

ON NONEXISTENCE OF GLOBAL IN TIME SOLUTION
FOR A MIXED PROBLEM
FOR A NONLINEAR EVOLUTION EQUATION
WITH MEMORY
GENERALIZING THE VOIGT–KELVIN
RHEOLOGICAL MODEL

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Abstract. The paper deals with investigating of the first mixed problem for a fifth-order nonlinear evolutionary equation which generalizes well known equation of the vibrations theory. We obtain sufficient conditions of nonexistence of a global solution in time variable.

Keywords: boundary value problem, beam vibrations, nonlinear evolution equation, Voigt-Kelvin model, memory, blowup.

Mathematics Subject Classification: 35G20, 35G31.

1. INTRODUCTION

We study the nonlinear evolution fifth-order equation with second-order temporal derivative which is a multidimensional nonlinear generalization of the well known one-dimensional linear equation of beam vibrations in the Timoshenko model [7]. Equations of such a type describe propagation of perturbations in a viscoelastic material under action of external ultrasonic aerodynamical forces [8]. Investigation of mixed problems for these equations and systems can be explained by the worn-out contact surfaces [7]. In paper [7] there is investigated the existence of weak solutions

for the mixed problems in the bounded domain D for a system of two linear evolution equations with partial one- and second-order temporal derivatives, where one of unknown functions describes a vertical displacement of a beam.

General mathematical models of contact dynamics for the elastic structures, described by such equations and systems, have been studied recently in many papers [2, 13, 22]. In paper [22] there was formulated a mathematical problem for dynamical viscoelastic friction with worn-out. Dynamical contact between the beam and movable surface was investigated in [2], thermoelastic contact was analyzed in [13].

A general equation that has finite speed of propagation compatible with Einstein's theory of special relativity is investigated in the paper [5]. Both stationary and evolutionary problems are considered.

Boundary value problems for the differential equations of such a type with odd order partial derivatives were also a topic of modern research [1, 3, 4, 6, 9, 11, 14, 17–19, 24]. The mixed problem for a strongly nonlinear equation of beam vibrations in a bounded domain was in detail studied in [17]. The case of a weakly nonlinear equation in an unbounded space domain was, in particular, considered in [18, 19]. The question of existence of the unique generalized solution to the mixed problem for a strongly nonlinear beam vibrations type equation in the domain $\Omega \times [0, +\infty)$ (Ω is a bounded domain) and a behavior of this solution as $t \rightarrow \infty$ were analyzed in [4]. The mixed problem for the nonlinear third-order equation was also investigated in the same domain in [6]. The existence of a unique classical solution, stable under perturbations of the initial data, was there proved, as well as the behavior of this solution as $t \rightarrow \infty$ was described. The conditions for existence of local and global solutions to the mixed problem in Sobolev spaces were formulated in [1]. The case, where the degree of nonlinearity in the main part is a function of space variables was studied in [3].

The phenomena of nonexistence of solutions global in time (also known as blowup) was considered in [14, 24], in particular, for the hyperbolic fourth-order equation it was studied in [11]. In [9] the sufficient conditions for existence of local and nonexistence of global in time solutions to a mixed problem for a hyperbolic third-order equation with the integral term were discussed. This integral term simulates the well-known phenomena of “memory” in oscillation processes. The description of mathematical model of propagating longitudinal waves in the inhomogenous rod one can consult [23]. The mixed problem for some nonlinear fifth-order equation similar to the previous view without integral term was proposed in [21].

Important questions of existence and stability as $t \rightarrow +\infty$ of solutions to nonlinear Hamilton-Jacobi equation in suitable functional spaces were studied in [15, 16], where there were devised effective tools for investigating nonlinear evolution problems based on the fixed point approach stemming from [10]. A related general method for studying the solution existence, based on the Leray-Schauder fixed point approach within the Calogero type projection-algebraic scheme of discrete approximations, was suggested for linear and nonlinear differential-operator equations in Banach spaces in [12].

The main aim of our paper is to establish sufficient conditions for the nonexistence of global solution to a mixed problem for some fifth-order partial differential equation

with a fourth order spatial derivative. As a main tool, the method of estimating the energy functional for the mechanical oscillation system will be used.

2. PROBLEM STATEMENT. EXISTENCE OF LOCAL SOLUTION

Let $T > 0$ be an arbitrary number, $\Omega \subset R^n$ ($n \geq 1$) be a bounded domain with the smooth bound $\partial\Omega$ of class C^1 . Denote $Q_\tau = \Omega \times (0, \tau)$, $S_\tau = \partial\Omega \times (0, \tau)$, $\Omega_\tau = \{(x, t) : x \in \Omega, t = \tau\}$, $\tau \in [0, T]$.

We will consider the following nonlinear evolution fifth-order equation in the domain Q_T :

$$\begin{aligned}
 & u_{tt} + \sum_{|\alpha|=|\beta|=2} D^\beta (a_{\alpha\beta}(x)D^\alpha u_t) + \sum_{|\alpha|=|\beta|=2} D^\beta (b_{\alpha\beta}(x)D^\alpha u) \\
 & + \sum_{|\alpha|=2} D^\alpha (b_\alpha(x)|D^\alpha u|^{q-2}D^\alpha u) \\
 & - \int_0^t g(t-\theta) \sum_{|\alpha|=2} D^\alpha (d_\alpha(x)D^\alpha u(x, \theta)) d\theta = c(x)|u|^{p-2}u,
 \end{aligned} \tag{2.1}$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in N \cup \{0\}$, $i = 1, \dots, n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, with initial conditions

$$u|_{t=0} = u_0(x), \tag{2.2}$$

$$u_t|_{t=0} = u_1(x) \tag{2.3}$$

and boundary conditions

$$u|_S = 0, \quad \frac{\partial u}{\partial \nu} \Big|_S = 0, \tag{2.4}$$

ν is the external normal unit vector of the surface $\partial\Omega$.

Problem (2.1)–(2.4) is multidimensional generalization of rheological nonlinear Voigt-Kelvin model. An influence of the internal friction as a result of waves dispersion on the accidental inhomogeneous material is investigated in this model [23].

Assume the next conditions are satisfied.

(A) $a_{\alpha\beta} \in L^\infty(\Omega)$, $|\alpha| = |\beta| = 2$,

$$\sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq A_2 \sum_{|\alpha|=2} |\xi_\alpha|^2, \quad A_2 > 0,$$

for arbitrary real numbers ξ_α , $|\alpha| = 2$, and almost all $x \in \Omega$.

(B) $b_{\alpha\beta} \in L^\infty(\Omega)$, $|\alpha| = |\beta| = 2$,

$$\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)\xi_\alpha\xi_\beta \geq B_2 \sum_{|\alpha|=2} |\xi_\alpha|^2, \quad B_2 > 0,$$

for arbitrary real numbers ξ_α , $|\alpha| = 2$, and for almost all $x \in \Omega$; $b_{\alpha\beta}(x) = b_{\beta\alpha}(x)$ for almost all $x \in \Omega$.

- (B1) $b_\alpha \in L^\infty(\Omega)$, $b_\alpha(x) \geq b_2 > 0$ for almost all $x \in \Omega$ and for all α , $|\alpha| = 2$.
- (C) $c \in L^\infty(\Omega)$, $c(x) \geq c_0 > 0$ for almost all $x \in \Omega$.
- (D) $d_\alpha \in L^\infty(\Omega)$, $d_\alpha(x) \geq d_2 \geq 0$ for almost all $x \in \Omega$ and for all α , $|\alpha| = 2$,
 $d_3 = \text{ess sup}_{\Omega} \sum_{|\alpha|=2} d_\alpha^2(x)$.
- (G) $g(t) \geq 0$, $g'(t) \leq 0$ for all $t \in [0, +\infty)$,

$$G(t) := \int_0^t g(\tau) d\tau \geq 0, \quad G(+\infty) = G < \frac{2p-4}{2p-3} B_2, \quad l(t) := B_2 - G(t)d_3 > 0,$$

$$l(+\infty) = B_2 - Gd_3 \equiv l > 0.$$

- (PQ) $p > q > 2$.
- (U) $u_0, u_1 \in W_0^{2,q}(\Omega)$.

Definition 2.1. Function $u : \Omega \times [0, T) \rightarrow \mathbb{R}$ (T is a positive number or $+\infty$) such that

$$u \in C([0, T_0]; W_0^{2,q}(\Omega)) \cap L^p((0, T_0); L^p(\Omega)), \quad u_t \in C([0, T_0]; W_0^{2,q}(\Omega)),$$

$$u_{tt} \in L^\infty((0, T_0); L^2(\Omega))$$

for arbitrary number T_0 from $(0, T)$ denote generalized solution of the problem (2.1)–(2.4) in the domain Q_T , if it satisfies the initial conditions (2.2), (2.3) and an integral identity

$$\begin{aligned} & \int_{\Omega_t} \left[u_{tt}v + \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x) D^\alpha u_t D^\beta v \right. \\ & + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta v + \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^{q-2} D^\alpha u D^\alpha v \\ & \left. - \int_0^t g(t-\theta) \sum_{|\alpha|=2} d_\alpha(x) D^\alpha u(x, \theta) D^\alpha v d\theta - c_0(x) |u|^{p-2} uv \right] dx = 0 \end{aligned} \tag{2.5}$$

for almost all $t \in (0, T)$ and for all $v \in W_0^{2,q}(\Omega) \cap L^p(\Omega)$. In case $T = +\infty$, solution is called global.

Remark 2.2. If $T < +\infty$, then solution is called local. Under some conditions on the coefficients, right part of equation and initial data it is possible to find a finite time moment T (depending on the coefficients, right part of equation and initial data), such that the local solution u of the problem (2.1)–(2.4) exists in the domain Q_T . Sufficient conditions of local in time solution existence of the previous problem are proved via Faedo-Galerkin method [10] in [20] (see Theorem 1 therein).

3. THE MAIN RESULT. SUFFICIENT CONDITIONS OF BLOWUP

Further we will use the following notation:

$$\|v\|_r := \|v\|_{L^r(\Omega)} = \left(\int_{\Omega} |v|^r dx \right)^{1/r}, \quad r > 1; \quad \|D^2 v\|_2 := \left(\int_{\Omega} \sum_{|\alpha|=2} |D^\alpha v|^2 dx \right)^{1/2}.$$

Taking into account $p \leq \frac{2n}{n-4}$, $n > 4$, via Sobolev imbedding theorem the next is true $H^2(\Omega) \subset L^p(\Omega)$, i.e.

$$\|u\|_p \leq B \|D^2 u\|_2, \quad B > 0.$$

Denote

$$B_1 = Bl^{-1/2}, \quad C = \text{ess sup}_{\Omega} c(x), \quad A = \text{ess sup}_{\Omega} \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}^2(x).$$

Let us consider the functional (energy functional) of the form

$$\begin{aligned} E(t) &= \frac{1}{2} \int_{\Omega_t} \left[u_t^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \right] dx \\ &+ \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx - \frac{1}{p} \int_{\Omega_t} c(x) |u|^p dx \\ &+ \frac{1}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\ &- \frac{1}{2} G(t) \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, t)|^2 dx, \quad t \in [0, T]. \end{aligned} \tag{3.1}$$

Let us denote

$$\begin{aligned} E_0 &:= \frac{1}{2} \int_{\Omega} \left[u_1^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u_0 D^\beta u_0 \right] dx \\ &+ \frac{1}{q} \int_{\Omega} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u_0|^q dx - \frac{1}{p} \int_{\Omega} c(x) |u_0|^p dx, \\ E_1 &:= \frac{p-2}{2p} C^{-\frac{2}{p-2}} B_1^{-\frac{2p}{p-2}}. \end{aligned}$$

Theorem 3.1. *Suppose that the conditions indicated above are satisfied and, furthermore, $p \leq \frac{2n}{n-4}$ if $n > 4$; $2 < q < \frac{p+2}{2}$; $E_0 < E_1$, $\|D^2 u_0\|_2 > \sqrt{\frac{E_1}{B_2} \cdot \frac{2p}{p-2}}$. Then a global solution of problem (2.1)–(2.4) does not exist.*

4. THE MAIN RESULT PROOF

Assume the contrary, i.e., assume that global solutions of problem (2.1)–(2.4) exist. It follows from the definition of generalized solution that the function $E(t)$, $t \in [0, +\infty)$, is continuous and its restriction to an arbitrary segment $[0, \tau)$, $\tau > 0$, is an absolutely continuous function. Moreover, it is obvious that $E(0) = E_0$. Let function u is the global solution of the problem (2.1)–(2.4). Then it satisfies (2.5). Considering $v = u_t$ in (2.5), one can obtain identity

$$\begin{aligned} & \frac{d}{dt} \left[\frac{1}{2} \int_{\Omega_t} \left[u_t^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \right] dx + \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \right] \\ & - \frac{d}{dt} \left[\frac{1}{p} \int_{\Omega_t} c(x) |u|^p dx \right] + \int_{\Omega_t} \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x) D^\alpha u_t D^\beta u_t dx \\ & - \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) D^\alpha u(x, \theta) D^\alpha u_t(x, t) dx d\theta = 0 \end{aligned} \tag{4.1}$$

for almost all $t \in [0, +\infty)$. We will transform integral

$$\begin{aligned} & \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) D^\alpha u(x, \theta) D^\alpha u_t(x, t) dx d\theta \\ & = \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) D^\alpha u_t(x, t) [D^\alpha u(x, \theta) - D^\alpha u(x, t)] dx d\theta \\ & + \int_0^t g(t-\theta) dt \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) D^\alpha u(x, t) D^\alpha u_t(x, t) dx \\ & = -\frac{1}{2} \int_0^t g(t-\theta) \frac{d}{dt} \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\ & + \frac{1}{2} \int_0^t g(\theta) d\theta \frac{d}{dt} \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, t)|^2 dx \\ & = -\frac{1}{2} \frac{d}{dt} \left[\int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right] \\ & + \frac{1}{2} \int_0^t g'(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\ & + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(\theta) d\theta \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, t)|^2 dx \right] - \frac{1}{2} g(t) \int_{\Omega_t} \sum_{i,j=1}^n d_\alpha(x) |D^\alpha u(x, t)|^2 dx. \end{aligned}$$

Based on the last equality it follows from (4.1) that

$$\begin{aligned}
 & \frac{d}{dt} \left[\frac{1}{2} \int_{\Omega_t} \left[u_t^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \right] dx + \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \right] \\
 & - \frac{d}{dt} \left[\frac{1}{p} \int_{\Omega_t} c(x) |u|^p dx \right] \\
 & = - \int_{\Omega_t} \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x) D^\alpha u_t D^\beta u_t dx \\
 & - \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right] \\
 & + \frac{1}{2} \int_0^t g'(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\
 & + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(\theta) d\theta \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, t)|^2 dx \right] \\
 & - \frac{g(t)}{2} \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, t)|^2 dx \\
 & \leq - \int_{\Omega_t} \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x) D^\alpha u_t D^\beta u_t dx \\
 & - \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right] \\
 & + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(\theta) d\theta \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, t)|^2 dx \right].
 \end{aligned}$$

One can obtain

$$\begin{aligned}
 E'(t) &= \frac{d}{dt} \left[\frac{1}{2} \int_{\Omega_t} \left[u_t^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \right] dx + \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \right. \\
 &\quad \left. - \frac{1}{p} \int_{\Omega_t} c(x) |u|^p dx \right] \\
 &\quad + \frac{1}{2} \frac{d}{dt} \left[\int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right. \\
 &\quad \left. - \int_0^t g(\theta) d\theta \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, t)|^2 dx \right] \\
 &\leq - \int_{\Omega_t} \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x) D^\alpha u_t D^\beta u_t dx < 0
 \end{aligned}$$

for almost all $t \in [0, +\infty)$. From (3.1) it follows the conclusion

$$\begin{aligned}
 E(t) &\geq \frac{1}{2} l(t) \|D^2 u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \\
 &\quad - \frac{1}{p} \int_{\Omega_t} c(x) |u|^p dx + \frac{1}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\
 &\geq \frac{1}{2} l(t) \|D^2 u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \\
 &\quad + \frac{1}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta - \frac{C}{p} B_1^p l^{\frac{p}{2}} \|D^2 u\|_2^p \\
 &\geq \frac{1}{2} l(t) \|D^2 u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \\
 &\quad + \frac{1}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\
 &\quad - \frac{CB_1^p}{p} \left[l(t) \|D^2 u\|_2^2 + \frac{2}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \right. \\
 &\quad \left. + \|u_t\|_2^2 + \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right]^{\frac{p}{2}}.
 \end{aligned} \tag{4.2}$$

Therefore,

$$E(t) \geq h(\xi(t)),$$

where $h(y) = \frac{1}{2}y^2 - \frac{CB_1^p}{p}y^p$ and

$$\begin{aligned} \xi(t) = & \left[l(t)\|D^2u\|_2^2 + \|u_t\|_2^2 + \frac{2}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q dx \right. \\ & \left. + \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x)|D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 dx d\theta \right]^{\frac{1}{2}}. \end{aligned}$$

Obviously, $y_0 = C^{-\frac{1}{p-2}} B_1^{-\frac{p}{p-2}}$ is the maximum of the function h , because of $h'(y) = y - CB_1^p y^{p-1}$. Accordingly,

$$h(y_0) = \left(\frac{1}{2} - \frac{1}{p}\right) C^{-\frac{2}{p-2}} B_1^{-\frac{2p}{p-2}} = E_1 > E_0.$$

So there exists such $\beta > y_0$, that $h(\beta) = E_0$.

If

$$\beta_0 = \left[\int_{\Omega} \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u_0 D^\beta u_0 dx \right]^{\frac{1}{2}},$$

then

$$\begin{aligned} h(\beta_0) & \leq \frac{1}{2} \int_{\Omega} \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u_0 D^\beta u_0 dx - \frac{CB_1^p B_2^{\frac{p}{2}}}{p} \left[\left(\int_{\Omega} \sum_{|\alpha|=2} |D^\alpha u_0|^2 dx \right)^{\frac{1}{2}} \right]^p \\ & \leq \frac{1}{2} \int_{\Omega} \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u_0 D^\beta u_0 dx - \frac{CB_1^p B_2^{\frac{p}{2}}}{pB^p} \|u_0\|_p^p \\ & \leq \frac{1}{2} \int_{\Omega} \left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u_0 D^\beta u_0 + |u_1|^2 \right] dx + \frac{1}{q} \int_{\Omega} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u_0|^q dx \\ & \quad - \frac{1}{p} \int_{\Omega_t} c(x) |u_0|^p dx + \frac{1}{p} \int_{\Omega_t} c(x) |u_0|^p dx - \frac{CB_1^p B_2^{\frac{p}{2}}}{pB^p} \|u_0\|_p^p. \end{aligned}$$

Since $B_2 > l$, then $h(\beta_0) \leq E_0 = h(\beta)$. Obviously, $\beta_0 \geq \sqrt{B_2} \|D^2 u_0\|_2$. According to the theorem

$$\|D^2 u_0\|_2 > \sqrt{\frac{E_1}{B_2} \cdot \frac{2p}{p-2}},$$

consequently

$$\beta_0 > \sqrt{E_1 \cdot \frac{2p}{p-2}} \geq \sqrt{C^{-\frac{2}{p-2}} B_1^{-\frac{2p}{p-2}}} = y_0.$$

As function $h(y)$ is monotonically decreasing while $\beta > y_0$, taking into account the last estimations $h(\beta_0) \leq h(\beta)$, $\beta_0 > y_0$, one can get $\beta_0 > \beta > y_0$.

Further we will assume existence of $t_0 \in [0, +\infty)$, such that $\xi(t_0) < \beta$. Since ξ is continuous function, t_0 can be choosed as following $\xi(t_0) > y_0$, then $y_0 < \beta < \beta_0$. Hence, $E(t_0) \geq h(y_0) > h(\xi(t_0)) > h(\beta) = E_0$. That is impossible, because $E(t) \leq E_0$ for all $t \in [0, T)$ by the reason of strongly monotonically decreasing function $E(t)$.

So it is proved, that in case

$$E_0 < E_1, \quad \|D^2 u_0\|_2 > \sqrt{\frac{E_1}{B_2} \cdot \frac{2p}{p-2}}$$

exists $\beta \geq y_0$, such that

$$\begin{aligned} \xi(t) = & \left[l(t) \|D^2 u\|_2^2 + \|u_t\|_2^2 + \frac{2}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \right. \\ & \left. + \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right]^{\frac{1}{2}} \geq \beta, \quad t \in [0, +\infty). \end{aligned} \tag{4.3}$$

Moreover, since $E(t) < E_0$ on $(0, +\infty)$, based on (4.2) we obtain

$$\begin{aligned} & \frac{1}{2} l(t) \|D^2 u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \\ & + \frac{1}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta < E_0 + \frac{C}{p} \int_{\Omega_t} |u|^p dx. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4) it follows that

$$\frac{C}{p} \|u\|_p^p \geq \frac{1}{2} \beta^2 - E_0 \geq \frac{1}{2} \beta^2 - h(\beta) = \frac{B_1^p C}{p} \beta^p \quad \text{or} \quad \|u\|_p \geq B_1 \beta.$$

If $\|u\|_p \leq 1$, then $\|u\|_p^s \leq \|u\|_p^2 \leq B \|D^2 u\|_2^2$ as $2 \leq s \leq p$. If $\|u\|_p > 1$, then $\|u\|_p^s \leq \|u\|_p^p$ as $2 \leq s \leq p$. So there is an obvious estimation

$$\|u\|_p^s \leq \kappa_1 (\|D^2 u\|_2^2 + \|u\|_p^p), \quad \kappa_1 = \max\{B, 1\}, \quad s \in [2, p]. \tag{4.5}$$

Let $H(t) = E_1 - E(t)$. Using (4.2) one can get

$$\begin{aligned} \frac{1}{2} \left(B_2 - l \right) \|D^2 u\|_2^2 &\leq \frac{1}{2} l(t) \|D^2 u\|_2^2 \\ &\leq E(t) - \frac{1}{2} \|u_t\|_2^2 - \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \\ &\quad - \frac{1}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\ &\quad + \frac{1}{p} \int_{\Omega_t} c(x) |u|^p dx. \end{aligned} \tag{4.6}$$

Additionally

$$\begin{aligned} \|u\|_p^p &\geq B_1^p \beta^p > B_1^p y_0^p = B_1^p B_1^{-\frac{p^2}{p-2}} C^{-\frac{p}{p-2}} \\ &= B_1^{-\frac{2p}{p-2}} C^{-\frac{p}{p-2}} = B_1^{-\frac{2p}{p-2}} C^{-\frac{2}{p-2}} C^{-1} = \frac{2p}{(p-2)C} E_1, \end{aligned}$$

therefore $E_1 \leq \frac{(p-2)C}{2p} \|u\|_p^p$ and taking into consideration (4.6),

$$\begin{aligned} \frac{1}{2} \left(B_2 - l \right) \|D^2 u\|_2^2 &\leq -E_1 + E_1 + E(t) - \frac{1}{2} \|u_t\|_2^2 - \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \\ &\quad - \frac{1}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\ &\quad + \frac{1}{p} \int_{\Omega_t} c(x) |u|^p dx \\ &\leq -H(t) - \frac{1}{2} \|u_t\|_2^2 - \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx - \frac{1}{2} \int_0^t g(t-\theta) \\ &\quad \times \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta + \left(\frac{1}{p} + \frac{p-2}{2p} \right) C \|u\|_p^p \\ &\leq \frac{1}{q} \left[-H(t) - \|u_t\|_2^2 - \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \right. \\ &\quad \left. - \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta + \frac{q}{2} \|u\|_p^p \right]. \end{aligned} \tag{4.7}$$

From inequalities (4.5) and (4.7) it follows that

$$\begin{aligned} \|u\|_p^s \leq & \kappa_2 \left[-H(t) - \|u_t\|_2^2 - \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \right. \\ & - \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\ & \left. + \frac{q}{2} \|u\|_p^p \right], \quad \kappa_2 > 0, s \in [2, p]. \end{aligned}$$

Since $H'(t) > 0$ almost everywhere on $[0, +\infty)$, then $H(t) \geq H(0) = E_1 - E_0 > 0$. From (4.4) additionally obtain the following

$$\begin{aligned} H(t) & \leq E_1 - \frac{1}{2} \left[l(t) \|D^2 u\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{q} \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q dx \right. \\ & \quad \left. + \frac{1}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right] + \frac{C}{p} \|u\|_p^p \\ & \leq E_1 - \frac{1}{2} \beta^2 + \frac{C}{p} \|u\|_p^p \\ & \leq E_1 - \frac{1}{2} \xi_0^2 + \frac{C}{p} \|u\|_p^p \\ & \leq \frac{p-2}{2p} \xi_0^2 - \frac{1}{2} \xi_0^2 + \frac{C}{p} \|u\|_p^p \\ & \leq \frac{C}{p} \|u\|_p^p. \end{aligned}$$

Accordingly,

$$0 < H(0) \leq H(t) \leq \frac{C}{p} \|u\|_p^p, \quad t \in [0, T]. \quad (4.8)$$

Hereafter, let

$$L(t) := H^{1-\alpha}(t) + \varepsilon \int_{\Omega_t} u u_t dx,$$

where $\varepsilon > 0$ and $\alpha \in (0, 1)$ are arbitrary numbers. Then

$$\begin{aligned}
 L'(t) &= (1 - \alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_{\Omega_t} [u_t^2 + uu_{tt}] dx = (1 - \alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_{\Omega_t} u_t^2 dx \\
 &+ \varepsilon \int_{\Omega_t} \left[c(x)|u|^p - \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x)D^\alpha u_t D^\beta u - \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u \right] dx \\
 &- \varepsilon \int_{\Omega_t} \left[\sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \right] dx \\
 &+ \varepsilon \int_0^t g(t - \theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x)D^\alpha u(x, \theta)D^\alpha u(x, t) dx d\theta \\
 &= (1 - \alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_{\Omega_t} u_t^2 dx + \varepsilon \int_{\Omega_t} \left[c(x)|u|^p \right. \\
 &- \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta}(x)D^\alpha u_t D^\beta u - \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u - \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \left. \right] dx \\
 &+ \varepsilon \int_0^t g(t - \theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} d_\alpha(x) \left[D^\alpha u(x, \theta) - D^\alpha u(x, t) \right] D^\alpha u(x, t) dx d\theta \\
 &+ \varepsilon \int_0^t g(\theta) d\theta \int_{\Omega_t} \sum_{|\alpha|=2} d_\alpha(x) |D^\alpha u(x, t)|^2 dx \geq (1 - \alpha)H^{-\alpha}(t)H'(t) + \varepsilon \int_{\Omega_t} u_t^2 dx \\
 &+ \varepsilon \int_{\Omega_t} c(x)|u|^p dx - \frac{\varepsilon\delta_1}{2} \int_{\Omega_t} \sum_{|\alpha|=2} |D^\alpha u|^2 dx - \frac{\varepsilon C_1}{2\delta_1} \int_{\Omega_t} \sum_{|\alpha|=2} |D^\alpha u_t|^2 dx \\
 &- \varepsilon \int_{\Omega_t} \left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \right] dx \\
 &- \frac{\delta_2 C_2 \varepsilon}{2} \int_0^t g(t - \theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \\
 &- \frac{\varepsilon C_2}{2\delta_2} \int_0^t g(\theta) d\theta \int_{\Omega_t} \sum_{|\alpha|=2} |D^\alpha u(x, t)|^2 dx + \varepsilon C_3 \int_0^t g(\theta) d\theta \int_{\Omega_t} \sum_{|\alpha|=2} |D^\alpha u(x, t)|^2 dx,
 \end{aligned}$$

where δ_1, δ_2 are arbitrary positive constants, positive constant C_1 depends on A , positive constant C_2 depends on $\text{ess sup}_\Omega \sum_{|\alpha|=2} d_\alpha^2(x)$, positive constant C_3 depends on d_2 .

Therefore,

$$\begin{aligned}
L'(t) &\geq \left[(1-\alpha)H^{-\alpha}(t)A_2 - \frac{\varepsilon C_1}{2\delta_1} \right] \|D^2u_t\|_2^2 + \varepsilon \int_{\Omega_t} u_t^2 dx + \varepsilon \int_{\Omega_t} c(x)|u|^p dx \\
&\quad - \varepsilon \int_{\Omega_t} \left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \right] dx \\
&\quad + \varepsilon \left[\left(1 - \frac{C_2}{2\delta_2}\right) \int_0^t g(\theta) d\theta - \frac{\delta_1}{2} \right] \|D^2u\|_2^2 \\
&\quad - \frac{\varepsilon\delta_2 C_2}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 dx d\theta \\
&\geq \left[(1-\alpha)H^{-\alpha}(t)A_2 - \frac{\varepsilon C_1}{2\delta_1} \right] \|D^2u_t\|_2^2 + \varepsilon \int_{\Omega_t} u_t^2 dx + \varepsilon p \left[-H(t) + H(t) + \frac{1}{p} \int_{\Omega_t} c(x)|u|^p dx \right] \\
&\quad - \varepsilon \int_{\Omega_t} \left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \right] dx + \varepsilon \left[\left(1 - \frac{C_2}{2\delta_2}\right) \right. \\
&\quad \times \left. \int_0^t g(\theta) d\theta - \frac{\delta_1}{2} \right] \|D^2u\|_2^2 - \frac{\varepsilon\delta_2 C_2}{2} \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 dx d\theta \\
&\geq \left[(1-\alpha)H^{-\alpha}(t)A_2 - \frac{\varepsilon C_1}{2\delta_1} \right] \|D^2u_t\|_2^2 + \varepsilon \|u_t\|^2 + \varepsilon p \left[-H(t) \right. \\
&\quad \left. + \frac{\delta_3}{2} \int_{\Omega_t} \left[|u_t|^2 + \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \frac{2}{q} \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \right] dx + \frac{\delta_3 \varepsilon p}{2} \right. \\
&\quad \times \left. \int_0^t g(t-\theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 dx d\theta \right. \\
&\quad \left. - \frac{\delta_3 \varepsilon p}{2} \int_0^t g(\theta) d\theta \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u|^2 dx \right. \\
&\quad \left. + \varepsilon(1-\delta_3) \int_{\Omega_t} c(x)|u|^p dx - \varepsilon \int_{\Omega_t} \left[\sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x)D^\alpha u D^\beta u + \sum_{|\alpha|=2} b_\alpha(x)|D^\alpha u|^q \right] dx \right. \\
&\quad \left. + \varepsilon \left[\left(1 - \frac{C_2}{2\delta_2}\right) \int_0^t g(\theta) d\theta - \frac{\delta_1}{2} \right] \|D^2u\|_2^2 - \frac{\varepsilon\delta_2 C_2}{2} \int_0^t g(t-\theta) \right. \\
&\quad \times \left. \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x,\theta) - D^\alpha u(x,t)|^2 dx d\theta = \left[(1-\alpha)H^{-\alpha}(t)A_2 - \frac{\varepsilon C_1}{2\delta_1} \right] \right. \\
&\quad \left. \times \|D^2u_t\|_2^2 + \varepsilon \left(1 + \frac{\delta_3 p}{2} \right) \|u_t\|^2 dx - \varepsilon p H(t) + \varepsilon \left(\frac{\delta_3 p}{2} - 1 \right) \right]
\end{aligned}$$

$$\begin{aligned}
 & \times \int_{\Omega_t} \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \, dx + \varepsilon \left(\frac{\delta_3 p}{q} - 1 \right) \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \, dx + \varepsilon(1 - \delta_3) \\
 & \times \int_0^t c(x) |u|^p \, dx + \varepsilon \left(\frac{\delta_3 p}{2} - \frac{\delta_2}{2} \right) \int_0^t g(t - \theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 \, dx \, d\theta \\
 & - \varepsilon \left(\left(\frac{\delta_3 p}{2} - 1 + \frac{C_2}{2\delta_2} \right) \int_0^t g(\theta) \, d\theta + \frac{\delta_1}{2} \right) \|D^2 u\|_2^2,
 \end{aligned} \tag{4.9}$$

and $0 < \delta_3 < 1$.

Choosing $\delta_1 = H^\alpha(t)\delta_4$ in (4.9) one can get

$$\begin{aligned}
 L'(t) & \geq \left[(1 - \alpha)A_2 - \frac{\varepsilon C_1}{2\delta_4} \right] H^{-\alpha}(t) \|D^2 u_t\|_2^2 + \varepsilon \left(1 + \frac{\delta_3 p}{2} \right) \|u_t\|^2 - \varepsilon p H(t) + \varepsilon \\
 & \times \left(\frac{\delta_3 p}{2} - 1 \right) \int_{\Omega_t} \sum_{|\alpha|=|\beta|=2} b_{\alpha\beta}(x) D^\alpha u D^\beta u \, dx \\
 & + \varepsilon \left(\frac{\delta_3 p}{q} - 1 \right) \int_{\Omega_t} \sum_{|\alpha|=2} b_\alpha(x) |D^\alpha u|^q \, dx \\
 & + \varepsilon(1 - \delta_3) \int_0^t c(x) |u|^p \, dx + \varepsilon \left(\frac{\delta_3 p}{2} - \frac{\delta_2}{2} \right) \int_0^t g(t - \theta) \\
 & \times \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 \, dx \, d\theta \\
 & - \varepsilon \left(\frac{\delta_3 p}{2} - 1 + \frac{C_2}{2\delta_2} \right) \int_0^t g(\theta) \, d\theta \|D^2 u\|_2^2 - \frac{\varepsilon \delta_4}{2} H^\alpha(t) \|D^2 u\|_2^2.
 \end{aligned} \tag{4.10}$$

Let us set $\alpha = \frac{q-2}{p}$. By (4.8) and the spaces embeddings $L^{2+p\alpha}(\Omega) \subset L^2(\Omega)$, we get

$$\begin{aligned}
 H^\alpha(t) \|D^2 u\|_2^2 & \leq \left(\frac{C}{p} \right)^\alpha \|u\|_q^{p\alpha} \|D^2 u\|_2^2 \leq \left(\frac{C}{p} \right)^\alpha B^{p\alpha} \|D^2 u\|_2^{p\alpha} \|D^2 u\|_2^2 \\
 & \leq \left(\frac{CB^p}{p} \right)^\alpha \|D^2 u\|_2^{p\alpha+2} \leq \left(\frac{CB^p}{p} \right)^\alpha C_3 \|D^2 u\|_q^{p\alpha+2} = C_4 \|D^2 u\|_q^q,
 \end{aligned}$$

$$C_4 = \left(\frac{C}{p} \right)^\alpha B^{p\alpha} C_3, \quad C_3 > 0.$$

Inequality (4.10) can be rewritten as follows:

$$\begin{aligned}
 L'(t) \geq & \left[(1 - \alpha)A_2 - \frac{\varepsilon C_1}{2\delta_4} \right] H^{-\alpha}(t) \|D^2 u_t\|_2^2 + \varepsilon \left(1 + \frac{\delta_3 p}{2} \right) \|u_t\|^2 dx - \varepsilon p H(t) \\
 & + \varepsilon \left[\left(\frac{\delta_3 p}{2} - 1 \right) B_2 - \left(\frac{\delta_3 p}{q} - 1 + \frac{C_2}{2\delta_2} \right) (B_2 - l) \right] \|D^2 u\|_2^2 \\
 & + \varepsilon \left[\left(\frac{\delta_3 p}{2} - 1 \right) b_2 - \frac{C_4 \delta_4}{2} \right] \|D^2 u\|_q^q + \varepsilon (1 - \delta_3) c_0 \|u\|_p^p \\
 & + \varepsilon \left(\frac{\delta_3 p}{2} - \frac{\delta_2}{2} \right) \int_0^t g(t - \theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta.
 \end{aligned} \tag{4.11}$$

Due to the conditions of the theorem we can choose the parameters $\delta_i, i = 2, 3, 4$, so that inequality (4.11) yields

$$\begin{aligned}
 L'(t) \geq & C_5 \left[-H(t) + \|D^2 u\|_2^2 + \|D^2 u\|_q^q + \|u_t\|_2^2 + \|u\|_p^p \right. \\
 & \left. + \int_0^t g(t - \theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right], \quad C_5 > 0.
 \end{aligned} \tag{4.12}$$

The next point under consideration

$$\begin{aligned}
 \left[L(t) \right]^{\frac{1}{1-\alpha}} &= \left[H^\alpha(t) + \varepsilon \int_{\Omega_t} u u_t dx \right]^{\frac{1}{1-\alpha}} \leq C_6 \left[H(t) + \|u_t\|_2^{\frac{1}{1-\alpha}} \|u\|_2^{\frac{1}{1-\alpha}} \right] \\
 &\leq C_7 \left[H(t) + \|u_t\|_2^2 + \|u\|_2^{\frac{2}{1-2\alpha}} \right], \quad C_6 > 0, \quad C_7 > 0.
 \end{aligned} \tag{4.13}$$

Since $q < \frac{p+2}{2}$ and $\alpha = \frac{q-2}{p}$, then $\alpha < \frac{1}{2}$ and $2 \leq \frac{2}{1-2\alpha} \leq p$. Hence

$$\|u\|_2^{\frac{2}{1-2\alpha}} \leq C_8 \left(\|u\|_2^2 + \|u\|_p^p \right), \quad \|u\|_2^2 \leq C_9 \|D^2 u\|_2^2 \leq C_9 \left[-H(t) + C_{10} \|u\|_p^p \right],$$

$C_8 > 0, C_9 > 0, C_{10} > 0$. Thereby from (4.13)

$$\begin{aligned}
 L'(t) \leq & C_{11} \left[-H(t) + \|D^2 u\|_2^2 + \|D^2 u\|_q^q + \|u_t\|_2^2 + \|u\|_p^p \right. \\
 & \left. + \int_0^t g(t - \theta) \int_{\Omega_\theta} \sum_{|\alpha|=2} |D^\alpha u(x, \theta) - D^\alpha u(x, t)|^2 dx d\theta \right], \quad C_{11} > 0.
 \end{aligned} \tag{4.14}$$

Taking into account (4.12), (4.14),

$$L'(t) \geq C_{12} \left[L(t) \right]^{\frac{1}{1-\alpha}}, \quad C_{12} > 0. \tag{4.15}$$

Integrating the both sides of (4.15) by variable τ from 0 to t , obtain the following

$$L(t) \geq \frac{1}{\left[L(0)^{\frac{\alpha}{\alpha-1}} - \frac{C_{12}\alpha}{1-\alpha} t \right]^{\frac{1-\alpha}{\alpha}}}. \quad (4.16)$$

Since

$$L(0) = H^{1-\alpha}(0) + \varepsilon \int_{\Omega_0} u_0(x) u_1(x) dx,$$

then, in virtue of $H(0) > 0$, by choosing sufficiently small $\varepsilon > 0$ it is possible to obtain $L(0) > 0$. Then from (4.16) we deduce the existence of such $T^* > 0$ that

$$\lim_{t \rightarrow T^* - 0} L(t) = +\infty.$$

We arrive at a contradiction with the statement that the function $L(t)$ is continuous on $[0, +\infty)$. So u cannot be a global solution of the problem (2.1)–(2.4) in the domain Q_T . The theorem is proved.

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