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SOLUTIONS TO $p(x)$-LAPLACE TYPE EQUATIONS VIA NONVARIATIONAL TECHNIQUES

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Abstract. In this article, we consider a class of nonlinear Dirichlet problems driven by a Leray-Lions type operator with variable exponent. The main result establishes an existence property by means of nonvariational arguments, that is, nonlinear monotone operator theory and approximation method. Under some natural conditions, we show that a weak limit of approximate solutions is a solution of the given quasilinear elliptic partial differential equation involving variable exponent.

Keywords: Leray–Lions type operator, nonlinear monotone operator, approximation, variable Lebesgue spaces.

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1. INTRODUCTION

We are concerned with the following quasilinear elliptic partial differential equation

$$\begin{cases} -\nabla \cdot a(x, \nabla u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a vector-valued function, and $f \in W^{-1,p'(x)}(\Omega)$.

Equations of type (1.1) is an interesting topic of research due to its significant role in the interplay between pure and applied nonlinear analysis as well as in many fields of mathematics such as nonlinear partial differential equations, calculus of variations, non-linear potential theory, non-Newtonian fluids, image processing to name a few (see, e.g., [8, 9, 26, 29] and references therein).

Operator $-\nabla \cdot a(x, \cdot)$ appearing in problem (1.1) is a Leray–Lions type operator (see [25]) and can be particularised to many well-known operators. To be more precise, in case of $a(x,\xi) = |\xi|^{p(x)-2}\xi$, where $p(x) \geq 2$, we get the $p(x)$-Laplace operator, an operator acting from $W_0^{1,p(x)}(\Omega)$ to its dual $W^{-1,p'(x)}(\Omega)$, defined by

$$\Delta_{p(x)} u := \nabla \cdot \left(|\nabla u|^{p(x)-2} \nabla u \right) = \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \left[\left| \frac{\partial u}{\partial x_k} \right|^{p(x)-2} \frac{\partial u}{\partial x_k} \right].$$

There are many classes of problems which are driven by the $p(x)$-Laplace-type operators, for example,

$$\begin{cases} -\nabla \cdot (|\nabla u|^{p(x)-2} \nabla u) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

which is the $p(x)$-Laplace Poisson equation.

If we set $a_i(x,\xi) = |\xi|^{p_i(x)-2}\xi$ for all $i \in \{1,\dots,N\}$, where $\overrightarrow{p} : \overline{\Omega} \to \mathbb{R}^N$ is a vectorial function $\overrightarrow{p}(x) = (p_1(x),\dots,p_N(x))$ with $p_i \in C(\overline{\Omega})$, $p_i(x) \geq 2$, problem (1.1) turns into

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i} (|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

which is the anisotropic $\overrightarrow{p}(\cdot)$-Laplace Poisson equation.

Moreover, if we set $a(x,\xi) = (1+|\xi|^2)^{(p(x)-2)/2}\xi$, where $\xi \in \mathbb{R}^N$ and $p(x) \geq 2$, then we obtain the generalized mean curvature operator

$$\nabla \cdot ((1+|\nabla u|^2)^{(p(x)-2)/2} \nabla u)$$

which leads to the equation

$$\begin{cases} -\nabla \cdot ((1+|\nabla u|^2)^{(p(x)-2)/2} \nabla u) = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

See [6, 27, 28] for further examples and applications for the operator $a(x,\cdot)$.

We want to remark that application of nonlinear monotone operator theory and approximation method to the problems of type (1.1) is not a new topic. For example, in [5] the authors dealt with the equation

$$A(u) + g(x,u,\nabla u) = h, \tag{1.5}$$

where A is a Leray-Lions type operator acting from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, $h \in W^{-1,p'}(\Omega)$ and g is a nonlinear lower order term with natural growth of order p with respect to $|\nabla u|$. Using nonlinear monotone operator theory and approximation method, the authors obtained unbounded solutions to problem (1.4).

Moreover, in [14] the authors studied the equation

$$\sum_{k=1}^{N} \frac{\partial}{\partial x_k} a_k(x,u,\nabla u) - c_0 |u|^{p-2} u = f(x,u,\nabla u) \tag{1.6}$$

where $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, f is a nonlinear Carathéodory function which has the growth of order p with respect to $|\nabla u|$, $1 < p < \infty$ is a real number and c_0 is positive constant. Using nonlinear monotone operator theory, the authors obtained bounded solutions to problem (1.5).

Recently, in [4], the authors studied the Dirichlet problem for multivalued elliptic equations of the form

$$-\nabla \cdot a(x, \nabla u) \ni \nabla \cdot h, \quad u|_{\partial\Omega} = 0, \tag{1.7}$$

where $a(x, \cdot) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is maximal monotone operator for almost all $x \in \Omega$, h is a given vector function in Musielak-Orlicz space $W^{-1,\varphi'}(\Omega)$. In this paper, the authors consider the so-called Lavrentiev phenomenon (see [32]). By applying maximal monotone operator theory and approximation method, they obtained H- and W-solutions for the multivalued problem (1.6) and its singlevalued version.

We would like to notice that there are many papers dealt with equation of the form (1.1) in which nonlinearity is given under the natural growth of order p via monotone operator methods and approximation methods (see, e.g., [2, 11, 13, 23] and references therein). However, this is not the case for the equations of the form (1.1) involving variable exponent of nonlinearity, that is, growth of order $p(x)$. Rather, the authors have usually applied variational methods along with critical point theory (see, e.g., [3, 4, 6, 18, 21, 22, 31] and references therein).

2. PRELIMINARIES

We start with some basic concepts of variable Lebesgue spaces. For more details we refer the readers to the monographs [1, 10, 12, 28], and the papers [15, 19, 24].

For any $p \in C(\overline{\Omega})$ with $p^- > 1$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u(x)|^{p(x)} dx < \infty \right\},$$

then $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

becomes a Banach space.

Proposition 2.1 (Hölder Inequality)**.** *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv| dx \leq C(p^-, (p^-)')|u|_{p(x)}|v|_{p'(x)}.$$

Proposition 2.2 (Young Inequality). *Let $p(x) > 1$ for all $x \in \overline{\Omega}$ and $p(x)^{-1} + p'(x)^{-1} = 1$ or $p'(x) = \frac{p(x)}{p(x)-1}$. Then, for all $a, b \in \mathbb{R}^N$*

$$|a||b| \leq p(x)^{-1}|a|^{p(x)} + p'(x)^{-1}|b|^{p'(x)}.$$

The convex functional $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$$

is called modular on $L^{p(x)}(\Omega)$.

Proposition 2.3. *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, \ldots)$, we have*

(i) $|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho(u) < 1 (= 1, > 1)$,
(ii) $|u|_{p(x)} > 1 \implies |u|_{p(x)}^{p^-} \leq \rho(u) \leq |u|_{(x)}^{p^+}$, $|u|_{p(x)} \leq 1 \implies |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$,
(iii) $\lim\limits_{n\to\infty} |u_n|_{p(x)} = 0 \Leftrightarrow \lim\limits_{n\to\infty} \rho(u_n) = 0$; $\lim\limits_{n\to\infty} |u_n|_{p(x)} = \infty \Leftrightarrow \lim\limits_{n\to\infty} \rho(u_n) = \infty$.

Proposition 2.4. *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, \ldots)$, then the following statements are equivalent:*

(i) $\lim\limits_{n\to\infty} |u_n - u|_{p(x)} = 0$,
(ii) $\lim\limits_{n\to\infty} \rho(u_n - u) = 0$,
(iii) *$u_n \to u$ in measure in Ω and $\lim\limits_{n\to\infty} \rho(u_n) = \rho(u)$.*

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \Big\{ u \in L^{p(x)}(\Omega) : \nabla u \in \prod_{i=1}^N L^{p(x)}(\Omega) \Big\},$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

or equivalently

$$\|u\|_{1,p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} + \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\}$$

for all $u \in W^{1,p(x)}(\Omega)$. The space $W_0^{1,p(x)}(\Omega)$ is defined as $\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,p(x)}} = W^{1,p(x)}(\Omega)$, and hence, $u \in W_0^{1,p(x)}(\Omega)$ iff there exists a sequence (u_n) of $C_0^\infty(\Omega)$ such that $\|u_n - u\|_{1,p(x)} \to 0$.

As a consequence of the Poincaré inequality, $\|u\|_{1,p(x)}$ and $|\nabla u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. Therefore, for any $u \in W_0^{1,p(x)}(\Omega)$ we can define an equivalent norm $\|u\|$ such that

$$\|u\| = |\nabla u|_{p(x)}.$$

Proposition 2.5. *If* $1 < p^- \leq p^+ < \infty$*, then the spaces* $L^{p(x)}(\Omega)$ *and* $W^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

Proposition 2.6. *Let* $q \in C(\overline{\Omega})$*. If* $1 \leq q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$*, then the embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

3. MAIN RESULTS

In the present paper, we show that if $(u_n) \subset W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u \in W_0^{1,p(x)}(\Omega)$ is a sequence of solutions to the approximate problem

$$\begin{cases} -\nabla \cdot a(x, \nabla u_n) = f \text{ in } \Omega, \\ u_n = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

then u is a weak solution to problem (1.1) provided that $f \in W^{-1,p'(x)}(\Omega)$.

To this end, we assume the following hypotheses.

Let us define $p \in C(\overline{\Omega})$ as

$$1 < p^- := \min_{x \in \overline{\Omega}} p(x) \leq p(x) \leq p^+ := \max_{x \in \overline{\Omega}} p(x) < \infty.$$

(a0) $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.

(a1) There exists a positive constant c_0 such that

$$(a(x, \xi) - a(x, \zeta)) \cdot (\xi - \zeta) \geq c_0(|\xi| + |\zeta|)^{p(x)-2}|\xi - \zeta|^2$$

for all $\xi, \zeta \in \mathbb{R}^N$ and $x \in \Omega$.

(a2) The following inequality holds

$$|a(x, \xi)| \leq c_1(h_0(x) + |\xi|^{p(x)-1}),$$

for all $\xi \in \mathbb{R}^N$ and $x \in \Omega$, where $p \in C(\overline{\Omega})$ such that $1 < p(x) < N$, c_1 is a positive real number, $h_0 \in L^{p'(x)}(\Omega)$ is a nonnegative function.

(a3)

$$a(x, 0) = 0, \text{ a.e. in } \Omega.$$

(a4) The following inequality holds

$$|a(x, \xi) - a(x, \zeta)| \leq c_2(1 + |\xi| + |\zeta|)^{p(x)-1-\alpha}|\xi - \zeta|^{\alpha},$$

for all $\xi, \zeta \in \mathbb{R}^N$ and $x \in \Omega$, where c_2 is a positive real number and $0 < \alpha < p^- - 1$ is a constant.

Remark 3.1. We want to notice that condition $(a2)$ is a particular case of $(a4)$ since putting $\zeta = 0$ in $(a4)$ leads to $(a2)$. Therefore, $(a4)$ is consistent with the nonlinear growth condition $(a2)$ which is accepted as a natural growth of order $p(x)$. We would also like to mention that condition $(a1)$ has been previously assumed in papers [20, 30].

Let $\mathcal{A}$ be an operator from $W_0^{1,p(x)}(\Omega)$ to its dual $W^{-1,p'(x)}(\Omega)$ defined by

$$\mathcal{A}(u) = -\nabla \cdot a(x, \nabla u).$$

Then the operator $\mathcal{A}$ is bounded, continuous and monotone due to the conditions $(a0)$–$(a3)$ (see [17, Theorem 2.1]).

Definition 3.2. We say that a function $u \in W_0^{1,p(x)}(\Omega)$ is a solution of the operator equation

$$\mathcal{A}(u) = f \tag{3.2}$$

provided that for given any $f \in W^{-1,p'(x)}(\Omega)$ we have

$$\langle \mathcal{A}(u), \varphi \rangle = \int\limits_\Omega a(x, \nabla u) \cdot \nabla \varphi dx = \int\limits_\Omega f \varphi dx = \langle f, \varphi \rangle, \text{ for all } \varphi \text{ in } W_0^{1,p(x)}(\Omega), \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality map between $W_0^{1,p(x)}(\Omega)$ and $W^{-1,p'(x)}(\Omega)$.

Remark 3.3. Equation (3.2) means that we have an equality between $\mathcal{A}(u)$ and f in $W^{-1,p'(x)}(\Omega)$, where $\mathcal{A}(u)$ is capable of acting on any $\varphi \in W_0^{1,p(x)}(\Omega)$ as

$$\langle \mathcal{A}(u), \varphi \rangle = \langle f, \varphi \rangle$$

In conclusion, this means that we can understand the nature of $\mathcal{A}(u) \in W^{-1,p'(x)}(\Omega)$ through its effect on $\varphi \in W_0^{1,p(x)}(\Omega)$.

First, we provide a-priori estimate.

Lemma 3.4. *Let* $(u_n) \subset W_0^{1,p(x)}(\Omega)$ *be a sequence of solutions to* (3.1). *If* $f \in W^{-1,p'(x)}(\Omega)$, *then there exists a positive constant* K *such that*

$$\|u_n\| \leq K, \ for\ n = 1, 2, \ldots \tag{3.4}$$

that is, any sequence of solutions to problem (3.1) *is uniformly bounded in* $W_0^{1,p(x)}(\Omega)$.

Proof. Since $(u_n) \subset W_0^{1,p(x)}(\Omega)$ is a sequence of solutions to problem (3.1), then by Definition 3.2, for $n = 1, 2, \ldots$ we must have

$$\int\limits_\Omega a(x, \nabla u_n) \cdot \nabla u_n dx = \int\limits_\Omega f u_n dx. \tag{3.5}$$

If $\|u_n\| \leq 1$ for any $n = 1, 2, \ldots$, then there is nothing to prove. Therefore, without loss of generality, we may assume that $\|u_n\| > 1$ for all $n = 1, 2, \ldots$. Then, applying condition $(a1)$ for $\zeta = 0$ and using Proposition 2.3, the Hölder inequality and the continuous imbedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ it reads

$$c_3\|u_n\|^{p^-} \leq \int_\Omega c_0|\nabla u_n|^{p(x)}dx \leq \int_\Omega f u_n dx,$$

$$c_3\|u_n\|^{p^-} \leq |f|_{p'(x)}|u_n|_{p(x)} \leq c_4\|u_n\|$$

which shows the uniform boundedness of sequence $(\|u_n\|)$ since $p^- > 1$, where c_3, c_4 are positive constants. □

Theorem 3.5. *Suppose that* $(u_n) \subset W_0^{1,p(x)}(\Omega)$ *is a sequence of solutions to the approximate problem* (3.1). *If* $u_n \rightharpoonup u \in W_0^{1,p(x)}(\Omega)$ *and* $f \in W^{-1,p'(x)}(\Omega)$, *then* u *is a weak solution to problem* (1.1), *that is, for every* $\varphi \in W_0^{1,p(x)}(\Omega)$, $u \in W_0^{1,p(x)}(\Omega)$ *must satisfy the identity*

$$\int_\Omega a(x, \nabla u) \cdot \nabla\varphi dx = \int_\Omega f\varphi dx. \tag{3.6}$$

Proof. Let us assume that $(u_n) \subset W_0^{1,p(x)}(\Omega)$ is a sequence of solutions to problem (3.1). By Lemma 3.4 and reflexivity of $W_0^{1,p(x)}(\Omega)$, we can extract a subsequence (not relabelled) $(u_n) \subset W_0^{1,p(x)}(\Omega)$ such that $u_n \rightharpoonup u \in W_0^{1,p(x)}(\Omega)$ and $u_n \to u \in L^{p(x)}(\Omega)$. From $(a2)$, and the Hölder inequality, we have

$$|a(x, \nabla u_n)| \leq c_1(h_0(x) + |\nabla u_n|^{p(x)-1}) \leq c|h_0|_{p'(x)}||u_n|^{p(x)-1}|_{p(x)} \leq c\|u_n\|^{p_M-1}$$

which means that $(a(x, \nabla u_n))$ is bounded in $L^{p'(x)}(\Omega)^N$, where $p_M = \max\{p^-, p^+\}$. Therefore, up to a subsequence if necessary, we have

$$a(x, \nabla u_n) \rightharpoonup \xi \in L^{p'(x)}(\Omega)^N, \tag{3.7}$$

for some $\xi \in L^{p'(x)}(\Omega)^N = \prod_{i=1}^N L^{p'(x)}(\Omega)$. This writing is mandatory because we can not conclude directly the convergence $a(x, \nabla u_n) \rightharpoonup a(x, \nabla u)$ since nonlinearities are not continuous with respect to weak convergence in general. Thus, combining (3.7) and (3.5), we obtain

$$\lim_{n\to\infty} \int_\Omega a(x, \nabla u_n) \cdot \nabla\varphi dx = \int_\Omega \xi \cdot \nabla\varphi dx = \int_\Omega f\varphi dx, \quad \varphi \in W_0^{1,p(x)}(\Omega), \tag{3.8}$$

where we put $u_n = \varphi$. On the other hand, employing monotonicity condition $(a1)$ we obtain that

$$\int_\Omega (a(x, \nabla u_n) - a(x, \nabla\omega)) \cdot (\nabla u_n - \nabla\omega)dx \geq 0 \tag{3.9}$$

for all $\omega \in W_0^{1,p(x)}(\Omega)$. Next, if we substitute identity (3.5) in (3.9), and apply some elementary calculations, we obtain

$$\int_\Omega [fu_n - a(x,\nabla u_n)\cdot\nabla\omega - a(x,\nabla\omega)\cdot(\nabla u_n - \nabla\omega)]dx \geq 0. \tag{3.10}$$

Considering $u_n \rightharpoonup u \in W_0^{1,p(x)}(\Omega)$ and $u_n \to u \in L^{p(x)}(\Omega)$ along with (3.6) lead us to the inequality

$$\int_\Omega [fu - \xi\cdot\nabla\omega - a(x,\nabla\omega)\cdot(\nabla u - \nabla\omega)]dx \geq 0. \tag{3.11}$$

Since (3.8) holds for every $\varphi \in W_0^{1,p(x)}(\Omega)$, we can put $\varphi = u$. Then

$$\int_\Omega (\xi - a(x,\nabla\omega))\cdot(\nabla u - \nabla\omega)dx \geq 0. \tag{3.12}$$

Next, we apply Minty's trick allowing us passing to weak limit within the nonlinearity. To this end, let us put $\omega = u + \varepsilon v$ for $\varepsilon > 0$, and let $0 \neq v \in W_0^{1,p(x)}(\Omega)$ be fixed. Then, if we let $\varepsilon \to 0$, it reads

$$\int_\Omega (\xi - a(x,\nabla u))\cdot\nabla v dx \leq 0. \tag{3.13}$$

Applying the same argument for $-v$ leads to

$$\int_\Omega (\xi - a(x,\nabla u))\cdot\nabla v dx \geq 0. \tag{3.14}$$

Then (3.13) and (3.14) yield $\xi = a(x,\nabla u)$. Using this information along with (3.8), we obtain

$$\int_\Omega a(x,\nabla u)\cdot\nabla v dx = \int_\Omega f v dx, \quad v \in W_0^{1,p(x)}(\Omega), \tag{3.15}$$

which shows that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (1.1). $\square$

Lemma 3.6 ([7]). *Let X be a reflexive real Banach space. Moreover, let $T : X \to X^*$ be an operator satisfying the conditions:*

(i) *T is coercive, that is,*

$$\lim_{\|u\|_X\to\infty} \frac{\langle T(u), u\rangle}{\|u\|_X} = +\infty,$$

(ii) *T is hemicontinuous, that is, T is directionally weakly continuous, iff the function*

$$\Psi(\gamma) = \langle T(u+\gamma w), v\rangle$$

is continuous in γ on $[0,1]$ for every $u, w, v \in W_0^{1,p(x)}(\Omega)$,

(iii) *T is monotone on the space X, that is, for all $u, v \in X$ we have*

$$\langle T(u) - T(v), u - v\rangle \geq 0. \tag{3.16}$$

Then the operator equation

$$T(u) = g \tag{3.17}$$

has at least one solution $u \in X$ for every $g \in X^*$. If, moreover, the inequality (3.16) *is strict for all $u, v \in X$, $u \neq v$, then the equation* (3.17) *has precisely one solution $u \in X$ for every $g \in X^*$.*

In the sequel, we show that problem (1.1) has a unique solution.

Theorem 3.7 (Existence and uniqueness)**.** *Assume that* (a0)–(a4) *hold. Then, for given any $f \in W^{-1,p'(x)}(\Omega)$ the operator equation* (3.2) *has a unique solution $u \in W_0^{1,p(x)}(\Omega)$ which in turn becomes a weak solution to problem* (1.1)*.*

Proof. First, we show that operator $\mathcal{A}$ is coercive. Without loss of generality, we may suppose that $\|u\| > 1$. Then, by $(a1)$ for $\zeta = 0$, we have

$$\langle \mathcal{A}(u) - \mathcal{A}(0), u \rangle = \int_\Omega a(x, \nabla u) \cdot \nabla u dx \geq c_0 \int_\Omega |\nabla u|^{p(x)} dx \geq c_0 \|u\|^{p^+},$$

$$\lim_{\|u\| \to \infty} \frac{\langle \mathcal{A}(u), u \rangle}{\|u\|} \geq \lim_{\|u\| \to \infty} c_0 \|u\|^{p^+ - 1} = +\infty,$$

that is, $\mathcal{A}$ is coercive.

Next, we show that operator $\mathcal{A}$ is hemicontinuous. To this end, using $(a4)$, the Hölder inequality and the inequality

$$|w + v|^m \leq 2^m (|w|^m + |v|^m), \text{ for all } w, v \in \mathbb{R}^N \text{ and } m > 0, \tag{3.18}$$

we have

$$\begin{aligned} |\Psi(\gamma_1) - \Psi(\gamma_2)| &= |\langle \mathcal{A}(u + \gamma_1 w) - \mathcal{A}(u + \gamma_2 w), v \rangle| \\ &\leq \int_\Omega |(a(x, \nabla(u + \gamma_1 w)) - a(x, \nabla(u + \gamma_2 w))||\nabla v| dx \\ &\leq c|\gamma_1 - \gamma_2|^\alpha \int_\Omega (1 + |\nabla u| + |\nabla w|)^{p(x)-1-\alpha} |\nabla w|^\alpha |\nabla v| dx. \end{aligned}$$

Let us define a function $\Theta : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$ by

$$\Theta(u, w) = 1 + |\nabla u| + |\nabla w|.$$

Then, for every $u, w \in W_0^{1,p(x)}(\Omega)$, we obtain

$$\begin{aligned} &\int_\Omega (1 + |\nabla u| + |\nabla w|)^{p(x)-1-\alpha} |\nabla w|^\alpha |\nabla v| dx \\ &\leq \int_\Omega |\Theta|^{p(x)-1-\alpha} |\Theta|^\alpha |\nabla v| dx = \int_\Omega |\Theta|^{p(x)-1} |\nabla v| dx \\ &\leq ||\Theta|^{p(x)-1}|_{\frac{p(x)}{p(x)-1}} |\nabla v|_{p(x)} \leq |\Theta|_{p(x)}^{p_M - 1} |\nabla v|_{p(x)} < \infty. \end{aligned}$$

So we are multiplying a bounded function with a function getting very small. The end result should be small, namely, we obtain

$$|\Psi(\gamma_1) - \Psi(\gamma_2)| \to 0 \text{ as } \gamma_1 \to \gamma_2$$

which means that $\mathcal{A}$ is hemicontinuous. As for monotonicity of $\mathcal{A}$, it is enough to apply condition $(a1)$, that is,

$$\langle \mathcal{A}(u) - \mathcal{A}(v), u - v\rangle = \int\limits_{\Omega} (a(x, \nabla u) - a(x, \nabla v)) \cdot (\nabla u - \nabla v) dx > 0$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$ provided $u \neq v$.

In conclusion, the operator equation (3.2) has a unique solution. □

Remark 3.8 ($W_0^{2,2}(\Omega)$ regularity)**.** In case of $p(x) = 2$, the assumption (a1) turns into

$$(a(x, \xi) - a(x, \zeta)) \cdot (\xi - \zeta) \geq c_0|\xi - \zeta|^2 \tag{3.19}$$

or

$$\sum_{i=1}^{N}(a^i(x, \xi) - a^i(x, \eta)) \cdot (\xi_i - \zeta_i) \geq c_0 \sum_{i=1}^{N}(\xi_i - \zeta_i)^2,$$

where $a = (a^1, a^2, \ldots, a^N)$. Using (3.19), it can be obtained that any weak solution to problem (1.1) belongs to $W_0^{2,2}(\Omega)$, and hence, satisfies

$$-\nabla \cdot a(x, \nabla u) = f \quad \text{in } \Omega. \tag{3.20}$$

As for the illustration, pick $\eta \in \mathbb{R}^N$ and $h \in \mathbb{R}$ with $h \neq 0$ and put $\xi = \zeta + h\eta$ in (3.19). Then, dividing the obtained inequality by h^2, it leads to

$$\sum_{i=1}^{N} \frac{(a^i(x, \zeta + h\eta) - a^i(x, \zeta))}{h}\eta_i \geq c_0 \sum_{i=1}^{N} \eta_i^2. \tag{3.21}$$

If we take the derivative of a^i in the direction η, that is, letting $h \to 0$ in (3.21), we have

$$\sum_{i,j=1}^{N} a_j^i(x, \zeta)\eta_j\eta_i \geq c_0 \sum_{i=1}^{N} \eta_i^2, \tag{3.22}$$

where $\nabla_{\xi_j} a^i = a_j^i$. Therefore, (3.20) is an uniformly elliptic equation. For the proof of $W_0^{2,2}(\Omega)$ regularity of the weak solution, one can follow the same arguments and steps given in Theorem 1 in [16, §6.3.1], so it is omitted here.

4. EXAMPLES

In the sequel we provide an example to illustrate the use of results obtained in the previous chapter.

Theorem 4.1. *Assume that (a0)–(a4) hold, $p(x) > 2$ and $h \in L^\infty(\Omega)$. Then, the following problem*

$$\begin{cases} -\nabla \cdot a(x, \nabla u) = -u|\nabla u|^{p(x)-2} + h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{4.1}$$

has a unique solution $u \in W_0^{1,p(x)}(\Omega)$.

Proof. We set the nonlinear term $g(x, u, \nabla u) = u|\nabla u|^{p(x)-2}$ and approximate g by functions

$$g_n = g_n(x, u, \nabla u) := \frac{g(x, u, \nabla u)}{1 + (1/n)|g(x, u, \nabla u)|}$$

which is bounded, that is, $|g_n| \leq n$. To this end, we consider the following approximate problem of (4.1)

$$\begin{cases} -\nabla \cdot a(x, \nabla u_n) = -g_n(x, u_n, \nabla u_n) + h \text{ in } \Omega, \\ u_n = 0 \text{ on } \partial\Omega, \end{cases}$$

and assume a sequence of solutions $(u_n) \subset W_0^{1,p(x)}(\Omega)$ satisfying

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla \varphi dx = \int_\Omega g_n(x, u_n, \nabla u_n) \varphi dx + \int_\Omega h \varphi dx \tag{4.2}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$ provided $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$.

First, we want to notice that $g_n(x, u_n, \nabla u_n)$ is $L^{p'(x)}(\Omega)$-norm bounded. Indeed, applying Young's inequality for $\varepsilon = p(x) - 1$ and $\varepsilon' = \frac{p(x)-1}{p(x)-2}$ and considering $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega |g_n(x, u_n, \nabla u_n)|^{p'(x)} dx &= \int_\Omega |u_n|^{p'(x)} |\nabla u_n|^{p'(x)(p(x)-2)} dx \\ &\leq \int_\Omega \left[\frac{1}{\varepsilon} |u_n|^{p(x)} + \frac{1}{\varepsilon'} |\nabla u_n|^{p(x)} \right] dx \leq K_1. \end{aligned}$$

Therefore, according to Theorem 3.5, $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (4.1).

To proceed, we need to show that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. From the weak lower semicontinuity of modular ρ, we already have

$$\liminf_{n\to\infty} \rho(\nabla u_n) \geq \rho(\nabla u) \tag{4.3}$$

Without loss of generality we may choose $a(x,\xi)=|\xi|^{p(x)-2}\xi$ in problem (4.1) since it satisfies all conditions (a0)-(a4) under some proper constants. Then, for $t\in(0,1)$ and by $(a1)$ we can write

$$\begin{aligned}
&|-\nabla u|^{p(x)}-|-\nabla u_n|^{p(x)}-p(x)a(x,-\nabla u_n)\cdot(\nabla u_n-\nabla u)\\
&=p(x)\int\limits_0^1\left(\frac{d}{dt}\frac{|-\nabla u_n+t(\nabla u_n-\nabla u)|^{p(x)}}{p(x)}-a(x,-\nabla u_n)\cdot(\nabla u_n-\nabla u)\right)dt\\
&=p(x)\int\limits_0^1(a(x,-\nabla u_n+t(\nabla u_n-\nabla u))-a(x,-\nabla u_n))\cdot(t(\nabla u_n-\nabla u))\frac{1}{t}dt\\
&\geq c_0p(x)\int\limits_0^1(|-\nabla u_n+t(\nabla u_n-\nabla u)|+|-\nabla u_n|)^{p(x)-2}|\nabla u_n-\nabla u|^2tdt\geq 0
\end{aligned}$$

and hence

$$\int\limits_\Omega|-\nabla u_n|^{p(x)}dx-\int\limits_\Omega|-\nabla u|^{p(x)}dx\leq -p_m\int\limits_\Omega a(x,-\nabla u_n)\cdot(\nabla u_n-\nabla u)dx, \tag{4.4}$$

where $p_m=\min\{p^-,p^+\}$. On the other hand, since $g_n(x,u_n,\nabla u_n)$ is $L^{p'(x)}(\Omega)$-norm bounded, $u_n\to u$ in $L^{p(x)}(\Omega)$, and $u_n\to u$ a.e. in Ω, by applying the Hölder inequality, we obtain

$$\begin{aligned}
\int\limits_\Omega a(x,-\nabla u_n)\cdot(\nabla u_n-\nabla u)dx&=\int\limits_\Omega g_n(x,u_n,-\nabla u_n)(u_n-u)dx+\int\limits_\Omega h(u_n-u)dx\\
&\leq|g_n|_{p'(x)}|u_n-u|_{p(x)}+c\int\limits_\Omega|u_n-u|dx\\
&\to 0 \quad \text{as} \quad n\to\infty
\end{aligned}$$

which, along with (4.4), implies that

$$\limsup_{n\to\infty}\int\limits_\Omega|-\nabla u_n|^{p(x)}dx\leq\int\limits_\Omega|-\nabla u|^{p(x)}dx$$

or

$$\limsup_{n\to\infty}\rho(\nabla u_n)\leq\rho(\nabla u). \tag{4.5}$$

Then, (4.3) and (4.5) lead to

$$\lim_{n\to\infty}\rho(\nabla u_n)=\rho(\nabla u).$$

From condition $(a1)$ and assumption $u_n\rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, (∇u_n) converges in measure to ∇u in Ω. In conclusion, by Proposition 2.4, we obtain that

$$\lim_{n\to\infty}|\nabla u_n-\nabla u|_{p(x)}=0 \tag{4.6}$$

that is,

$$u_n \to u \text{ in } W_0^{1,p(x)}(\Omega) \text{ and } \nabla u_n \to \nabla u \text{ a.e. in } \Omega \tag{4.7}$$

Because of (4.6) and (4.7), the function $g(x, u, \nabla u) = u|\nabla u|^{p(x)-2}$ is continuous in the two last arguments. Therefore, by (4.7), we obtain

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ a.e. in } \Omega. \tag{4.8}$$

On the other hand, by the Hölder inequality we have

$$\begin{aligned} 0 \leq \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx &= \int_\Omega |u_n|^2 |\nabla u_n|^{p(x)-2} dx \\ &\leq ||u_n|^2|_{\frac{p(x)}{2}} ||\nabla u_n|^{p(x)-2}|_{\frac{p(x)}{p(x)-2}} \leq K_2. \end{aligned}$$

Let $E \subset \Omega$ be a measurable subset. Let us choose an arbitrary real number $\delta > 0$ such that $|E| < \delta$. Then, by considering the fact that sequences $(u_n), (\nabla u_n)$ converge strongly in $L^{p(x)}(\Omega)$ and $L^{p(x)}(\Omega)^N$, respectively, we have

$$\begin{aligned} &\int_E |g_n(x, u_n, \nabla u_n)| dx \\ &= \int_{E \cap \{|u_n(x)| \leq k\}} |g_n(x, u_n, \nabla u_n)| dx + \int_{E \cap \{|u_n(x)| > k\}} |g_n(x, u_n, \nabla u_n)| dx \\ &\leq \int_{E \cap \{|u_n(x)| \leq k\}} |g_n(x, u_n, \nabla u_n)| dx + \frac{1}{k} \int_\Omega g_n(x, u_n, \nabla u_n) u_n dx \\ &\leq \int_{E \cap \{|u_n(x)| \leq k\}} |u_n| |\nabla u_n|^{p(x)-2} dx + \frac{1}{k} K_2 \end{aligned}$$

which means the equi-integrability of $g_n(x, u_n, \nabla u_n)$. In conclusion, by (4.8) and Vitali's theorem, we obtain

$$g_n(x, u_n, \nabla u_n) \to g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{4.9}$$

Therefore, by (4.7) and (4.9), if we pass the limit in (4.2), we obtain that

$$\int_\Omega a(x, \nabla u) \cdot \nabla \varphi dx = \int_\Omega g(x, u, \nabla u) \varphi dx + \int_\Omega h \varphi dx$$

which means that $u \in W_0^{1,p(x)}(\Omega)$ is a nontrivial weak solution to problem (4.1). Moreover, because of assumption $(a1)$, this solution is unique. $\square$

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