Opuscula Math. 38, no. 1 (2018), 41–80
https://doi.org/10.7494/OpMath.2018.38.1.41 Opuscula Mathematica

ASYMPTOTIC PROFILES FOR A CLASS OF PERTURBED BURGERS EQUATIONS IN ONE SPACE DIMENSION

F. Dkhil, M.A. Hamza, and B. Mannoubi

Communicated by Vicentiu D. Radulescu

Abstract. In this article we consider the Burgers equation with some class of perturbations in one space dimension. Using various energy functionals in appropriate weighted Sobolev spaces rewritten in the variables $\frac{\xi}{\sqrt{\tau}}$ and $\log \tau$, we prove that the large time behavior of solutions is given by the self-similar solutions of the associated Burgers equation.

Keywords: Burgers equation, self-similar variables, asymptotic behavior, self-similar solutions.

Mathematics Subject Classification: 35B20, 35B40, 35C20, 35K55, 35L05, 35L60.

1. INTRODUCTION

This paper is devoted to the study of the long time behavior of solutions to the following equation

$$\varepsilon u_{\tau\tau} + u_\tau = (a(\xi)u_\xi)_\xi - \left(u^2 + \mathcal{N}(u)\right)_\xi \quad \text{in } \mathbb{R} \times \mathbb{R}_+, \tag{1.1}$$

where ε is a positive, not necessarily small parameter. We assume that the diffusion coefficient $a(\xi)$ is positive and satisfies $\lim_{\xi \to \pm\infty} a(\xi) = 1$. In addition, we suppose that there exists $\gamma \geq 3$ and $C > 0$, such that

$$|\mathcal{N}(u)| + |u\mathcal{N}'(u)| \leq C|u|^\gamma. \tag{1.2}$$

Equations of the form (1.1) arise as mathematical models describing various natural phenomena, especially in genetics and population dynamics (see for example [15] and [24]). In the case of an inhomogeneous medium, the diffusion coefficients in such equations may depend on the space variable ξ.

In one space dimension, the viscous Burgers equation is given by

$$u_\tau = u_{\xi\xi} - \left(u^2\right)_\xi \quad \text{in } \mathbb{R} \times \mathbb{R}_+. \tag{1.3}$$

This is the simplest PDE combining both nonlinear propagation effects and diffusive effects and it was proposed as a model of turbulent fluid motion by J.M. Burgers in a series of several articles (see, for example, [5]). Burgers equation could serve as a nonlinear analog of the Navier-Stokes equations. Moreover, it arises in the study of pattern formation and in the context of modulations of spatially-periodic waves (see for example [6]). Noted that the parabolic equation (1.3) can be explicitly solved by means of the Hopf-Cole transformation and the exact formula reveals that the solution is regular for all times (see for example [19] for more details on the subject).

The unperturbed case (1.3) is considered in the mathematical community as a lab model. In fact, in more physical situations, the models are often more rich, hence more complicated, with respectively hyperbolic, laplacian and regular terms ($\varepsilon u_{\tau\tau}$, $a(\xi)$ and $\mathcal{N}(u)$). Therefore, it is completely meaningful for the mathematician to try to extend his methods and results to perturbations of the lab models, since the perturbed models are more encountered in the real-worlds models.

Starting with the hyperbolic perturbation of the Burgers equation (1.3). This hyperbolic modification consists in adding the term $\varepsilon u_{\tau\tau}$ to the equation (1.3), therefore we obtain the following equation

$$\varepsilon u_{\tau\tau} + u_\tau = u_{\xi\xi} - \left(u^2\right)_\xi \quad \text{in } \mathbb{R} \times \mathbb{R}_+. \tag{1.4}$$

We note that the effect of relaxation of various equations has been studied extensively, we take a moment to recall some known results. The problems of existence and uniqueness of solution of equation (1.4) have been studied by Escudero in [10]. Particularly in [25], Orive and Zuazua proved that the equation (1.4) has the same behavior in large time as the corresponding viscous Burgers equation. In [2], Brenier, Natalini and Puel have considered a hyperbolic perturbation of the classical Navier-Stokes equations consisting in adding the term εu_{tt} to the Euler equations in $\mathbb{R}^2$ after appropriate rescaling variables, the global existence and uniqueness for solutions are proved with regular initial data, for sufficiently small ε. In [26], Paicu and Raugel considered the same relaxed model, they proved, again if ε is small enough, global existence and uniqueness results of a mild solution only for much less regular in itial data, they thus improved the global existence results of [2]. Moreover, in the three-dimensional case, the global existence results for small initial data and sufficiently small ε in analogy to the classical case, are successfully proved.

On the other hand, we suppose that the laplacian term $u_{\xi\xi}$ in equation (1.4) is replaced by $(a(\xi)u_\xi)_\xi$, where $a(\xi)$ is a positive diffusion coefficient which satisfies $\lim_{\xi\to\pm\infty} a(\xi) = 1$. Moreover, for physical reasons, we consider a regular disturbance $\mathcal{N}(u)$ in equation (1.4) which verifies the hypothesis (1.2). Specifically, in most physical applications, we find a polynomial nonlinearity equivalent to u^2 close to zero.

Therefore, by combining the three previous perturbations we obtain equation (1.1).

Since in this paper we are going to analyse the asymptotic behavior of the solution of equation (1.1), it is interesting to introduce this field. We recall that the asymptotic stability of small solutions of the damped wave equation has been studied by Gallay and Raugel in [11] when the nonlinearity $\mathcal{N}(u, u_\xi, u_\tau)$ fulfills certain conditions and vanishes sufficiently as $u \to 0$, under appropriate assumptions on the function $a(\xi)$. They rely on an introduction of scaling variables and used energy estimates. Likewise, Hamza in [16] has treated the asymptotic stability of the solution of the damped hyperbolic equation when the nonlinearity is equal to $-|u|^{p-1}u$ and $p \in (1,3)$, under appropriate hypotheses on the diffusion $a(\xi)$. In [20], Jaffal-Mourtada has studied the asymptotic behavior of the solutions for the seconde grade fluids equations in two-dimensional space. Recently, Hamza in [17], has also considered the third grade fluids equation in one space dimension, where he shows that the large time behavior of solutions is given by the very singular self-similar solutions of the associated Burgers equation, note that in this case he proves that the nonlinear term does not disappear any more.

In the case $\varepsilon = 0$, $a(\xi) \equiv 1$ and $\mathcal{N} \equiv 0$, equation (1.1) reduces to the preceding Burgers equation (1.3).

Our purpose in this paper is to obtain that a small perturbation of the self-similar solution of equation (1.1) converges to the self-similar solution of the equation (1.3) with mass

$$M = \int_{\mathbb{R}} (u(\xi,\tau) + \varepsilon u_\tau(\xi,\tau))\, d\xi = \int_{\mathbb{R}} (u_0(\xi) + \varepsilon u_1(\xi))\, d\xi.$$

The mass M is conserved along the trajectory and should play a crucial role when describing the large time behavior of the solutions.

As we see above, the description of asymptotic behavior is given by the equation (1.3). It would be useful to recall some known results. First, we remark that if $u(\xi,\tau)$ is solution of equation (1.3), then for all $\lambda > 0, u_\lambda(\xi,\tau) = \lambda u(\lambda^2\tau, \lambda\xi)$ is also a solution. A solution $u \not\equiv 0$ is said to be self-similar, when $u_\lambda \equiv u$, for all $\lambda > 0$. Such a solution has the form $u(\xi,\tau) = \tau^{-\frac{1}{2}} f(\frac{\xi}{\sqrt{\tau}})$, where f is a positive function which satisfies the ordinary differential equation

$$f''(x) + \frac{x}{2} f'(x) + \frac{1}{2} f(x) - 2f(x)f'(x) = 0 \quad \text{in } \mathbb{R}. \tag{1.5}$$

A simple calculation yields

$$f_M(x) = \frac{e^{\frac{-x^2}{4}}}{C_M - \int\limits_{-\infty}^{x} e^{\frac{-t^2}{4}}\, dt}, \tag{1.6}$$

where $C_M = \frac{2\sqrt{\pi}}{1-e^{-M}}$ is a constant so that $\int_{\mathbb{R}} f_M(x)dx = M$. We denote by g_M the self-similar positive solution,

$$g_M(\xi,\tau) = \frac{1}{\sqrt{\tau}} \frac{e^{\frac{-\xi^2}{4\tau}}}{C_M - \int\limits_{-\infty}^{\frac{\xi}{\sqrt{\tau}}} e^{\frac{-t^2}{4}}\, dt}. \tag{1.7}$$

We notice that the mass of the function g_M is conserved since it is easy to prove that

$$\int_{\mathbb{R}} g_M(\xi,\tau)d\xi = M \quad \text{for all} \quad \tau > 0.$$

Although, g_M is not a solution of the hyperbolic equation (1.1), we can study its asymptotic stability in the following sense. Let $\tau_0 > 1$ be a fixed, sufficiently large real number that will be chosen later; for $(u(\xi,\tau_0), u_\tau(\xi,\tau_0))$ near $(g_M(\xi,\tau_0), \partial_\tau g_M(\xi,\tau_0))$, the corresponding solution $u(\xi,\tau)$ of equation (1.1) converges to $g_M(\xi,\tau)$ in an appropriate norm, when $\tau \to +\infty$.

Remark 1.1. We notice that, g_M is only an asymptotic solution. That is why we choose $\tau_0 > 1$ sufficiently large in this paper. This technical choice was made in many similar case (see for example in [11, 12, 16, 17] and [20]).

Now, we can rewrite (1.1) conveniently in terms of the variables

$$x = \frac{\xi}{\sqrt{\tau}} \quad \text{and} \quad t = \log\tau. \tag{1.8}$$

These similarity variables have been introduced before for proving the convergence to self-similar solutions in the case of the parabolic equation $u_\tau = u_{\xi\xi} - |u|^{p-1}u$ (see [3, 4, 7–9] and [21]). These techniques work even for a large similar class of equations (for more detail, see for example ([22, 28] and [32])). As we see above, the method of scaling variables coupled with energy estimates and various weighted energy estimates was successfully used by Gallay and Raugel in [11] and [12]. Similarly, Gallay and Wayne used these scaling variables to obtain the asymptotic behavior of the solution for the Navier-Stokes equations in the full space of the d-multidimensional case ($d = 2, 3$) (see [13] and [14] for example for more details on the subject).

Writing equation (1.1) as a first order system of equations for the functions u, u_τ and rescaling the two components independently as in [11], we are led to set

$$u(\xi,\tau) = \tau^{\frac{-1}{2}} v\left(\xi\tau^{\frac{-1}{2}}, \log\tau\right) \quad \text{and} \quad u_\tau(\xi,\tau) = \tau^{\frac{-3}{2}} w\left(\xi\tau^{\frac{-1}{2}}, \log\tau\right), \tag{1.9}$$

or equivalently

$$v(x,t) = e^{\frac{t}{2}} u\left(xe^{\frac{t}{2}}, e^t\right) \quad \text{and} \quad w(x,t) = e^{\frac{3t}{2}} u_\tau\left(xe^{\frac{t}{2}}, e^t\right).$$

It is easy to show that $u(\xi,\tau)$ is a solution of (1.1) if and only if the new functions $v(x,t), w(x,t)$ satisfy the system

$$\begin{cases} w = v_t - \frac{x}{2}v_x - \frac{1}{2}v, \\ \varepsilon e^{-t}\left[w_t - \frac{x}{2}w_x - \frac{3}{2}w\right] + w = \left(a\left(xe^{\frac{t}{2}}\right)v_x\right)_x - 2vv_x - e^t\left(\mathcal{N}\left(e^{-\frac{t}{2}}v\right)\right)_x, \end{cases} \tag{1.10}$$

where $x \in \mathbb{R}, t \geq t_0 = \log\tau_0$.

The initial data for (v, w) at time $t = t_0$ are related to those of u at time $\tau = \tau_0$ by

$$v(x, t_0) = e^{\frac{t_0}{2}} u\left(xe^{\frac{t_0}{2}}, e^{t_0}\right) \quad \text{and} \quad w(x, t_0) = e^{\frac{3t_0}{2}} u_\tau\left(xe^{\frac{t_0}{2}}, e^{t_0}\right).$$

Since we study the stability of the solutions $u(\xi, \tau)$ of equation (1.1), with initial data $(u(\xi, \tau_0), u_\tau(\xi, \tau_0))$ near $(g_M(\xi, \tau_0), \partial\tau g_M(\xi, \tau_0))$, it is convenient to perform the following change of functions.

$$F(x, t) = v(x, t) - f_M(x) \quad \text{and} \quad G(x, t) = w(x, t) + \frac{1}{2} f_M(x) + \frac{x}{2} f'_M(x).$$

The functions $F(x, t), G(x, t)$ satisfy the system:

$$\begin{cases} G = F_t - \frac{x}{2} F_x - \frac{1}{2} F, \\ \varepsilon e^{-t}\left[G_t - \frac{x}{2} G_x - \frac{3}{2} G\right] + F_t = \mathbf{L}(F) + \mathbf{N}(F) - \varepsilon e^{-t} r(x), \end{cases} \tag{1.11}$$

with the initial data

$$\begin{cases} F(x, t_0) = e^{\frac{t_0}{2}} u\left(xe^{\frac{t_0}{2}}, e^{t_0}\right) - f_M(x) = F_0(x), \\ G(x, t_0) = e^{\frac{3t_0}{2}} u_\tau\left(xe^{\frac{t_0}{2}}, e^{t_0}\right) + \frac{1}{2} f_M(x) + \frac{x}{2} f'_M(x) = G_0(x), \end{cases} \tag{1.12}$$

where

$$\begin{cases} \mathbf{L}(F) = \left(a\left(xe^{\frac{t}{2}}\right) F_x\right)_x + \frac{x}{2} F_x + \frac{1}{2} F - 2(F f_M)_x + \left[\left(a\left(xe^{\frac{t}{2}}\right) - 1\right) f'_M(x)\right]_x, \\ \mathbf{N}(F) = -(F^2)_x - e^t\left(\mathcal{N}\left(e^{-\frac{t}{2}}(F + f)\right)\right)_x, \\ r(x) = \frac{3}{2}\left(\frac{x}{2} f_M(x)\right)' + \left(\frac{x^2}{4} f'_M(x)\right)''. \end{cases} \tag{1.13}$$

We now give the precise assumptions on the diffusion $a(\xi)$. We will assume that the diffusion coefficient $a(\xi) : \mathbb{R} \to \mathbb{R}$ is a C^1-functions satisfying

$$\bar{a} > a(\xi) > \underline{a} \quad \text{for all } \xi \in \mathbb{R} \quad \text{and} \quad \lim_{|\xi| \to +\infty} a(\xi) = 1.$$

We set $b(\xi) = a(\xi) - 1$ and assume that

$$b \in L^2(\mathbb{R}) \quad \text{and} \quad \xi b'(\xi) \in L^2(\mathbb{R}). \tag{1.14}$$

Also, we define

$$|||b|||^2 = \|b\|^2_{L^2} + \int_{\mathbb{R}} \xi^2 |b'(\xi)|^2 d\xi. \tag{1.15}$$

We next introduce the Hilbert spaces in which we shall study the solutions of our problem (1.11). For any real number m, we define $L^2(m)$ the weighted Lebesgue space as

$$L^2(m) = \left\{u \in L^2(\mathbb{R}), u(1 + |x|^m) \in L^2(\mathbb{R})\right\},$$

equipped with the norm

$$\|u\|^2_{L^2(m)} = \int_{\mathbb{R}} u^2(1+|x|^{2m})dx.$$

Also, we define the following weighted Sobolev space

$$H^1(m) = \left\{u \in L^2(m), u_x \in L^2(m)\right\},$$

equipped with the following norm

$$\|u\|^2_{H^1(m)} = \|u\|^2_{L^2(m)} + \|u_x\|^2_{L^2(m)}.$$

In particular, we define the product space

$$X^m = H^1(m) \times L^2(m),$$

equipped with the standard norm

$$\|(v,w)\|^2_{X^m} = \|v\|^2_{H^1(m)} + \|w\|^2_{L^2(m)}.$$

Also, given $\varepsilon > 0$, we shall often endow the space X^m with the ε-dependent norm associated with the quadratic form

$$\Phi_m(\varepsilon, v, w) = \|v\|^2_{H^1(m)} + \varepsilon\|w\|^2_{L^2(m)}. \tag{1.16}$$

This ε-dependent norm will be useful to state existence results and estimates which are uniform in ε, as $\varepsilon \to 0$.

Here we proclaim our main result.

Theorem 1.2. *Let $m > 1$, $\gamma \geq 3$ and $\varepsilon_0 > 0$ be fixed. There exist $t_0 > 0$ and $\delta_0 > 0$, such that, for all $\varepsilon \in (0, \varepsilon_0]$ and for all $(F_0, G_0) \in X^m$ with*

$$\int_{\mathbb{R}} \left(F_0(x) + \varepsilon e^{-t_0} G_0(x)\right) dx = 0$$

and $\Phi_m(\varepsilon e^{-t_0}, F_0, G_0) \leq \delta_0^2$, the equation (1.11) has a unique solution $(F,G) \in \mathcal{C}^0([t_0, +\infty), X^m)$ satisfying

$$(F(t_0), G(t_0)) = (F_0, G_0).$$

Moreover, there exist $\mu_0 > 0$ and $C > 0$, such that for all $t \geq t_0$,

$$\begin{aligned} &\|F(t)\|^2_{H^1(m)} + \varepsilon e^{-t}\|G(t)\|^2_{L^2(m)} + \int_{t_0}^{t} e^{-\left(\frac{1}{2}+\mu_0\right)(t-s)}\|G(s)\|^2_{L^2(m)} ds \\ &\leq C\left[\Phi_m\left(\varepsilon e^{-t_0}, F_0, G_0\right) + \|b\|^2 e^{\frac{-t_0}{2}} + \varepsilon e^{-t_0} + e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right] e^{-\frac{1}{2}(t-t_0)}. \end{aligned} \tag{1.17}$$

Moreover, we have

$$\varepsilon\|G(t)\|_{L^2}^2 \leq C\left[\|(F_0,G_0)\|_{X^m}^2+\varepsilon\|b\|^2 e^{\frac{-t_0}{2}}+\varepsilon^2 e^{-t_0}+\varepsilon e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right]e^{-\frac{1}{2}(t-t_0)}, \tag{1.18}$$

where $\|b\|^2$ *is defined in* (1.15).

In the original variables, Theorem 1.2 implies the following result.

Corollary 1.3. *For all* $(u(\cdot,\tau_0),u_\tau(\cdot,\tau_0))\in X^m$ *such that*

$$\int_{\mathbb{R}}(u(\xi,\tau_0)+\varepsilon u_\tau(\xi,\tau_0))\,d\xi = M$$

and $\|u(\cdot,\tau_0)-g_M(\cdot,\tau_0)\|_{H^1(m)}+\varepsilon\|u_\tau(\cdot,\tau_0)-\partial_\tau g_M(\cdot,\tau_0)\|_{L^2(m)}$ *is small enough, the solution* u *of* (1.1) *belongs to* $\mathcal{C}^0([\tau_0,+\infty),H^1(m))\cap\mathcal{C}^1([\tau_0,+\infty),L^2(m))$ *and satisfies in particular the following estimate*

$$\|\sqrt{\tau}u(\xi\sqrt{\tau},\tau)-f_M(\xi)\|_{L^\infty}=\mathcal{O}\left(\tau^{-\frac{1}{4}}\right),\qquad \tau\to+\infty. \tag{1.19}$$

Moreover, by (1.18) *we have*

$$\|\sqrt{\tau}u_\tau(\xi\sqrt{\tau},\tau)+\frac{1}{2\tau}f_M(\xi)+\frac{\xi}{2\tau}f_M'(\xi)\|_{L^2}=\mathcal{O}\left(\tau^{-1}\right),\qquad \tau\to+\infty. \tag{1.20}$$

This result can be improved in the case where $b(\xi)=0$ and $\gamma>3$, in the following sense:

Theorem 1.4. *Let* $m>1$, $\gamma>3$ *and* $\varepsilon_0>0$ *be fixed. We suppose that* $b(\xi)=0$. *There exist* $t_0>0$ *and* $\delta_0>0$, *such that, for all* $\varepsilon\in(0,\varepsilon_0]$ *and for all* $(F_0,G_0)\in X^m$ *with*

$$\int_{\mathbb{R}}\left(F_0(x)+\varepsilon e^{-t_0}G_0(x)\right)dx=0$$

and

$$\Phi_m\left(\varepsilon e^{-t_0},F_0,G_0\right)\leq\delta_0^2,$$

the equation (1.11) *has a unique solution* $(F,G)\in\mathcal{C}^0\left([t_0,+\infty),X^m\right)$ *satisfying*

$$(F(t_0),G(t_0))=(F_0,G_0).$$

Moreover, there exist $\mu_0>0$ *and* $C>0$, *such that for all* $t\geq t_0$,

$$\begin{aligned}&\|F(t)\|_{H^1(m)}^2+\varepsilon e^{-t}\|G(t)\|_{L^2(m)}^2+\int_{t_0}^{t}e^{-\left(\frac{1}{2}+\mu_0\right)(t-s)}\|G(s)\|_{L^2(m)}^2ds\\&\leq C\left[\Phi_m\left(\varepsilon e^{-t_0},F_0,G_0\right)+\varepsilon e^{-t_0}+e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right]e^{-\min\left(\frac{\gamma}{2}-1,\frac{1}{2}+\mu_0\right)(t-t_0)}.\end{aligned} \tag{1.21}$$

Moreover, we have

$$\varepsilon\|G(t)\|_{L^2}^2\leq C\left[\|(F_0,G_0)\|_{X^m}^2+\varepsilon^2e^{-t_0}+\varepsilon e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right]e^{-\min\left(\frac{\gamma}{2}-1,\frac{1}{2}+\mu_0\right)(t-t_0)}. \tag{1.22}$$

Also, in the original variables, Theorem 1.4 implies the following corollary.

Corollary 1.5. *For all* $(u(\cdot,\tau_0),u_\tau(\cdot,\tau_0))\in X^m$ *such that*

$$\int_{\mathbb{R}} (u(\xi,\tau_0)+\varepsilon u_\tau(\xi,\tau_0))\,d\xi = M$$

and $\|u(\cdot,\tau_0)-g_M(\cdot,\tau_0)\|_{H^1(m)}+\varepsilon\|u_\tau(\cdot,\tau_0)-\partial_\tau g_M(\cdot,\tau_0)\|_{L^2(m)}$ *is small enough, the solution* u *of* (1.1) *belongs to* $\mathcal{C}^0([\tau_0,+\infty),H^1(m))\cap\mathcal{C}^1([\tau_0,+\infty),L^2(m))$ *and satisfies in particular the following estimate*

$$\|\sqrt{\tau}u(\xi\sqrt{\tau},\tau)-f_M(\xi)\|_{L^\infty}=\mathcal{O}\left(\tau^{-\min\left(\frac{\gamma}{4}-\frac{1}{2},\frac{1}{4}+\frac{\mu_0}{2}\right)}\right),\qquad \tau\to+\infty. \tag{1.23}$$

Moreover, by (1.22) *we have*

$$\|\sqrt{\tau}u_\tau(\xi\sqrt{\tau},\tau)+\frac{1}{2\tau}f_M(\xi)+\frac{\xi}{2\tau}f_M'(\xi)\|_{L^2}=\mathcal{O}\left(\tau^{-\min\left(\frac{\gamma}{4}+\frac{1}{4},\frac{\mu_0}{2}+1\right)}\right),\qquad \tau\to+\infty. \tag{1.24}$$

Remark 1.6. We notice that, if we study the classical functional $E_1(t)$ as in [11], then we obtain in the energy estimates some positive terms that we can not control (unlike the case in [11]). In particular, this does not allow us to obtain "good estimates" of the energy. To overcome this problem, we introduce a new functional $E_2(t)$, with a weight $q(x)$ (defined as below), in the way that the new functional is "equivalent" to $E_1(t)$. This choice is due to a purely technical reason. The construction of the weight $q(x)$ in the estimate (2.43) (below), use the fact that $f_M=f(x)$ and we deduce the crucial differential equality (defined in (2.49) below).

We now pass to a short discussion on the parabolic case. If $\varepsilon=0$, equation (1.1) is reduced to

$$u_\tau=(a(\xi)u_\xi)_\xi-\left(u^2+\mathcal{N}(u)\right)_\xi\quad \text{in } \mathbb{R}\times\mathbb{R}_+.$$

In the variable (x,t), the corresponding problem is

$$F_t=\mathbf{L}(F)+\mathbf{N}(F). \tag{1.25}$$

Since all the estimates in Theorem 1.2 are shown to be uniform in ε, for $0<\varepsilon\le\varepsilon_0$, we have the following result

Theorem 1.7. *Let* $m>1$ *and* $\gamma\ge 3$ *be fixed. There exist* $t_0>0$ *and* $\delta_0>0$*, such that, for all* $F_0\in H^1(m)$ *with* $\int_{\mathbb{R}}F_0(x)dx=0$ *and* $\|F_0\|_{H^1(m)}<\delta_0$*, the equation* (1.25) *has a unique solution* $F\in\mathcal{C}^0\left([t_0,+\infty),H^1(m)\right)$ *satisfying* $F(t_0)=F_0$*. Moreover, there exist* $C>0$*, such that for all* $t>t_0$*,*

$$\|F(t)\|_{H^1(m)}\le C\left[\|F_0\|^2_{H^1(m)}+\|\!|b|\!\|^2e^{\frac{-t_0}{2}}+e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right]^{\frac{1}{2}}e^{-\frac{1}{4}(t-t_0)}.$$

In this type of problem, the first step in the study of the asymptotic behavior of the solutions of equation (1.11) is to prove a local existence in weighted Sobolev spaces. In [11] a local existence result of the solutions of equation (1.11) in the energy space X^m was given for more general situations. Here we will state a specific version of the result as a lemma.

Lemma 1.8. *Let $\varepsilon > 0$, $m \in \mathbb{N}$, $\gamma \geq 3$ and $\delta > 0$ be given. There exists $t_{max} > 0$ such that, for all initial data $(F_0, G_0) \in X^m$, with*

$$\int\limits_{\mathbb{R}} \left(F_0(x) + \varepsilon e^{-t_0} G_0(x)\right) dx = 0$$

and $\|(F_0, G_0)\|_{X^m} < \delta$, the equation (1.11) has a unique (mild) solution $(F, G) \in \mathcal{C}^0([t_0, t_0 + t_{max}), X^m)$ satisfying

$$(F(t_0), G(t_0)) = (F_0, G_0).$$

The solution $(F(t), G(t))$ depends continuously on the initial data in X^m, uniformly in $t \in [t_0, t_0 + t_{max})$.

The proof of this lemma is based on the semi-group method given by Pazy in [27] in order to show that the Cauchy problem for (1.11) is locally well-posed in the space X^m.

Remark 1.9. We call a mild solution of a differential equation a continuous solution of the corresponding integral equation. In particular, every mild solution is a weak solution of the differential equation. For more details, see for example the works of Pazy in [27] and Ball in [1].

Remark 1.10. The theorem of the local existence, uniqueness and estimates of solutions in Hölder spaces for some nonlinear differential evolutionary system with initial conditions has been formulated and proved by Sapa in [31].

Remark 1.11. If $t_{max} < +\infty$, we have

$$\lim_{t \to \to t_0 + t_{max}} \| \left((F(t), G(t)\right) \|_{X^m} = +\infty.$$

This paper is divided as follows; in Section 2, we use various energy estimates to control the behavior in time of the solutions of system (1.11) in the space X^m. In Section 3, we show the global existence of the solutions and obtain the decay estimates (1.17) and (1.21). Finally in Section 4, we prove better estimates of the time derivative which allow to obtain (1.18) and (1.22).

2. ENERGY ESTIMATES

Throughout the section, in order to simplify the notations, we write the function f_M by f.

In this section, we make energy estimates on solutions of the equation (1.11). These estimates are independent on ε. Thus, we fix $\varepsilon_0 > 0$, $m > 1$ and $\gamma \geq 3$. We assume that for $\varepsilon \in (0, \varepsilon_0]$, $t_0 > 0$ and $T > 0$, we are given a solution $(F, G) \in \mathcal{C}^0([t_0, t_0 + T], X^m)$ of (1.11) with initial data $(F_0, G_0) \in X^m$, $\|(F_0, G_0)\|_{X^m} \leq \delta_0$ and

$$\int_{\mathbb{R}} \left(F_0(x) + \varepsilon e^{-t_0} G_0(x)\right) dx = 0,$$

which satisfies

$$\|F(t)\|_{H^1(m)} < \kappa\delta_0 < 1, \qquad t \in [t_0, t_0 + T] \tag{2.1}$$

where κ is a real number which will be fixed later and δ_0 is small enough such that $\kappa\delta_0 < 1$.

Our aim is to control the behavior of the solution (F, G) on the time interval $[t_0, t_0 + T]$ using energy functionals. To do that, we first decompose the solution (F, G). Then, we introduce the functions

$$\widetilde{F}(x,t) = F(x,t) - \alpha(t)\varphi(x) \quad \text{and} \quad \widetilde{G}(x,t) = G(x,t) - \beta(t)\varphi(x) - \alpha(t)\psi(x), \tag{2.2}$$

where $\varphi(x)$ satisfies the differential equation

$$\varphi'(x) + \frac{x}{2}\varphi(x) - 2\varphi(x)f(x) = 0, \tag{2.3}$$

and

$$\psi(x) = \frac{x}{2}\varphi'(x) + \frac{1}{2}\varphi(x). \tag{2.4}$$

A simple calculation yields

$$\varphi(x) = -\frac{2\sqrt{\pi}}{(1 - e^M)(1 - e^{-M})} \frac{e^{\frac{-x^2}{4}}}{(C_M - \int\limits_{-\infty}^{x} e^{\frac{-t^2}{4}} dt)^2}. \tag{2.5}$$

Clearly, we have

$$\int_{\mathbb{R}} \varphi(x)dx = 1 \quad \text{and} \quad \int_{\mathbb{R}} \psi(x)dx = 0. \tag{2.6}$$

We also set

$$\alpha(t) = \int_{\mathbb{R}} F(x,t)dx \quad \text{and} \quad \beta(t) = \int_{\mathbb{R}} G(x,t)dx. \tag{2.7}$$

Using (1.11), we remark that $\alpha(t)$ and $\beta(t)$ satisfy the following relation

$$\beta(t) = \dot{\alpha}(t) \quad \text{and} \quad \dot{\alpha}(t) + \varepsilon e^{-t}\left(\dot{\beta}(t) - \dot{\alpha}(t)\right) = 0. \tag{2.8}$$

Then, we obtain

$$\alpha(t) = \alpha(t_0)e^{-\frac{1}{\varepsilon}\left(e^t - e^{t_0}\right)} \quad \text{and} \quad \beta(t) = -\frac{\alpha(t_0)}{\varepsilon}e^t e^{-\frac{1}{\varepsilon}\left(e^t - e^{t_0}\right)}. \tag{2.9}$$

Remark 2.1. It should be noted that from (2.9), we see easily that the growth of $\alpha(t)$ and $\beta(t)$ is much smaller than $e^{-\zeta t}$ for all $\zeta > 0$.

It follows from (2.2), (2.6) and (2.7) that

$$\int_{\mathbb{R}} \widetilde{F}(y,t)dy = \int_{\mathbb{R}} \widetilde{G}(y,t)dy = 0.$$

Therefore we can define the primitive functions

$$V(x,t) = \int_{-\infty}^{x} \widetilde{F}(y,t)dy \quad \text{and} \quad W(x,t) = \int_{-\infty}^{x} \widetilde{G}(y,t)dy. \tag{2.10}$$

Let us recall the following Hardy type inequality (for more details on this subject, we refer the reader to Section 9.9 in [18])

$$\|V\|_{L^2} \le 2\|x\widetilde{F}\|_{L^2} \quad \text{and} \quad \|W\|_{L^2} \le 2\|x\widetilde{G}\|_{L^2}. \tag{2.11}$$

Using (1.11) and (2.2), we remark that $(\widetilde{F},\widetilde{G}) \in C^0\left([t_0, t_0+T], X^m\right)$ is a solution of the system

$$\begin{cases} \widetilde{G} = \widetilde{F}_t - \frac{x}{2}\widetilde{F}_x - \frac{1}{2}\widetilde{F}, \\ \varepsilon e^{-t}\left[\widetilde{G}_t - \frac{x}{2}\widetilde{G}_x - \frac{3}{2}\widetilde{G}\right] + \widetilde{G} = \left(a\left(xe^{\frac{t}{2}}\right)\widetilde{F}_x\right)_x - 2(\widetilde{F}f)_x - \left(\left(\widetilde{F}+\alpha(t)\varphi\right)^2\right)_x, \\ -e^t\left(\mathcal{N}\left(e^{-\frac{t}{2}}\left(\widetilde{F}+\alpha(t)\varphi+f\right)\right)\right)_x + \alpha(t)\left(b\left(xe^{\frac{t}{2}}\right)\varphi'\right)_x + h(x,t) - \varepsilon e^{-t}r(x), \end{cases} \tag{2.12}$$

where

$$h(x,t) = \left[\varepsilon e^{-t}\left(x\beta(t)\varphi - \frac{x}{2}\alpha(t)(\psi-\varphi)\right) + b\left(xe^{\frac{t}{2}}\right)f'\right]_x. \tag{2.13}$$

We notice that the term $b(xe^{\frac{t}{2}})\varphi'$ in system (2.12) is obtained by our choice of the function φ.

Since $(F,G) \in \mathcal{C}^0\left([t_0, t_0+T], X^m\right)$ and $m > 1$, then $(V,W) \in \mathcal{C}^0\left([t_0, t_0+T], X^0\right)$ is a classical solution of the system

$$\begin{cases} W = V_t - \frac{x}{2}V_x, \\ \varepsilon e^{-t}[W_t - \frac{x}{2}W_x - W] + W = a\left(xe^{\frac{t}{2}}\right)V_{xx} - 2V_x f - (V_x+\alpha(t)\varphi)^2, \\ -e^t(\mathcal{N}\left(e^{-\frac{t}{2}}(V_x+\alpha(t)\varphi+f)\right)) + \alpha(t)b\left(xe^{\frac{t}{2}}\right)\varphi' + H(x,t) - \varepsilon e^{-t}R(x), \end{cases} \tag{2.14}$$

where

$$H(x,t) = \int_{-\infty}^{x} h(y,t)dy = \varepsilon e^{-t}\left(x\beta(t)\varphi(x) - \frac{x}{2}\alpha(t)(\psi(x)-\varphi(x))\right) + b\left(xe^{\frac{t}{2}}\right)f'(x), \tag{2.15}$$

and

$$R(x)=\int_{-\infty}^{x} r(y)dy=\frac{3x}{4}f(x)+\left(\frac{x^2}{4}f'(x)\right)'. \tag{2.16}$$

In the sequel C denotes positive constants which can change from one step to another that are independent of $\kappa\delta_0$ and ε. In order to obtain some energy estimates, we need some properties of the function f defined in (1.6). More precisely, by exploiting the expression (1.6), we easily remark that we have the following estimate

$$\sum_{i=0}^{2}\sup_{x\in\mathbb{R}}\left|\left(1+|x|^{2m}\right)f^{(i)}(x)\right|\le C. \tag{2.17}$$

To control the norm of $(F,G)\in X^m$, we first make an estimate of (V,W) by introducing the following functional

$$E_1(t)=\frac{1}{2}\int_{\mathbb{R}} V^2(x,t)dx+\varepsilon e^{-t}\int_{\mathbb{R}} V(x,t)W(x,t)dx.$$

2.1. ESTIMATE OF (V,W)

Lemma 2.2. *Assume that* $(F,G)\in\mathcal{C}^0([t_0,t_0+T],X^m)$ *is a solution of* (1.11) *satisfying* (2.1) *and*

$$\int_{\mathbb{R}}\left(F_0(x)+\varepsilon e^{-t_0}G_0(x)\right)dx=0.$$

Then $E_1\in\mathcal{C}^1([t_0,t_0+T])$ *and there exists* $C>0$ *such that for all* $t\in[t_0,t_0+T]$,

$$\frac{d}{dt}E_1(t)+\frac{E_1(t)}{2}\le-\frac{1}{2}\int_{\mathbb{R}}F^2dx+\int_{\mathbb{R}}V^2f'dx+\Sigma_1(t), \tag{2.18}$$

where Σ_1 *satisfies for all* $\omega>0$,

$$\begin{aligned}\Sigma_1(t)\le{}& C\kappa\delta_0\int_{\mathbb{R}}(F^2+F_x^2)dx+C\|F_0\|_{L^2}^2e^{-2(t-t_0)}+C\varepsilon e^{-t}\int_{\mathbb{R}}G^2\left(1+|x|^{2m}\right)dx\\&+C\omega\int_{\mathbb{R}}F^2\left(1+|x|^{2m}\right)dx+C\left(1+\frac{1}{\omega}\right)e^{\frac{-t}{2}}\|b\|_{L^2}^2+C\varepsilon e^{-t}+Ce^{(-\frac{\gamma}{2}+1)t}.\end{aligned} \tag{2.19}$$

Proof. The functional E_1 is differentiable for $t\in[t_0,t_0+T]$ and

$$\frac{d}{dt}E_1(t)=\int_{\mathbb{R}}VV_tdx+\varepsilon e^{-t}\int_{\mathbb{R}}(VW_t+V_tW-VW)dx.$$

We note

$$\mathcal{M}(V) = a\left(xe^{\frac{t}{2}}\right)V_{xx} + \frac{x}{2}V_x - 2V_x f - (V_x + \alpha(t)\varphi)^2 - e^t\mathcal{N}\left(e^{-\frac{t}{2}}(V_x + \alpha(t)\varphi + f)\right)$$
$$+ \alpha(t)b\left(xe^{\frac{t}{2}}\right)\varphi' + H(x,t). \tag{2.20}$$

Using the identity,

$$VV_t + \varepsilon e^{-t}(VW_t + V_tW - VW) = V\mathcal{M}(V) + \varepsilon e^{-t}\left[W^2 - VR + \frac{x}{2}(VW)_x\right], \tag{2.21}$$

we obtain

$$\frac{d}{dt}E_1(t) = \int_{\mathbb{R}} V\mathcal{M}(V)dx + \varepsilon e^{-t}\int_{\mathbb{R}}\left[W^2 - VR + \frac{x}{2}(VW)_x\right]dx.$$

Then by integrating by parts, we have

$$\frac{d}{dt}E_1(t) = -\int_{\mathbb{R}}\widetilde{F}^2dx - \frac{1}{4}\int_{\mathbb{R}}V^2dx + \int_{\mathbb{R}}V^2f'dx + \Sigma_1^1(t),$$

where

$$\begin{aligned}\Sigma_1^1(t) \quad &= \int_{\mathbb{R}} b\left(xe^{\frac{t}{2}}\right)VV_{xx}dx - \int_{\mathbb{R}} V(V_x + \alpha(t)\varphi)^2dx \\ &-e^t\int_{\mathbb{R}} V\mathcal{N}\left(e^{-\frac{t}{2}}(V_x + \alpha(t)\varphi + f)\right)dx + \alpha(t)\int_{\mathbb{R}} Vb\left(xe^{\frac{t}{2}}\right)\varphi'dx.\end{aligned}$$

Hence

$$\frac{d}{dt}E_1(t) + \frac{E_1(t)}{2} = -\int_{\mathbb{R}}\widetilde{F}^2dx + \int_{\mathbb{R}}V^2f'dx + \Sigma_1^2(t), \tag{2.22}$$

where

$$\begin{aligned}\Sigma_1^2(t) &= \Sigma_1^1(t) + \frac{1}{2}\varepsilon e^{-t}\int_{\mathbb{R}}VWdx \\ &= \int_{\mathbb{R}} b\left(xe^{\frac{t}{2}}\right)VV_{xx}dx - \int_{\mathbb{R}}V(V_x + \alpha(t)\varphi)^2dx \\ &\quad - e^t\int_{\mathbb{R}}V\mathcal{N}\left(e^{-\frac{t}{2}}(V_x + \alpha(t)\varphi + f)\right)dx + \alpha(t)\int_{\mathbb{R}}Vb\left(xe^{\frac{t}{2}}\right)\varphi' \\ &\quad + \int_{\mathbb{R}}VH(x,t)dx + \varepsilon e^{-t}\int_{\mathbb{R}}\left(W^2 - VR\right)dx.\end{aligned}$$

Using the fact that $ab \le \frac{1}{4}a^2 + 4b^2$ and (2.9), we prove easily

$$-\int_{\mathbb{R}}\widetilde{F}^2dx \le -\frac{1}{2}\int_{\mathbb{R}}F^2dx + C\|F_0\|_{L^2}^2e^{-2(t-t_0)}. \tag{2.23}$$

By the Sobolev inequality, we get

$$\|V\|_{L^\infty} \le C\|V\|_{L^2}^{\frac{1}{2}} \|\widetilde{F}\|_{L^2}^{\frac{1}{2}}. \tag{2.24}$$

From (2.7), a simple calculation yields

$$|\alpha(t)| \le C\|F\|_{H^1(m)} < C\kappa\delta_0. \tag{2.25}$$

Using (2.2), the Hardy inequality (2.11), (2.24) and (2.25), we obtain

$$\|V\|_{L^\infty} \le C\|x\widetilde{F}\|_{L^2}^{\frac{1}{2}} \|\widetilde{F}\|_{L^2}^{\frac{1}{2}} \le C\|\widetilde{F}\|_{H^1(m)} \le C\|F\|_{H^1(m)}. \tag{2.26}$$

By (2.1), (2.5), (2.25) and (2.26), we get

$$\int_{\mathbb{R}} V(V_x + \alpha(t)\varphi)^2 dx \le \|V\|_{L^\infty} \int_{\mathbb{R}} F^2 dx \le C\kappa\delta_0 \int_{\mathbb{R}} F^2 dx. \tag{2.27}$$

By the Hardy inequality (2.11) we have the estimate

$$\|V\|_{L^2} \le C\|F\|_{L^2(m)}. \tag{2.28}$$

On the other hand, by Sobolev inequalities and (2.1), we find

$$\|F(t)\|_{L^\infty} \le C\|F(t)\|_{H^1(m)} < C\kappa\delta_0. \tag{2.29}$$

A simple calculation yields

$$\|(F+f)^\gamma\|_{L^2} \le C\|F^\gamma\|_{L^2} + C\|f^\gamma\|_{L^2} \le C\|F\|_{L^\infty}^{\gamma-1}\|F\|_{L^2} + C\|f^\gamma\|_{L^2}. \tag{2.30}$$

From (2.17), (2.29) and (2.30), we deduce that

$$\|(F+f)^\gamma\|_{L^2} \le C. \tag{2.31}$$

Also, by (1.2), we have

$$e^t \int_{\mathbb{R}} V\mathcal{N}\left(e^{-\frac{t}{2}}(F+f)\right) dx \le Ce^{\left(-\frac{\gamma}{2}+1\right)t} \int_{\mathbb{R}} |V||(F+f)^\gamma| dx. \tag{2.32}$$

Therefore, by using the fact that $ab \le a^2 + b^2$, (2.1), (2.28), (2.31) and (2.32), we obtain

$$\begin{aligned} e^t \int_{\mathbb{R}} V\mathcal{N}\left(e^{-\frac{t}{2}}(F+f)\right) dx &\le Ce^{\left(-\frac{\gamma}{2}+1\right)t} \left[\int_{\mathbb{R}} V^2 dx + \int_{\mathbb{R}} (F+f)^{2\gamma} dx\right] \\ &\le Ce^{\left(-\frac{\gamma}{2}+1\right)t}. \end{aligned} \tag{2.33}$$

Also, the fact that $ab \le \omega a^2 + \frac{1}{\omega} b^2$ for all $\omega > 0$, together with the fact that $\varphi' \in L^\infty(\mathbb{R})$ and (1.14), implies that

$$\left|\alpha(t) \int_{\mathbb{R}} Vb\left(xe^{\frac{t}{2}}\right)\varphi'\right| \le |\alpha(t)| \|\varphi'\|_{L^\infty} \left[\omega \int_{\mathbb{R}} V^2 dx + \frac{1}{\omega} \int_{\mathbb{R}} b^2\left(xe^{\frac{t}{2}}\right) dx\right].$$

Then, by (2.25) and (2.28), we get, for all $\omega > 0$,

$$\left|\alpha(t)\int\limits_{\mathbb{R}} Vb\left(xe^{\frac{t}{2}}\right)\varphi'\right| \leq C\omega\int\limits_{\mathbb{R}} F^2\left(1+|x|^{2m}\right)+\frac{C}{\omega}e^{\frac{-t}{2}}\|b\|^2_{L^2}. \tag{2.34}$$

By (2.2), (2.7) and the Hardy inequality (2.11) we have the estimate

$$\|W\|_{L^2} \leq C\left[\|F\|_{H^1(m)}+\|G\|_{L^2(m)}\right]. \tag{2.35}$$

Also, from (2.16), we find

$$\|R\|_{L^2} \leq C. \tag{2.36}$$

Hence, by combining (2.28), (2.35), (2.36) and the Cauchy-Schwarz inequality, we obtain

$$\varepsilon e^{-t}\int\limits_{\mathbb{R}}\left(W^2-VR\right)dx \leq C\left[\varepsilon e^{-t}\int\limits_{\mathbb{R}} G^2\left(1+|x|^{2m}\right)dx+\varepsilon e^{-t}\right]. \tag{2.37}$$

By the fact that $ab \leq a^2+b^2$ and (2.26), we conclude that

$$\int\limits_{\mathbb{R}} b\left(xe^{\frac{t}{2}}\right)VV_{xx}dx \leq \|V\|_{L^\infty}\left[\int\limits_{\mathbb{R}}\widetilde{F}_x^2dx+\int\limits_{\mathbb{R}} b^2\left(xe^{\frac{t}{2}}\right)dx\right]. \tag{2.38}$$

Therefore, by (1.14), (2.1), (2.9), (2.38), together with the fact that $\varphi' \in L^\infty(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$, we obtain

$$\int\limits_{\mathbb{R}} b\left(xe^{\frac{t}{2}}\right)VV_{xx}dx \leq C\kappa\delta_0\int\limits_{\mathbb{R}} F_x^2dx+C\|F_0\|^2_{L^2}e^{-2(t-t_0)}+Ce^{\frac{-t}{2}}\|b\|^2_{L^2}. \tag{2.39}$$

Now we control the term $\int\limits_{\mathbb{R}} VH(x,t)dx$.

By (2.17) and as in (2.34) we have the estimate, for all $\omega > 0$,

$$\int\limits_{\mathbb{R}} Vb\left(xe^{\frac{t}{2}}\right)f'dx \leq C\omega\int\limits_{\mathbb{R}} F^2\left(1+|x|^{2m}\right)+\frac{C}{\omega}e^{\frac{-t}{2}}\|b\|^2_{L^2}. \tag{2.40}$$

Also, we have

$$\begin{aligned}&\int\limits_{\mathbb{R}} V\left(x\beta(t)\varphi(x)-\frac{x}{2}\alpha(t)(\psi(x)-\varphi(x))\right)dx\\ &\leq \|V\|_{L^\infty}\int\limits_{\mathbb{R}}\left|x\beta(t)\varphi(x)-\frac{x}{2}\alpha(t)(\psi(x)-\varphi(x))\right|dx.\end{aligned} \tag{2.41}$$

Therefore, by (2.4), (2.5), (2.9), (2.15), (2.24), (2.40) and (2.41), we conclude that, for all $\omega > 0$,

$$\begin{aligned}\int_{\mathbb{R}} VH(x,t)dx \leq \frac{C}{\omega} e^{\frac{-t}{2}} \|b\|_{L^2}^2 + C\varepsilon e^{-t} \int_{\mathbb{R}} G^2 \left(1+|x^{2m}|\right) dx \\ + C\left[\varepsilon e^{-t} + \omega \int_{\mathbb{R}} F^2 \left(1+|x|^{2m}\right)\right].\end{aligned} \tag{2.42}$$

Combining (2.22), (2.23), (2.27), (2.33), (2.34), (2.37), (2.39) and (2.42), one easily obtains (2.18) and (2.19) where

$$\Sigma_1(t) = \Sigma_1^2(t) + C\|F_0\|_{L^2}^2 e^{-2(t-t_0)}.$$

□

Remark 2.3. We notice that the term $\int_{\mathbb{R}} V^2 f'$ in the inequality (2.18) is not necessarily negative, which does not allow to prove decay estimate of (F,G) in X^m. To overcome this difficulty, we construct a new functional $E_2(t)$ with a weight q by

$$E_2(t) = \frac{1}{2}\int_{\mathbb{R}} V^2(x,t)q(x)dx + \varepsilon e^{-t} \int_{\mathbb{R}} V(x,t)W(x,t)q(x)dx,$$

where q will be given below. We have $E_2(t)$ is equivalent to $E_1(t)$ that is, there exists a constant $C > 1$ such that we have $C^{-1}E_1(t) \leq E_2(t) \leq CE_1(t)$. The introduction of this new functional $E_2(t)$ is a crucial step to obtain optimal energy estimates.

2.2. BETTER ESTIMATE OF (V,W)

An appropriate choice for the weight q is, for example,

$$q(x) = C_M - \int_{-\infty}^{x} e^{\frac{-t^2}{4}} dt = f(x) e^{\frac{x^2}{4}} e^{-2H(x)}, \tag{2.43}$$

where

$$H(x) = -\log\left(C_M - \int_{-\infty}^{x} e^{\frac{-t^2}{4}} dt\right) = \int_{-\infty}^{x} f(y)dy.$$

Clearly, we have $q \in C^\infty(\mathbb{R})$, positive function and satisfying the bounds

$$\frac{1}{c_1} \leq q(x) \leq c_1, \quad \text{for all} \quad x \in \mathbb{R}, \tag{2.44}$$

where $c_1 > 1$. Moreover, we easily obtain the following estimate

$$\|q\|_{L^\infty(\mathbb{R})} + \sup_{x\in\mathbb{R}} \left|q'(x)\left(1+|x|\right)\right| \leq C, \tag{2.45}$$

which is useful to obtain the following lemma.

Lemma 2.4. *Assume that* $(F,G) \in \mathcal{C}^0\left([t_0, t_0+T], X^m\right)$ *is a solution of* (1.11) *satisfying* (2.1) *and*

$$\int_{\mathbb{R}} \left(F_0(x) + \varepsilon e^{-t_0} G_0(x)\right) dx = 0.$$

Then $E_2 \in \mathcal{C}^1\left([t_0, t_0+T]\right)$ *and there exists* $C>0$ *such that for all* $t \in [t_0, t_0+T]$,

$$\frac{d}{dt}E_2(t) + \frac{E_2(t)}{2} \leq -\frac{1}{2}\int_{\mathbb{R}} F^2 q(x) dx + \Sigma_2(t), \tag{2.46}$$

where Σ_2 *satisfies for all* $\omega > 0$,

$$\begin{aligned}\Sigma_2(t) \leq C\kappa\delta_0 \int_{\mathbb{R}} (F^2 + F_x^2) dx + C\|F_0\|_{L^2}^2 e^{-2(t-t_0)} + C\varepsilon e^{-t} \int_{\mathbb{R}} G^2\left(1+|x|^{2m}\right) dx \\ + C\omega \int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right) dx + C\left(1+\frac{1}{\omega}\right) e^{\frac{-t}{2}} \|b\|_{L^2}^2 + C\varepsilon e^{-t} + Ce^{\left(-\frac{\gamma}{2}+1\right)t}.\end{aligned} \tag{2.47}$$

Proof. The functional E_2 is differentiable for $t \in [t_0, t_0+T]$ and

$$\frac{d}{dt}E_2(t) = \int_{\mathbb{R}} VV_t q(x) dx + \varepsilon e^{-t} \int_{\mathbb{R}} \left(VW_t + V_t W - VW\right) q(x) dx.$$

Using the identity (2.21) and (2.20), we obtain

$$\frac{d}{dt}E_2(t) = \int_{\mathbb{R}} V\mathcal{M}(V) q dx + \varepsilon e^{-t} \int_{\mathbb{R}} \left[W^2 - VR + \frac{x}{2}(VW)_x\right] q dx.$$

Integrating by parts, we have

$$\begin{aligned}\frac{d}{dt}E_2(t) = \int_{\mathbb{R}} V\mathcal{M}(V) q(x) dx - \varepsilon \frac{e^{-t}}{2} \int_{\mathbb{R}} VWxq'(x) \\ + \varepsilon e^{-t} \int_{\mathbb{R}} \left(W^2 - VR - \frac{1}{2}VW\right) q(x) dx.\end{aligned}$$

Therefore

$$\frac{d}{dt}E_2(t) = \int_{\mathbb{R}} V\left[a\left(xe^{\frac{t}{2}}\right) V_{xx} + \frac{x}{2}V_x - 2fV_x\right] q(x) dx + \Sigma_2^1(t),$$

where

$$\Sigma_2^1(t) = -\int_{\mathbb{R}} V(V_x+\alpha(t)\varphi)^2 q dx - e^t \int_{\mathbb{R}} V\mathcal{N}\left(e^{-\frac{t}{2}}(V_x+\alpha(t)\varphi+f)\right) q dx$$
$$+\alpha(t)\int_{\mathbb{R}} Vb\left(xe^{\frac{t}{2}}\right)\varphi' q dx + \int_{\mathbb{R}} VH(x,t) q dx - \varepsilon\frac{e^{-t}}{2}\int_{\mathbb{R}} VWxq'(x)dx$$
$$+\varepsilon e^{-t}\int_{\mathbb{R}}\left(W^2 - VR - \frac{1}{2}VW\right) q(x)dx.$$

A simple calculation yields

$$\int_{\mathbb{R}} V\left[a\left(xe^{\frac{t}{2}}\right)V_{xx} + \frac{x}{2}V_x - 2fV_x\right] q(x)dx = \int_{\mathbb{R}} V\left[V_{xx} + \frac{x}{2}V_x - 2fV_x\right] q(x)dx + \Sigma_2^2(t),$$

where

$$\Sigma_2^2(t) = \int_{\mathbb{R}} b(xe^{\frac{t}{2}})VV_{xx}q(x)dx.$$

Let us recall that $q(x) = f(x)e^{\frac{x^2}{4}-2H(x)}$. Then, we deduce

$$\int_{\mathbb{R}} V\left[V_{xx} + \frac{x}{2}V_x - 2fV_x\right] q(x)dx = \int_{\mathbb{R}} V\left(V_x e^{\frac{x^2}{4}-2H}\right)_x f(x)dx$$
$$= -\int_{\mathbb{R}} V_x^2 q(x) + \frac{1}{2}\int_{\mathbb{R}} V^2\left(e^{\frac{x^2}{4}-2H}f'(x)\right)' dx. \tag{2.48}$$

Note that by (1.5) we may write

$$\left(e^{\frac{x^2}{4}-2H(x)}f'(x)\right)' = e^{\frac{x^2}{4}-2H(x)}\left(\frac{x}{2}f'(x) - 2f(x)f'(x) + f''(x)\right) = -\frac{1}{2}q(x). \tag{2.49}$$

We infer from (2.48) and (2.49) that

$$\int_{\mathbb{R}} V\left[V_{xx}dx + \frac{x}{2}V_x - 2fV_x\right] q(x)dx = -\int_{\mathbb{R}} \widetilde{F}^2 q(x)dx - \frac{1}{4}\int_{\mathbb{R}} V^2 q(x). \tag{2.50}$$

Using (2.23) and (2.44), we write

$$-\int_{\mathbb{R}} \widetilde{F}^2 q(x)dx \le -\frac{1}{2}\int_{\mathbb{R}} F^2 q(x)dx + C\|F(t_0)\|_{L^2}^2 e^{-2(t-t_0)}. \tag{2.51}$$

Therefore

$$\frac{d}{dt}E_2(t) + \frac{E_2(t)}{2} \le -\frac{1}{2}\int_{\mathbb{R}} F^2 q(x)dx + \Sigma_2(t), \tag{2.52}$$

where

$$\Sigma_2(t) = \Sigma_2^1(t) + \Sigma_2^2(t) + \frac{1}{2}\varepsilon e^{-t}\int\limits_{\mathbb{R}} VWq(x)dx + C\|F(t_0)\|_{L^2}^2 e^{-2(t-t_0)}. \tag{2.53}$$

By (2.1), (2.5), (2.25) (2.26) and (2.45), we get

$$\int\limits_{\mathbb{R}} V(V_x + \alpha(t)\varphi)^2 q dx \leq \|V\|_{L^\infty}\|q\|_{L^\infty}\int\limits_{\mathbb{R}} F^2 dx \leq C\kappa\delta_0 \int\limits_{\mathbb{R}} F^2 dx. \tag{2.54}$$

As in (2.33) and by (2.45), we have

$$e^t\int\limits_{\mathbb{R}} V\mathcal{N}\left(e^{-\frac{t}{2}}(V_x + \alpha(t)\varphi + f)\right) q dx \leq Ce^{(-\frac{\gamma}{2}+1)t}. \tag{2.55}$$

By the fact that $ab \leq a^2 + b^2$, (2.26) and (2.45), we conclude that

$$\int\limits_{\mathbb{R}} b(xe^{\frac{t}{2}})VV_{xx}q(x)dx \leq \|V\|_{L^\infty}\|q\|_{L^\infty}\left[\int\limits_{\mathbb{R}} \widetilde{F}_x^2 dx + \int\limits_{\mathbb{R}} b^2(xe^{\frac{t}{2}})dx\right]. \tag{2.56}$$

Therefore, by (1.14), (2.1), (2.9), (2.56), together with the fact that $\varphi' \in L^\infty(\mathbb{R})$ and $\varphi \in L^2(\mathbb{R})$, we obtain

$$\int\limits_{\mathbb{R}} b(xe^{\frac{t}{2}})VV_{xx}q(x)dx \leq C\kappa\delta_0\int\limits_{\mathbb{R}} F_x^2 dx + C\|F_0\|_{L^2}^2 e^{-2(t-t_0)} + Ce^{\frac{-t}{2}}\|b\|_{L^2}^2. \tag{2.57}$$

The fact that $ab \leq \omega a^2 + \frac{1}{\omega}b^2$ for all $\omega > 0$, together with the fact that $\varphi' \in L^\infty(\mathbb{R})$, (1.14) and (2.45), implies that

$$\left|\alpha(t)\int\limits_{\mathbb{R}} Vb(xe^{\frac{t}{2}})\varphi' q\right| \leq |\alpha(t)|\|\varphi'\|_{L^\infty}\|q\|_{L^\infty}\left[\omega\int\limits_{\mathbb{R}} V^2 dx + \frac{1}{\omega}\int\limits_{\mathbb{R}} b^2(xe^{\frac{t}{2}})dx\right].$$

Furthermore, by (2.1), (2.25) and (2.28), we get, for all $\omega > 0$,

$$\left|\alpha(t)\int\limits_{\mathbb{R}} Vb(xe^{\frac{t}{2}})\varphi' q\right| \leq C\omega\int\limits_{\mathbb{R}} F^2\left(1 + |x|^{2m}\right) + \frac{C}{\omega}e^{\frac{-t}{2}}\|b\|_{L^2}^2. \tag{2.58}$$

In a similar way, by (2.28), (2.35) and (2.45), we have

$$\int\limits_{\mathbb{R}} VWxq'dx \leq \|xq'\|_{L^\infty}\|V\|_{L^2}\|W\|_{L^2},$$

this estimate together with (2.37), implies that

$$\begin{aligned}&\varepsilon\frac{e^{-t}}{2}\int\limits_{\mathbb{R}}VWxq'(x)dx+\varepsilon e^{-t}\int\limits_{\mathbb{R}}(W^2-VR)q(x)dx\\&\le C\left[\varepsilon e^{-t}\int\limits_{\mathbb{R}}G^2\left(1+|x^{2m}|\right)dx+\varepsilon e^{-t}\right].\end{aligned}\tag{2.59}$$

Similarly as in (2.42) together with (2.45), we obtain, for all $\omega>0$,

$$\begin{aligned}\int\limits_{\mathbb{R}}VH(x,t)qdx\le\frac{C}{\omega}e^{\frac{-t}{2}}\|b\|_{L^2}^2+C\varepsilon e^{-t}\int\limits_{\mathbb{R}}G^2\left(1+|x^{2m}|\right)dx\\+C\left[\varepsilon e^{-t}+\omega\int\limits_{\mathbb{R}}F^2\left(1+|x|^{2m}\right)\right].\end{aligned}\tag{2.60}$$

In the same way to (2.39), we have

$$\Sigma_2^2(t)\le C\kappa\delta_0\int\limits_{\mathbb{R}}F_x^2dx+C\|F_0\|_{L^2}^2e^{-2(t-t_0)}+Ce^{\frac{-t}{2}}\|b\|_{L^2}^2.\tag{2.61}$$

Consequently, combining (2.52), (2.53), (2.54), (2.55), (2.57), (2.58),(2.59) and (2.60), one easily obtains (2.46) and (2.47). This concludes the proof of Lemma 2.4. □

2.3. ESTIMATES IN X^m OF (F,G)

To control the time behavior of $(F(t),G(t))$ defined in (1.11) in X^m, we introduce the following energy functionals

$$\begin{aligned}E_3(t)&=\frac{1}{2}\int\limits_{\mathbb{R}}F^2(x,t)dx+\varepsilon e^{-t}\int\limits_{\mathbb{R}}F(x,t)G(x,t)dx,\\E_4(t)&=\frac{1}{2}\int\limits_{\mathbb{R}}F^2(x,t)|x|^{2m}dx+\varepsilon e^{-t}\int\limits_{\mathbb{R}}F(x,t)G(x,t)|x|^{2m}dx,\\E_5(t)&=E_3(t)+E_4(t).\end{aligned}$$

Lemma 2.5. *Assume that* $(F,G)\in\mathcal{C}^0\left([t_0,t_0+T],X^m\right)$ *is a solution of* (1.11) *satisfying* (2.1) *and*

$$\int\limits_{\mathbb{R}}\left(F_0(x)+\varepsilon e^{-t_0}G_0(x)\right)dx=0.$$

Then $E_5 \in \mathcal{C}^1([t_0, t_0+T])$ *and there exists* $C>0$ *such that for all* $t \in [t_0, t_0+T]$ *and* $\varrho > 0$,

$$\begin{aligned}\frac{d}{dt}E_5(t) + \frac{E_5(t)}{2} \le & -\underline{a}\int_{\mathbb{R}} F_x^2\left(1+|x|^{2m}\right)dx + \left(\frac{3}{4} - \frac{2m+1}{4}\right)\int_{\mathbb{R}} F^2|x|^{2m}dx \\ & + \frac{1}{2}\int_{\mathbb{R}} F^2 dx + m(2m-1)\overline{a}\int_{\mathbb{R}} F^2|x|^{2m-2}dx \\ & + C\left(1+\frac{\varrho}{\underline{a}}\right)\int_{\mathbb{R}} F^2 dx + \frac{C\underline{a}}{\varrho}\int_{\mathbb{R}} F_x^2 dx + \Sigma_5(t),\end{aligned} \tag{2.62}$$

where Σ_5 *satisfies for all* $\omega > 0$,

$$\begin{aligned}\Sigma_5(t) \le & C\kappa\delta_0 \int_{\mathbb{R}} \left(F^2+F_x^2\right)\left(1+|x|^{2m}\right)dx + C\varepsilon e^{-t}\int_{\mathbb{R}} G^2\left(1+|x|^{2m}\right)dx \\ & + \frac{C}{\omega}\|\!|b|\!\|^2 e^{\frac{-t}{2}} + C\varepsilon e^{-t} + C\omega\int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right)dx + C\omega\int_{\mathbb{R}} F_x^2 dx + Ce^{\left(-\frac{\gamma}{2}+1\right)t}.\end{aligned} \tag{2.63}$$

Proof. The functional E_3 is differentiable for $t \in [t_0, t_0+T]$ and

$$\frac{d}{dt}E_3(t) = \int_{\mathbb{R}} FF_t dx + \varepsilon e^{-t}\int_{\mathbb{R}} (FG_t + F_tG - FG)dx.$$

Using the identity,

$$FF_t + \varepsilon e^{-t}(FG_t + F_tG - FG) = F\mathbf{L}(F) + F\mathbf{N}(F) + \varepsilon e^{-t}\left[\frac{x}{2}(FG)_x + G^2 - Fr\right]. \tag{2.64}$$

We obtain

$$\frac{d}{dt}E_3(t) = \int_{\mathbb{R}} F\mathbf{L}(F)dx + \int_{\mathbb{R}} F\mathbf{N}(F)dx + \varepsilon e^{-t}\int_{\mathbb{R}}\left(G^2 - Fr + \frac{1}{2}FG\right)dx.$$

Then by integrating by parts, we have

$$\frac{d}{dt}E_3(t) = -\int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right)F_x^2 dx + \frac{1}{4}\int_{\mathbb{R}} F^2 dx - 2\underbrace{\int_{\mathbb{R}} F(Ff)_x dx}_{A_0(t)} + \Sigma_3(t),$$

where

$$\Sigma_3(t) = \underbrace{-\int_{\mathbb{R}} b\left(xe^{\frac{t}{2}}\right) f' F_x dx}_{A_1(t)} \underbrace{-\int_{\mathbb{R}} F\left(F^2\right)_x dx}_{A_2(t)}$$
$$\underbrace{-\int_{\mathbb{R}} Fe^t\left(\mathcal{N}\left(e^{-\frac{t}{2}}(F+f)\right)\right)_x dx}_{A_3(t)} + \varepsilon e^{-t}\int_{\mathbb{R}}\left(G^2 - Fr + \frac{1}{2}FG\right) dx.$$

In a similar way, E_4 is a differentiable function for $t \in [t_0, t_0+T]$ and

$$\frac{d}{dt}E_4(t) = \int_{\mathbb{R}} FF_t|x|^{2m} dx + \varepsilon e^{-t}\int_{\mathbb{R}}(FG_t + F_tG - FG)|x|^{2m} dx.$$

Using (2.64) and integrating by parts, we have

$$\frac{d}{dt}E_4(t) = -\int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right) F_x^2|x|^{2m} dx + \left(\frac{1}{2} - \frac{2m+1}{4}\right)\int_{\mathbb{R}} F^2|x|^{2m} dx$$
$$+ m(2m-1)\int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right) F^2|x|^{2m-2} dx \underbrace{- 2\int_{\mathbb{R}} F(Ff)_x|x|^{2m} dx}_{B_0(t)} + \Sigma_4(t),$$

where

$$\Sigma_4(t) = \underbrace{m\int_{\mathbb{R}} b'\left(xe^{\frac{t}{2}}\right) F^2 xe^{\frac{t}{2}}|x|^{2m-2} dx}_{B_1(t)} + \underbrace{\int_{\mathbb{R}} e^{\frac{t}{2}} b'\left(xe^{\frac{t}{2}}\right) f'F|x|^{2m} dx}_{B_2(t)}$$
$$+ \underbrace{\int_{\mathbb{R}} b\left(xe^{\frac{t}{2}}\right) f''F|x|^{2m} dx}_{B_3(t)} \underbrace{- \int_{\mathbb{R}} F\left(F^2\right)_x |x|^{2m} dx}_{B_4(t)}$$
$$\underbrace{-\int_{\mathbb{R}} Fe^t\left(\mathcal{N}\left(e^{-\frac{t}{2}}(F+f)\right)\right)_x |x|^{2m} dx}_{B_5(t)} + \varepsilon e^{-t}\int_{\mathbb{R}}\left(G^2 - Fr + \frac{1}{2}FG\right)|x|^{2m} dx.$$

We note

$$\Sigma_5^1(t) = \Sigma_3(t) + \Sigma_4(t). \tag{2.65}$$

The remaining of the proof of this lemma is devoted to the estimate of these terms. Using the inequalities $ab \le \omega a^2 + \frac{1}{\omega}b^2$ for all $\omega > 0$, together with (1.6) and (1.14), we obtain

$$A_1(t) \le C\omega\int_{\mathbb{R}} F_x^2 dx + \frac{C}{\omega}e^{\frac{-t}{2}}\|b\|_{L^2}^2. \tag{2.66}$$

Now, we estimate the expression $B_1(t)$. Using the inequalities $\left\||x|^{m-1}F\right\|_{L^\infty} \leq C$ and $ab \leq \omega a^2 + \frac{1}{\omega}b^2$ for all $\omega > 0$, together with (1.14) and the fact that $|x|^{2m-2} < 1+|x|^{2m}$, we obtain

$$\begin{aligned} B_1(t) &\leq \left\||x|^{m-1}F\right\|_{L^\infty} \int_{\mathbb{R}} \left|b'\left(xe^{\frac{t}{2}}\right)Fx\right| e^{\frac{t}{2}}|x|^{m-1}dx \\ &\leq C\omega \int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right) + \frac{C}{\omega}e^{\frac{-t}{2}} \int_{\mathbb{R}} \xi^2|b'(\xi)|^2 d\xi. \end{aligned} \tag{2.67}$$

Using (1.6), we have

$$\left\|f'(x)\sqrt{1+|x|^{2m}}\right\|_{L^\infty} \leq C.$$

This property together with (1.14), (2.1) and the fact that $|x|^{2m-1} < 1+|x|^{2m}$ implies that, for all $\omega > 0$,

$$\begin{aligned} B_2(t) &= \int_{\mathbb{R}} xe^{\frac{t}{2}}b'\left(xe^{\frac{t}{2}}\right)f'F|x|^{2m-1}dx \\ &\leq C\omega \int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right) + \frac{C}{\omega}e^{\frac{-t}{2}} \int_{\mathbb{R}} \xi^2|b'(\xi)|^2 d\xi. \end{aligned} \tag{2.68}$$

Similarly, by using (1.6), we have

$$\left\|f''(x)\sqrt{1+|x|^{2m}}\right\|_{L^\infty} \leq C$$

and thus, for all $\omega > 0$,

$$B_3(t) \leq C\omega \int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right) + \frac{C}{\omega}e^{\frac{-t}{2}}\|b\|_{L^2}^2. \tag{2.69}$$

Now, we estimate the expression $A_2(t) + B_4(t)$. The estimate $ab \leq a^2 + b^2$ together with (2.29), implies that

$$\begin{aligned} A_2(t) + B_4(t) &\leq 2\|F(t)\|_{L^\infty} \int_{\mathbb{R}} |FF_x|\left(1+|x|^{2m}\right) \\ &\leq C\kappa\delta_0 \int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right)dx + C\kappa\delta_0 \int_{\mathbb{R}} F_x^2\left(1+|x|^{2m}|\right)dx. \end{aligned} \tag{2.70}$$

Now we control the term $A_3(t)+B_5(t)$. A simple calculation yields

$$\begin{aligned} A_3(t)+B_5(t) = & \underbrace{-\,e^t\int_{\mathbb{R}} \mathcal{N}\left(e^{-\frac{t}{2}}(F+f)\right)F_x\left(1+|x|^{2m}\right)dx}_{K_1(t)} \\ & \underbrace{-\,2m\,e^t\int_{\mathbb{R}} \mathcal{N}\left(e^{-\frac{t}{2}}(F+f)\right)F|x|^{2m-1}dx}_{K_2(t)}\,. \end{aligned} \tag{2.71}$$

By using the fact that $ab \le a^2+b^2$ together with (1.2), we get

$$K_1(t) \le Ce^{\left(-\frac{\gamma}{2}+1\right)t}\left[\int_{\mathbb{R}} F_x^2\left(1+|x|^{2m}\right)dx + \int_{\mathbb{R}}(F+f)^{2\gamma}\left(1+|x|^{2m}\right)dx\right]. \tag{2.72}$$

In the same way, by adding the estimate $|x|^{2m-1} \le 1+|x|^{2m}$, we obtain

$$K_2(t) \le Ce^{\left(-\frac{\gamma}{2}+1\right)t}\left[\int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right)dx + \int_{\mathbb{R}}(F+f)^{2\gamma}\left(1+|x|^{2m}\right)dx\right]. \tag{2.73}$$

From (2.17) and (2.29), we deduce

$$\left\|(F+f)^{\gamma}\sqrt{1+|x|^{2m}}\right\|_{L^2} \le C. \tag{2.74}$$

By combining (2.1), (2.71), (2.72), (2.73) and (2.74), we have

$$A_3(t)+B_5(t) \le Ce^{\left(-\frac{\gamma}{2}+1\right)t}. \tag{2.75}$$

Likewise, we estimate the term $A_0(t)+B_0(t)$. Indeed, by (2.17) together with the estimate $ab \le \frac{\varrho}{\underline{a}}a^2+\frac{\underline{a}}{\varrho}b^2$ for all $\varrho>0$, we get

$$\int_{\mathbb{R}} |FF_xf|\left(1+|x|^{2m}\right) \le \frac{\varrho C}{\underline{a}}\int_{\mathbb{R}} F^2dx + \frac{C\underline{a}}{\varrho}\int_{\mathbb{R}} F_x^2dx. \tag{2.76}$$

Therefore, by (2.76), we obtain, for all $\varrho>0$,

$$A_0(t)+B_0(t) \le C\left(1+\frac{\varrho}{\underline{a}}\right)\int_{\mathbb{R}} F^2dx + \frac{C\underline{a}}{\varrho}\int_{\mathbb{R}} F_x^2dx. \tag{2.77}$$

By (2.1), (2.65), (2.66), (2.67), (2.68), (2.69), (2.70), (2.75), (2.77), by the Cauchy–
–Schwarz inequality together with the fact that $\|r(x)\|_{L^2(m)} = C$, one easily obtains, for all $\varrho > 0$,

$$\begin{aligned}\frac{d}{dt}E_5(t) \le -\underline{a}\int_{\mathbb{R}} F_x^2\left(1+|x|^{2m}\right)dx + \left(\frac{1}{2}-\frac{2m+1}{4}\right)\int_{\mathbb{R}} F^2|x|^{2m}dx \\ +\frac{1}{4}\int_{\mathbb{R}} F^2dx + m(2m-1)\overline{a}\int_{\mathbb{R}} F^2|x|^{2m-2}dx \\ +C\left(1+\frac{\varrho}{\underline{a}}\right)\int_{\mathbb{R}} F^2dx + \frac{C\underline{a}}{\varrho}\int_{\mathbb{R}} F_x^2dx + \Sigma_5^1(t),\end{aligned} \tag{2.78}$$

where Σ_5^1 satisfies, for all $\omega > 0$,

$$\begin{aligned}\Sigma_5^1(t) \le C\kappa\delta_0\int_{\mathbb{R}}\left(F^2+F_x^2\right)\left(1+|x|^{2m}\right)dx + C\varepsilon e^{-t}\int_{\mathbb{R}} G^2\left(1+|x|^{2m}\right)dx \\ +\frac{C}{\omega}\|b\|^2 e^{\frac{-t}{2}} + C\varepsilon e^{-t} + C\omega\int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right).\end{aligned} \tag{2.79}$$

By adding the term $\frac{E_5(t)}{2}$ in (2.78), we get to (2.62) and (2.63), where

$$\Sigma_5(t) = \Sigma_5^1(t) + \frac{1}{2}\varepsilon e^{-t}\int_{\mathbb{R}} F(x,t)G(x,t)\left(1+|x|^{2m}\right)dx.$$

Lemma 2.5 is thus shown. □

Remark 2.6. We notice that the term

$$\frac{1}{2}\int_{\mathbb{R}} F^2dx + m(2m-1)\overline{a}\int_{\mathbb{R}} F^2|x|^{2m-2}dx + C\Big(1+\frac{\varrho}{\underline{a}}\Big)\int_{\mathbb{R}} F^2dx$$

in the inequality (2.62) is positive, which does not allow to prove decay estimate of (F, G) in X^m. To overcome this difficulty, we introduce the following functional

$$E_{6,\rho}(t) = E_5(t) + \rho E_2(t),$$

where ρ is a sufficiently large constant that will be determined later.

The results of the preceding lemmas can be summarized as follows.

Lemma 2.7. *Assume that* $(F, G) \in \mathcal{C}^0\left([t_0, t_0+T], X^m\right)$ *is a solution of* (1.11) *satisfying* (2.1) *and*

$$\int_{\mathbb{R}}\left(F_0(x) + \varepsilon e^{-t_0}G_0(x)\right)dx = 0.$$

Then $E_{6,\rho} \in \mathcal{C}^1([t_0, t_0+T])$ *and there exists* $C_1 > 0$ *such that for all* $t \in [t_0, t_0+T]$,

$$\begin{aligned}\frac{d}{dt}E_{6,\rho}(t) + \frac{E_{6,\rho}(t)}{2} \le &-\frac{\underline{a}}{2}\int_{\mathbb{R}} F_x^2\left(1+|x|^{2m}\right)dx - \lambda_0 \int_{\mathbb{R}} F^2\left(1+|x|^{2m}\right)dx \\ &+ C_1\varepsilon e^{-t}\int_{\mathbb{R}} G^2\left(1+|x|^{2m}\right)dx + C_1\|b\|^2 e^{\frac{-t}{2}} \\ &+ C_1\varepsilon e^{-t} + C_1\|F_0\|_{L^2}^2 e^{-2(t-t_0)} + C_1 e^{\left(-\frac{\gamma}{2}+1\right)t}.\end{aligned} \tag{2.80}$$

Proof. We know that, for all $\rho > 0$, $E_{6,\rho} \in \mathcal{C}^1([t_0, t_0+T])$. Then, we get for all $\rho > 0$ and for all $\varrho > 0$,

$$\begin{aligned}&\frac{d}{dt}E_{6,\rho}(t) + \frac{E_{6,\rho}(t)}{2} \le -\underline{a}\int_{\mathbb{R}} F_x^2\left(1+|x|^{2m}\right)dx + \frac{C\underline{a}}{\varrho}\int_{\mathbb{R}} F_x^2 dx \\ &+ \underbrace{\left(\frac{3}{4} - \frac{2m+1}{4}\right)\int_{\mathbb{R}} F^2|x|^{2m}dx + \frac{1}{2}\int_{\mathbb{R}} F^2 dx + m(2m-1)\overline{a}\int_{\mathbb{R}} F^2|x|^{2m-2}dx}_{L_1(t)} \\ &+ \underbrace{C\left(1+\frac{\varrho}{\underline{a}}\right)\int_{\mathbb{R}} F^2 dx - \frac{1}{2}\rho\int_{\mathbb{R}} F^2 q(x)dx}_{L_2(t)} + \Sigma_5(t) + \rho\Sigma_2(t).\end{aligned} \tag{2.81}$$

Since $\frac{3}{4} - \frac{2m+1}{4} < 0$, we can choose R_0 large enough so that, for all $\varrho > 0$,

$$\left(\frac{3}{4} - \frac{2m+1}{4}\right) + \frac{m(2m-1)\overline{a}}{R_0^2} + \frac{1}{2}\frac{1}{R_0^{2m}} + \left(1+\frac{\varrho}{\underline{a}}\right)\frac{C}{R_0^{2m}} \le \frac{1}{2}\left(\frac{3}{4} - \frac{2m+1}{4}\right). \tag{2.82}$$

For all $\rho > 0$ and for all $\varrho > 0$, we have

$$\begin{aligned}L_1(t) + L_2(t) \le &\left[\frac{1}{2} + m(2m-1)\overline{a}R_0^{2m-2} + C\left(1+\frac{\varrho}{\underline{a}}\right)\right]\int_{|x|<R_0} F^2 dx \\ &- \frac{1}{2}\rho\int_{|x|<R_0} F^2 q(x)dx \\ &+ \left[\left(\frac{3}{4} - \frac{2m+1}{4}\right) + \frac{m(2m-1)\overline{a}}{R_0^2} + \frac{1}{2}\frac{1}{R_0^{2m}} + \left(1+\frac{\varrho}{\underline{a}}\right)\frac{C}{R_0^{2m}}\right]\int_{|x|>R_0} F^2|x|^{2m}dx.\end{aligned} \tag{2.83}$$

Now we choose $\rho = \rho_0$ large enough, so that, for all $\varrho > 0$,

$$\frac{1}{2} + m(2m-1)\overline{a}R_0^{2m-2} + C\left(1+\frac{\varrho}{\underline{a}}\right) - \frac{1}{2c_1}\rho_0 \le -1. \tag{2.84}$$

We recall that c_1 is the constant defined in (2.44).
From (2.44), (2.81), (2.82), (2.83) and (2.84) we deduce that, for all $\varrho > 0$,

$$\begin{aligned} L_1(t) + L_2(t) &\le \frac{1}{2}\left(\frac{3}{4} - \frac{2m+1}{4}\right) \int\limits_{|x|>R_0} F^2 |x|^{2m} dx - \int\limits_{|x|<R_0} F^2 dx \\ &\le -2\lambda_0 \int\limits_{\mathbb{R}} F^2 \left(1 + |x|^{2m}\right) dx, \end{aligned} \tag{2.85}$$

where $\lambda_0 > 0$.

If we choose $\kappa\delta_0$ and ω small enough and satisfies $C\kappa\delta_0 < \frac{a}{8}$, we deduce that

$$L_3(t) \le \lambda_0 \int\limits_{\mathbb{R}} F^2 \left(1 + |x|^{2m}\right) dx, \tag{2.86}$$

where

$$\begin{aligned} L_3(t) = C\kappa\delta_0 &\int\limits_{\mathbb{R}} F^2 \left(1 + |x|^{2m}\right) dx \\ &+ \rho_0 C\kappa\delta_0 \int\limits_{\mathbb{R}} F^2 dx + C\omega(1+\rho_0) \int\limits_{\mathbb{R}} F^2 \left(1 + |x|^{2m}\right) dx, \end{aligned}$$

and

$$C\omega \int\limits_{\mathbb{R}} F_x^2 dx + C\kappa\delta_0 \int\limits_{\mathbb{R}} F_x^2 \left(1 + |x|^{2m}\right) dx \le \frac{a}{4} \int\limits_{\mathbb{R}} F_x^2 \left(1 + |x|^{2m}\right) dx. \tag{2.87}$$

In a similar way, if we choose also $\varrho > 8C$, we obtain

$$C\left(\frac{a}{\varrho} + \rho_0\kappa\delta_0\right) \int\limits_{\mathbb{R}} F_x^2 dx \le \frac{a}{4} \int\limits_{\mathbb{R}} F_x^2 \left(1 + |x|^{2m}\right) dx. \tag{2.88}$$

Combining (2.81), (2.85), (2.86), (2.87) and (2.88), we get (2.80). □

Since we want to control $(F, G) \in X^m$, it is natural to introduce the following functionals:

$$\begin{aligned} E_7(t) &= \frac{1}{2} \int\limits_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right) F_x^2(x,t) dx + \frac{\varepsilon e^{-t}}{2} \int\limits_{\mathbb{R}} G^2(x,t) dx, \\ E_8(t) &= \frac{1}{2} \int\limits_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right) F_x^2(x,t) |x|^{2m} dx + \frac{\varepsilon e^{-t}}{2} \int\limits_{\mathbb{R}} G^2(x,t) |x|^{2m} dx, \\ E_9(t) &= E_7(t) + E_8(t). \end{aligned}$$

Lemma 2.8. *Assume that* $(F,G) \in \mathcal{C}^0([t_0, t_0+T], X^m)$ *is a solution of* (1.11) *satisfying* (2.1) *and*

$$\int_{\mathbb{R}} \left(F_0(x) + \varepsilon e^{-t_0} G_0(x)\right) dx = 0.$$

Then $E_9 \in \mathcal{C}^1([t_0, t_0+T])$ *and there exists* $C_2 > 0$ *such that for all* $t \in [t_0, t_0+T]$,

$$\begin{aligned}
&\frac{d}{dt}E_9(t) + \frac{E_9(t)}{2} \\
&\leq -\frac{1}{2}\int_{\mathbb{R}} G^2\left(1+|x|^{2m}\right)dx + C_2\int_{\mathbb{R}}\left[F^2+F_x^2\right]\left(1+|x|^{2m}\right)dx \\
&\quad + C_2 \|b\|^2 e^{\frac{-t}{2}} + C_2\varepsilon e^{-t}\int_{\mathbb{R}} G^2\left(1+|x|^{2m}\right)dx + C_2\varepsilon e^{-t} + C_2 e^{\left(-\frac{\gamma}{2}+1\right)t}.
\end{aligned} \tag{2.89}$$

Proof. The functional E_7 is of class $\mathcal{C}^1([t_0, t_0+T])$ and

$$\frac{d}{dt}E_7(t) = \int_{\mathbb{R}} a(xe^{\frac{t}{2}})F_x F_{xt}dx + \varepsilon e^{-t}\int_{\mathbb{R}}\left(GG_t - \frac{G^2}{2}\right)dx + \frac{1}{4}\int_{\mathbb{R}} xe^{\frac{t}{2}}b'(xe^{\frac{t}{2}})F_x^2.$$

Using the fact that (F,G) is a solution of (1.11), we can write

$$\begin{aligned}
\frac{d}{dt}E_7(t) &= \int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right)F_x\left[G_x + \frac{x}{2}F_{xx} + F_x\right]dx + \int_{\mathbb{R}} G\left[\mathbf{L}(F) - F_t - \varepsilon e^{-t}r(x)\right]dx \\
&\quad + \varepsilon e^{-t}\int_{\mathbb{R}} G\left(\frac{x}{2}G_x + G\right)dx + \int_{\mathbb{R}} G\mathbf{N}(F)dx + \frac{1}{4}\int_{\mathbb{R}} xe^{\frac{t}{2}}b'\left(xe^{\frac{t}{2}}\right)F_x^2.
\end{aligned}$$

Integrating by parts, therefore

$$\begin{aligned}
\frac{d}{dt}E_7(t) &= -\int_{\mathbb{R}} G^2dx + \frac{3}{4}\int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right)F_x^2dx + \frac{3}{4}\varepsilon e^{-t}\int_{\mathbb{R}} G^2dx \\
&\quad - \varepsilon e^{-t}\int_{\mathbb{R}} Grdx - 2\int_{\mathbb{R}} G(Ff)_xdx + \int_{\mathbb{R}} G\mathbf{N}(F)dx + \int_{\mathbb{R}} G\left[b\left(xe^{\frac{t}{2}}\right)f'\right]_x dx.
\end{aligned}$$

Using the Cauchy-Schwarz inequality together with the estimate $ab \leq a^2+b^2$ and (2.75), we show that

$$\begin{aligned}
\frac{d}{dt}E_7(t) + \frac{E_7(t)}{2} &\leq -\frac{1}{2}\int_{\mathbb{R}} G^2dx + C\int_{\mathbb{R}}\left[F^2+F_x^2\right]dx + C\varepsilon e^{-t}\int_{\mathbb{R}} G^2dx \\
&\quad + C\|b\|^2 e^{\frac{-t}{2}} + C\varepsilon e^{-t} + Ce^{\left(-\frac{\gamma}{2}+1\right)t}.
\end{aligned} \tag{2.90}$$

In a similar way,

$$\frac{d}{dt}E_8(t) = \int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right) F_x F_{xt}|x|^{2m}dx + \varepsilon e^{-t}\int_{\mathbb{R}} \left(GG_t - \frac{G^2}{2}\right)|x|^{2m}dx + \frac{1}{4}\int_{\mathbb{R}} xe^{\frac{t}{2}}b'\left(xe^{\frac{t}{2}}\right)F_x^2|x|^{2m}dx.$$

Arguing as above, we obtain

$$\begin{aligned}\frac{d}{dt}E_8(t) = &\int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right) F_x\left[G_x + \frac{x}{2}F_{xx} + F_x\right]|x|^{2m}dx \\ &+ \int_{\mathbb{R}} G\left[\mathbf{L}(F) - F_t - \varepsilon e^{-t}r(x)\right]|x|^{2m}dx + \varepsilon e^{-t}\int_{\mathbb{R}} G\left(\frac{x}{2}G_x + G\right)|x|^{2m}dx \\ &+ \int_{\mathbb{R}} G\mathbf{N}(F)|x|^{2m}dx + \frac{1}{4}\int_{\mathbb{R}} xe^{\frac{t}{2}}b'\left(xe^{\frac{t}{2}}\right)F_x^2|x|^{2m}dx.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}E_8(t) = &-\int_{\mathbb{R}} G^2|x|^{2m}dx + \left(1 - \frac{2m+1}{4}\right)\int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right)F_x^2|x|^{2m}dx + \int_{\mathbb{R}} G\mathbf{N}(F)|x|^{2m}dx \\ &- \varepsilon e^{-t}\int_{\mathbb{R}} Gr|x|^{2m}dx - 2\int_{\mathbb{R}} G(Ff)_x|x|^{2m}dx \\ &- 2m\int_{\mathbb{R}} a\left(xe^{\frac{t}{2}}\right)F_xG|x|^{2m-1}dx + \int_{\mathbb{R}} G\left[b\left(xe^{\frac{t}{2}}\right)f'\right]_x|x|^{2m}dx. \\ &+ \left(1 - \frac{2m+1}{4}\right)\varepsilon e^{-t}\int_{\mathbb{R}} G^2|x|^{2m}dx.\end{aligned}$$

In the same way, we prove that

$$\begin{aligned}\frac{d}{dt}E_8(t) + \frac{E_8(t)}{2} \le &-\frac{1}{2}\int_{\mathbb{R}} G^2|x|^{2m}dx \\ &+ C\int_{\mathbb{R}}\left[F^2 + F_x^2\right]|x|^{2m}dx + C\varepsilon e^{-t}\int_{\mathbb{R}} G^2|x|^{2m}dx \\ &+ C|||b|||^2 e^{\frac{-t}{2}} + C\varepsilon e^{-t} + Ce^{\left(-\frac{\gamma}{2}+1\right)t}.\end{aligned} \tag{2.91}$$

Finally, combining (2.90) and (2.91), we get (2.89). This concludes the proof of Lemma 2.8. □

3. PROOF OF THEOREM 1.2 AND THEOREM 1.4 (FIRST PARTS)

We introduce the following functional

$$E_{10,\sigma}(t) = E_{6,\rho_0}(t) + \sigma E_9(t),$$

where $\sigma > 0$ will be chosen later.

Proposition 3.1. *Let $\kappa\delta_0$ be small enough. There exists a positive constant $\mu_0 > 0$, so that, for any solution $(F, G) \in \mathcal{C}^0([t_0, t_0 + T], X^m)$ of (1.11) with $(F(t_0), G(t_0)) = (F_0, G_0) \in X^m$, chosen so that*

$$\int_{\mathbb{R}} \left(F_0(x) + \varepsilon e^{-t_0} G_0(x)\right) dx = 0$$

and $\Phi_m(\varepsilon e^{-t_0}, F_0, G_0) < \delta_0^2$, we have for all $t \in [t_0, t_0 + T]$

$$\begin{aligned} &\|F(t)\|^2_{H^1(m)} + \varepsilon e^{-t}\|G(t)\|^2_{L^2(m)} + \int_{t_0}^{t} e^{-\left(\frac{1}{2}+\mu_0\right)(t-s)}\|G(s)\|^2_{L^2(m)} ds \\ &\le C\left[\Phi_m\left(\varepsilon e^{-t_0}, F_0, G_0\right) + \|\!|b|\!\|^2 e^{\frac{-t_0}{2}} + \varepsilon e^{-t_0} + e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right] e^{-\frac{1}{2}(t-t_0)}. \end{aligned} \tag{3.1}$$

Proof. The functional $E_{10,\sigma}$ is of class $\mathcal{C}^1([t_0, t_0 + T])$. Moreover, by combining (2.80) and (2.89), we can write easily

$$\begin{aligned} &\frac{d}{dt}E_{10,\sigma}(t) + \frac{E_{10,\sigma}(t)}{2} \\ &\le -\lambda_0 \int_{\mathbb{R}} F^2\left(1 + |x|^{2m}\right) dx - \frac{a}{2}\int_{\mathbb{R}} F_x^2\left(1 + |x|^{2m}\right) \\ &\quad - \frac{\sigma}{2}\int_{\mathbb{R}} G^2\left(1 + |x|^{2m}\right) dx + (\sigma C_2 + C_1)\varepsilon e^{-t}\int_{\mathbb{R}} G^2\left(1 + |x|^{2m}\right) dx \\ &\quad + \sigma C_2 \int_{\mathbb{R}} \left[F^2 + F_x^2\right]\left(1 + |x|^{2m}\right) dx + (\sigma C_2 + C_1)\|\!|b|\!\|^2 e^{\frac{-t}{2}} \\ &\quad + (\sigma C_2 + C_1)\varepsilon e^{-t} + (\sigma C_2 + C_1)e^{\left(-\frac{\gamma}{2}+1\right)t} + C_1\|F_0\|^2_{L^2} e^{-2(t-t_0)}. \end{aligned}$$

Also, there exists $K > 1$ such that for all $t \in [t_0, t_0 + T]$, where t_0 is large enough, we have

$$\frac{1}{K}\Phi_m\left(\varepsilon e^{-t}, F(t), G(t)\right) \le E_{10,\sigma}(t) \le K\Phi_m\left(\varepsilon e^{-t}, F(t), G(t)\right). \tag{3.2}$$

We then conclude that

$$\frac{d}{dt}E_{10,\sigma}(t)+\left(\frac{1}{2}+\mu_0\right)E_{10,\sigma}(t)\leq-(\lambda_1-B_1)\int_{\mathbb{R}}\left[F^2+F_x^2\right]\left(1+|x|^{2m}\right)dx$$
$$-\left(\frac{\sigma}{2}-B_2\right)\int_{\mathbb{R}}G^2\left(1+|x|^{2m}\right)dx \tag{3.3}$$
$$+(\sigma C_2+C_1)\left[\varepsilon e^{-t}+\|b\|^2e^{\frac{-t}{2}}+e^{\left(-\frac{\gamma}{2}+1\right)t}\right]$$
$$+C_1\|F_0\|_{L^2}^2e^{-2(t-t_0)},$$

where $\lambda_1=\inf\left(\frac{a}{2},\lambda_0\right), B_1=\sigma C_2+K\mu_0$ and $B_2=(\sigma C_2+K\mu_0+C_1)\varepsilon e^{-t}$. We choose σ and μ_0 small enough and t_0 large enough, so that

$$B_1<\frac{\lambda_1}{2},B_2<\frac{\sigma}{4}.$$

Then, we deduce from (3.3) that for all $t\in[t_0,t_0+T]$,

$$\frac{d}{dt}E_{10,\sigma}(t)+\left(\frac{1}{2}+\mu_0\right)E_{10,\sigma}(t)+\frac{\sigma}{4}\|G(t)\|_{L^2(m)}^2\leq C\varepsilon e^{-t}+Ce^{\left(-\frac{\gamma}{2}+1\right)t}+C\|b\|^2e^{\frac{-t}{2}}$$
$$+C\|F_0\|_{L^2}^2e^{-2(t-t_0)}. \tag{3.4}$$

Integrating (3.4), we obtain for all $t\in[t_0,t_0+T]$,

$$E_{10,\sigma}(t)+\frac{\sigma}{4}\int_{t_0}^{t}e^{-\left(\frac{1}{2}+\mu_0\right)(t-s)}\|G(s)\|_{L^2(m)}^2ds$$
$$\leq\left[E_{10,\sigma}(t_0)+C\|b\|^2e^{\frac{-t_0}{2}}+C\varepsilon e^{-t_0}+Ce^{\left(-\frac{\gamma}{2}+1\right)t_0}+C\|F_0\|_{L^2}^2\right]e^{-\frac{1}{2}(t-t_0)}. \tag{3.5}$$

Now (3.1) is a direct consequence of (3.2) and (3.5). This concludes the proof of Proposition 3.1. □

Proof of Theorem 1.2 (First part). Let $\varepsilon_0>0$ be fixed. We choose $t_0>0$ large enough and δ_0 small enough. For $\varepsilon\in(0,\varepsilon_0]$, if $(F_0,G_0)\in X^m$ satisfies

$$\int_{\mathbb{R}}\left(F_0(x)+\varepsilon e^{-t_0}G_0(x)\right)dx=0$$

and $\Phi_m\left(\varepsilon e^{-t_0},F_0,G_0\right)\leq\delta_0^2$, then the equation (1.11) has a unique solution $(F,G)\in\mathcal{C}^0\left([t_0,t_0+t_{max}),X^m\right)$ satisfying $(F(t_0),G(t_0))=(F_0,G_0)$.
To prove that this solution is global, we argue by contradiction. Assume that there exists $\widetilde{T}>0$ such that

$$\|F(t)\|_{H^1(m)}<\kappa\delta_0,\quad\text{for all}\quad t\in[t_0,t_0+\widetilde{T}) \tag{3.6}$$

and

$$\|F(\widetilde{T})\|_{H^1(m)} = \kappa\delta_0. \tag{3.7}$$

If t_0 is large enough, so that

$$\|b\|^2 e^{\frac{-t_0}{2}} + \varepsilon e^{-t_0} + e^{\left(-\frac{\gamma}{2}+1\right)t_0} \leq \delta_0^2$$

and if $\kappa > \sqrt{8C}$, we have by (3.1), for all $t \in [t_0, t_0 + \widetilde{T}]$,

$$\|F(t)\|^2_{H^1(m)} \leq C\left[\delta_0^2 + \|b\|^2 e^{\frac{-t_0}{2}} + \varepsilon e^{-t_0} + e^{(-\frac{\gamma}{2}+1)t_0}\right].$$

Then

$$\|F(t)\|^2_{H^1(m)} \leq 2C\delta_0^2 \leq \frac{\kappa^2\delta_0^2}{4} \leq \frac{\kappa\delta_0}{4},$$

which contradicts (3.7). Thus we have

$$\|F(t)\|_{H^1(m)} < \kappa\delta_0 \quad \text{for all} \quad t \in [t_0, t_0 + t_{max}).$$

By Proposition 3.1, we conclude that

$$\Phi_m\left(\varepsilon e^{-t}, F(t), G(t)\right) \leq \frac{\kappa\delta_0}{4} \leq \frac{1}{4} \quad \text{for all} \quad t \in [t_0, t_0 + t_{max}).$$

Then, Remark 1.11 implies that the solution can be continued to $[t_0, +\infty)$. By Proposition 3.1, we conclude the proof of the first part of Theorem 1.2. □

Proof of Theorem 1.4 (*First part*). Similarly, in the case where $b(\xi) = 0$ and $\gamma > 3$, we find

$$\begin{aligned} &\frac{d}{dt}E_{10,\sigma}(t) + \left(\frac{1}{2} + \mu_0\right)E_{10,\sigma}(t) + \frac{\sigma}{4}\|G(t)\|^2_{L^2(m)} \\ &\leq C\varepsilon e^{-t} + Ce^{\left(-\frac{\gamma}{2}+1\right)t} + C\|F_0\|^2_{L^2} e^{-2(t-t_0)}. \end{aligned} \tag{3.8}$$

Integrating (3.8), we obtain for all $t \in [t_0, t_0 + T]$,

$$\begin{aligned} &E_{10,\sigma}(t) + \frac{\sigma}{4}\int_{t_0}^{t} e^{-\left(\frac{1}{2}+\mu_0\right)(t-s)}\|G(s)\|^2_{L^2(m)}ds \\ &\leq \left[E_{10,\sigma}(t_0) + C\varepsilon e^{-t_0} + Ce^{\left(-\frac{\gamma}{2}+1\right)t_0} + C\|F_0\|^2_{L^2}\right] e^{-\min\left(\frac{\gamma}{2}-1, \frac{1}{2}+\mu_0\right)(t-t_0)}. \end{aligned} \tag{3.9}$$

Arguing as above, we show that the first part of Theorem 1.4 is a direct consequence of (3.2) and (3.9). □

4. FURTHER ESTIMATES ON THE TIME DERIVATIVE

This section is devoted to prove the second parts of Theorem 1.2 and Theorem 1.4.

Proof of Theorem 1.2 (Second part). As before, let $(F, G) \in \mathcal{C}^0([t_0, t_0+T], X^m)$ be a solution of (1.11) satisfying

$$\int_{\mathbb{R}} \left(F_0(x) + \varepsilon e^{-t_0} G_0(x)\right) dx = 0$$

and the bound (2.1). We define the new function $M(x,t) = W_t(x,t) - \frac{x}{2} W_x(x,t)$, where W is given by (2.10). It is straightforward to show that the function $(W, M) \in \mathcal{C}^0([t_0, t_0+T], X^0)$ and satisfies the system

$$\begin{cases} M = W_t - \frac{x}{2} W_x, \\ \varepsilon e^{-t}[M_t - \frac{x}{2} M_x - 2M] + M = a\left(xe^{\frac{t}{2}}\right)(V_{xx} + W_{xx}) + J(x,t), \end{cases} \tag{4.1}$$

where

$$J(x,t) = -\varepsilon e^{-t} W - \frac{x}{2}\frac{dH_1}{dx} + \frac{dH_1}{dt}, \tag{4.2}$$

and

$$\begin{aligned} H_1(x,t) = & -2V_x f - (V_x + \alpha(t)\varphi)^2 - e^t \mathcal{N}\left(e^{-\frac{t}{2}}(V_x + \alpha(t)\varphi + f)\right) \\ & + \alpha(t) b\left(xe^{\frac{t}{2}}\right)\varphi' + H(x,t) - \varepsilon e^{-t} R(x). \end{aligned} \tag{4.3}$$

In analogy with the preceding section, we introduce the energy functional

$$E_{11}(t) = \frac{1}{2}\int_{\mathbb{R}} W_x^2(x,t)dx + \frac{\varepsilon e^{-t}}{2}\int_{\mathbb{R}} \frac{M^2(x,t)}{a\left(xe^{\frac{t}{2}}\right)}dx.$$ □

Lemma 4.1. *Under the hypotheses of Theorem 1.2, there exists $C > 0$ such that for t_0 large enough, for all $t \geq t_0$,*

$$\varepsilon\|\widetilde{G}(t)\|_{L^2}^2 \leq C\left[\|(F_0, G_0)\|_{X^m}^2 + \varepsilon\|b\|^2 e^{\frac{-t_0}{2}} + \varepsilon^2 e^{-t_0} + \varepsilon e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right] e^{-\frac{1}{2}(t-t_0)}. \tag{4.4}$$

Proof. Clearly, the functional E_{11} is differentiable for all $t \geq t_0$ and

$$\begin{aligned} \frac{d}{dt}E_{11}(t) = & \frac{1}{4}\int_{\mathbb{R}} W_x^2 dx - \int_{\mathbb{R}} \frac{M^2}{a\left(xe^{\frac{t}{2}}\right)}dx \\ & + \int_{\mathbb{R}} \frac{MJ}{a\left(xe^{\frac{t}{2}}\right)}dx + \int_{\mathbb{R}} MV_{xx}dx + \frac{5}{4}\varepsilon e^{-t}\int_{\mathbb{R}} \frac{M^2}{a\left(xe^{\frac{t}{2}}\right)}dx. \end{aligned} \tag{4.5}$$

By exploiting the Cauchy–Schwarz inequality and (4.5), we get

$$\begin{aligned}\frac{d}{dt}E_{11}(t)+\frac{E_{11}(t)}{2}\leq \frac{3}{4}\int_{\mathbb{R}} W_x^2dx-\frac{1}{2}\int_{\mathbb{R}}\frac{M^2}{a\left(xe^{\frac{t}{2}}\right)}dx+C\int_{\mathbb{R}} J^2dx\\ +C\int_{\mathbb{R}} V_{xx}^2dx+C\varepsilon e^{-t}\int_{\mathbb{R}}\frac{M^2}{a\left(xe^{\frac{t}{2}}\right)}dx.\end{aligned}\tag{4.6}$$

Using (4.2), we have

$$\|J\|_{L^2}\leq C\left(\left\|x\frac{dH_1}{dx}\right\|_{L^2}+\varepsilon e^{-t}\|W\|_{L^2}+\left\|\frac{dH_1}{dt}\right\|_{L^2}\right).\tag{4.7}$$

By (4.3), we get

$$\begin{aligned}\frac{dH_1}{dx}(x,t)=&-2V_{xx}f-2V_xf'-2(V_x+\alpha(t)\varphi)(V_{xx}+\alpha(t)\varphi')\\ &-\underbrace{e^{\frac{t}{2}}\left(F_x+f'\right)\mathcal{N}'\left(e^{-\frac{t}{2}}(F+f)\right)}_{I_1(\cdot,t)}+\underbrace{\alpha(t)e^{\frac{t}{2}}b'\left(xe^{\frac{t}{2}}\right)\varphi'}_{I_2(\cdot,t)}\\ &+\underbrace{\alpha(t)b\left(xe^{\frac{t}{2}}\right)\varphi''}_{I_3(\cdot,t)}+h(x,t)-\varepsilon e^{-t}r(x).\end{aligned}\tag{4.8}$$

To estimate $xI_1(x,t)$, we use (1.2) to obtain

$$\|xI_1(x,t)\|_{L^2}^2\leq Ce^{(-\gamma+2)t}\int_{\mathbb{R}}(F+f)^{2\gamma-2}\left(F_x+f'\right)^2x^2dx.\tag{4.9}$$

The estimate $x^2\leq 1+x^{2m}$ implies that we have

$$\|xI_1(x,t)\|_{L^2}^2\leq Ce^{(-\gamma+2)t}\,\|F+f\|_{L^\infty}^{2\gamma-2}\int_{\mathbb{R}}\left(F_x+f'\right)^2\left(1+x^{2m}\right)dx.\tag{4.10}$$

From (2.17) and (2.29) we deduce that

$$\|F+f\|_{L^\infty}^{2\gamma-2}\leq C\quad\text{and}\quad\left\|\left(F_x+f'\right)\sqrt{1+x^{2m}}\right\|_{L^2}\leq C.\tag{4.11}$$

Therefore, by adding (4.10) and (4.11), we conclude

$$\|xI_1(x,t)\|_{L^2}\leq Ce^{\left(-\frac{\gamma}{2}+1\right)t}.\tag{4.12}$$

On the other hand, since $\|\varphi'\|_{L^\infty}+\|x\varphi''\|_{L^\infty}\leq C$ and by (2.25), we can write

$$\|xI_2(x,t)\|_{L^2}+\|xI_3(x,t)\|_{L^2}\leq C|\!|\!|b|\!|\!|e^{\frac{-t}{4}}.\tag{4.13}$$

By (4.8), (4.12) and (4.13), we obtain

$$\left\| x \frac{dH_1}{dx} \right\|_{L^2} + \varepsilon e^{-t} \|W\|_{L^2} \leq C \left[\|F\|_{H^1(m)} + \varepsilon e^{-t} \|G\|_{L^2(m)} \right] + C \left[\|\|b\|\| e^{\frac{-t}{4}} + \varepsilon e^{-t} + e^{\left(-\frac{\gamma}{2}+1\right)t} + \|F_0\|^2_{L^2} e^{-2(t-t_0)} \right]. \tag{4.14}$$

Differentiating $H_1(x,t)$, we get

$$\begin{aligned} \frac{dH_1}{dt}(x,t) = & -2 \left(G + \frac{x}{2} F_x + \frac{1}{2} F \right) f - 2FG - xFF_x - F^2 + 2\beta(t)\varphi f \\ & - \underbrace{e^t \mathcal{N}\left(e^{-\frac{t}{2}}(F+f) \right)}_{I_4(\cdot,t)} - \underbrace{e^{\frac{t}{2}} \left(G + \frac{x}{2} F_x + \frac{1}{2} F \right) \mathcal{N}'\left(e^{-\frac{t}{2}}(F+f) \right)}_{I_5(\cdot,t)} \\ & + \underbrace{\frac{1}{2} e^{\frac{t}{2}} (F+f) \mathcal{N}'\left(e^{-\frac{t}{2}}(F+f) \right)}_{I_6(\cdot,t)} + \underbrace{\beta(t) b\left(x e^{\frac{t}{2}} \right) \varphi'}_{I_7(\cdot,t)} \\ & + \underbrace{\frac{\alpha(t)}{2} x e^{\frac{t}{2}} b'\left(x e^{\frac{t}{2}} \right) \varphi'}_{I_8(\cdot,t)} \\ & - \varepsilon e^{-t} \left(x\beta(t)\varphi - \frac{x}{2}\alpha(t)(\psi - \varphi) \right) + \varepsilon e^{-t} \left(x\dot{\beta}(t)\varphi - \frac{x}{2}\beta(t)(\psi - \varphi) \right) \\ & + \underbrace{\frac{1}{2} x e^{\frac{t}{2}} b'\left(x e^{\frac{t}{2}} \right) f'}_{I_9(\cdot,t)} + \varepsilon e^{-t} R(x). \end{aligned} \tag{4.15}$$

As in (4.12), we have

$$\|I_4(\cdot,t)\|_{L^2} + \|I_5(\cdot,t)\|_{L^2} + \|I_6(\cdot,t)\|_{L^2} \leq C e^{\left(-\frac{\gamma}{2}+1\right)t}. \tag{4.16}$$

On the other hand, by (2.9) together with the fact that $\varphi' \in L^\infty(\mathbb{R})$, we get

$$\|I_7(\cdot,t)\|_{L^2} + \|I_8(\cdot,t)\|_{L^2} \leq C \|\|b\|\| e^{\frac{-t}{4}}. \tag{4.17}$$

In the same way, by (2.17), we have

$$\|I_9(\cdot,t)\|_{L^2} \leq C \left(\int_{\mathbb{R}} \xi^2 |b'(\xi)|^2 d\xi \right)^{\frac{1}{2}} e^{\frac{-t}{4}}. \tag{4.18}$$

Thus, we conclude from (4.15), (4.16), (4.17) and (4.18) that

$$\left\| \frac{dH_1}{dt} \right\|_{L^2} \leq C \left[\|F\|_{H^1(m)} + \|G\|_{L^2(m)} + \|\|b\|\| e^{\frac{-t}{4}} + \varepsilon e^{-t} + e^{\left(-\frac{\gamma}{2}+1\right)t} \right]. \tag{4.19}$$

Combining (4.7), (4.14) and (4.19), we obtain

$$\begin{aligned}\|J\|_{L^2} \leq C\left[\|F\|_{H^1(m)} + \|G\|_{L^2(m)} + \|\|b\|\|e^{\frac{-t}{4}} + \varepsilon e^{-t} + e^{\left(-\frac{\gamma}{2}+1\right)t}\right]\\ + C\|F_0\|^2_{L^2}e^{-2(t-t_0)}.\end{aligned} \tag{4.20}$$

By adding (4.6), (4.20) and the fact that $\|W_x\|_{L^2} \leq C\left[\|F\|_{H^1(m)} + \|G\|_{L^2(m)}\right]$, we conclude

$$\begin{aligned}\frac{d}{dt}E_{11}(t) + \frac{E_{11}(t)}{2} \leq -\frac{1}{2}\int_{\mathbb{R}} \frac{M^2}{a(xe^{\frac{t}{2}})}dx + C\varepsilon e^{-t}\int_{\mathbb{R}} \frac{M^2}{a(xe^{\frac{t}{2}})}dx\\ +C\Big[\|F\|^2_{H^1(m)} + \|G\|^2_{L^2(m)} + \|\|b\|\|^2 e^{\frac{-t}{2}} + \varepsilon^2 e^{-2t}\\ + e^{2\left(-\frac{\gamma}{2}+1\right)t} + \|F_0\|^4_{L^2}e^{-4(t-t_0)}\Big].\end{aligned} \tag{4.21}$$

By exploiting (4.21), we have

$$\begin{aligned}\frac{d}{dt}E_{11}(t) + \left(\frac{1}{2}+\mu_0\right)E_{11}(t) \leq \left(-\frac{1}{2} + C\varepsilon e^{-t} + \frac{1}{2}\varepsilon e^{-t}\right)\int_{\mathbb{R}} \frac{M^2}{a(xe^{\frac{t}{2}})}dx\\ +C\Big[\|F\|^2_{H^1(m)} + \|G\|^2_{L^2(m)} + \|\|b\|\|^2 e^{\frac{-t}{2}}\\ + \varepsilon^2 e^{-2t} + e^{2\left(-\frac{\gamma}{2}+1\right)t} + \|F_0\|^4_{L^2}e^{-4(t-t_0)}\Big].\end{aligned} \tag{4.22}$$

For t_0 large enough, we conclude that

$$\begin{aligned}&\frac{d}{dt}E_{11}(t) + \left(\frac{1}{2}+\mu_0\right)E_{11}(t)\\ &\leq C\left[\|F\|^2_{H^1(m)} + \|G\|^2_{L^2(m)}\right]\\ &\quad + C\left[\|\|b\|\|^2 e^{\frac{-t}{2}} + \varepsilon^2 e^{-2t} + e^{2\left(-\frac{\gamma}{2}+1\right)t} + \|F_0\|^4_{L^2}e^{-4(t-t_0)}\right].\end{aligned} \tag{4.23}$$

As in (3.5), integrating in t, we obtain

$$\begin{aligned}E_{11}(t) \leq C\int_{t_0}^{t} e^{-\left(\frac{1}{2}+\mu_0\right)(t-s)}\left(\|F(s)\|^2_{H^1(m)} + \|G(s)\|^2_{L^2(m)}\right)ds\\ + \left[E_{11}(t_0) + C\|\|b\|\|^2 e^{\frac{-t_0}{2}} + C\varepsilon^2 e^{-2t_0} + Ce^{2\left(-\frac{\gamma}{2}+1\right)t_0} + C\|F_0\|^4_{L^2}\right]e^{-\frac{1}{2}(t-t_0)}.\end{aligned} \tag{4.24}$$

Therefore, by (1.17) and (4.24), we get

$$E_{11}(t) \leq \left[E_{11}(t_0) + C\Phi_m\left(\varepsilon e^{-t_0}, F_0, G_0\right) + C\|\|b\|\|^2 e^{\frac{-t_0}{2}} + C\varepsilon e^{-t_0} + Ce^{\left(-\frac{\gamma}{2}+1\right)t_0}\right]e^{-\frac{1}{2}(t-t_0)}.$$

By (2.14), we can write

$$M = W - \frac{e^t}{\varepsilon}\left[-W + a\left(xe^{\frac{t}{2}}\right)V_{xx} + J(x,t)\right].$$

Hence, we have

$$\|M(t_0)\|_{L^2} \leq C\left(1+\frac{1}{\varepsilon}\right)\left(\left\|\widetilde{F}_0\right\|_{H^1(m)}+\left\|\widetilde{G}_0\right\|_{L^2(m)}\right),$$

which implies that

$$\varepsilon E_{11}(t_0) \leq C\left(\|F_0\|_{H^1(m)}^2+\|G_0\|_{L^2(m)}^2\right).$$

Consequently, one easily obtains

$$\varepsilon E_{11}(t) \leq C\left[\|(F_0,G_0)\|_{X^m}^2+\varepsilon\|b\|^2 e^{\frac{-t_0}{2}}+\varepsilon^2 e^{-t_0}+\varepsilon e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right] e^{-\frac{1}{2}(t-t_0)}.$$

This concludes the proof of Lemma 4.1. □

By Remark 2.1, we have the appropriate estimate of $\beta(t)$, then by combining Lemma 4.1 and (2.2), one easily obtains (1.18). This concludes the proof of the second part of Theorem 1.2.

Proof of Theorem 1.4 (Second part). Similarly, in the case where $b(\xi)=0$ and $\gamma>3$, integrating (4.23) in t, we obtain

$$\begin{aligned} E_{11}(t) \leq{}& C\int_{t_0}^{t} e^{-\left(\frac{1}{2}+\mu_0\right)(t-s)}\left(\|F(s)\|_{H^1(m)}^2+\|G(s)\|_{L^2(m)}^2\right)ds \\ &+\left[E_{11}(t_0)+C\varepsilon^2 e^{-2t_0}+Ce^{2\left(-\frac{\gamma}{2}+1\right)t_0}+C\|F_0\|_{L^2}^4\right]e^{-\left(\frac{1}{2}+\mu_0\right)(t-t_0)}. \end{aligned} \tag{4.25}$$

Therefore, by (1.21) and (4.25), we get

$$\begin{aligned} E_{11}(t) \leq{}& \left[E_{11}(t_0)+C\Phi_m\left(\varepsilon e^{-t_0},F_0,G_0\right)+C\varepsilon e^{-t_0}+Ce^{\left(-\frac{\gamma}{2}+1\right)t_0}\right] \\ &\times e^{-\min\left(\frac{\gamma}{2}-1,\frac{1}{2}+\mu_0\right)(t-t_0)}. \end{aligned} \tag{4.26}$$

Arguing as above, we get

$$\varepsilon E_{11}(t) \leq C\left[\|(F_0,G_0)\|_{Z^m}^2+\varepsilon^2 e^{-t_0}+\varepsilon e^{\left(-\frac{\gamma}{2}+1\right)t_0}\right]e^{-\min\left(\frac{\gamma}{2}-1,\frac{1}{2}+\mu_0\right)(t-t_0)}. \tag{4.27}$$

Combining (2.2), (4.26) and (4.27), one easily obtains (1.22). This concludes the proof of the second part of Theorem 1.4. □

REFERENCES

[1] J.M. Ball, *Strongly continuous semigroups, weak solutions and the variation of constants formula*, Proc. Amer. Math. Soc. **63** (1977), 370–373.

[2] Y. Brenier, R. Natalini, M. Puel, *On a relaxation approximation of the incompressible Navier-Stokes equations*, Proc. Amer. Math. Soc. **132** (2004), 1021–1028.

[3] J. Bricmont, A. Kupiainen, *Stable non Gaussian diffusive profiles*, Nonlinear Anal. Theory Methods Appl. **26** (1996), 583–593.

[4] J. Bricmont, A. Kupiainen, G. Lin, *Renormalization group and asymptotics of solutions of nonlinear parabolic equations*, Communications on Pure and Applied Mathematics, **47** (1994), 893–922.

[5] J.M. Burgers, *A mathematical model illustrating the theory of turbulence*, Advances in Applied Mechanics (1948), 171–199.

[6] A. Doelman, B. Sandstede, A. Scheel, G. Schneider, *The dynamics of modulated wave trains*, Mem. Amer. Math. Soc. **199** (2009), 934.

[7] M. Escobedo, O. Kavian, *Variational problems related to self-similar solutions of the heat equation*, Nonlinear Analysis T.M.A. **11** (1987), 1103–1133.

[8] M. Escobedo, O. Kavian, H. Matano, *Large time behavior of solutions of a dissipative semi-linear heat equation*, Comm. Partial Differential Equations **20** (1995), 1427–1452.

[9] M. Escobedo, E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^n$, J. Funct. Anal. **100** (1991), 119–161.

[10] C. Escudero, *Blow-up of the hyperbolic Burgers equation*, J. Stat. Phys. **127** (2007), 327–338.

[11] Th. Gallay, G. Rauguel, *Scaling variables and asymptotic expansions in damped wave equations*, J. Differential Equations **150** (1998), 42–97.

[12] Th. Gallay, G. Rauguel, *Scaling variables and stability of hyperbolic fronts*, SIAM J. Math. Anal. **32** (2000), 1–29.

[13] Th. Gallay, C.E. Wayne, *Invariant Manifolds and the Long Time Asymptotics of the Navier-Stokes and Vorticity equations on* $\mathbb{R}^2$, Arch. Rational Mech. Anal **163** (2002), 209–258.

[14] Th. Gallay, C.E. Wayne, *Long-time asymptotics of the Navier-Stokes equation in* $\mathbb{R}^2$ *and* $\mathbb{R}^3$, Z. Angew. Math. Mech **86** (2006), 256–267.

[15] K.P. Hadeler, *Reaction Telegraph Equations and Random Walk Systems*, [in:] Stochastic and Spatial Structures of Dynamical Systems, S. van Strien and S. Verduyn Lunel (eds), Royal Academy of the Netherlands, North Holland, Amsterdam, 1996.

[16] M.A. Hamza, *Asymptotically self-similar solutions of the damped wave equation*, Nonlinear Anal. **73** (2010), 2897–2916.

[17] M.A. Hamza, *Asymptotic profiles for the third-grade fluids equations in one space dimension*, Differential and Integral Equations **28** (2015), 155–200.

[18] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, UK, 1934; reprinted, 1964.

[19] E. Hopf, *The partial differential equation $u_t + uu_x = \mu u_{xx}$*, Comm. Pure Appl. Math. **3** (1950), 201–230.

[20] B. Jaffal-Mourtada, *Long time asymptotics of the second grade fluid equations on $\mathbb{R}^2$*, Dyn. of PDE **8** (2011), 185–223.

[21] S. Kamin, L.A. Peletier, *Large time behavior of solutions of the heat equation with absorption*, Ann. Sc. Norm. Sup. Piza **12** (1985), 393–408.

[22] G. Karch, *Self-similar large time behavior of solutions to Korteweg-de Vries-Burgers equation*, Nonlinear Anal. **35** (1991), 199–219.

[23] M. Mihailescu, V. Rădulescu, D. Repovš, *On a non-homogeneous eigenvalue problem involving a potential: an Orlicz-Sobolev space setting*, J. Math. Pures Appl. **93** (2010), 132–148.

[24] J.D. Murray, *Mathematical Biology*, 2nd ed., Biomathematics, vol. 19, Springer-Verlag, New York, Berlin, 1993.

[25] R. Orive, E. Zuazua, *Long-time behavior of solutions to a non-linear hyperbolic relaxation system*, J. Differential Equations **228** (2006), 17–38.

[26] M. Paicu, G. Raugel, *A hyperbolic perturbation of the Navier–Stokes equations*, ESAIM Proceedings **21** (2007), 65–87.

[27] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci., vol. 44, Springer-Verlag, New York, 1983.

[28] R. Prado, E. Zuazua, *Asymptotic expansion for the generalized Benjamin–Bona–Mahony––Burger equation*, Differential and Integral Equations **15** (2002), 1409–1434.

[29] V. Rădulescu, D. Repovš, *Combined effects in nonlinear problems arising in the study of anisotropic continuous media*, Nonlinear Anal. **75** (2012), 1524–1530.

[30] V. Rădulescu, D. Repovš, *Partial Differential Equations with Variable Exponents. Variational Methods and Qualitative Analysis*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015.

[31] L. Sapa, *Existence, uniqueness and estimates of classical solutions to some evolutionary system*, Opuscula Math. **35** (2015) 6, 935–956.

[32] J.L. Vazquez, E. Zuazua, *Complexity of large time behavior of evolutions equations with bounded data*, Chinese Annals of Mathematics **23**, ser. B (2002), 293–310.

F. Dkhil
fathi.dkhil@isi.rnu.tn

Université Tunis El Manar
Département de Mathématiques
Institut Supérieur d'Informatique
2 rue Abou Raihan Bayrouni, 2080 Ariana, Tunisia

Université de Tunis El Manar
Faculté des Sciences de Tunis
UR13ES32 Analyse non linéaire et géometrie
2092 Tunis, Tunisia

M.A. Hamza
mahamza@iad.edu.sa

Department of Basic Sciences
Deanship of Preparatory and Supporting Studies
Imam Abdulrahman Bin Faisal University
P.O. Box 1982 Dammam, Saudi Arabia

B. Mannoubi
bechir_ca@live.com

Université de Tunis El Manar
Faculté des Sciences de Tunis
UR13ES32 Analyse non linéaire et géometrie
2092 Tunis, Tunisia

Received: November 23, 2016.
Accepted: March 18, 2017.