected graph such that $25 \leq \tau(G) \not\equiv 0 \pmod 6$ and $\alpha(G) \geq 3$. Then G is a 3-TEPC graph.*

Proof. As $25 \leq \tau(G) \not\equiv 0 \pmod 6$, G is a graph of order at least 7 and there is an odd integer $t_0 \geq 9$ such that $t_0 \in \{\lfloor \tau(G)/3 \rfloor, \lceil \tau(G)/3 \rceil\}$. Moreover, according to Lemma 2.6, it is enough to consider $|E(G)| < 5(|V(G)|-1)$. Then, $\tau(G) < 6|V(G)|-5$ and $t_0 \leq 2|V(G)| - 3$. As t_0 is odd, there is a positive integer s such that $t_0 = 2s+3$. Clearly, $3 \leq s \leq |V(G)| - 3$.

Since $\alpha(G) \geq 3$, there are three pairwise nonadjacent edges in G. Any minimal connected subgraph of G containing these edges is a tree whose each pendant edge is some of these three edges. Therefore, it is either a path of length at least 5 or a tree with precisely three (pairwise nonadjacent) pendant edges. In the both cases there exists a subtree T of size 5 with $\alpha(T) = 3$.

The edges of T denote by e_i, $i \in [1, 5]$, in such a way that $\{e_1, e_3, e_5\}$ is a matching in T (also in G) and subgraphs induced by $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ are connected. Moreover, there is a spanning tree T' of G which contains T. Put $p = |V(G)| - 1$ and denote by $e_6, \dots, e_p$ the edges of $E(T') - E(T)$ in such a way that the subgraph of G induced by $\{e_1, \dots, e_j\}$ is a connected graph for each $j \in [1, p]$. Evidently, the subgraph of G induced by $\{e_i : i \in [1, p]\}$ is its spanning tree. Set $A = \{e_1, e_3, e_5, e_6, \dots e_{s+2}\}$. The graph which we obtain from $G[A]$ by adding the edges e_2 and e_4 is a tree. Therefore, $G[A]$ is a forest with three connected components and so $|E(G[A])| = s$, $|V(G[A])| = s + 3$, i.e., $\tau(G[A]) = t_0$. Moreover, e_2 and e_4 are AA-edges. Thus, according to Lemma 2.5, G is a 3-TEPC graph. □

Lemma 2.9. *Let G be a connected graph containing a cycle of length k. If*

$$\max\{16, 6k - 8\} \leq \tau(G) \not\equiv 3 \pmod 6,$$

then G is a 3-TEPC graph.

Proof. As $16 \leq \tau(G) \not\equiv 3 \pmod 6$, G is a graph of order at least 6 and there is an even integer $t_0 \geq 6$ such that $t_0 \in \{\lfloor \tau(G)/3 \rfloor, \lceil \tau(G)/3 \rceil\}$. Moreover, according to Lemma 2.6, it is enough to consider $|E(G)| < 5(|V(G)| - 1)$. Then, $\tau(G) < 6|V(G)| - 5$ and $t_0 \leq 2|V(G)| - 2$. As t_0 is even, there is a positive integer s such that $t_0 = 2s - 2$. Clearly, $\max\{4, k\} \leq s \leq |V(G)|$.

Suppose that C is an assumed cycle of length k in G. As G is connected and $s \geq k$, there is a connected subgraph of G on s vertices which contains C. Let H be such subgraph with the minimal number of edges. Clearly, H is an unicyclic graph of order (and size) s. Deleting any edge $e \in E(C)$ from H we get a tree $H - \{e\}$ of order s. Evidently, there is an edge $e_1 \in E(C)$ such that $H - \{e_1\}$ is no star. Then there is an edge e_2 in $H - \{e_1\}$ which is not a pendant edge of $H - \{e_1\}$. Now consider the set $A := E(H) - \{e_1, e_2\} \subset E(G)$. Obviously, $G[A] = H - \{e_1, e_2\}$ and so $\tau(G[A]) = 2s - 2 = t_0$. As e_1 and e_2 are AA-edges of G, by Lemma 2.5, G is a 3-TEPC graph. □

Corollary 2.10. *Let G be a connected graph containing a cycle of length k. If $k \geq 6$ and $\tau(G) \geq 6k - 11$, then G is a 3-TEPC graph.*

Proof. Since G contains a cycle of length at least 6, $\alpha(G) \geq 3$. Moreover, $\tau(G) \geq 6k - 11 \geq 25$ and by Lemma 2.8, G is a 3-total edge product cordial graph for $\tau(G) \not\equiv 0 \pmod 6$.

Now suppose that $\tau(G) \equiv 0 \pmod 6$. Then $\tau(G) \geq 6k - 6 > 16$ and according to Lemma 2.9, G is a 3-TEPC graph. □

Lemma 2.11. *Let G be a connected graph containing a path of length 7. If $\tau(G) > 30$, then G is a 3-TEPC graph.*

Proof. As G contains a path of length 7, $\alpha(G) \geq 4$. Moreover, $\tau(G) > 30$ and by Lemma 2.8, G is a 3-TEPC graph for $\tau(G) \not\equiv 0 \pmod 6$.

Now suppose that $\tau(G) \equiv 0 \pmod 6$. According to Lemma 2.6, it is enough to consider $|E(G)| < 5\big(|V(G)| - 1\big)$. Then, $36 \leq \tau(G) \leq 6|V(G)| - 6$. As $\tau(G)/3$ is even, there is an integer s such that $\tau(G)/3 = 2s + 4$. Clearly, $4 \leq s \leq |V(G)| - 3 = p - 2$, where $p = |V(G)| - 1$.

Let P be an assumed path of length 7 in G. Denote by $e_1, e_2, \dots, e_7$ the edges of P in such a way that e_i and e_{i+1} are adjacent edges for each $i \in [1, 6]$. Moreover, there is a spanning tree T of G which contains P. If $p > 7$, then denote by $e_8, \dots, e_p$ the edges of $E(T) - E(P)$ in such a way that the subgraph of G induced by $\{e_1, \dots, e_j\}$ is a connected graph (tree) for each $j \in [1, p]$. Set

$$A = \begin{cases} \big\{e_i : i \in [1, s+3] - \{2, 4, 6\}\big\} & \text{when } \ s \leq p - 3, \\ \big\{e_i : i \in [1, p] - \{2\}\big\} & \text{when } \ s = p - 2. \end{cases}$$

If $s \leq p-3$, then $G[A]$ is a forest with four connected components and so $|E(G[A])| = s$, $|V(G[A])| = s + 4$, i.e., $\tau(G[A]) = \tau(G)/3$. Moreover, e_2 and e_4 are AA-edges and so, according to Lemma 2.5, G is a 3-TEPC graph. Similarly, if $s = p - 2$, then $G[A]$ is a forest with two connected components and so $|E(G[A])| = p-1$, $|V(G[A])| = p+1$, i.e., $\tau(G[A]) = 2p = \tau(G)/3$. As $V(G[A]) = V(G)$, every edge of $E(G) - A$ is an AA-edge. Therefore, by Lemma 2.5, G is a 3-TEPC graph. □

3. MAIN RESULTS

Theorem 3.1. *Let T be a tree of order at least 12. Then T is a 3-TEPC graph if and only if $T \neq K_{1,n}$ for $n \equiv 1 \pmod 3$.*

Proof. According to Proposition 1.1, it is enough to prove that T is a 3-TEPC graph when $\alpha(T) > 1$.

As $\delta(T) = 1$, $\alpha(T) \geq 2$ and $\tau(T) = 2|V(T)| - 1 \geq 23$, by Lemma 2.7, T is a 3-TEPC graph when $\tau(T) \not\equiv 3 \pmod 6$.

Suppose now that $\tau(T) \equiv 3 \pmod 6$. Thus, $14 \leq |V(T)| \equiv 2 \pmod 3$ and $\tau(T) \geq 27$. If $\alpha(T) \geq 3$ then, according to Lemma 2.8, T is a 3-TEPC graph. If $\alpha(T) = 2$ then, by Kőnig theorem [4], there are vertices u_0 and v_0 such that every edge of T is incident with at least one of this vertices. Therefore, there are two edge-disjoint stars S_u and S_v (subgraphs of T) such that $E(T) = E(S_u) \cup E(S_v)$. Let

$$\begin{aligned} V(S_u) &= \{u_i : i \in [0, r]\}, & E(S_u) &= \{u_0 u_j : j \in [1, r]\}, \\ V(S_v) &= \{v_i : i \in [0, s]\}, & E(S_v) &= \{v_0 v_j : j \in [1, s]\}, \end{aligned}$$

where $2 \leq s \leq r$ and either $v_1 = u_0$ (when $u_0 v_0 \in E(T)$) or $v_1 = u_1$ (when $u_0 v_0 \notin E(T)$). Clearly, $r + s \equiv 1 \pmod 3$ in this case. Thus, there is a positive integer t such that $r + s = 3t + 1$. Evidently, $r > t$. Let q be the largest even integer satisfying

$q \leq \min\{s, t+1\}$. Clearly, $q \geq 2$. Now consider the mapping φ from $E(T)$ to $[0,2]$ given by

$$\varphi(e) = \begin{cases} 0 & \text{when } \ e = u_0u_i,\ i \in [1,t], \\ 2 & \text{when } \ e = v_0v_i,\ i \in [1,q], \\ 2 & \text{when } \ e = u_0u_i,\ i \in [1+t, 1+2t-q], \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to see that for any $w \in V(T)$ we have

$$\varphi^*(w) = \begin{cases} 0 & \text{when } \ w = u_i,\ i \in [0,t], \\ 2 & \text{when } \ w = v_i,\ i \in [2,q], \\ 2 & \text{when } \ w = u_i,\ i \in [1+t, 1+2t-q], \\ 1 & \text{otherwise.} \end{cases}$$

Thus, $\mu_\varphi(i) = 2t+1$ for each $i \in [0,2]$, i.e., φ is a 3-TEPC labeling of T. □

Theorem 3.2. *Let G be an unicyclic graph of order at least 8. Then G is a 3-TEPC graph.*

Proof. According to Proposition 1.2, it is enough to consider that G is not a cycle, i.e., $\delta(G) = 1$. Moreover, $\alpha(G) \geq 2$ and $\tau(G) = 2|V(G)| \geq 16$ in this case. Therefore, by Lemma 2.7, G is a 3-TEPC graph. □

Theorem 3.3. *Let G be a connected graph of order at least 15. Then G is a 3-TEPC graph if and only if $G \neq K_{1,n}$ for $n \equiv 1 \pmod 3$.*

Proof. According to Theorem 3.1 and Theorem 3.2, it is enough to prove that G is a 3-TEPC graph when $|E(G)| > |V(G)|$. By Lemma 2.6, it is sufficient to consider $|V(G)| < |E(G)| < 5\big(|V(G)| - 1\big)$.

As $|E(G)| > |V(G)|$, $\tau(G) \geq 15 + 16 = 31$ and there are at least two distinct cycles in G. The length of a longest cycle in G denote by ℓ. Consider the following cases.

Case A. $\ell \geq 8$. In this case, G contains a path of length 7. Therefore, by Lemma 2.11, G is a 3-TEPC graph.

Case B. $6 \leq \ell \leq 7$. According to Corollary 2.10, G is a 3-TEPC graph.

Case C. $\ell = 5$. The edges of a cycle of length 5 together with an edge which is not a chord of this cycle contain a 3-matching. Thus, $\alpha(G) \geq 3$ in this case. Therefore, by Lemma 2.9 (when $\tau(G) \not\equiv 3 \pmod 6$) or by Lemma 2.8 (when $\tau(G) \equiv 3 \pmod 6$), G is a 3-TEPC graph.

Case D. $\ell \leq 4$. According to Lemma 2.9, G is a 3-TEPC graph whenever $\tau(G) \not\equiv 3 \pmod 6$. Thus, next suppose that $\tau(G) \equiv 3 \pmod 6$. Then there is an integer t such that $\tau(G) = 6t + 3$. As $|V(G)| < |E(G)| < 5\big(|V(G)| - 1\big)$, $30 \leq 2|V(G)| < \tau(G) < 6|V(G)| - 5$ and consequently $5 \leq t < |V(G)| - 1$.

By Lemma 2.8, G is a 3-TEPC graph when $\alpha(G) \geq 3$. So, it remains to consider that $\alpha(G) = 2$.

Let C and C' be two distinct cycles in G. If C and C' are vertex disjoint, then for any edge of a path joining C and C' there are two edges (the first from C and the second from C') such that they altogether form a 3-matching, a contradiction to $\alpha(G) = 2$. So, $V(C) \cap V(C') \neq \emptyset$. Moreover, if both cycles have length 4, then at least one end vertex of any edge of C' belongs to $V(C)$. Therefore, the subgraph of G induced by $E(C) \cup E(C')$ is a connected graph of order at most 6 with at least two distinct cycles. Then there is a connected subgraph H of G such that $|V(H)| = 6$ and $|E(H)| = 7$. Let T_H be a spanning tree of H. Then there are two distinct edges a_1 and a_2 of H such that $E(H) = E(T_H) \cup \{a_1, a_2\}$. As G is connected, there is a spanning tree T of G which contains T_H. Denote by $e_1, e_2, \dots, e_p$ ($p = |V(G)| - 1$) the edges of T in such a way that $e_i \in E(T_H)$ for each $i \in [1, 5]$ and the subgraph of G induced by $\{e_1, \dots, e_j\}$ is a connected graph (tree) for each $j \in [1, p]$. Set

$$A = \big\{e_i : i \in [1, t]\big\}.$$

Then $G[A]$ is a tree and so $|E(G[A])| = t$, $|V(G[A])| = t + 1$. Therefore, $\tau(G[A]) = 2t + 1 = \tau(G)/3$. Moreover, a_1 and a_2 are AA-edges and so, according to Lemma 2.5, G is a 3-TEPC graph. □

We believe that the following conjecture is true.

Conjecture 3.4. *Let G be a connected graph of order at least 4. Then G is a 3-TEPC graph if and only if*

$$\tau(G) \neq 12 \quad \textit{and} \quad G \neq K_{1,n} \ \textit{ for } \ n \equiv 1 \pmod 3.$$

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