Opuscula Math. 38, no. 1 (2018), 117–131
https://doi.org/10.7494/OpMath.2018.38.1.117 Opuscula Mathematica

STUDY OF ODE LIMIT PROBLEMS FOR REACTION-DIFFUSION EQUATIONS

Jacson Simsen, Mariza Stefanello Simsen,
and Aleksandra Zimmermann

Communicated by Bruno Welfert

Abstract. In this work we study ODE limit problems for reaction-diffusion equations for large diffusion and we study the sensitivity of nonlinear ODEs with respect to initial conditions and exponent parameters. Moreover, we prove continuity of the flow and weak upper semicontinuity of a family of global attractors for reaction-diffusion equations with spatially variable exponents when the exponents go to 2 in $L^\infty(\Omega)$ and the diffusion coefficients go to infinity.

Keywords: ODE limit problems, shadow systems, reaction-diffusion equations, parabolic problems, variable exponents, attractors, upper semicontinuity.

Mathematics Subject Classification: 35B40, 35B41, 35K57, 35K59.

1. INTRODUCTION

Reaction-Diffusion systems for which the flow is essentially determined by an ordinary differential equation have been studied by many researchers and they often appear as shadow systems, see for example [2, 4, 5, 8, 12, 14, 15, 22, 23, 26]. It is a well-known fact that many models of chemical, biological and ecological problems involve reaction-diffusion systems. When variable exponents are included these models often appear in applications in electrorheological fluids [9, 10, 19–21] and image processing [6, 11].

In [24–26] the authors investigated in which way the exponent parameter $p(x)$ affects the dynamic of PDEs involving the $p(x)$-Laplacian. In [24, 25] the limit problem was also a PDE and in [26] the limit problem was an ODE.

In this paper we study a problem of the form

$$\begin{cases} \frac{\partial u_s}{\partial t}(t) - \operatorname{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + |u_s|^{p_s(x)-2}u_s = B(u_s(t)), \ \ t > 0, \\ u_s(0) = u_{0s}, \end{cases} \tag{1.1}$$

under homogeneous Neumann boundary conditions, $u_{0s} \in H := L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a smooth bounded domain, $B : H \to H$ is a globally Lipschitz map with Lipschitz constant $L \geq 0$, $D_s \in [1, \infty)$, $p_s(\cdot) \in C(\bar{\Omega})$, $p_s^- := \min_{x\in\bar{\Omega}} p_s(x) > 2$, and there exists a constant $a > 2$ such that $p_s^+ := \max_{x\in\bar{\Omega}} p_s(x) \leq a$, for all $s \in \mathbb{N}$. We assume that $D_s \to \infty$ and $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$ as $s \to \infty$.

The aim of this work is to study the asymptotic behavior as $s \to \infty$. We prove continuity of the flows and weak upper semicontinuity of the family of global attractors $\{\mathcal{A}_s\}_{s\in\mathbb{N}}$ as s goes to infinity for the problem (1.1) with respect to the couple of parameters (D_s, p_s), where p_s is the variable exponent and D_s is the diffusion coefficient.

Problem (1.1) has a strong solution u_s, i.e., $u_s \in C([0, T]; H)$ is absolutely continuous in any compact subinterval of $(0, T)$, $u_s(t) \in \mathcal{D}(A^s)$ for a.e. $t \in (0, T)$, and

$$\frac{du_s}{dt}(t) + A^s(u_s(t)) = B(u_s(t)) \text{ for a.e. } t \in (0, T),$$

where

$$A^s(u_s) := -\mathrm{div}(D_s|\nabla u_s|^{p_s(x)-2}\nabla u_s) + |u_s|^{p_s(x)-2}u_s$$

and problem (1.1) has a global attractor $\mathcal{A}_s$ (see [26]). The authors in [26] had considered the problem (1.1) with $p_s(\cdot) \to p > 2$ in $L^\infty(\Omega)$ and proved continuity of the flows and upper semicontinuity of the family of global attractors. In this work we want to give one step more and reach the linear case, i.e., consider $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$. It is worth mentioning that we were not able to obtain upper semicontinuity of the family of global attractors, but only a weak upper semicontinuity of the family of global attractors $\{\mathcal{A}_s\}_{s\in\mathbb{N}}$ as s goes to infinity for the problem (1.1), weak in the sense that we developed an algorithm which said to us how to control the gaps between two consecutive exponent functions for a given δ_0 (see condition $(H2)$ in Section 4) in order to obtain $\mathcal{A}_s \subset O_{\delta_0}(\mathcal{A}_\infty)$, for s large enough, where $\mathcal{A}_\infty$ will be the global attractor of the limit problem. To obtain upper semicontinuity of the family of global attractors it would be necessary to find general conditions on the exponents independent of δ_0. We developed in this work a new technique by using arguments with Hausdorff semi-distances, numerical series and limit processes (see Lemma 4.5, Lemma 4.6 and Theorem 4.7).

Considering $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$ and large diffusion, a fast redistribution process of the solution occurs having homogenization, any spatial variation of the solution is reduced to zero; i.e.; the only relevant parameter at the limit of the dynamics of the problem becomes the time. In other words, the limit problem will be the ODE (3.2). Going directly to 2 when varying the exponents is technically difficult or even impossible because of the lack of a uniform estimate of the solutions as the parameter of the problem converges to its limit, for this reason we will use what was done in [26] and from that limit problem with a constant exponent $p > 2$ we go to 2. For this reason we will consider a family (in p) of ODE's reaching the same limit problem (3.2) when p goes to 2. So, we will consider the following hypothesis

(H) There exists $\epsilon_0 > 0$ such that if $p_s \in F_{\epsilon_0}(2) := \{g; \|g - 2\|_{L^\infty(\Omega)} \leq \epsilon_0\}$, then p_s is a constant function.

The paper is organized as follows. In Section 2 we collect some definitions and results on semigroup theory. In Section 3 we prove a uniform estimate for the solutions of nonlinear ODEs and we prove continuity of the solutions with respect to initial conditions and exponent parameters. In Section 4 we prove that the solutions $\{u_s\}$ of the PDE (1.1) converge for $s \to \infty$ to the solution u of the limit problem (3.2) which is an ODE, and, after that, we obtain a weak upper semicontinuity of the global attractors for the problem (1.1). As a consequence we obtain that the attractors of problem (1.1) are included in a neighborhood of an interval.

2. PRELIMINARIES

For convenience to the reader we recall some definitions from Ladyzhenskaya [17] on (nonlinear) semigroup theory.

Definition 2.1. Let (X, d) be a complete metric space. A semigroup is a family $\{T(t) : X \to X, t \geq 0\}$ of single-valued continuous operators $T(t) : X \to X$ depending on a parameter $t \in \mathbb{R}^+$ and satisfying the semigroup property:

$$T(t_1)T(t_2)(x) = T(t_1 + t_2)(x), \text{ for all } t_1, t_2 \in \mathbb{R}^+ \text{ and } x \in X;$$

and $T(0) = I_d$.

Definition 2.2. Let A and M be subsets of X. We say that A attracts M or M is attracted to A by the semigroup $\{T(t)\}_{t\geq 0}$ if for every $\epsilon > 0$ there exists a $t_1(\epsilon, M) \in \mathbb{R}^+$ such that

$$T(t)M \subset O_\epsilon(A) := \{x \in X; d(x, A) < \epsilon\}$$

for all $t \geq t_1(\epsilon, M)$.

Definition 2.3. A is called a global B-attractor if A attracts each bounded set in X.

Definition 2.4. A semigroup is called bounded dissipative or B-dissipative if it has a bounded global B-attractor.

Definition 2.5. A set $A \subset X$ is called invariant (relative to semigroup $\{T(t)\}$) if $T(t)A = A$, for all $t \in \mathbb{R}^+$.

Definition 2.6. A semigroup $\{T(t)\}_{t\geq 0}$ belongs to the class $\mathcal{K}$ if for each $t > 0$ the operator $T(t)$ is compact, i.e., for any bounded set $B \subset X$ its image $T(t)B$ is precompact.

Theorem 2.7. *Let $\{T(t) : X \to X, t \geq 0\}$ be a semigroup of class $\mathcal{K}$. If it is B-dissipative, then $\{T(t) : X \to X, t \geq 0\}$ has a minimal closed global B-attractor $\mathcal{M}$, which is compact and invariant.*

Definition 2.8. A point $x \in X$ is said to be an equilibrium (or fixed point) of the semigroup $\{T(t)\}_{t\geq 0}$ if $x = T(t)x$, for all $t \geq 0$.

Definition 2.9. A complete trajectory $\gamma(x)$ of the point x is the curve $x(t)$, $-\infty < t < +\infty$, satisfying the following conditions: $x(t) \in X$ for all $t \in \mathbb{R}$, $x(0) = x$, $T(\tau)(x(t)) = x(t+\tau)$ for all $t \in \mathbb{R}$ and $\tau \in \mathbb{R}^+$.

For more details on (nonlinear) semigroup theory see [13, 17, 18, 28].

Now we rewrite some definitions and results about monotone (or order-preserving) semigroups as particular cases of the results in [3] (see also [1, 7]).

Definition 2.10. A semigroup $\{T(t) : X \to X, t \geq 0\}$ is said to be monotone if there exists an order relation " $\leq$ " in X such that, if $x_0 \leq y_0$, then $T(t)x_0 \leq T(t)y_0$, for all $t \geq 0$.

We assume that the order is compatible with the topology (see [3, p. 303]). The next result provides information on the structure of the global B-attractor with upper and lower asymptotically stable equilibria.

Theorem 2.11 ([3]). *Let $\{T(t) : X \to X, t \geq 0\}$ be a B-dissipative monotone semigroup of class $\mathcal{K}$ and $\mathcal{A}$ its associated maximal compact invariant global B-attractor. Then, there exist equilibria $x_*, y^* \in \mathcal{A}$ such that:*

(1) *$x_* \leq y^*$ and $\mathcal{A} \subseteq [x_*, y^*] := \{x \in X; x_* \leq x \leq y^*\}$,*
(2) *x_* (resp. y^*) is minimal (resp. maximal) in the sense that any other fixed points are contained in the interval $[x_*, y^*]$,*
(3) *x_* is globally attracting from below, that is, for all $v \in X$ with $v \leq x_*$, we have that $\lim_{t\to+\infty} T(t)v = x_*$,*
(4) *y^* is globally attracting from above, that is, for all $v \in X$ with $y^* \leq v$, we have that $\lim_{t\to+\infty} T(t)v = y^*$.*

3. THE FAMILY OF ODES AND ITS LIMIT PROBLEM

Now consider the following family (in p) of ODEs

$$\begin{cases} \dot{u}_p(t) + |u_p(t)|^{p-2}u_p(t) = f(u_p(t)), \quad t > 0, \\ u_p(0) = u_{0p} \in \mathbb{R}, \end{cases} \tag{3.1}$$

with $f : \mathbb{R} \to \mathbb{R}$ a globally Lipschitz map with Lipschitz constant $L > 0$ and $p \in (2, 3]$ a constant.

By Lemma 3.2 and Theorem 3.1 in [22] problem (3.1) has a unique global solution u_p and defines a B-dissipative semigroup of class $\mathcal{K}$ which has a maximal compact invariant global $B-$attractor $\mathcal{M}_p$, given as the union of all bounded complete trajectories in $\mathbb{R}$.

Now, we intend to study the sensitivity of problem (3.1) when the constant exponent p goes to 2. We guess and will prove that the limit problem is

$$\begin{cases} \dot{u}(t) + u(t) = f(u(t)), \quad t > 0, \\ u(0) = u_0 \in \mathbb{R}. \end{cases} \tag{3.2}$$

From Picard's Theorem problem (3.2) has a unique global solution u. Moreover, given $T > 0$ and $u_0 \in \mathbb{R}$, there exists a constant $K_\infty = K_\infty(u_0, T) > 0$ such that $|u(t)| \leq K_\infty$, for all $t \in [0, T]$.

In the next result we prove the continuity of the solutions of (3.1) with respect to the initial data and exponent parameter.

Theorem 3.1. *Let u_p be a solution of* (3.1) *with $u_p(0) = u_{0p}$ and let u be the solution of* (3.2) *with $u(0) = u_0$. If $u_{0p} \to u_0$ in $\mathbb{R}$ as $p \to 2$, then for each $T > 0$, $u_p \to u$ in $C([0,T];\mathbb{R})$ as $p \to 2$.*

Proof. Let $T > 0$ be fixed and suppose that $u_{0p} \to u_0$ in $\mathbb{R}$ as $p \to 2$. Subtracting the two equations in (3.1) and (3.2) and making the product with $u_p - u$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_p(t) - u(t)|^2 + [|u_p(t)|^{p-2}u_p(t) - u(t)][u_p(t) - u(t)]\\ &= [f(u_p(t)) - f(u(t))][u_p(t) - u(t)].\end{aligned}$$

Adding $\pm|u(t)|^{p-2}u(t)$, using that f is Lipschitz and that for any $\xi, \eta \in \mathbb{R}^n$,

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq 0,$$

we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_p(t) - u(t)|^2 &\leq L|u_p(t) - u(t)|^2 - (|u(t)|^{p-2} - 1)u(t)(u_p(t) - u(t))\\ &\leq L|u_p(t) - u(t)|^2 + \big||u(t)|^{p-1} - |u(t)|\big|\,|u_p(t) - u(t)|,\end{aligned}$$

for all $t \in (0, T)$.

Now, let us estimate the term

$$\big||u(t)|^{p-1} - |u(t)|\big|\,|u_p(t) - u(t)|.$$

By the Mean Value Theorem, for each $p > 2$ there is a $q \in (2, p)$ such that

$$\big||u(t)|^{p-1} - |u(t)|\big| = \big||u(t)|^{q-1}\ln|u(t)|\big|\,|p - 2|$$

provided that $u(t) \neq 0$. Consider the continuous function $g_\theta : [0, K_\infty] \to \mathbb{R}$ given by

$$g_\theta(w) = \begin{cases} w^\theta \ln w & \text{if } w \in (0, K_\infty], \\ 0 & \text{if } w = 0, \end{cases}$$

where $\theta \geq 1$ is a given number. Using this continuous function defined in the compact set $[0, K_\infty]$ with $\theta = 1$ when $|u(t)| < 1$ and with $\theta = 2$ when $|u(t)| \geq 1$, there exists a positive constant R such that

$$\big||u(t)|^{q-1}\ln|u(t)|\big| \leq R,$$

for all $t \in [0, T]$ with $u(t) \neq 0$. So,

$$\big||u(t)|^{p-1} - |u(t)|\big| \leq R|p - 2|,$$

for all $t \in [0,T]$. Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_p(t)-u(t)|^2 \leq & L|u_p(t)-u(t)|^2 + R|p-2||u_p(t)-u(t)| \\ \leq & L|u_p(t)-u(t)|^2 + \frac{1}{2}[R|p-2|]^2 + \frac{1}{2}|u_p(t)-u(t)|^2,\end{aligned}$$

for all $t \in (0,T)$.

Integrating from 0 to t, $t \leq T$, we obtain

$$|u_p(t)-u(t)|^2 \leq |u_{0p}-u_0|^2 + [R|p-2|]^2T + \int_0^t (2L+1)|u_p(\tau)-u(\tau)|^2 d\tau.$$

So, by Gronwall-Bellman's Lemma we obtain

$$|u_p(t)-u(t)|^2 \leq \left(|u_{0p}-u_0|^2 + [R|p-2|]^2T\right) e^{(2L+1)T},$$

for all $t \in [0,T]$. Therefore, $u_p \to u$ in $C([0,T];\mathbb{R})$ as $p \to 2$. □

If we restrict the initial conditions to a bounded set $M \subset \mathbb{R}$ in problem (3.1) and consider $L < 1$ then we can obtain the following uniform estimates of the solutions of problem (3.1).

Proposition 3.2. *Consider f with Lipschitz constant $L < 1$. Let M be a bounded set and u_p be a solution of* (3.1) *with $u_p(0) = u_{0p} \in M$. There exists a positive number r_0 such that $|u_p(t)| \leq r_0$, for each $t \geq 0$ and for all $p \in (2,3]$.*

Proof. Let $\tau > 0$. Multiplying the equation on (3.1) by $u_p(\tau)$ we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}|u_p(\tau)|^2 &\leq -|u_p(\tau)|^p + |f(u_p(\tau))||u_p(\tau)| \\ &\leq -|u_p(\tau)|^p + |f(u_p(\tau)) - f(0)||u_p(\tau)| + |f(0)||u_p(\tau)|.\end{aligned}$$

So,

$$\frac{1}{2}\frac{d}{d\tau}|u_p(\tau)|^2 \leq -|u_p(\tau)|^p + L|u_p(\tau)|^2 + C_0|u_p(\tau)|, \tag{3.3}$$

where $C_0 := |f(0)| \geq 0$.

If $|u_p(\tau)| > 1$, $-|u_p(\tau)|^p \leq -|u_p(\tau)|^2$, then from (3.3)

$$\frac{1}{2}\frac{d}{d\tau}|u_p(\tau)|^2 \leq (L-1)|u_p(\tau)|^2 + C_0|u_p(\tau)|.$$

Consider $\epsilon > 0$ arbitrary. Using Young's inequality we obtain

$$\frac{1}{2}\frac{d}{d\tau}|u_p(\tau)|^2 \leq \left(-1 + L + \frac{1}{2}\epsilon^2\right)|u_p(\tau)|^2 + \frac{1}{2}\left(\frac{C_0}{\epsilon}\right)^2.$$

Now, choose $\epsilon = \epsilon_1 > 0$ sufficiently small such that $0 < \epsilon_1 < (1-L)^{1/2}$ we obtain

$$\frac{1}{2}\frac{d}{d\tau}|u_p(\tau)|^2 \leq -\alpha|u_p(\tau)|^2 + C_1,$$

where $\alpha := \frac{1}{2} - \frac{1}{2}L > 0$ and $C_1 := \frac{1}{2}\left(\frac{C_0}{\epsilon_1}\right)^2$. Then

$$\frac{d}{d\tau}[|u_p(\tau)|^2]e^{2\alpha\tau} + 2\alpha|u_p(\tau)|^2 e^{2\alpha\tau} \le 2C_1 e^{2\alpha\tau}. \tag{3.4}$$

If $|u_p(\tau)| \le 1$, then from (3.3),

$$\frac{d}{d\tau}|u_p(\tau)|^2 \le 2(L + C_0) =: C_2.$$

Thus,

$$\frac{d}{d\tau}[|u_p(\tau)|^2]e^{2\alpha\tau} + 2\alpha|u_p(\tau)|^2 e^{2\alpha\tau} \le C_2 e^{2\alpha\tau} + 2\alpha|u_p(\tau)|^2 e^{2\alpha\tau} \le (C_2 + 2\alpha)e^{2\alpha\tau}. \tag{3.5}$$

Considering $y_p(\tau) := |u_p(\tau)|^2$ and $C_3 := \max\{2C_1, C_2 + 2\alpha\}$, we obtain from (3.4) and (3.5) that

$$\frac{d}{d\tau}[y_p(\tau)e^{2\alpha\tau}] \le C_3 e^{2\alpha\tau}, \text{ for all } \tau > 0.$$

Integrating from 0 to t, we have

$$y_p(t)e^{2\alpha t} \le y_p(0) + \frac{C_3}{2\alpha}e^{2\alpha t} - \frac{C_3}{2\alpha} \le |u_{0p}|^2 + \frac{C_3}{2\alpha}e^{2\alpha t}.$$

Multiplying by $e^{-2\alpha t}$, we obtain

$$|u_p(t)|^2 = y_p(t) \le |u_{0p}|^2 e^{-2\alpha t} + \frac{C_3}{2\alpha} \le |u_{0p}|^2 e^0 + \frac{C_3}{2\alpha}, \text{ for all } t \ge 0.$$

Since $u_{0p} \in M$ and M is bounded, there exists $K \ge 0$ such that $|u_{0p}| \le K$ for all $p \in (2, 3]$. Thus,

$$|u_p(\tau)| \le r_0 := \left(K^2 + \frac{C_3}{2\alpha}\right)^{1/2}, \text{ for all } t \ge 0 \text{ and } p \in (2, 3]. \qquad \square$$

4. CONTINUITY OF THE FLOWS AND WEAK UPPER SEMICONTINUITY OF ATTRACTORS

Our objective in this section is to prove that the limit problem of problem (1.1) as D_s increases to infinity and $p_s(\cdot) \to 2$ in $L^\infty(\Omega)$ as $s \to \infty$ is described by the ordinary differential equation in (3.2).

The next result guarantees that (3.2) is in fact the limit problem for (1.1), as $s \to \infty$. The proof is analogous to the proof of Theorem 4.2 in [26].

Theorem 4.1. *Let u_s be a solution of* (1.1) *with $u_s(0) = u_{0s}$ and let u be the solution of* (3.2) *with $f = B_{|\mathbb{R}}$ and $u(0) = u_0$. If $u_{0s} \to u_0$ in H as $s \to \infty$, then for each $T > 0$, $u_s \to u$ in $C([0, T]; H)$ as $s \to +\infty$.*

Theorem 4.2. *The problem* (3.2) *defines a semigroup of class* $\mathcal{K}$.

Proof. We define $S(t) : \mathbb{R} \to \mathbb{R}$ by $S(t)u_0 = u(t)$ with u being the unique global solution of the problem (3.2) with $u(0) = u_0$. It is easy to see that $\{S(t) : \mathbb{R} \to \mathbb{R}, t \geq 0\}$ verifies the semigroup properties. Consider $F(u) := f(u) - u$. We will show that $\{S(t) : \mathbb{R} \to \mathbb{R}, t \geq 0\}$ is of class $\mathcal{K}$. In fact, multiplying the equation in (3.2) by $u(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 \leq \Big(L + \frac{1}{2}\Big)|u(t)|^2 + \frac{|F(0)|^2}{2} \tag{4.1}$$

Let $T > 0$ fixed, integrating (4.1) from 0 to τ, $\tau < T$, we obtain

$$|u(\tau)|^2 \leq |u(0)|^2 + |F(0)|^2 T + \int_0^\tau (2L+1)|u(s)|^2 ds.$$

So, by Gronwall-Bellman's Lemma it follows that

$$|u(\tau)|^2 \leq (|u_0|^2 + |F(0)|^2 T)e^{(2L+1)T}, \quad \text{for all } \tau \leq T.$$

Thus, we conclude that for each $t > 0$, $S(t)$ maps bounded sets into bounded sets. As a result we conclude that for each $t > 0$ the operator $S(t) : \mathbb{R} \to \mathbb{R}$ is compact. ☐

Observe that the semigroup of class $\mathcal{K}$ defined by the problem (3.2) is not necessarily B-dissipative. For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(u) = \alpha u$ with $\alpha > 1$ a real number and so the solution of (3.2) is $u(t) = u_0 e^{(\alpha-1)t}$ and $|u(t)| \to \infty$ as $t \to \infty$. In this case a global B-attractor for the problem (3.2) does not exist. If the semigroup defined by the limit problem (3.2) is B-dissipative then, Theorem 2.7 guarantees that the semigroup $S(t)$ has a maximal compact invariant global $B-$attractor $\mathcal{A}^\infty$. By Proposition 2.2 in [17], $\mathcal{A}^\infty$ is given as the union of all bounded complete trajectories in $\mathbb{R}$. There are examples that provide situations where the semigroup defined by the limit problem (3.2) is B-dissipative. If $f : \mathbb{R} \to \mathbb{R}$ is given by $f(u) = \alpha u$ with $\alpha < 1$ a real number and so the solution of (3.2) is $u(t) = u_0 e^{(\alpha-1)t}$ and $u(t) \to 0$ as $t \to \infty$. So, the semigroup defined by the limit problem (3.2) is B-dissipative.

Now, we suppose that $f = B_{|\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, which is globally Lipschitz, is such that the limit problem (3.2) has a B-dissipative semigroup. So, let $\mathcal{A}_\infty$ be the maximal compact invariant global B-attractor for (3.2) with $f = B_{|\mathbb{R}}$.

We need to use the following theorem.

Theorem 4.3 ([26]). *Let* $\mathcal{A}_s$ *be the global attractor associated with problem* (1.1) *and* $\mathcal{M}_p$ *the global attractor for problem* (3.1) *with* $f = B_{|\mathbb{R}}$. *Then,* $\operatorname{dist}(\mathcal{A}_s; \mathcal{M}_p) \to 0$ *in the topology of* H*, when* $p_s(\cdot) \to p > 2$ *in* $L^\infty(\Omega)$.

The condition (H) is needed in the proof of the weak upper semicontinuity of the family of global attractors for problem (1.1) as $p_s \to 2$ in $L^\infty(\Omega)$. Moreover, after the functions $p_s(\cdot)$ enter into $F_{\epsilon_0}(2)$, given $\delta_0 > 0$, in order to show $\mathcal{A}_s \subset O_{\delta_0}(\mathcal{A}_\infty)$ for $s > 0$ large enough, we have to control the gap between two consecutive functions p_s and p_{s+1} by an appropriate term which depends on s and δ_0 (see hypothesis (H2) below).

Consider $p := 2+\epsilon_0$, where $\epsilon_0 > 0$ is from hypothesis (H). Then there exists $s_1 \in \mathbb{N}$ large enough such that $2 < p_{s_1} < p$ and $2 < p_s \leq p_{s_1}$ is constant for all $s \geq s_1$. Thus, let us call, $\{p_s\}_{s\geq s_1}$ simply $\{p_j\}_{j\geq 1}$, where $p_j := p_{s_j}$. Consider from now on $L < 1$ and the constant $r_0 = r_0(M)$ in Proposition 3.2 for $M = \mathcal{M}_{s_1}$, where $\mathcal{M}_{s_1}$ is the global attractor for problem (3.1) with the exponent parameter p_{s_1}. The set $\mathcal{M}_{s_1}$ is compact, in particular bounded, so given δ_0 there exists $t_0 = t_0(\delta_0, \mathcal{M}_{s_1}) > 0$ such that

$$\operatorname{dist}_{\mathbb{R}}(S(t)\mathcal{M}_{s_1}; \mathcal{A}_\infty) < \frac{\delta_0}{4|\Omega|^{1/2}}, \quad \text{for all } t \geq t_0, \tag{4.2}$$

where $S(t)u_0 := u(t, u_0)$ is the solution of (3.2) and $\operatorname{dist}_{\mathbb{R}}(S(t)\mathcal{M}_{s_1}; \mathcal{A}_\infty)$ is the Hausdorff semi-distance between $S(t)\mathcal{M}_{s_1}$ and $\mathcal{A}_\infty$ in $\mathbb{R}$. Let $\psi_0 \in \mathcal{M}_{s_1}$ be arbitrarily fixed. Let $\{S^j(t)\}$ be the semigroup defined by problem (3.1) with the exponent parameter p_j and consider $u_j(\tau) := S^j(\tau)\psi_0$.

Let us first prove the following three technical lemmas and then we present our main result.

Lemma 4.4. *There exists a positive constant κ such that*

$$\left||u_{j+1}(\tau)|^{p_j-1} - |u_{j+1}(\tau)|^{p_{j+1}-1}\right| \leq \kappa|p_j - p_{j+1}|,$$

for all $j \in \mathbb{N}$ and $\tau \in [0, t_0]$.

Proof. By the Mean Value Theorem we conclude that

$$\left||u_{j+1}(\tau)|^{p_j-1} - |u_{j+1}(\tau)|^{p_{j+1}-1}\right| = \left||u_{j+1}(\tau)|^{\theta_j} \ln|u_{j+1}(\tau)|\right| p_j - p_{j+1}|,$$

for some $\theta_j \in (p_{j+1}, p_j)$. Consider the continuous function $g_\theta : [0, r_0] \to \mathbb{R}$ given by

$$g_\theta(x) = \begin{cases} x^\theta \ln x & \text{if } x \in (0, r_0], \\ 0 & \text{if } x = 0, \end{cases}$$

where $r_0 = r_0(\mathcal{M}_{s_1})$ is as in Proposition 3.2 and $\theta \geq 1$ is a given number. Using this continuous function defined in the compact set $[0, r_0]$ with $\theta = 2$ when $|u_{j+1}(\tau)| < 1$ and with $\theta = 2 + \epsilon_0$ when $|u_{j+1}(\tau)| \geq 1$, there exists a positive constant κ such that

$$\left||u_{j+1}(\tau)|^\theta \ln|u_{j+1}(\tau)|\right| \leq \kappa,$$

for all $j \in \mathbb{N}$ and $\tau \in [0, t_0]$ and the result follows. □

Now, we can establish the following hypothesis

(H2) For each $j \in \mathbb{N}$, $|p_j - p_{j+1}| < \left[\frac{\delta_0^2}{5^{2j}\kappa^2|\Omega|e^{(2L+1)t_0}t_0}\right]^{1/2}$.

Lemma 4.5. *If condition* (H2) *is fulfilled for a given $\delta_0 > 0$, then*

$$\operatorname{dist}_{\mathbb{R}}(S^j(t_0)\mathcal{M}_{s_1}; S^{j+1}(t_0)\mathcal{M}_{s_1}) < \frac{\delta_0}{5^j|\Omega|^{1/2}},$$

for all $j \in \mathbb{N}$.

Proof. Subtracting the two equations in (3.1) and making the product with $u_j - u_{j+1}$ we obtain

$$\frac{1}{2}\frac{d}{dt}|u_j(t) - u_{j+1}(t)|^2 + [|u_j(t)|^{p_j-2}u_j(t) - |u_{j+1}(t)|^{p_{j+1}-2}u_{j+1}(t)][u_j(t) - u_{j+1}(t)]$$
$$= [f(u_j(t)) - f(u_{j+1}(t))][u_j(t) - u_{j+1}(t)].$$

Adding $\pm|u_{j+1}(t)|^{p_j-2}u_{j+1}(t)$, using that f is Lipschitz and that for any $\xi, \eta \in \mathbb{R}^n$, $(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) \geq 0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|u_j(t) - u_{j+1}(t)|^2 &\leq L|u_j(t) - u_{j+1}(t)|^2 \\ &\quad - \left[|u_{j+1}(t)|^{p_j-2} - |u_{j+1}(t)|^{p_{j+1}-2}\right] u_{j+1}(t)\left(u_j(t) - u_{j+1}(t)\right) \\ &\leq L|u_j(t) - u_{j+1}(t)|^2 \\ &\quad + \left||u_{j+1}(t)|^{p_j-1} - |u_{j+1}(t)|^{p_{j+1}-1}\right| |u_j(t) - u_{j+1}(t)| \\ &\leq \left(L + \frac{1}{2}\right) |u_j(t) - u_{j+1}(t)|^2 \\ &\quad + \frac{1}{2}\left||u_{j+1}(t)|^{p_j-1} - |u_{j+1}(t)|^{p_{j+1}-1}\right|^2,\end{aligned}$$

for all $t \in [0, t_0]$. Using Lemma 4.4 we obtain

$$\frac{d}{dt}|u_j(t) - u_{j+1}(t)|^2 \leq (2L+1)\,|u_j(t) - u_{j+1}(t)|^2 + \kappa^2|p_j - p_{j+1}|^2,$$

for all $t \in [0, t_0]$. From condition (H2),

$$|p_j - p_{j+1}|^2 < \frac{\delta_0^2}{5^{2j}\kappa^2|\Omega|e^{(2L+1)t_0}t_0}.$$

Then,

$$\frac{d}{dt}|u_j(t) - u_{j+1}(t)|^2 \leq (2L+1)\,|u_j(t) - u_{j+1}(t)|^2 + \kappa^2\frac{\delta_0^2}{5^{2j}\kappa^2|\Omega|e^{(2L+1)t_0}t_0},$$

for all $t \in [0, t_0]$. Integrating from 0 to t_0 and using that $u_j(0) = u_{j+1}(0) = \psi_0$, we obtain

$$|u_j(t) - u_{j+1}(t)|^2 \leq \frac{\delta_0^2}{5^{2j}|\Omega|e^{(2L+1)t_0}} + \int_0^{t_0} (2L+1)\,|u_j(t) - u_{j+1}(t)|^2 dt.$$

So, by Gronwall-Bellman's Lemma we obtain

$$|u_j(t) - u_{j+1}(t)| \leq \frac{\delta_0}{5^j|\Omega|^{1/2}},$$

for all $j \in \mathbb{N}$ and $t \in [0, t_0]$. Thus,

$$\begin{aligned} \operatorname{dist}_{\mathbb{R}}(S^j(t_0)\psi_0; S^{j+1}(t_0)\mathcal{M}_{s_1}) &= \inf_{b \in S^{j+1}(t_0)\mathcal{M}_{s_1}} \operatorname{dist}_{\mathbb{R}}(S^j(t_0)\psi_0; b) \\ &\leq \operatorname{dist}_{\mathbb{R}}(S^j(t_0)\psi_0; S^{j+1}(t_0)\psi_0) \\ &= |u_j(t_0) - u_{j+1}(t_0)| \leq \frac{\delta_0}{5^j|\Omega|^{1/2}}. \end{aligned}$$

Since $\psi_0 \in \mathcal{M}_{s_1}$ was arbitrary, we conclude that

$$\operatorname{dist}_{\mathbb{R}}(S^j(t_0)\mathcal{M}_{s_1}; S^{j+1}(t_0)\mathcal{M}_{s_1}) = \sup_{\psi_0 \in \mathcal{M}_{s_1}} \operatorname{dist}_{\mathbb{R}}(S^j(t_0)\psi_0; S^{j+1}(t_0)\mathcal{M}_{s_1}) \leq \frac{\delta_0}{5^j|\Omega|^{1/2}}.$$

□

Lemma 4.6. *Given $\delta_0 > 0$, we have*

$$\operatorname{dist}_{\mathbb{R}}(S^\ell(t_0)\mathcal{M}_{s_1}; S(t_0)\mathcal{M}_{s_1}) < \frac{\delta_0}{4|\Omega|^{1/2}},$$

for ℓ large enough.

Proof. Let $\psi_0 \in \mathcal{M}_{s_1}$ arbitrarily fixed. From Theorem 3.1,

$$|S^\ell(t_0)\psi_0 - S(t_0)\psi_0| = |u_\ell(t_0) - u(t_0)| < \frac{\delta_0}{4|\Omega|^{1/2}},$$

for ℓ large enough. So,

$$\begin{aligned} \operatorname{dist}_{\mathbb{R}}(S^\ell(t_0)\psi_0; S(t_0)\mathcal{M}_{s_1}) &= \inf_{b \in S(t_0)\mathcal{M}_{s_1}} \operatorname{dist}_{\mathbb{R}}(S^\ell(t_0)\psi_0; b) \\ &\leq \operatorname{dist}_{\mathbb{R}}(S^\ell(t_0)\psi_0; S(t_0)\psi_0) < \frac{\delta_0}{4|\Omega|^{1/2}}. \end{aligned}$$

Since $\psi_0 \in \mathcal{M}_{s_1}$ was arbitrary, we conclude that

$$\operatorname{dist}_{\mathbb{R}}(S^\ell(t_0)\mathcal{M}_{s_1}; S(t_0)\mathcal{M}_{s_1}) = \sup_{\psi_0 \in \mathcal{M}_{s_1}} \operatorname{dist}_{\mathbb{R}}(S^\ell(t_0)\psi_0; S(t_0)\mathcal{M}_{s_1}) \leq \frac{\delta_0}{4|\Omega|^{1/2}},$$

for ℓ large enough. □

Theorem 4.7. *Consider $f = B_{|\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ with $L < 1$ and such that the limit problem* (3.2) *has a B-dissipative semigroup. Assume condition* (H)*. If condition* (H2) *is fulfilled for a given $\delta_0 > 0$, then*

$$\mathcal{A}_s \subset O_{\delta_0}(\mathcal{A}_\infty) = \{z \in H; \inf_{a \in \mathcal{A}_\infty} \|z - a\|_H < \delta_0\}$$

for s large enough.

Proof. Consider the sequence of functions $\{\tilde{p_s}(\cdot)\}_{s\in\mathbb{N}}$ defined by $\tilde{p_1}(\cdot) = p_1(\cdot)$, $\tilde{p_2}(\cdot) = p_2(\cdot), \dots, \tilde{p}_{s_1-1}(\cdot) = p_{s_1-1}(\cdot)$, $\tilde{p_{s_1}}(\cdot) \equiv p_{s_1}$, $\tilde{p}_{s_1+1}(\cdot) \equiv p_{s_1}, \dots$ Applying Theorem 4.3 for this sequence of exponent functions and for the original sequence of diffusion coefficients, we have that

$$\mathrm{dist}(\mathcal{A}_s; \mathcal{M}_{s_1}) < \delta_0/4$$

for s large enough. Here $\mathrm{dist}(\mathcal{A}_s; \mathcal{M}_{s_1})$ is the Hausdorff semi-distance between $\mathcal{A}_s$ and $\mathcal{M}_{s_1}$ in the Hilbert space H. So,

$$\begin{aligned}\mathrm{dist}(\mathcal{A}_s; \mathcal{A}_\infty) &\le \mathrm{dist}(\mathcal{A}_s; \mathcal{M}_{s_1}) + \mathrm{dist}(\mathcal{M}_{s_1}; \mathcal{A}_\infty)\\ &< \delta_0/4 + |\Omega|^{1/2}\mathrm{dist}_{\mathbb{R}}(\mathcal{M}_{s_1}; \mathcal{A}_\infty),\end{aligned} \tag{4.3}$$

for s large enough.

By the invariance of the global attractor $\mathcal{M}_{s_1}$ we have $S^1(t_0)\mathcal{M}_{s_1}{=}\mathcal{M}_{s_1}$. Then,

$$\begin{aligned}\mathrm{dist}_{\mathbb{R}}(\mathcal{M}_{s_1}; \mathcal{A}_\infty) \le &\sum_{j=1}^{\ell} \mathrm{dist}_{\mathbb{R}}(S^j(t_0)\mathcal{M}_{s_1}; S^{j+1}(t_0)\mathcal{M}_{s_1})\\ &+ \mathrm{dist}_{\mathbb{R}}(S^{\ell+1}(t_0)\mathcal{M}_{s_1}; S(t_0)\mathcal{M}_{s_1}) + \mathrm{dist}_{\mathbb{R}}(S(t_0)\mathcal{M}_{s_1}; \mathcal{A}_\infty),\end{aligned} \tag{4.4}$$

for all $\ell \in \mathbb{N}$. Using (4.2), Lemma 4.5, Lemma 4.6 and letting $\ell \to +\infty$ in (4.4), we obtain

$$\mathrm{dist}_{\mathbb{R}}(\mathcal{M}_{s_1}; \mathcal{A}_\infty) \le \sum_{j=1}^{+\infty} \frac{\delta_0}{5^j|\Omega|^{1/2}} + \frac{\delta_0}{4|\Omega|^{1/2}} + \frac{\delta_0}{4|\Omega|^{1/2}} = \frac{3\delta_0}{4|\Omega|^{1/2}}. \tag{4.5}$$

Using (4.5) in (4.3) the result follows. □

By using Theorem 4.1 in [16] we obtain the following result.

Theorem 4.8. *The problem* (3.2) *defines a monotone semigroup.*

So, using Theorem 4.2, Theorem 4.8, Theorem 2.11 and Theorem 4.7 we obtain the following interesting result

Theorem 4.9. *Consider $f = B_{|\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ with $L < 1$ and such that the limit problem* (3.2) *has a B-dissipative semigroup. Assume condition* (H)*. If condition* (H2) *is fulfilled for a given $\delta_0 > 0$, then there exist equilibria $x_*, y^* \in \mathcal{A}_\infty$ with $\mathcal{A}_\infty \subseteq [x_*, y^*]$ satisfying* (1)–(4) *from Theorem 2.11 and there exists $s_0 = s_0(\delta_0)$ such that $\mathcal{A}_s \subset O_{\delta_0}([x_*, y^*])$, for all $s \ge s_0$.*

Remark 4.10. If problem (3.2) has only one equilibrium x_*, for example, when $F = f - I_d$ is linear, then $\mathcal{A}_\infty = \{x_*\}$.

Remark 4.11. One can also obtain some information about equilibria and attractivity inside the interval $[x_*, y^*]$ in the previous theorem by using Theorem 2.2 at p.17 in [27].

5. FINAL REMARKS

A natural question that raise is why is technically difficult to go directly to 2? In order to prove upper semicontinuity of a family of global attractors $\{\mathcal{A}_s;\ s \in \mathbb{N}\}$, generally compactness of the set $\mathcal{A} := \overline{\bigcup_{s\in\mathbb{N}} \mathcal{A}_s}$ is needed. To obtain this compactness generally is used the invariance of the attractors and a uniform estimate of the solutions with a constant which does not depend on the initial data and on the parameter which is varying. When you go directly to 2, this is a problem. Usually to obtain this uniform estimates of the solutions without dependence on the initial conditions the Lemma 5.1 at p. 163 in [28] is used. If we stop in a stage before, that is, at $p > 2$, we can choose the constant uniformly for the family of exponents p_s (see [24–26]), but not when we go directly to 2, because

$$\lim_{p\to 2^+} \frac{1}{\left(\frac{p-2}{2}\right)^{2/(p-2)}} = +\infty.$$

The same difficulty would appear if we wanted to prove upper semicontinuity of the family of attractors $\{\mathcal{M}_p;\ p > 2\}$ of problem (3.1) when $p \to 2$. Observe that Proposition 3.2 is not enough to obtain compactness of the set $\overline{\bigcup_{p>2} \mathcal{M}_p}$ because of the restriction on the considered initial data.

Acknowledgements

This work has been partially supported by Science without Borders-CAPES-PVE-Process 88881.030388/2013-01. The first author was supported with CNPq scholarship – process 202645/2014-2 (Brazil). The second author was supported with CAPES scholarship – process 99999.006893/2014-07 (Brazil).

REFERENCES

[1] L. Arnold, I. Chueshov, *Order-preserving random dynamical systems: Equilibria, attractors, applications*, Dynamics and Stability of Systems **13** (1998) 3, 265–280.

[2] J.M. Arrieta, A.N. Carvalho, A. Rodríguez-Bernal, *Upper semicontinuity for attractors of parabolic problems with localized large diffusion and nonlinear boundary conditions*, J. Differential Equations **168** (2000), 33–59.

[3] T. Caraballo, J.A. Langa, J. Valero, *Asymptotic behaviour of monotone multi-valued dynamical systems*, Dyn. Syst. **20** (2005) 3, 301–321.

[4] A.N. Carvalho, *Infinite dimensional dynamics described by ordinary differential equations*, J. Differential Equations **116** (1995), 338–404.

[5] A.N. Carvalho, J.K. Hale, *Large diffusion with dispersion*, Nonlinear Anal. **17** (1991) 12, 1139–1151.

[6] Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Math. **66** (2006), 1383–1406.

[7] J.W. Cholewa, A. Rodriguez-Bernal, *Extremal equilibria for monotone semigroups in ordered spaces with applications to evolutionary equations*, J. Differential Equations **249** (2010), 485–525.

[8] E. Conway, D. Hoff, J. Smoller, *Large time behavior of solutions of systems of non-linear reaction-diffusion equations*, SIAM J. Appl. Math. **35** (1978) 1, 1–16.

[9] L. Diening, P. Harjulehto, P. Hästö, M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer-Verlag, Berlin, Heidelberg, 2011.

[10] F. Ettwein, M. Ružička, *Existence of local strong solutions for motions of electrorheological fluids in three dimensions*, Computers and Mathematics with Applications **53** (2007), 595–604.

[11] Z. Guo, Q. Liu, J. Sun, B. Wu, *Reaction-diffusion systems with $p(x)$-growth for image denoising*, Nonlinear Anal. Real World Appl. **12** (2011), 2904–2918.

[12] J.K. Hale, *Large diffusivity and asymptotic behavior in parabolic systems*, J. Math. Anal. Appl. **118** (1986), 455–466.

[13] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, vol. 25, American Mathematical Society, Providence, RI, 1988.

[14] J.K. Hale, C. Rocha, *Varying boundary conditions with large diffusivity*, J. Math. Pures Appl. **66** (1987), 139–158.

[15] J.K. Hale, K. Sakamoto, *Shadow systems and attractors in reaction-diffusion equations*, Appl. Anal. **32** (1989), 287–303.

[16] Ph. Hartman, *Ordinary Differential Equations*, Classics Appl. Math., vol. 38, SIAM, Philadelphia, 2002.

[17] O. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Lezioni Lincee, 1991.

[18] De. Liu, *The critical forms of the attractors for semigroups and the existence of critical attractors*, Proc. Roy Soc. Lond. Ser. A Math. Phys. Eng. Sci. **454** (1998), 2157–2171.

[19] K. Rajagopal, M. Ružička, *Mathematical modelling of electrorheological materials*, Contin. Mech. Thermodyn. **13** (2001), 59–78.

[20] M. Ružička, *Flow of shear dependent electrorheological fluids*, C.R. Acad. Sci. Paris Sér. I **329** (1999), 393–398.

[21] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lectures Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000.

[22] J. Simsen, C.B. Gentile, *Well-posed p-Laplacian problems with large diffusion*, Nonlinear Anal. **71** (2009), 4609–4617.

[23] J. Simsen, M.S. Simsen, *PDE and ODE limit problems for $p(x)$-Laplacian parabolic equations*, J. Math. Anal. Appl. **383** (2011), 71–81.

[24] J. Simsen, M.S. Simsen, M.R.T. Primo, *Continuity of the flows and upper semicontinuity of global attractors for $p_s(x)$-Laplacian parabolic problems*, J. Math. Anal. Appl. **398** (2013), 138–150.

[25] J. Simsen, M.S. Simsen, M.R.T. Primo, *On $p_s(x)$-Laplacian parabolic problems with non-globally Lipschitz forcing term*, Z. Anal. Anwend. **33** (2014), 447–462.

[26] J. Simsen, M.S. Simsen, M.R.T. Primo, *Reaction-diffusion equations with spatially variable exponents and large diffusion*, Commun. Pure Appl. Anal. **15** (2016) 2, 495–506.

[27] H.L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Mathematical Surveys and Monographs, vol. 41, American Mathematical Society, Providence, 1995.

[28] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer--Verlag, New York, 1988.

Jacson Simsen
jacson@unifei.edu.br

Universidade Federal de Itajubá
Instituto de Matemática e Computação
Av. BPS n. 1303, Bairro Pinheirinho
37500-903, Itajubá – MG – Brazil

Universität of Duisburg-Essen
Fakultät für Mathematik
Thea-Leymann-Str. 9, 45127 Essen, Germany

Mariza Stefanello Simsen
mariza@unifei.edu.br

Universidade Federal de Itajubá
Instituto de Matemática e Computação
Av. BPS n. 1303, Bairro Pinheirinho
37500-903, Itajubá – MG – Brazil

Universität of Duisburg-Essen
Fakultät für Mathematik
Thea-Leymann-Str. 9, 45127 Essen, Germany

Aleksandra Zimmermann
aleksandra.zimmermann@uni-due.de

Universität of Duisburg-Essen
Fakultät für Mathematik
Thea-Leymann-Str. 9, 45127 Essen, Germany

Received: December 1, 2016.
Revised: July 10, 2017.
Accepted: August 1, 2017.