

BANACH *-ALGEBRAS GENERATED BY SEMICIRCULAR ELEMENTS INDUCED BY CERTAIN ORTHOGONAL PROJECTIONS

Ilwoo Cho and Palle E.T. Jorgensen

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Abstract. The main purpose of this paper is to study structure theorems of Banach *-algebras generated by semicircular elements. In particular, we are interested in the cases where given semicircular elements are induced by orthogonal projections in a C^* -probability space.

Keywords: free probability, orthogonal projections, weighted-semicircular elements, semicircular elements.

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1. INTRODUCTION

The main purposes of this paper are (i) to introduce a way to construct *weighted-semicircular*, and *semicircular elements* generated by *orthogonal projections* in a certain *Banach *-probability space*, (ii) to study *operator-algebraic properties* of *Banach *-algebras* generated by our weighted-semicircular or semicircular elements, and (iii) to apply the results of (ii) to *operator-theoretic* (or *spectral*) data of operators in such Banach *-algebras via *free probability theory*.

There are different approaches to establish semicircular elements (e.g., [1, 11, 13] and [17]) in topological *-probability spaces (e.g., C^* -probability spaces, or W^* -probability spaces, or *Banach *-probability spaces*, etc). Our construction of semicircular elements is different from those already known. It is highly motivated by *weighted-semicircularity* and *semicircularity* in the sense of [3, 4] and [5].

1.1. MOTIVATION

Semicircularity is extremely important not only in mathematical (classical, or operator-valued) *probability theory* and *statistics*, but also in *mathematical physics* (quantum physics, quantum chaos theory, etc, e.g., see [8] and [10]). In particular,

in *free probability theory*, studying semicircularity of *free random variables* is one of the main branches (e.g., [1, 9, 11–14, 16] and [17]).

Independently, *p-adic* and *Adelic analysis* provide a fundamental tools, and play important roles not only in *number theory* and *geometry* at “very small” distance (e.g., [7] and [15]), but also in related mathematical or scientific fields (e.g., [2–4] and [5]). So, we cannot help emphasizing their importances and valuable applications.

In [5], the we studied certain semicircular-like elements, called *weighted-semicircular elements*, and corresponding *semicircular elements* induced from *p-adic number fields* \mathbb{Q}_p , motivated by the earlier studies of free-probabilistic structures over \mathbb{Q}_p (e.g., see [2] and cited papers therein). In [4], the author extended the “*p-adic local*” weighted-semicircular laws and the semicircular law of [5] to “*Adelic universal*” weighted-semicircular laws and the semicircular laws under *free product* of Banach $*$ -probability spaces in the sense of [5], over *primes p*. As application of the main results in [4], we studied *free stochastic integration for free stochastic processes* determined by the weighted-semicircular laws and the semicircular law of [4] in [3]. Also, see [6] and [12].

In this paper, we apply the construction of weighted-semicircular elements, and that of semicircular element of [3, 4] and [5] to construct such free random variables in certain Banach $*$ -probability spaces induced by $|\mathbb{Z}|$ -many orthogonal projections in a C^* -algebra.

1.2. OVERVIEW

Our study starts from the assumption that there exists integer-many orthogonal projections $\{q_j\}_{j \in \mathbb{Z}}$ in an arbitrarily given C^* -algebra A . One can have such cases artificially (e.g., see Section 3 below), or naturally (e.g., [3, 4] and [5]). Define a C^* -subalgebra Q of A generated by the family. And then construct a suitable *radial operator l acting on Q*. Define a cyclic Banach $*$ -algebra $\mathfrak{L} = \overline{\mathbb{C}\{l\}}$ generated by $\{l\}$. Then, each element of \mathfrak{L} is a Banach-space operator acting on Q . Then construct the tensor product algebra

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q.$$

Generating operators $u_j = l \otimes q_j$ of \mathfrak{L}_Q form a weighted-semicircular elements in the free product Banach $*$ -probability space,

$$\star_{j \in \mathbb{Z}} (\mathfrak{L}_Q, \tau_j),$$

for suitable linear functionals τ_j on \mathfrak{L}_Q .

And hence, there exist $s_j \in \mathbb{R}$ in \mathbb{C} , such that $s_j u_j$ become semicircular elements, for all $j \in \mathbb{Z}$.

The construction of our weighted-semicircular elements is one of the main results of this very paper. As we discussed above, the construction is very similar to those in [4]. From our (weighted-)semicircular elements, we construct several different types of Banach $*$ -algebras, and study operator-algebraic properties of such $*$ -algebras, and consider operator-theoretic properties of certain operators of these Banach $*$ -algebras under *spectral data*, expressed by *free distributions* of operators.

1.3. SKETCH OF MAIN RESULTS

In Sections 3 and 4, we established self-adjoint operators $\{u_j\}_{j \in \mathbb{Z}}$ in a certain Banach *-algebra $\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q$ induced by the C^* -algebra Q generated by a family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections q_j 's. By defining a family $\{\tau_j\}_{j \in \mathbb{Z}}$ of suitable linear functionals τ_j 's on \mathfrak{L}_Q , we show our self-adjoint Banach-space operators u_j are weighted-semicircular in the Banach *-probability spaces (\mathfrak{L}_Q, τ_j) , for all $j \in \mathbb{Z}$, i.e., the algebra \mathfrak{L}_Q is sectionized by $\{\tau_j\}_{j \in \mathbb{Z}}$, and in each sectionized Banach *-probability space of \mathfrak{L}_Q , our operator u_j is weighted-semicircular. The construction of our weighted-semicircular elements, itself, is one of the main results of this paper.

In Section 5, we construct semicircular elements $\{U_j\}_{j \in \mathbb{Z}}$ in \mathfrak{L}_Q generated by our weighted-semicircular elements $\{u_j\}_{j \in \mathbb{Z}}$ of Section 4. It shows that whenever one can take a family of $|\mathbb{Z}|$ -many mutually orthogonal projections, the corresponding semicircular elements U_j exist.

In Section 6, based on the constructions of weighted-semicircularity and semicircularity in Sections 4 and 5, we study weighted-semicircularity induced by different types of Banach *-algebras generated by “finite” copies of \mathfrak{L}_Q 's. In particular, we are interested in the weighted-semicircularity on Banach *-algebras generated by finite-copies of \mathfrak{L}_Q 's under *direct product*, *tensor product*, and *free product*. Check the weighted-semicircularity (6.12) on direct products of \mathfrak{L}_Q 's, and the weighted-semicircularity (6.32), and the semicircularity (6.34) on tensor products of \mathfrak{L}_Q 's, and the semicircularity (6.49) on free products of \mathfrak{L}_Q 's.

In Section 7, we consider an example from our main results of Sections 4, 5 and 6. Especially, we study weighted-semicircular elements and semicircular elements determined by the mutually orthogonal *rank-1 projections* on the Hilbert space $l^2(\mathbb{Z})$.

2. PRELIMINARIES

Readers can check fundamental analytic-and-combinatorial free probability theory from [14] and [16] (and the cited papers therein). *Free probability* is understood as the noncommutative operator-algebraic version of classical probability theory and statistics. The classical *independence* is replaced by the *freeness*. It has various applications not only in pure mathematics (e.g., [13]), but also in related applied topics (e.g., [2–4] and [5]). In particular, we will use combinatorial free probabilistic approach of *Speicher* (e.g., [14]). *Free moments* and *free cumulants* of operators will be computed without introducing detailed concepts.

Also, readers can check the weighted-semicircularity and semicircularity induced from p -adic number fields \mathbb{Q}_p in [4] and [5]. The main results, including the constructions, of [4] and [5] motivate our study.

3. FUNDAMENTAL SETTINGS

In this section, we establish backgrounds of our study. Let (A, ψ) be a C^* -probability space, where A is a *unital C^* -algebra* in the *operator algebra* $B(H)$ consisting of all (bounded linear) operators on a *Hilbert space* H , and ψ is a *bounded linear functional* on A , satisfying $\psi(1_A) = 1$, where 1_A is the *identity operator* of A .

An operator a of A is said to be a *free random variable* whenever it is regarded as an element of (A, ψ) . As usual, we say a is *self-adjoint*, if $a^* = a$ in A , where a^* is the *adjoint* of a .

We say a self-adjoint free random variable $a \in (A, \psi)$ is an *even element*, if it satisfies

$$\psi(a^{2n-1}) = 0, \text{ for all } n \in \mathbb{N}.$$

Definition 3.1. A self-adjoint free random variable a is said to be *weighted-semicircular* in (A, ψ) with its weight $t_0 \in \mathbb{C}$, (or in short, t_0 -semicircular), if a satisfies the free cumulant computation,

$$k_n(a, \dots, a) = \begin{cases} k_2(a, a) = t_0 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where $k_n(\dots)$ means the free cumulant on A in terms of ψ (in the sense of [14]).

If $t_0 = 1$ in (3.1), the 1-semicircular element a is simply said to be *semicircular* in (A, ψ) , i.e., a is *semicircular* in (A, ψ) , if a satisfies

$$k_n(a, \dots, a) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$.

By the Möbius inversion of [14], one can characterize the weighted-semicircularity (3.1) as follows: a self-adjoint operator a is t_0 -semicircular in (A, ψ) , if and only if

$$\psi(a^{2n-1}) = 0, \text{ i.e., it is even in } (A, \psi),$$

and

$$\psi(a^{2n}) = t_0^n c_n$$

for all $n \in \mathbb{N}$, where

$$c_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

is the n -th *Catalan number*.

In short, a is t_0 -semicircular in (A, ψ) , if and only if

$$\psi(a^n) = \omega_n \left(t_0^{\frac{n}{2}} c_{\frac{n}{2}} \right), \tag{3.3}$$

where

$$\omega_n \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$.

Similarly, a free random variable a is *semicircular* in (A, ψ) , if and only if

$$\psi(a^n) = \omega_n c_{\frac{n}{2}}, \tag{3.4}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (3.3).

So, we will use the t_0 -semicircularity (3.1) and its characterization (3.3) alternatively; and similarly, one can use the semicircularity (3.2) and its characterization (3.4), alternatively.

Note that, if a is a self-adjoint free random variable in (A, ψ) , then the free moments $\{\psi(a^n)\}_{n=1}^\infty$, and the free cumulants $\{k_n(a, \dots, a)\}_{n=1}^\infty$ provide equivalent *free distribution of a* in (A, ψ) , which are also equivalent to the *spectral data of a* (e.g., [14] and [16]). Indeed, the *Möbius inversion* of [14] satisfies

$$\psi(a^n) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} k_{|V|}(a, \dots, a) \right),$$

and

$$k_n(a, \dots, a) = \sum_{\pi \in NC(n)} \left(\prod_{V \in \theta} \psi(a^{|V|}) \right) \mu(\pi, 1_n),$$

where $NC(n)$ is the *lattice* consisting of all *noncrossing partitions* over $\{1, \dots, n\}$, and “ $V \in \pi$ ” means “ V is a *block* of π ,” and where

$$\mu(0_k, 1_k) = (-1)^{k+1} c_{k+1}, \text{ and } \sum_{\pi \in NC(n)} \mu(\pi, 1_n) = 0,$$

where c_m are m -th Catalan numbers for all $m \in \mathbb{N}$.

Therefore, the free-moment formula (3.3) (resp., (3.4)) is equivalent to the free-cumulant definition (3.1) (resp., (3.2)).

Now, let (A, ψ) be a fixed C^* -probability space, and let $q_j \in A$ be a *projection* in the sense that

$$q_j^* = q_j = q_j^2 \text{ in } A, \tag{3.5}$$

for all $j \in \mathbb{Z}$, where \mathbb{Z} is the set of all *integers*. Moreover, assume that the projections $\{q_j\}_{j \in \mathbb{Z}}$ are *mutually orthogonal* in A , in the sense that:

$$q_i q_j = \delta_{i,j} q_j \text{ in } A, \text{ for all } i, j \in \mathbb{Z}, \tag{3.6}$$

where δ is the *Kronecker delta*. Also, suppose, for convenience, that

$$\psi(q_j) \neq 0 \text{ in } \mathbb{C}, \text{ for all } j \in \mathbb{Z}. \tag{3.7}$$

Now, we fix the family $\{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections of A , and we denote it by \mathbf{Q} ,

$$\mathbf{Q} = \{q_j : j \in \mathbb{Z}, q_j\text{'s satisfy (3.5), (3.6) and (3.7)}\}, \tag{3.8}$$

in A .

Remark 3.2. Assume that one can find mutually orthogonal projections $\{q_1, \dots, q_N\}$ in a C^* -algebra A_0 , for some $N \in \mathbb{N} \cup \{\infty\}$. Assume that $N < \infty$, for instance, A_0 is a matricial algebra $M_N(\mathbb{C})$, and

$$q_j = [t_{ij}]_{N \times N}, \text{ with } t_{jj} = 1, \text{ and } t_{kl} = 0,$$

for all $(k, l) \neq (j, j)$ in $\{1, \dots, N\}^2$, etc.

Then one can construct a C^* -algebra

$$A = A_0^{\oplus|\mathbb{Z}|} = C^*(\dots \sqcup A_0 \sqcup A_0 \sqcup \dots),$$

under suitable product topology. Then one can obtain the mutually orthogonal projections,

$$\{\dots, q_1^{(-1)}, \dots, q_N^{(-1)}, q_1^{(0)}, \dots, q_N^{(0)}, q_1^{(1)}, \dots, q_N^{(1)}, \dots, \}$$

in A , with identity:

$$q_j^{(k)} = q_j \text{ in the direct summand } A_0 \text{ of } A, \text{ for all } k \in \mathbb{Z},$$

for all $j = 1, \dots, N$.

Similarly, if one can find mutually orthogonal projections $\{q_1, q_2, q_3, \dots\}$ in a C^* -algebra A_0 (i.e., $N = \infty$), then one can construct a C^* -algebra

$$A = A_0 \oplus A_0,$$

and the corresponding orthogonal projections,

$$\{\dots, q_3^{(1)}, q_2^{(1)}, q_1^{(1)}, q_1^{(2)}, q_2^{(2)}, q_3^{(2)}, \dots\}$$

in A , with

$$q_j^{(k)} = q_j \text{ in the direct summand } A_0 \text{ of } A, \text{ for } k = 1, 2,$$

for all $j \in \mathbb{N}$.

Thus, we can assume the existence of mutually orthogonal $|\mathbb{Z}|$ -many projections induced by a fixed C^* -algebra A .

And let Q be the C^* -subalgebra of A generated by \mathbf{Q} of (3.8),

$$Q \stackrel{def}{=} C^*(\mathbf{Q}) \subseteq A, \tag{3.9}$$

where \mathbf{Q} is in the sense of (3.8).

Proposition 3.3. *Let Q be a C^* -subalgebra (3.9) of a given C^* -algebra A generated by \mathbf{Q} of (3.8). Then*

$$Q \stackrel{*iso}{=} \bigoplus_{j \in \mathbb{Z}} (\mathbb{C} \cdot q_j) \stackrel{*iso}{=} \mathbb{C}^{\oplus|\mathbb{Z}|}, \text{ in } A. \tag{3.10}$$

Proof. The proof of the isomorphism theorem (3.10) is straightforward by the mutually-orthogonality (3.5) of the generator set \mathbf{Q} of Q in A . □

Define now linear functionals ψ_j on Q by

$$\psi_j(q_i) = \delta_{ij}\psi(q_j), \text{ for all } i \in \mathbb{Z}, \tag{3.11}$$

for all $j \in \mathbb{Z}$, where ψ is a fixed linear functional in the fixed C^* -probability space (A, ψ) . The linear functionals $\{\psi_j\}_{j \in \mathbb{Z}}$ of (3.11) are well-defined on Q by the structure theorem (3.10) of the C^* -subalgebra Q of (3.9) in A .

So, if $q \in Q$, then

$$q = \sum_{j \in \mathbb{Z}} t_j q_j \quad (\text{with } t_j \in \mathbb{C}),$$

by (3.10), and hence,

$$\psi_j(q) = \psi_j \left(\sum_{j \in \mathbb{Z}} t_j q_j \right) = t_j \psi(q_j),$$

by (3.11), for all $j \in \mathbb{Z}$.

Definition 3.4. The C^* -probability spaces (Q, ψ_j) are called the j -th C^* -probability spaces of Q in a fixed C^* -probability space (A, ψ) , where Q is in the sense of (3.9), and ψ_j are in the sense of (3.11), for all $j \in \mathbb{Z}$.

Now, let us define a *Banach-space operator* c and a “acting on the C^* -algebra Q ” by a bounded linear transformations satisfying that

$$c(q_j) = q_{j+1}, \text{ and } a(q_j) = q_{j-1}, \tag{3.12}$$

for all $j \in \mathbb{Z}$. Then c and a are indeed well-defined bounded linear operators “on Q ”.

Definition 3.5. We call these Banach-space operators c and a of (3.12), the *creation*, respectively, the *annihilation* on Q . Define now a new Banach-space operator l acting on Q , by

$$l = c + a. \tag{3.13}$$

This operator l of (3.13) is said to be the *radial operator* on Q .

By the definition (3.13), one has

$$l \left(\sum_{j \in \mathbb{Z}} t_j q_j \right) = \sum_{j \in \mathbb{Z}} t_j (q_{j+1} + q_{j-1}), \text{ on } Q.$$

Now, define a cyclic *Banach algebra*

$$\mathfrak{L} \stackrel{def}{=} \overline{\mathbb{C}\{l\}}^{\|\cdot\|}, \tag{3.14}$$

generated the radial operator l of (3.13), equipped with the *operator norm*,

$$\|T\| = \sup\{\|Tq\|_Q : \|q\|_Q = 1\},$$

where $\|\cdot\|_Q$ is the C^* -norm on Q , where $\overline{X}^{\|\cdot\|}$ mean the operator-norm closures of subsets X of the *operator space* $B(Q)$, consisting of all bounded linear transformations on Q .

On the Banach algebra \mathfrak{L} of (3.14), define a unary operation $(*)$ by

$$\left(\sum_{n=0}^{\infty} t_n l^n \right)^* = \sum_{n=0}^{\infty} \bar{t}_n l^n \text{ in } \mathfrak{L}, \tag{3.15}$$

where \bar{z} mean the *conjugates* of z , for all $z \in \mathbb{C}$.

Then it is not difficult to check that the operation (3.15) forms an *adjoint on \mathfrak{L}* , i.e., the Banach algebra \mathfrak{L} of (3.14) forms a *Banach *-algebra* with its adjoint (3.15).

Definition 3.6. We call the Banach *-algebra \mathfrak{L} of (3.14), the *radial (Banach *-)-algebra on Q* .

Now, let \mathfrak{L} be the radial algebra on Q . Construct now the *tensor product Banach *-algebra*,

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q. \tag{3.16}$$

Since the radial algebra \mathfrak{L} of Q is a Banach *-algebra, and since Q is a C^* -algebra, the topological tensor product \mathfrak{L}_Q of them is a well-determined Banach *-algebra under the product topology.

Definition 3.7. We call the tensor product Banach *-algebra \mathfrak{L}_Q of (3.16), the radial projection (Banach *-)-algebra on Q .

4. WEIGHTED-SEMICIRCULAR ELEMENTS INDUCED BY \mathbf{Q}

Throughout this section, we use the same settings, notations and concepts of Section 3. We here construct weighted-semicircular elements induced by the family \mathbf{Q} of mutually orthogonal projections in the sense of (3.8). Let (Q, ψ_j) be j -th C^* -probability space of Q in (A, ψ) , where ψ_j are in the sense of (3.11), for all $j \in \mathbb{Z}$, and let \mathfrak{L}_Q be the radial projection algebra (3.16) on Q .

Remark that, if

$$u_j = l \otimes q_j \in \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.1}$$

then

$$u_j^n = (l \otimes q_j)^n = l^n \otimes q_j, \text{ for all } n \in \mathbb{N},$$

since $q_j^n = q_j$, for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

One can construct a linear functional φ_j on the radial projection algebra \mathfrak{L}_Q by a linear morphism satisfying that

$$\varphi_j((l \otimes q_i)^n) = \varphi_j(l^n \otimes q_i) \stackrel{def}{=} \psi_j(l^n(q_i)), \tag{4.2}$$

for all $n \in \mathbb{N}$, for all $i, j \in \mathbb{Z}$.

We call the Banach *-probability spaces

$$(\mathfrak{L}_Q, \varphi_j), \text{ for all } j \in \mathbb{Z}, \tag{4.3}$$

the j -th (Banach-*)-probability spaces on Q .

Now, observe the elements $l^n(q_i)$ in Q , for all $n \in \mathbb{N}, i \in \mathbb{Z}$. If c and a are the creation, respectively, the annihilation on Q in the sense of (3.12), then

$$ca = ac = 1_Q, \text{ the identity operator on } Q, \tag{4.4}$$

where

$$1_Q(q_j) = q_j, \text{ for all } j \in \mathbb{Z},$$

and hence,

$$1_Q(x) = x, \text{ for all } x \in Q.$$

Indeed, for any $q_j \in \mathbf{Q}$ in Q ,

$$ca(q_j) = c(a(q_j)) = c(q_{j-1}) = q_{j-1+1} = q_j,$$

and

$$ac(q_j) = a(c(q_j)) = a(q_{j+1}) = q_{j+1-1} = q_j,$$

for all $j \in \mathbb{Z}$. More generally, one has

$$c^n a^n = 1_Q = a^n c^n, \text{ for all } n \in \mathbb{N}, \tag{4.5}$$

by induction on (4.4).

By (4.4) and (4.5), we also obtain that

$$c^{n_1} a^{n_2} = a^{n_2} c^{n_1}, \text{ for all } n_1, n_2 \in \mathbb{N}, \tag{4.6}$$

too.

Thus, one obtains that

$$l^n = (c + a)^n = \sum_{k=0}^n \binom{n}{k} c^k a^{n-k}, \tag{4.7}$$

by (4.4), (4.5), and (4.4)'', for all $n \in \mathbb{N}$.

Note that, for any $n \in \mathbb{N}$,

$$l^{2n-1} = \sum_{k=0}^{2n-1} \binom{2n-1}{k} c^k a^{n-k}, \tag{4.8}$$

by (4.7). So, the formula (4.8) does not contain 1_Q -terms, by (4.4) and (4.5).

Note also that, for any $n \in \mathbb{N}$, one has

$$l^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} c^k a^{n-k} = \binom{2n}{n} c^n a^n + [\text{Rest terms}], \tag{4.9}$$

by (4.4), (4.5), and (4.7).

Proposition 4.1. *Let l be the radial operator (3.13) generating the radial algebra \mathfrak{L} on Q . Then*

$$l^{2n-1} \text{ does not contain } 1_Q\text{-terms in } \mathfrak{L}, \tag{4.10}$$

$$l^{2n} \text{ contains } \binom{2n}{n} \cdot 1_Q \text{ in } \mathfrak{L}. \tag{4.11}$$

Proof. The statement (4.10) is proven by (4.8), with help of (4.4), (4.5) and (4.7). And, the statement (4.11) is proven by (4.9) and (4.5). \square

Since

$$u_j^n = (l \otimes q_j)^n = l^n \otimes q_j,$$

one can get that

$$\varphi_j (u_j^{2n-1}) = \psi_j (l^{2n-1} (q_j)) = 0, \tag{4.12}$$

for all $n \in \mathbb{N}$, and $j \in \mathbb{Z}$, by (3.11) and (4.10).

Similarly, we have

$$\varphi_j (u_j^{2n}) = \psi_j (l^{2n} (q_j)) = \psi_j \left(\binom{2n}{n} q_j + [\text{Rest terms}]q_j \right)$$

by (4.9)

$$= \binom{2n}{n} \psi_j (q_j) = \binom{2n}{n} \psi (q_j), \tag{4.13}$$

by (3.11), for all $n \in \mathbb{N}$, i.e., by (4.10) and (4.11), we obtain the following free-distributional data on the j -th probability space $(\mathfrak{L}_Q, \varphi_j)$, for $j \in \mathbb{Z}$.

Theorem 4.2. Fix $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the corresponding generating operator (4.1) of the j -th probability space $(\mathfrak{L}_Q, \varphi_j)$. Then

$$\varphi_j (u_j^n) = \omega_n \left(\left(\frac{n}{2} + 1 \right) \psi (q_j) \right) c_{\frac{n}{2}}, \tag{4.14}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (3.3), and c_k are the k -th Catalan numbers, for all $k \in \mathbb{N}$.

Proof. By (4.12) and (4.13), one can get that:

$$\varphi_j (u_j^{2n-1}) = 0,$$

and

$$\begin{aligned} \varphi_j (u_j^{2n}) &= \binom{2n}{n} \psi (q_j) = \binom{n+1}{n+1} \binom{2n}{n} \psi (q_j) \\ &= ((n+1)\psi (q_j)) \binom{1}{n+1} \binom{2n}{n} \\ &= ((n+1)\psi (q_j)) c_n, \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, the formula (4.14) holds. □

Motivated by the free-distributional data (4.14) of the generating operator u_j of \mathfrak{L}_Q , we define the following morphism

$$E_{j,Q} : \mathfrak{L}_Q \rightarrow \mathfrak{L}_Q$$

by a surjective linear transformation satisfying

$$E_{j,Q} (u_i^n) = \delta_{j,i} \left(\frac{\psi (q_j)^{n-1}}{\left(\left[\frac{n}{2} \right] + 1 \right)} u_j^n \right), \tag{4.15}$$

for all $n \in \mathbb{N}, i, j \in \mathbb{Z}$, where δ is the Kronecker delta, and $\lceil \frac{n}{2} \rceil$ mean the *minimal integers* greater than or equal to $\frac{n}{2}$, for example,

$$\lceil \frac{3}{2} \rceil = 2 = \lceil \frac{4}{2} \rceil.$$

The linear transformations $E_{j,Q}$ of (4.15) are well-defined linear transformations on \mathfrak{L}_Q , because of the construction (3.16) of \mathfrak{L}_Q , and by the structure theorem (3.10) of Q , i.e., all elements T of \mathfrak{L}_Q has its form,

$$\sum_{k=1}^{\infty} \sum_{j \in \mathbb{Z}} t_{k,j} u_j^{n_k} \quad (\text{with } n_k \in \mathbb{N}, t_{k,j} \in \mathbb{C}).$$

So, under linearity, the morphisms $E_{j,Q}$ of (4.15) are indeed well-defined on the radial projection algebra \mathfrak{L}_Q , for all $j \in \mathbb{Z}$.

Define now a new linear functional τ_j on \mathfrak{L}_Q by

$$\tau_j \stackrel{\text{def}}{=} \varphi_j \circ E_{j,Q} \text{ on } \mathfrak{L}_Q, \text{ for all } j \in \mathbb{Z}, \tag{4.16}$$

where φ_j is the linear functional (4.2) on \mathfrak{L}_Q . By the linearity of φ_j and $E_{j,Q}$, the above morphisms τ_j are indeed well-defined linear functionals on \mathfrak{L}_Q , for all $j \in \mathbb{Z}$.

Definition 4.3. The well-defined Banach *-probability spaces

$$\mathfrak{L}_Q(j) \stackrel{\text{denote}}{=} (\mathfrak{L}_Q, \tau_j) \tag{4.17}$$

are called the j -th filtered (Banach-*)probability spaces of \mathfrak{L}_Q , for all $j \in \mathbb{Z}$.

On the j -th filtered probability space $\mathfrak{L}_Q(j)$ of (4.17), one can obtain that

$$\begin{aligned} \tau_j(u_j^n) &= \varphi_j(E_{j,Q}(u_j^n)) \\ &= \varphi_j\left(\frac{\psi(q_j)^{n-1}}{\lceil \frac{n}{2} \rceil + 1}(u_j^n)\right) = \frac{\psi(q_j)^{n-1}}{\lceil \frac{n}{2} \rceil + 1} \varphi_j(u_j^n) \\ &= \frac{\psi(q_j)^{n-1}}{\lceil \frac{n}{2} \rceil + 1} \omega_n\left(\left(\frac{n}{2} + 1\right) \psi(q_j)\right) c_{\frac{n}{2}} \end{aligned}$$

by (4.14), i.e., we can get that

$$\tau_j(u_j^n) = \begin{cases} \psi(q_j)^n c_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases} \tag{4.18}$$

for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

Theorem 4.4. Let $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \tau_j)$ be the j -th filtered probability space of \mathfrak{L}_Q , for $j \in \mathbb{Z}$. Then

$$\tau_j(u_k^n) = \delta_{j,k} (\omega_n \psi(q_j)^n c_{\frac{n}{2}}) \tag{4.19}$$

for all $n \in \mathbb{N}$, for all $k \in \mathbb{Z}$, where ω_n are in the sense of (3.3).

Proof. If $k = j$ in \mathbb{Z} , then the free-distributional data (4.19) holds true by (4.18), for all $n \in \mathbb{N}$.

If $k \neq j$ in \mathbb{Z} , then, by the very definition (4.15) of the j -th filterization $E_{j,Q}$, and also by the definition (4.2) of φ_j ,

$$\tau_j(u_k^n) = 0.$$

Therefore, the above formula (4.19) holds true, for all $k \in \mathbb{Z}$. □

The following theorem is the direct consequence of the above free distributional data (4.19).

Theorem 4.5. *Let $\mathfrak{L}_Q(j)$ be the j -th filtered probability space (\mathfrak{L}_Q, τ_j) of \mathfrak{L}_Q , for $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be the j -th generating operator of \mathfrak{L}_Q in $\mathfrak{L}_Q(j)$. Then u_j is $\psi(q_j)^2$ -semicircular in $\mathfrak{L}_Q(j)$.*

Proof. First of all, the generating operators u_k are self-adjoint in \mathfrak{L}_Q . Indeed,

$$u_k^* = (l \otimes q_k)^* = l^* \otimes q_k^* = l \otimes q_k = u_k,$$

by (3.15) and by the self-adjointness of projections q_k , for all $k \in \mathbb{Z}$.

Let us fix $j \in \mathbb{Z}$, and let $u_j = l \otimes q_j$ be a generating free random variable of the j -th filtered probability space $\mathfrak{L}_Q(j)$. Then it is self-adjoint in $\mathfrak{L}_Q(j)$.

Also, by (4.19), we have that

$$\tau_j(u_j^n) = \omega_n(\psi(q_j))^n c_{\frac{n}{2}} = \omega_n(\psi(q_j)^2)^{\frac{n}{2}} c_{\frac{n}{2}},$$

where ω_n are in the sense of (3.3), for all $n \in \mathbb{N}$.

Therefore, by the free-momental characterization (3.3) of weighted-semicircularity, the element u_j is $\psi(q_j)^2$ -semicircular in the j -th filtered probability space $\mathfrak{L}_Q(j)$ of \mathfrak{L}_Q . □

The above theorem shows that, for any $k \in \mathbb{Z}$, the k -th generating operator $u_k = l \otimes q_k$ of the k -th filtered probability space $\mathfrak{L}_Q(k)$ of Q is $\psi(q_k)^2$ -semicircular.

Observe also that if $k_n^j(\dots)$ is the free cumulant in terms of the linear functional τ_j on \mathfrak{L}_Q in the sense of [14], then

$$k_n^j(\underbrace{u_j, u_j, \dots, u_j}_{n\text{-times}}) = \begin{cases} k_2^j(u_j, u_j) = \psi(q_j)^2 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{4.20}$$

for all $n \in \mathbb{N}$, by (4.19), and by the Möbius inversion of [14]. So, by (3.1), the j -th generating operators u_j are $\psi(q_j)^2$ -semicircular in the j -th filtered probability spaces $\mathfrak{L}_Q(j)$, for all $j \in \mathbb{Z}$.

5. SEMICIRCULAR ELEMENTS INDUCED BY \mathbf{Q}

As in Section 4, we will keep working on the j -th filtered probability space $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$. The main results of Section 4 show that, for any fixed $j \in \mathbb{Z}$, the j -th generating operator $u_j = l \otimes q_j$ of \mathfrak{L}_Q is $\psi(q_j)^2$ -semicircular in $\mathfrak{L}_Q(j)$, satisfying that

$$\tau_j(u_j^n) = \omega_n \psi(q_j)^n c_{\frac{n}{2}},$$

equivalently,

$$k_n^j(u_j, \dots, u_j) = \begin{cases} \psi(q_j)^2 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

for all $n \in \mathbb{N}$, where $k_n^j(\dots)$ means the free cumulant on \mathfrak{L}_Q in terms of the linear functional τ_j of (4.16).

Corollary 5.1. *Let $u_j = l \otimes q_j$ be the j -th generating operator of the radial projection algebra \mathfrak{L}_Q , for $j \in \mathbb{Z}$. If the projection q_j of Q satisfies $\psi(q_j) = 1$, then u_j is semicircular in the j -th filtered probability space $\mathfrak{L}_Q(j)$.*

However, generally $\psi(q_j)$ may not be identical to 1, for $j \in \mathbb{Z}$. So, we consider the following semicircularity induced by our weighted-semicircularity in $\mathfrak{L}_Q(j)$.

Define operators U_j of $\mathfrak{L}_Q(j)$ by

$$U_j = \frac{1}{\psi(q_j)} u_j \in \mathfrak{L}_Q(j), \tag{5.2}$$

for all $j \in \mathbb{Z}$, where u_j are the $\psi(q_j)^2$ -semicircular elements in $\mathfrak{L}_Q(j)$. Then these operators U_j can be semicircular in $\mathfrak{L}_Q(j)$ under a certain additional condition.

Theorem 5.2. *Let $U_j = \frac{1}{\psi(q_j)} u_j$ be a free random variable (5.2) of the j -th filtered probability space $\mathfrak{L}_Q(j)$, for $j \in \mathbb{Z}$. If $\psi(q_j) \in \mathbb{R}$ in \mathbb{C} , then U_j is semicircular in $\mathfrak{L}_Q(j)$.*

Proof. Fix $j \in \mathbb{Z}$, and assume that $\psi(q_j) \in \mathbb{R}$ in \mathbb{C} . Then

$$U_j^* = \left(\frac{1}{\psi(q_j)} u_j \right)^* = U_j,$$

by the self-adjointness of u_j in \mathfrak{L}_Q . So, the operator U_j is self-adjoint in \mathfrak{L}_Q .

Consider now that: if $k_n^j(\dots)$ is the free cumulant on \mathfrak{L}_Q in terms of τ_j , then

$$\begin{aligned} k_n^j(\underbrace{U_j, U_j, \dots, U_j}_{n\text{-times}}) &= k_n^j \left(\frac{1}{\psi(q_j)} u_j, \dots, \frac{1}{\psi(q_j)} u_j \right) \\ &= \left(\frac{1}{\psi(q_j)} \right)^n k_n^j(\underbrace{u_j, u_j, \dots, u_j}_{n\text{-times}}) \end{aligned}$$

by the bimodule-map property of free cumulants (e.g., [14])

$$= \begin{cases} \left(\frac{1}{\psi(q_j)} \right)^2 k_2^j(u_j, u_j) & \text{if } n = 2, \\ 0 & \text{otherwise} \end{cases}$$

by the $\psi(q_j)^2$ -semicircularity of u_j in $\mathfrak{L}_Q(j)$

$$= \begin{cases} \left(\frac{1}{\psi(q_j)^2}\right) (\psi(q_j))^2 = 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{5.3}$$

for all $n \in \mathbb{N}$. Therefore, if $\psi(q_j) \in \mathbb{R}$ in \mathbb{C} , then the self-adjoint operator U_j satisfies the free-distributional data (5.3). Thus, by (5.3) and (3.2), this operator U_j is semicircular in $\mathfrak{L}_Q(j)$. \square

The above theorem shows that, from the $\psi(q_j)^2$ -semicircular elements $u_j = l \otimes q_j$ in $\mathfrak{L}_Q(j)$, one can construct semicircular elements $U_j = \frac{1}{\psi(q_j)} u_j$ in $\mathfrak{L}_Q(j)$, whenever $\psi(q_j) \in \mathbb{R}$, for all $j \in \mathbb{Z}$.

6. WEIGHTED-SEMICIRCULARITY ON BANACH *-ALGEBRAS GENERATED BY COPIES OF \mathfrak{L}_Q

Let (A, ψ) be a fixed C^* -probability space, and Q , the C^* -subalgebra of A generated by the family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of $|\mathbb{Z}|$ -many projections satisfying the conditions (3.4), (3.6) and (3.7). And let \mathfrak{L}_Q be the radial projection algebra (3.16), inducing its j -th filtered probability spaces $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \tau_j)$ in the sense of (4.17), for all $j \in \mathbb{Z}$. Then the free random variables

$$u_j = l \otimes q_j \in \mathfrak{L}_Q \tag{6.1}$$

are $\psi(q_j)^2$ -semicircular in $\mathfrak{L}_Q(j)$, for all $j \in \mathbb{Z}$.

If $\psi(q_j) \in \mathbb{R}$ in \mathbb{C} , for $j \in \mathbb{Z}$, then the free random variables

$$U_j = \frac{1}{\psi(q_j)} u_j \in \mathfrak{L}_Q \tag{6.2}$$

are semicircular in $\mathfrak{L}_Q(j)$.

In the rest of this section, we assume $\psi(q_j) \in \mathbb{R}$ in \mathbb{C} , for all $j \in \mathbb{Z}$, for convenience.

6.1. ON THE BANACH *-PROBABILITY SPACES $\mathfrak{L}_Q^{\oplus N}(j_1, \dots, j_N)$

Let $W = (j_1, \dots, j_N)$ be an N -tuple of integers j_1, \dots, j_N in \mathbb{Z} , for $N \in \mathbb{N}$. Remark that the chosen integer-entries j_1, \dots, j_N of W are not necessarily distinct in \mathbb{Z} , and define the *direct product Banach *-algebra* $\mathfrak{L}_Q^{\oplus N}$ by

$$\mathfrak{L}_Q^{\oplus N} \stackrel{def}{=} \underbrace{\mathfrak{L}_Q \oplus \dots \oplus \mathfrak{L}_Q}_{N\text{-times}}, \tag{6.3}$$

where \oplus means the *direct sum of Banach *-algebras* under product topology.

Then on a new Banach *-algebra $\mathfrak{L}_Q^{\oplus N}$ of (6.3), one can define a conditional expectation

$$E_W : \mathfrak{L}_Q^{\oplus N} \rightarrow \mathbb{C}^{\oplus N}$$

by the linear transformation satisfying that

$$E_W((T_1, \dots, T_N)) = (\tau_{j_1}(T_1), \dots, \tau_{j_N}(T_N)), \tag{6.4}$$

for all

$$(T_1, \dots, T_N) \stackrel{\text{denote}}{=} \bigoplus_{l=1}^N T_l \in \mathfrak{L}_Q^{\oplus N},$$

where $\mathbb{C}^{\oplus N}$ is the N -dimensional algebra (which is regarded as the diagonal subalgebra in the matricial algebra $M_N(\mathbb{C})$), and τ_{j_k} are the linear functionals (4.16) on \mathfrak{L}_Q , inducing the corresponding j_k -th filtered probability space $\mathfrak{L}_Q(j_k) = (\mathfrak{L}_Q, \tau_{j_k})$, for $k = 1, \dots, N$, where j_1, \dots, j_N are the entries of the fixed N -tuple W .

Then the morphism E_W of (6.4) is indeed a well-defined conditional expectation from $\mathfrak{L}_Q^{\oplus N}$ onto $\mathbb{C}^{\oplus N}$ (e.g., [14]).

Proposition 6.1. *The linear morphism $E_W : \mathfrak{L}_Q^{\oplus N} \rightarrow \mathbb{C}^{\oplus N}$ of (6.4), for a fixed N -tuple $W = (j_1, \dots, j_N)$ of integers j_1, \dots, j_N , is a conditional expectation.*

Proof. By the very definition (6.4) of the linear morphism E_W , it is a well-determined bounded linear transformation from $\mathfrak{L}_Q^{\oplus N}$ “onto” $\mathbb{C}^{\oplus N}$, because τ_{j_l} are bounded, for all $l = 1, \dots, N$.

Clearly, if one takes $T = (t_1 \cdot I, \dots, t_N \cdot I) \in \mathfrak{L}_Q^{\oplus N}$, for

$$I \stackrel{\text{def}}{=} 1_{\mathfrak{L}_Q} = 1_{\mathfrak{L}} \otimes 1_Q \text{ in } \mathfrak{L}_Q,$$

for $t_1, \dots, t_N \in \mathbb{C}$, where $1_{\mathfrak{L}}$ is the identity element l^0 of the radial algebra \mathfrak{L} , satisfying $1_{\mathfrak{L}}(l^n) = l^n$, for all $n \in \mathbb{N}$, then

$$E_W(T) = (t_1, \dots, t_N) = T,$$

by understanding $T \in \bigoplus_{k=1}^N (\mathbb{C} \cdot I) \stackrel{\text{*iso}}{=} \mathbb{C}^{\oplus N}$ in $\mathfrak{L}_Q^{\oplus N}$. Therefore, under linearity, we have

$$E_W(T) = T, \text{ whenever } T \in \mathbb{C}^{\oplus N} \text{ in } \mathfrak{L}_Q^{\oplus N}. \tag{6.5}$$

Now, let $S_1 = (t_1, \dots, t_N), S_2 = (t'_1, \dots, t'_N) \in \mathbb{C}^{\oplus N}$ in $\mathfrak{L}_Q^{\oplus N}$, with $t_k, t'_k \in \mathbb{C}$, for all $k = 1, \dots, N$, and let $T = (T_1, \dots, T_N) \in \mathfrak{L}_Q^{\oplus N}$. Then

$$\begin{aligned} E_W(S_1 T S_2) &= E_W((t_1 T_1 t'_1, \dots, t_1 T_N t'_N)) \\ &= (\tau_{j_1}(t_1 T_1 t'_1), \dots, \tau_{j_N}(t_N T_N t'_N)) \\ &= (t_1 \tau_{j_1}(T_1) t'_1, \dots, t_N \tau_{j_N}(T_N) t'_N) \\ &= (t_1, \dots, t_N) (\tau_{j_1}(T_1), \dots, \tau_{j_N}(T_N)) (t'_1, \dots, t'_N) \end{aligned}$$

in the algebra $\mathbb{C}^{\oplus N}$

$$= S_1 E_W(T) S_2.$$

So, for any $S_1, S_2 \in \mathbb{C}^{\oplus N}$, and $T \in \mathfrak{L}_Q^{\oplus N}$, one obtains that

$$E_W(S_1 T S_2) = S_1 E_W(T) S_2 \text{ in } \mathbb{C}^{\oplus N}. \tag{6.6}$$

Finally, observe that, if $T = (T_1, \dots, T_N) \in \mathfrak{L}_Q^{\oplus N}$, then $T^* = (T_1^*, \dots, T_N^*)$ in $\mathfrak{L}_Q^{\oplus N}$. So,

$$E_W(T^*) = (\tau_{j_1}(T_1^*), \dots, \tau_{j_N}(T_N^*)) = \left(\overline{\tau_{j_1}(T_1)}, \dots, \overline{\tau_{j_N}(T_N)} \right),$$

where \bar{z} are the conjugates of z , for all $z \in \mathbb{C}$

$$= (\tau_{j_1}(T_1), \dots, \tau_{j_N}(T_N))^*$$

in the algebra $\mathbb{C}^{\oplus N}$

$$= (E_W(T))^* .$$

Thus, for any $T \in \mathfrak{L}_Q^{\oplus N}$, one has that

$$E_W(T^*) = (E_W(T))^* \text{ in } \mathbb{C}^{\oplus N}. \tag{6.7}$$

Therefore, by (6.5), (6.6) and (6.7), the bounded surjective linear transformation is a conditional expectation from $\mathfrak{L}_Q^{\oplus N}$ onto $\mathbb{C}^{\oplus N}$. □

Let tr_N be the usual *normalized trace* on the matricial algebra $M_N(\mathbb{C})$, i.e.,

$$tr_N([t_{ij}]_{N \times N}) = \frac{1}{N} \sum_{j=1}^N t_{jj},$$

for all $(N \times N)$ -matrices with their (i, j) -entries t_{ij} in \mathbb{C} .

Since the N -dimensional algebra $\mathbb{C}^{\oplus N}$ is the diagonal subalgebra of $M_N(\mathbb{C})$ consisting of all $(N \times N)$ -diagonal matrices, one can naturally restrict the normalized trace tr_N on $M_N(\mathbb{C})$ to the linear functional $tr_N|_{\mathbb{C}^{\oplus N}}$, also denoted simply by tr_N , on $\mathbb{C}^{\oplus N}$, i.e.,

$$tr_N((t_1, \dots, t_N)) \stackrel{def}{=} \frac{1}{N} \left(\sum_{j=1}^N t_j \right), \tag{6.8}$$

for all $(t_1, \dots, t_N) \in \mathbb{C}^{\oplus N}$.

Now, define a linear functional τ_W on the direct product Banach $*$ -algebra $\mathfrak{L}_Q^{\oplus N}$ by

$$\tau_W \stackrel{def}{=} tr_N \circ E_W \text{ on } \mathfrak{L}_Q^{\oplus N}, \tag{6.9}$$

where tr_N is the restricted trace (6.8) on $\mathbb{C}^{\oplus N}$, and E_W is the conditional expectation (6.4) from $\mathfrak{L}_Q^{\oplus N}$ onto $\mathbb{C}^{\oplus N}$.

Then the linear morphism η_W of (6.9) is indeed a well-defined linear functional on $\mathfrak{L}_Q^{\oplus N}$, moreover, it is a *trace* in the sense that:

$$\tau_W(T_1T_2) = \tau_W(T_2T_1), \text{ for all } T_1, T_2 \in \mathfrak{L}_Q^{\oplus N}.$$

So, the pair $(\mathfrak{L}_Q^{\oplus N}, \tau_W)$ forms a well-defined tracial Banach *-probability space.

Definition 6.2. We denote the Banach *-probability space $(\mathfrak{L}_Q^{\oplus N}, \tau_W)$ simply by $\mathfrak{L}_Q^{\oplus N}(W)$, where η_W is the linear functional (6.9) on $\mathfrak{L}_Q^{\oplus N}$, i.e.,

$$\mathfrak{L}_Q^{\oplus N}(W) \stackrel{def}{=} (\mathfrak{L}_Q^{\oplus N}, \tau_W), \tag{6.10}$$

for all N -tuples W in \mathbb{Z} .

Let us take an operator u_W in $\mathfrak{L}_Q^{\oplus N}$, as a free random variable of the Banach *-probability space $\mathfrak{L}_Q^{\oplus N}(W)$ in the sense of (6.10), where

$$u_W \stackrel{def}{=} \frac{1}{N}(U_{j_1}, U_{j_2}, \dots, U_{j_N}) \in \mathfrak{L}_Q^{\oplus N}, \tag{6.11}$$

and

$$U_{j_k} \stackrel{def}{=} \frac{1}{\psi(q_{j_k})} u_{j_k} = \frac{1}{\psi(q_{j_k})} (l \otimes q_{j_k}) \in \mathfrak{L}_Q(j_k)$$

are our semicircular elements in the j_k -th filtered probability space $\mathfrak{L}_Q(j_k) = (\mathfrak{L}_Q, \tau_{j_k})$, for all $k = 1, \dots, N$. (Recall and remark that in the rest of this paper, we automatically assumed $\psi(q_j) \in \mathbb{R}$ in \mathbb{C} , for all $j \in \mathbb{Z}$. So, the direct summands U_{j_k} of the operator u_W of (6.11) in $\mathfrak{L}_Q^{\oplus N}(W)$ are well-determined semicircular elements in $\mathfrak{L}_Q(j_k)$, for $k = 1, \dots, N = |W|$.)

Theorem 6.3. *Let u_W be a free random variable (6.11) in $\mathfrak{L}_Q^{\oplus N}(W)$ of (6.10), for a fixed N -tuple $W = (j_1, \dots, j_N)$ of integers. Then u_W is $\frac{1}{N^2}$ -semicircular in $\mathfrak{L}_Q^{\oplus N}(W)$, i.e., for any $W \in \mathbb{Z}^N$, there exists $u_W \in \mathfrak{L}_Q^{\oplus |W|}$, such that*

$$u_W \text{ is } \frac{1}{|W|^2}\text{-semicircular in } \mathfrak{L}_Q^{\oplus |W|}(W). \tag{6.12}$$

Proof. Let u_W be in the sense of (6.11) in the Banach *-probability space $\mathfrak{L}_Q^{\oplus N}(W) = (\mathfrak{L}_Q^{\oplus N}, \tau_W)$ of (6.10), for a fixed $W = (j_1, \dots, j_N) \in \mathbb{Z}^N$. Clearly, the operator u_W is self-adjoint in $\mathfrak{L}_Q^{\oplus N}$, because every direct summand U_{j_k} is self-adjoint by its semicircularity, for all $k = 1, \dots, N$.

By the definition (6.9) of τ_W , one can get that

$$\begin{aligned} \tau_W(u_W^n) &= \tau_W\left(\frac{1}{N^n}(U_{j_1}, U_{j_2}, \dots, U_{j_N})^n\right) \\ &= \frac{1}{N^n} \tau_W((U_{j_1}^n, U_{j_2}^n, \dots, U_{j_N}^n)) \\ &= \frac{1}{N^n} \text{tr}_N(E_W((U_{j_1}^n, U_{j_2}^n, \dots, U_{j_N}^n))) \\ &= \frac{1}{N^n} \text{tr}_N((\tau_{j_1}(U_{j_1}^n), \dots, \tau_{j_N}(U_{j_N}^n))) \\ &= \frac{1}{N^n} \text{tr}_N((\omega_n c_{\frac{n}{2}}, \dots, \omega_n c_{\frac{n}{2}})) \end{aligned}$$

by the semicircularity of U_{j_k} in $(\mathfrak{L}_Q, \tau_{j_k})$, for $k = 1, \dots, N$

$$\begin{aligned} &= \frac{\omega_n}{N^n} \text{tr}_N((c_{\frac{n}{2}}, \dots, c_{\frac{n}{2}})) \\ &= \frac{\omega_n}{N^n} \left(\frac{1}{N} \sum_{k=1}^N c_{\frac{n}{2}}\right) \end{aligned}$$

by the definition (6.8) of the normalized trace tr_N on $\mathbb{C}^{\oplus N}$

$$= \frac{\omega_n}{N^n} \left(\frac{N}{N} c_{\frac{n}{2}}\right) = \frac{\omega_n}{N^n} c_{\frac{n}{2}} = \omega_n \left(\frac{1}{N}\right)^n c_{\frac{n}{2}}, \tag{6.13}$$

where ω_n are in the sense of (3.3), for all $n \in \mathbb{N}$.

Therefore, by (6.13) and (3.3), one can conclude that the free random variable T_W is $\frac{1}{N^2}$ -semicircular in $\mathfrak{L}_Q^{\oplus N}(W)$. Just for sure, we consider the following “extra” parts of the proof.

By the free-moment computation (6.13), we obtain that: if $k_n^W(\dots)$ is the free cumulant on $\mathfrak{L}_Q^{\oplus N}(W)$ with respect to the linear functional τ_W , then

$$k_2^W(u_W, u_W) = \tau_W(u_W^2) - (\tau_W(u_W))^2 = \tau_W(u_W^2) = \frac{1}{N^2},$$

under the Möbius inversion of [14], i.e., one obtains that

$$k_2^W(u_W, u_W) = \frac{1}{N^2}. \tag{6.14}$$

Now, for any $n \in \mathbb{N}$,

$$k_{2n-1}^W(u_W, \dots, u_W) = \sum_{\pi \in NC(2n-1)} \left(\prod_{V \in \pi} \tau_W(u_W^{|V|})\right) \mu(\pi, 1_{2n-1})$$

by the Möbius inversion of [14]

$$= 0,$$

because every noncrossing partition π over $\{1, \dots, 2n - 1\}$ contains at least one “odd” block V_0 , satisfying the block-depending free moment $\tau_W(u_W^{|V_0|}) = 0$, by the evenness from the formula (6.13), i.e.,

$$k_n^W(u_W, \dots, u_W) = 0, \text{ whenever } n \text{ is odd in } \mathbb{N}. \tag{6.15}$$

Observe that, for any $n \in \mathbb{N} \setminus \{1\}$,

$$k_{2n}^W(u_W, \dots, u_W) = \sum_{\theta \in NC_e(2n)} \left(\prod_{V \in \theta} \eta_W(u_W^{|V|}) \right) \mu(\theta, 1_{2n})$$

where

$$NC_e(2n) = \{\theta \in NC(2n) : V \in \theta \iff |V| \text{ is even}\},$$

and hence

$$= \sum_{\theta \in NC_e(2n)} \left(\prod_{V \in \theta} \frac{1}{N^{|V|/2}} c_{\frac{|V|}{2}} \right) \mu(\theta, 1_{2n})$$

by (6.13)

$$\begin{aligned} &= \sum_{\theta \in NC_e(2n)} \left(\prod_{V \in \theta} \frac{1}{N^{|V|}} \right) \left(\prod_{V \in \theta} c_{\frac{|V|}{2}} \right) \mu(\theta, 1_{2n}) \\ &= \sum_{\theta \in NC_e(2n)} \left(\frac{1}{N} \right)^{\sum_{V \in \theta} |V|} \left(\prod_{V \in \theta} c_{\frac{|V|}{2}} \right) \mu(\theta, 1_{2n}) \\ &= \sum_{\theta \in NC_e(2n)} \left(\frac{1}{N} \right)^n \left(\prod_{V \in \theta} c_{\frac{|V|}{2}} \right) \mu(\theta, 1_{2n}) \\ &= \left(\left(\frac{1}{N} \right)^n c_{n+1} \right) \sum_{\theta \in NC_e(2n)} \mu(\theta, 1_{2n}) \\ &= \left(\left(\frac{1}{N} \right)^n c_{n+1} \right) \left(\sum_{\pi \in NC(n)} \mu(\pi, 1_n) \right) = 0, \end{aligned}$$

by [14], i.e.,

$$k_{2n}^W(u_W, \dots, u_W) = 0, \text{ whenever } n > 1 \text{ in } \mathbb{N}. \tag{6.16}$$

Thus, by (6.14), (6.15) and (6.16), indeed, we obtain that

$$k_n^W(u_W, \dots, u_W) = \begin{cases} \frac{1}{N^2} & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{6.17}$$

for all $n \in \mathbb{N}$.

Therefore, by (6.17) and (3.1), the free random variable u_W is indeed $\frac{1}{N^2}$ -semicircular in $\mathfrak{L}_Q^{\oplus N}(W)$. □

The above theorem shows that, for any finite sequences W of integers, if we define an operator

$$u_W = \frac{1}{|W|} \left(\bigoplus_{j \in W} U_j \right) \text{ in } \mathfrak{L}_Q^{\oplus |W|},$$

then it is $\frac{1}{|W|^2}$ -semicircular in the Banach $*$ -probability space $\mathfrak{L}_Q^{\oplus |W|}(W)$, where $U_j = \frac{1}{\psi(q_j)} (l \otimes q_j)$ are the semicircular elements of (\mathfrak{L}_Q, τ_j) , for all $j \in W$ in \mathbb{Z} .

6.2. ON THE BANACH $*$ -PROBABILITY SPACES $\mathfrak{L}_Q^{\otimes N}(j_1, \dots, j_N)$

Like in Section 6.1, let $W = (j_1, \dots, j_N) \in \mathbb{Z}^N$ be an arbitrarily fixed N -tuple of integers j_1, \dots, j_N , which are not necessarily distinct from each other in \mathbb{Z} , and let \mathfrak{L}_Q be the radial projection algebra of the C^* -algebra Q of the fixed C^* -probability space (A, ψ) . Let τ_j be the linear functionals (4.16) on \mathfrak{L}_Q , inducing the corresponding j -th filtered probability space $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \tau_j)$ of Q . Also, let

$$U_j = \frac{1}{\psi(q_j)} u_j = \frac{1}{\psi(q_j)} (l \otimes q_j)$$

be the semicircular element in $\mathfrak{L}_Q(j)$, for all $j \in \mathbb{Z}$.

Now, define the tensor product Banach $*$ -algebra $\mathfrak{L}_Q^{\otimes N}$ by

$$\mathfrak{L}_Q^{\otimes N} \stackrel{def}{=} \underbrace{\mathfrak{L}_Q \otimes_{\mathbb{C}} \mathfrak{L}_Q \otimes_{\mathbb{C}} \dots \otimes_{\mathbb{C}} \mathfrak{L}_Q}_{N\text{-times}} \tag{6.18}$$

under product topology.

Define a linear transformation F_W from the tensor product Banach $*$ -algebra $\mathfrak{L}_Q^{\otimes N}$ of (6.18) onto the finite-dimensional algebra $\mathbb{C}^{\otimes N}$ by the linear morphism,

$$F_W \stackrel{denote}{=} \bigotimes_{k=1}^N \tau_{j_k} \text{ on } \mathfrak{L}_Q^{\otimes N}, \tag{6.19}$$

satisfying that

$$F_W \left(\bigotimes_{k=1}^N T_k \right) = \bigotimes_{k=1}^N (\tau_{j_k}(T_k) \cdot I),$$

for all $\bigotimes_{k=1}^N T_k \in \mathfrak{L}_Q^{\otimes N}$, where I is the identity element of \mathfrak{L}_Q introduced in Section 6.1.

Proposition 6.4. *Let $\mathfrak{L}_Q^{\otimes N}$ be in the sense of (6.18) for a fixed N -tuple W with $N = |W|$. Then the linear morphism F_W of (6.19) is a conditional expectation from $\mathfrak{L}_Q^{\otimes N}$ onto $\mathbb{C}^{\otimes N}$.*

Proof. By the boundedness of the linear functionals τ_{j_k} of (4.16) on \mathfrak{L}_Q , for $k = 1, \dots, N$, the linear transformation F_W is a bounded linear transformation from $\mathfrak{L}_Q^{\otimes N}$ to $\mathbb{C}^{\otimes N}$. Also, by the very definition (6.19), it is surjective.

Now, take $T = \bigotimes_{k=1}^N t_k I$ in $\mathfrak{L}_Q^{\otimes N}$, for $t_1, \dots, t_N \in \mathbb{C}$, i.e., $T \in \mathbb{C}^{\otimes N} \subset \mathfrak{L}_Q^{\otimes N}$. Then

$$F_W(T) = \bigotimes_{k=1}^N \tau_{j_k}(t_k I) = \bigotimes_{k=1}^N t_k \text{ in } \mathbb{C}^{\otimes N}.$$

So, under linearity, one has

$$F_W(S) = S \text{ in } \mathbb{C}^{\otimes N}, \text{ for all } S \in \mathfrak{L}_Q^{\otimes N}. \tag{6.20}$$

Now, let $T_1 = \bigotimes_{k=1}^N t_k I$ (with $t_k \in \mathbb{C}$), and $T_2 = \bigotimes_{k=1}^N t'_k I$ (with $t'_k \in \mathbb{C}$) be in $\mathbb{C}^{\otimes N}$, and $S = \bigotimes_{k=1}^N S_k \in \mathfrak{L}_Q^{\otimes N}$ (with $S_k \in \mathfrak{L}_Q$). Then

$$F_W(T_1 S T_2) = F_W\left(\bigotimes_{k=1}^N (t_k S_k t'_k)\right) = \bigotimes_{k=1}^N (t_k \tau_{j_k}(S_k) t'_k)$$

by (6.19)

$$\begin{aligned} &= \left(\bigotimes_{k=1}^N t_k I\right) \left(\bigotimes_{k=1}^N \tau(S_k)\right) \left(\bigotimes_{k=1}^N t'_k I\right) \\ &= T_1 F_W(S) T_2. \end{aligned}$$

Thus, under linearity, for all $T_1, T_2 \in \mathbb{C}^{\otimes N}$, and $S \in \mathfrak{L}_Q^{\otimes N}$, one obtains that

$$F_W(T_1 S T_2) = T_1 F_W(S) T_2 \text{ in } \mathbb{C}^{\otimes N}. \tag{6.21}$$

Take $T = \bigotimes_{k=1}^N T_k \in \mathfrak{L}_Q^{\otimes N}$. Then $T^* = \bigotimes_{k=1}^N T_k^*$ in $\mathfrak{L}_Q^{\otimes N}$. So, by (6.19),

$$F_W(T^*) = F_W(T)^*. \tag{6.22}$$

Therefore, the surjective bounded linear transformation F_W is a conditional expectation from $\mathfrak{L}_Q^{\otimes N}$ onto $\mathbb{C}^{\otimes N}$, by (6.20), (6.21) and (6.22). \square

On the N -dimensional tensor product algebra $\mathbb{C}^{\otimes N}$, define a linear functional φ_N by a linear morphism satisfying that:

$$\varphi_N\left(\bigotimes_{k=1}^N t_k\right) \stackrel{def}{=} \prod_{k=1}^N t_k, \text{ for all } \bigotimes_{k=1}^N t_k \in \mathbb{C}^{\otimes N}. \tag{6.23}$$

Define now a linear functional φ_W^o on $\mathfrak{L}_Q^{\otimes N}$ by

$$\varphi_W^o \stackrel{def}{=} \varphi_N \circ F_W \text{ on } \mathfrak{L}_Q^{\otimes N}, \tag{6.24}$$

where φ_N is in the sense of (6.23) and F_W is the conditional expectation (6.19).

By the definition (6.24) of φ_W^o , it is a well-determined linear functional on $\mathfrak{L}_Q^{\otimes N}$ satisfying

$$\varphi_W^o \left(\bigotimes_{k=1}^N T_k \right) = \prod_{k=1}^N \tau_{j_k} (T_k). \tag{6.25}$$

Define an element I_{W,j_k} of $\mathfrak{L}_Q^{\otimes N}$ by

$$I_{W,j_k} \stackrel{def}{=} (1_{\mathfrak{L}} \otimes \hat{q}_{j_k}), \tag{6.26}$$

with

$$\hat{q}_{j_k} = 1_Q \otimes \dots \otimes 1_Q \otimes \underset{k\text{-th}}{q_{j_k}} \otimes 1_Q \otimes \dots \otimes 1_Q,$$

for all $j_k \in W$ in \mathbb{Z} , for $k = 1, \dots, N$.

From the operators I_{W,j_k} of (6.26), define an operator I_W of $\mathfrak{L}_Q^{\otimes N}$ by

$$I_W = \sum_{k=1}^N I_{W,j_k} \tag{6.27}$$

Note that, if the chosen projections q_{j_1}, \dots, q_{j_N} are “mutually orthogonal” on Q , equivalently, if $W = (j_1, \dots, j_N)$ is an N -tuple of “mutually distinct” integers in \mathbb{Z}^N , then one can get that

$$\begin{aligned} I_W^n &= \left(\sum_{k=1}^N (1_{\mathfrak{L}} \otimes \hat{q}_{j_k}) \right)^n = \sum_{k=1}^N (1_{\mathfrak{L}} \otimes \hat{q}_{j_k})^n \\ &= \sum_{k=1}^N (1_{\mathfrak{L}}^n \otimes (\hat{q}_{j_k})^n) = \sum_{k=1}^N (1_{\mathfrak{L}} \otimes \hat{q}_{j_k}) = I_W, \end{aligned} \tag{6.28}$$

for all $n \in \mathbb{N}$, by (6.26) and (6.27).

So, if I_W is in the sense of (6.27) in $\mathfrak{L}_Q^{\otimes N}$, then one can get that

$$\varphi_W^o (I_W^n) = \varphi_W^o (I_W) = \prod_{k=1}^N \tau_{j_k} (1_{\mathfrak{L}} \otimes q_{j_k}) = \prod_{k=1}^N \psi (q_{j_k}),$$

and

$$\varphi_W^o (I_{W,j_k}^n) = \varphi_W^o (I_{W,j_k}) = \prod_{k=1}^N \tau_{j_k} (1_{\mathfrak{L}} \otimes q_{j_k}) = \psi (q_{j_k}), \tag{6.29}$$

in \mathbb{C} , by (6.25) and (6.28), for all $n \in \mathbb{N}$, where φ_W^o is in the sense of (6.24).

For convenience, we denote the \mathbb{C} -quantity $\varphi_W^o (I_W)$ of (6.29), simply by ψ_W , below. Note that, since $N < \infty$, the quantity ψ_W converges in \mathbb{C} .

Now, define new linear functionals $\varphi_{W,k}$'s on $\mathfrak{L}_Q^{\otimes N}$ by

$$\varphi_{W,k} \stackrel{\text{def}}{=} \frac{\psi(q_{j_k})}{\psi_W} \varphi_W^{\circ} \text{ on } \mathfrak{L}_Q^{\otimes N}, \tag{6.30}$$

for all $k = 1, \dots, N$, where φ_W° is the linear functional (6.24) on $\mathfrak{L}_Q^{\otimes N}$ satisfying (6.25), and where ψ_W is the \mathbb{C} -quantity (6.29).

Then the linear functionals $\varphi_{W,k}$ of (6.30) are well-defined on $\mathfrak{L}_Q^{\otimes N}$, and hence, we obtain the tensor product Banach *-probability spaces $\mathfrak{L}_Q^{\otimes N}(k, W)$,

$$\mathfrak{L}_Q^{\otimes N}(k, W) \stackrel{\text{def}}{=} \left(\mathfrak{L}_Q^{\otimes N}, \varphi_{W,k} \right), \tag{6.31}$$

where $\varphi_{W,k}$ are in the sense of (6.30), for all $k = 1, \dots, N$.

On our tensor product Banach *-probability space $\mathfrak{L}_Q^{\otimes N}(k, W)$ of (6.31), one can obtain the following semicircular elements in $\mathfrak{L}_Q^{\otimes N}$.

Theorem 6.5. *Let $W = (j_1, \dots, j_N)$ be an N -tuple of "mutually distinct" integers j_1, \dots, j_N in \mathbb{Z} , for $N \in \mathbb{N}$, and fix $k_0 \in \{1, \dots, N\}$. For a fixed k_0 , let $\mathfrak{L}_Q^{\otimes N}(k_0, W)$ be the corresponding Banach *-probability space (6.31) equipped with its linear functional $\varphi_{k_0, W}$ of (6.30). Define the free random variables U_{W, k_0} , for some $k_0 \in \{1, \dots, N\}$, by*

$$U_{W, k_0} \stackrel{\text{def}}{=} \left(\bigotimes_{k=1}^N S_{j_k} \right) \in \mathfrak{L}_Q^{\otimes N}(k_0, W) = (\mathfrak{L}_Q^{\otimes N}, \varphi_{k_0, W}), \tag{6.32}$$

where $\psi_W \in \mathbb{C}$ is in the sense of (6.29), and

$$S_{j_k} = \begin{cases} u_{j_{k_0}} & \text{if } k = k_0, \\ I_{W, j_k} & \text{otherwise,} \end{cases}$$

for all $k = 1, \dots, N$, where $u_{j_{k_0}} = (l \otimes q_{j_{k_0}})$ is our $\psi(q_{j_{k_0}})^2$ -semicircular element of the j_{k_0} -th filtered probability space $\mathfrak{L}_Q(j_{k_0})$ of Q , and I_{W, j_k} are in the sense of (6.26), for $k \neq k_0$ in $\{1, \dots, N\}$. Then U_{W, k_0} of (6.32) is a $\psi(q_{j_{k_0}})^2$ -semicircular element in $\mathfrak{L}_Q^{\otimes N}(W)$.

Proof. By the very definition (6.32) of U_{W, k_0} , it is self-adjoint in $\mathfrak{L}_Q^{\otimes N}$. Indeed, the tensor factors I_{W, j_k} and $u_{j_{k_0}}$ are self-adjoint in \mathfrak{L}_Q , and hence, the operator U_{W, k_0} is self-adjoint in $\mathfrak{L}_Q^{\otimes N}$.

Observe now that

$$\begin{aligned} \varphi_{k_0, W} (U_{W, k_0}^n) &= \varphi_{k_0, W} \left(\left(I_{W, j_1}^n \otimes \dots \otimes I_{W, j_{k_0-1}}^n \otimes \underset{k_0\text{-th}}{u_{j_{k_0}}^n} \otimes I_{W, j_{k_0+1}}^n \otimes \dots \otimes I_{W, j_N}^n \right) \right) \\ &= \varphi_{k_0, W} \left(\left(I_{W, j_1} \otimes \dots \otimes I_{W, j_{k_0-1}} \otimes \underset{k_0\text{-th}}{u_{j_{k_0}}^n} \otimes I_{W, j_{k_0+1}} \otimes \dots \otimes I_{W, j_N} \right) \right) \end{aligned}$$

by (6.27)

$$= \frac{\psi(q_{j_{k_0}})}{\psi_W} (\tau_{j_1} (I_{W, j_1})) \dots (\tau_{j_{k_0}} (u_{j_{k_0}}^n)) \dots (\tau_{j_N} (I_{W, j_N}))$$

by (6.25) and (6.30)

$$\begin{aligned} &= \frac{\psi(q_{j_{k_0}})}{\psi_W} (\psi(q_{j_1})) \dots (\omega_n \psi(q_{j_{k_0}})^n c_{\frac{n}{2}}) \dots (\psi(q_{j_N})) \\ &= \frac{\psi(q_{j_{k_0}}) \left(\prod_{k=1}^{k_0-1} \psi(q_{j_k}) \right) \left(\prod_{k=k_0+1}^N \psi(q_{j_k}) \right)}{\psi_W} (\omega_n \psi(q_{j_{k_0}})^n c_{\frac{n}{2}}) \end{aligned}$$

by (6.29)

$$= \omega_n (\psi(q_{j_{k_0}}))^n c_{\frac{n}{2}}, \tag{6.33}$$

for all $n \in \mathbb{N}$.

Therefore, by (6.33) and (3.3), the free random variable U_{W, k_0} is $\psi(q_{j_{k_0}})$ -semicircular in $\mathfrak{L}_Q^{\otimes N}(k_0, W)$. \square

The above theorem shows that, whenever W is a finite sequence of mutually distinct $|W|$ -many integers of \mathbb{Z} , then the operators $U_{W, k}$ of (6.32) in $\mathfrak{L}_Q^{\otimes |W|}$ are $\psi(q_{j_k})^2$ -semicircular in $\mathfrak{L}_Q^{\otimes |W|}(k, W)$, for all $k = 1, \dots, |W|$.

By the weighted-semicircularity (6.33) of the operator $U_{W, k}$ of (6.17), one can obtain the following corollary.

Corollary 6.6. *Let $W = (j_1, \dots, j_N)$ be the N -tuple of mutually distinct integers j_1, \dots, j_N in \mathbb{Z} , and let $\mathfrak{L}_Q^{\otimes N}(k, W)$ be the Banach $*$ -probability space in the sense of (6.31), for $k = 1, \dots, N$. Define a free random variable $T_{W, k}$ by*

$$T_{W, k} \stackrel{def}{=} I_{W, j_1} \otimes \dots \otimes I_{W, j_{k-1}} \otimes \underset{k\text{-th}}{U_{j_k}} \otimes I_{W, j_{k+1}} \otimes \dots \otimes I_{W, j_N}, \tag{6.34}$$

where I_{W, j_k} are in the sense of (6.26), and

$$U_{j_k} = \frac{1}{\psi(q_{j_k})} (l \otimes q_{j_k})$$

is our semicircular element in $\mathfrak{L}_Q(j_k)$, for all $k = 1, \dots, N$. Then the operator $T_{W, k}$ is semicircular in $\mathfrak{L}_Q^{\otimes N}(k, W)$, for $k = 1, \dots, N$.

Proof. Similar to (6.33), one can get that

$$\varphi_{k,W}(T_{W,k}^n) = \omega_n c_{\frac{n}{2}}, \text{ for all } n \in \mathbb{N}. \tag{6.35}$$

Thus, by (6.35) and (3.4), the free random variables $T_{W,k}$ are semicircular in $\mathfrak{L}_Q^{\otimes N}(k, W)$, for all $k = 1, \dots, N$. \square

6.3. ON THE BANACH *-PROBABILITY SPACES $\mathfrak{L}_Q^N(j_1, \dots, j_N)$

Let Q be the C^* -subalgebra of a fixed C^* -probability space (A, ψ) generated by the family $\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$ of mutually orthogonal projections satisfying

$$\psi(q_j) \in \mathbb{R} \text{ in } \mathbb{C}, \text{ for all } j \in \mathbb{Z},$$

and let \mathfrak{L}_Q be the radial projection algebra of Q , having the corresponding j -th filtered probability spaces $\mathfrak{L}_Q(j) = (\mathfrak{L}_Q, \tau_j)$, for all $j \in \mathbb{Z}$.

As in Sections 6.1 and 6.2, let $W = (j_1, \dots, j_N)$ be the N -tuple of integers j_1, \dots, j_N in \mathbb{Z} , for $N \in \mathbb{N}$. As in Section 6.1 (and different from the main results of Section 6.2), the integers j_1, \dots, j_N are not necessarily distinct from each other in \mathbb{Z} .

For a fixed N -tuple W , define the *free product Banach *-probability space* $\mathfrak{L}_Q^N(W)$ by the Banach *-probability space,

$$\begin{aligned} \mathfrak{L}_Q^N(W) &\stackrel{\text{denote}}{=} (\mathfrak{L}_Q^N, \tau^W) \stackrel{\text{def}}{=} \underset{k=1}{\star}^N \mathfrak{L}_Q(j_k) \\ &= \underset{k=1}{\star}^N (\mathfrak{L}_Q, \tau_{j_k}) = \left(\underset{k=1}{\star}^N \mathfrak{L}_Q, \underset{k=1}{\star}^N \tau_{j_k} \right) \end{aligned} \tag{6.36}$$

in the sense of [14] and [16], where

$$\tau^W = \underset{k=1}{\star}^N \tau_{j_k}$$

is the *free product linear functional* on

$$\mathfrak{L}_Q^N = \underset{k=1}{\star}^N \mathfrak{L}_Q = \underbrace{\mathfrak{L}_Q \star \mathfrak{L}_Q \star \dots \star \mathfrak{L}_Q}_{N\text{-times}}$$

Note that the above free product Banach *-probability space $\mathfrak{L}_Q^N(W)$ of (6.36) is highly dictated by the choice of linear functionals $\{\tau_{j_k}\}_{k=1}^N$ (e.g., see [14] and [16]).

Now, define a set \mathcal{N} by the family of all finite sequences in $\{1, \dots, N\}$, i.e.,

$$\mathcal{N} \stackrel{\text{def}}{=} \bigsqcup_{l=1}^{\infty} \mathcal{N}(l), \tag{6.37}$$

with its partition blocks

$$\mathcal{N}(l) \stackrel{\text{def}}{=} \{(i_1, \dots, i_l) : i_j \in \{1, \dots, N\} \text{ for all } j = 1, \dots, l\},$$

for all $l \in \mathbb{N}$.

For each $l \in \mathbb{N}$, define the subsets $\mathcal{AN}(l)$ of $\mathcal{N}(l)$ by

$$\mathcal{AN}(l) \stackrel{def}{=} \left\{ w \in \mathcal{N}(l) \left| \begin{array}{l} w \text{ is alternating in the sense that:} \\ \text{if } w = (i_1, \dots, i_l), \text{ then} \\ i_1 \neq i_2, i_2 \neq i_3, \dots, i_{l-1} \neq i_l \end{array} \right. \right\}. \tag{6.38}$$

Suppose $N = 3$, and let

$$w_1 = (1, 2, 2, 3, 1, 1) \in \mathcal{N}(6) \text{ in } \mathcal{N},$$

and

$$w_2 = (1, 3, 1, 2, 1, 2) \in \mathcal{N}(6) \text{ in } \mathcal{N},$$

where \mathcal{N} and $\mathcal{N}(6)$ are in the sense of (6.37). Then

$$w_1 \notin \mathcal{AN}(6), \text{ but } w_2 \in \mathcal{AN}(6),$$

in \mathcal{N} , where $\mathcal{AN}(6)$ is in the sense of (6.38).

Define now the subset \mathcal{AN} of \mathcal{N} by

$$\mathcal{AN} \stackrel{def}{=} \bigsqcup_{l=1}^{\infty} \mathcal{AN}(l), \text{ in } \mathcal{N}. \tag{6.39}$$

Now, let us take $w \in \mathcal{AN}(n)$ in \mathcal{AN} of (6.39), and choose an operator T_w ,

$$T_w = \prod_{l \in w} T_{j_l} \in \mathfrak{L}_Q^N(W), \tag{6.40}$$

as a free “reduced” word in $\{T_{j_1}, \dots, T_{j_N}\}$, where T_{j_k} are in the free blocks $\mathfrak{L}_Q(j_k)$ in $\mathfrak{L}_Q^N(W)$, for all $k = 1, \dots, N$.

Remark 6.7. By [14] and [16], if $\mathfrak{L}_Q^N(W)$ is identified with $(\mathfrak{L}_Q^N, \tau^W)$ in the sense of (6.36), the free product Banach $*$ -algebra \mathfrak{L}_Q^N of $\mathfrak{L}_Q(j_1), \dots, \mathfrak{L}_Q(j_N)$ has its Banach-space expression,

$$\mathfrak{L}_Q^N \stackrel{\text{Banach-Sp}}{=} \mathbb{C} \oplus \left(\bigoplus_{l=1}^{\infty} \bigoplus_{w \in \mathcal{AN}(l)} \mathfrak{L}^o(w) \right), \tag{6.41}$$

where

$$\mathfrak{L}^o(w) \stackrel{\text{Banach-Sp}}{=} \mathfrak{L}_Q^o(j_{i_1}) \otimes \mathfrak{L}_Q^o(j_{i_1}) \otimes \dots \otimes \mathfrak{L}_Q^o(j_{i_l}),$$

whenever $w = (i_1, \dots, i_l) \in \mathcal{AN}(l)$ in \mathcal{AN} , where

$$\mathfrak{L}_Q^o(j_{i_k}) \stackrel{\text{Banach-Sp}}{=} \mathfrak{L}_Q \ominus \mathbb{C}, \text{ for all } k = 1, \dots, l.$$

Here, “ $\stackrel{\text{Banach-Sp}}{=}$ ” means “being Banach-space isomorphic” and \oplus is the direct product, \otimes is the tensor product, and \ominus is the complement of Banach spaces under topology, i.e., if an operator T is contained in $\mathfrak{L}^o(w)$, and if $w \in \mathcal{AN}$, then T is understood as the free reduced words in \mathfrak{L}_Q^N .

By (6.36), (6.38) and (6.39), one obtains the following result.

Proposition 6.8. *If T_w is in the sense of (6.40) in $\mathfrak{L}_Q^N(W)$, then*

$$\tau^W(T_w) = \prod_{l \in w} \tau_{j_l}(T_{j_l}), \tag{6.42}$$

whenever $w \in \mathcal{AN}$.

Proof. The free-moment formula (6.42) is proven by (6.36), (6.38) and (6.39) (See [14] and [16]). □

By the free-moment formula (6.42), one can get that, if

$$w = (i_1, \dots, i_n) \in \mathcal{AN}(n) \text{ in } \mathcal{AN},$$

and if

$$M_w^{r_1, \dots, r_n} \stackrel{def}{=} \prod_{l \in w} U_{j_l}^{r_l} = \prod_{l=1}^n U_{j_{i_l}}^{r_l} \in \mathfrak{L}_Q^N(W), \tag{6.43}$$

for all $r_1, \dots, r_n \in \mathbb{N}$, where $U_j = \frac{1}{\psi(q_j)}(l \otimes q_j)$ are our semicircular elements of $\mathfrak{L}_Q(j)$, for all $j \in \mathbb{Z}$, then one has

$$\tau^W(M_w^{r_1, \dots, r_n}) = \prod_{l \in w} \tau_{j_l}(U_{j_l}^{r_l}) = \prod_{l=1}^n \tau_{j_{i_l}}(U_{j_{i_l}}^{r_l}),$$

by (6.42)

$$= \prod_{l=1}^n \omega_{r_l} c_{\frac{r_l}{2}} \tag{6.44}$$

by the semicircularity of U_{j_l} , for all $l = 1, \dots, n$, where c_k are the k -th Catalan numbers for all $k \in \mathbb{N}$.

Theorem 6.9. *Let $w = (j_{i_1}, \dots, j_{i_n}) \in \mathcal{AN}(n)$ in \mathcal{AN} , for some $n \in \mathbb{N}$, and let $M_w^{r_1, \dots, r_n}$ be a free random variable (6.43) in $\mathfrak{L}_Q^N(W)$ of (6.36). Then*

$$\tau^W(M_w^{r_1, \dots, r_n}) = \omega_{r_1, \dots, r_n} \prod_{l=1}^n c_{\frac{r_l}{2}}, \tag{6.45}$$

with

$$\omega_{r_1, \dots, r_n} = \begin{cases} 1 & \text{if all } r_1, \dots, r_n \text{ are even,} \\ 0 & \text{otherwise,} \end{cases}$$

for $r_1, \dots, r_n \in \mathbb{N}$.

Proof. The free-moment formula (6.45) is proven by (6.44). □

Similar to the free-moment formulas (6.45) for a free random variable $M_w^{r_1, \dots, r_n}$ of (6.43) in $\mathfrak{L}_Q^N(W)$, one can get the following result.

Corollary 6.10. *Let $w = (j_1, \dots, j_n) \in \mathcal{AN}(n)$ in \mathcal{AN} , for some $n \in \mathbb{N}$. For a fixed $w \in \mathcal{AN}(n)$, and fixed quantities $r_1, \dots, r_n \in \mathbb{N}$, define an operator $m_w^{r_1, \dots, r_n} \in \mathfrak{L}_Q^N$ by*

$$m_w^{r_1, \dots, r_n} \stackrel{\text{def}}{=} \prod_{l=1}^n u_{j_{i_l}}^{r_l} \in \mathfrak{L}_Q^N(W), \tag{6.46}$$

where $u_j = l \otimes q_j$ are our $\psi(q_j)^2$ -semicircular elements of (\mathfrak{L}_Q, τ_j) , for all $j \in \mathbb{Z}$. Then

$$\tau^W(m_w^{r_1, \dots, r_n}) = \omega_{r_1, \dots, r_n} \prod_{l=1}^n \left(\psi(q_{j_{i_l}})^{r_l} c_{\frac{r_l}{2}} \right), \tag{6.47}$$

for all $k \in \mathbb{N}$, where ω_{r_1, \dots, r_n} is in the sense of (6.45).

Proof. Let $m_w^{r_1, \dots, r_n} \in \mathfrak{L}_Q^N(W)$ be in the sense of (6.46), for $w \in \mathcal{AN}(n)$, and $n, r_1, \dots, r_n \in \mathbb{N}$. Observe that

$$\tau^W(m_w^{r_1, \dots, r_n}) = \tau^W \left(\prod_{l=1}^n u_{j_{i_l}}^{r_l} \right) = \tau^W \left(\prod_{l=1}^n u_{j_{i_l}}^{r_l} \right)$$

by understanding $m_w^{r_1, \dots, r_n}$ as a free reduced word in $\mathfrak{L}^o(w)$ of (6.41)

$$= \prod_{l=1}^n \tau_{j_{i_l}} \left(u_{j_{i_l}}^{r_l} \right) = \prod_{l=1}^n \omega_{r_l} \psi(q_{j_{i_l}})^{r_l} c_{\frac{r_l}{2}},$$

by the $\psi(q_{i_l})^2$ -semicircularity of $q_{j_{i_l}}$. Thus, we obtain the free-distributional data (6.47). □

The above free-moment formulas (6.45) and (6.47) provide ways to compute free moments of certain free random variables of $\mathfrak{L}_Q^N(W)$ generated by our semicircular elements U_j , and weighted-semicircular elements u_j of the j -th filtered probability spaces (\mathfrak{L}_Q, τ_j) , for $j \in \mathbb{Z}$.

Also, by (6.45) and (6.47), we obtain the following trivial, but interesting free-distributional data on $\mathfrak{L}_Q^N(W)$.

Theorem 6.11. *Let $\mathfrak{L}_Q^N(W) = (\mathfrak{L}_Q^N, \tau^W)$ be the Banach $*$ -probability space (6.36) for a fixed N -tuple $W = (j_1, \dots, j_N)$ in \mathbb{Z} . Let*

$$u_{j_k} = l \otimes q_{j_k} \in \mathfrak{L}_Q(j_k) \text{ in } \mathfrak{L}_Q^N(W), \text{ for all } k = 1, \dots, N, \tag{6.48}$$

where $\mathfrak{L}_Q(j_1), \dots, \mathfrak{L}_Q(j_N)$ are the free blocks of $\mathfrak{L}_Q^N(W)$. Then u_{j_k} are $\psi(q_{j_k})^2$ -semicircular in $\mathfrak{L}_Q^N(W)$, for all $k = 1, \dots, N$. Now, let

$$U_{j_k} = \frac{1}{\psi(q_{j_k})} u_{j_k} \in \mathfrak{L}_Q(j_k) \text{ in } \mathfrak{L}_Q^N(W), \text{ for all } k = 1, \dots, N. \tag{6.49}$$

Then they are semicircular in $\mathfrak{L}_Q^N(W)$.

Proof. Let u_{j_k} be in the sense of (6.48) in $\mathfrak{L}_Q^N(W)$, for $k \in \{1, \dots, N\}$. Then it forms a free reduced word with its length-1 in the free block $\mathfrak{L}_Q(j_k)$ of $\mathfrak{L}_Q^N(W)$, inducing free reduced words $u_{j_k}^n$ with their length-1 in $\mathfrak{L}_Q(j_k)$ in $\mathfrak{L}_Q^N(W)$, for all $n \in \mathbb{N}$. So, one obtains that

$$\tau^W(u_{j_k}^n) = \tau_{j_k}(u_{j_k}^n) = \omega_n(\psi(q_{j_k})^2)^{\frac{n}{2}} c_{\frac{n}{2}},$$

by (6.47), for all $n \in \mathbb{N}$, for all $k = 1, \dots, N$.

Therefore, the operators u_{j_k} of (6.48) are $\psi(q_{j_k})^2$ -semicircular in $\mathfrak{L}_Q^N(W)$, for all $k = 1, \dots, N$.

Similarly, one obtains that, if U_{j_k} are in the sense of (6.49) in $\mathfrak{L}_Q^N(W)$, then

$$\tau^W(U_{j_k}^n) = \tau_{j_k}(U_{j_k}^n) = \omega_n c_{\frac{n}{2}},$$

by (6.45), for all $n \in \mathbb{N}$, for all $k = 1, \dots, N$.

So, the operators U_{j_k} of (6.49) are semicircular in $\mathfrak{L}_Q^N(W)$, for all $k = 1, \dots, N$. \square

The above theorem shows that our weighted-semicircularity, and semicircularity in free blocks $\mathfrak{L}_Q(j)$, the j -th filtered probability spaces, for $j \in W$, are preserved by those in the free product Banach *-probability space $\mathfrak{L}_Q^{|W|}(W)$.

7. EXAMPLE: FROM ORTHOGONAL RANK-1 PROJECTIONS ON $l^2(\mathbb{Z})$

Let $H = l^2(\mathbb{Z})$ be the canonical l^2 -Hilbert space with its orthonormal basis $\mathcal{B}_H = \{\xi_j\}_{j \in \mathbb{Z}}$ satisfying

$$\langle \xi_{j_1}, \xi_{j_2} \rangle_2 = \delta_{j_1, j_2}, \text{ for all } j_1, j_2 \in \mathbb{Z}, \tag{7.1}$$

where

$$\xi_j = \left(\dots, 0, \dots, 0, \underset{j\text{-th}}{1}, 0, \dots, 0, \dots \right), \text{ for all } j \in \mathbb{Z},$$

and where $\langle \cdot, \cdot \rangle_2$ is the usual l^2 -inner product on H ,

$$\langle (\dots, t_{-1}, t_0, t_1, \dots), (\dots, s_{-1}, s_0, s_1, \dots) \rangle_2 = \sum_{k=-\infty}^{\infty} t_k \overline{s_k}.$$

Then on the operator algebra $B(H)$, one can naturally define the rank-1 projections q_j by

$$q_j = \langle \cdot, \xi_j \rangle_2 \xi_j, \text{ for all } j \in \mathbb{Z}. \tag{7.2}$$

i.e., for any $h \in H$, one has

$$q_j(h) = \langle h, \xi_j \rangle_2 \xi_j, \text{ for all } j \in \mathbb{Z},$$

and hence, if $h = \sum_{n \in \mathbb{Z}} t_n \xi_n \in H$, with $t_n \in \mathbb{C}$, then

$$q_j(h) = \left\langle \sum_{n \in \mathbb{Z}} t_n \xi_n, \xi_j \right\rangle_2 \xi_j = t_j \xi_j,$$

by (7.2), for all $j \in \mathbb{Z}$.

Thus, one can get the family

$$\mathbf{Q} = \{q_j\}_{j \in \mathbb{Z}}$$

of mutually orthogonal projections q_j 's of (7.2), and the corresponding C^* -subalgebra

$$Q = C^*(\mathbf{Q}) \tag{7.3}$$

of the C^* -algebra $B(H)$, which is $*$ -isomorphic to $\mathbb{C}^{\oplus|\mathbb{Z}|}$.

Now, fix an arbitrary vector h_0 of H ,

$$h_0 = \sum_{n \in \mathbb{Z}} t_n \xi_n \in H, \text{ with } t_n \in \mathbb{C}.$$

For the fixed vector h_0 of H , define now a linear functional ψ on $B(H)$ by

$$\psi(T) = \langle T(h_0), h_0 \rangle_2, \text{ for all } T \in B(H). \tag{7.4}$$

By the sesqui-linearity of the inner product $\langle \cdot, \cdot \rangle_2$, the morphism ψ of (7.4) is indeed a linear functional on $B(H)$, i.e., $(B(H), \psi)$ forms a well-defined C^* -probability space.

Observe that the linear functional ψ of (7.4) on $B(H)$ satisfies that

$$\begin{aligned} \psi(q_j) &= \langle q_j(h_0), h_0 \rangle_2 = \langle t_j \xi_j, h_0 \rangle_2 \\ &= t_j \langle \xi_j, h_0 \rangle_2 = t_j^2, \end{aligned}$$

by (7.1) and (7.2), i.e.,

$$\psi(q_j) = t_j^2, \text{ for all } j \in \mathbb{Z}, \tag{7.5}$$

where t_j is the j -th coefficient of a fixed vector h_0 of (7.3).

Clearly, one can restrict the linear functional ψ on $B(H)$ to that on the C^* -subalgebra Q of $B(H)$, in the sense of (7.3), also denoted by ψ . Then, for any

$$T = \sum_{j \in \mathbb{Z}} s_j q_j \in Q, \text{ with } s_j \in \mathbb{C},$$

one has

$$\psi(T) = \sum_{j \in \mathbb{Z}} s_j t_j^2 \text{ in } \mathbb{C},$$

by (7.5).

So, one can determine the family $\{\psi_j\}_{j \in \mathbb{Z}}$ of linear functionals providing sectionized free probability on Q , where they are linear functional satisfying that

$$\psi_j \left(\sum_{n \in \mathbb{Z}} s_n q_n \right) = \psi(s_j q_j) = s_j \psi(q_j) = s_j t_j^2, \tag{7.6}$$

by (7.5), for all $j \in \mathbb{Z}$, i.e., we obtain the following sectionized C^* -probability spaces,

$$\{(Q, \psi_j) : j \in \mathbb{Z}\},$$

for the linear functionals ψ_j of (7.6).

Now, construct the radial projection algebra \mathfrak{L}_Q ,

$$\mathfrak{L}_Q = \mathfrak{L} \otimes_{\mathbb{C}} Q \stackrel{\text{def}}{=} \overline{\mathbb{C}[\{l\}]^{B(Q)}} \otimes_{\mathbb{C}} Q \tag{7.7}$$

of Q , by defining the radial operator l on Q by the linear transformation satisfying

$$l(q_j) = c(q_j) + a(q_j) = q_{j+1} + q_{j-1},$$

for all $j \in \mathbb{Z}$, where c and a are the creation, respectively, the annihilation on Q .

On the Banach *-algebra \mathfrak{L}_Q of (7.7), define the sectionized linear functionals φ_j by linear morphisms satisfying that

$$\varphi_j((l \otimes q_i)^n) = \varphi_j(l^n \otimes q_i) = \psi_j(l^n(q_i)), \tag{7.8}$$

for all $i, j \in \mathbb{Z}$, for all $n \in \mathbb{N}$, where ψ_j are in the sense of (7.6).

Note that, by (7.6), one has that

$$\begin{aligned} \psi_j(l^n(q_i)) &= \psi_j\left(\sum_{k=0}^n \binom{n}{k} c^k a^{n-k}(q_i)\right) \\ &= \sum_{k=0}^n \binom{n}{k} \psi_j(q_{i+k-(n-k)}) \\ &= \sum_{k=0}^n \binom{n}{k} \psi_j(q_{i+2k-n}) \\ &= \sum_{k=0}^n \binom{n}{k} (\delta_{j,i+2k-n} t_j^2), \end{aligned}$$

with help of (7.5), for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

Define now a new linear functional τ_j on the radial projection algebra \mathfrak{L}_Q of (7.7) by a linear morphism satisfying

$$\tau_j((l \otimes q_j)^n) = \varphi_j\left(\delta_{j,i} \left(\frac{\psi(q_j)^{n-1}}{(\lfloor \frac{n}{2} \rfloor + 1)} (l \otimes q_j)^n\right)\right), \tag{7.9}$$

for all $n \in \mathbb{N}$, for all $j \in \mathbb{Z}$, where φ_j are in the sense of (7.8).

Then, by (4.18), if

$$u_j = l \otimes q_j \in (\mathfrak{L}_Q, \tau_j),$$

then we obtain that

$$\tau_j(u_j^n) = \omega_n \psi(q_j)^n c_{\frac{n}{2}} = \omega_n t_j^{2n} c_{\frac{n}{2}}, \tag{7.10}$$

by (7.5) and (7.6), for all $n \in \mathbb{N}$, where $\omega_n = 1$, if n is even, and $\omega_n = 0$, if n is odd in \mathbb{N} , for all $j \in \mathbb{Z}$. Recall again that t_j are the j -th entry of the fixed vector h_0 of (7.3) in $H = l^2(\mathbb{Z})$.

Observation 7.1. Let $u_j = l \otimes q_j$ be a free random variable of the Banach $*$ -probability space (\mathfrak{L}_Q, τ_j) , where τ_j are in the sense of (7.9), for all $j \in \mathbb{Z}$. If the j -th entry t_j of the vector h_0 of (7.3) is nonzero, then u_j is t_j^4 -semicircular in (\mathfrak{L}_Q, τ_j) , for all $j \in \mathbb{Z}$, by (7.10) and (3.3). By the Möbius inversion of [14], one obtains the equivalent free-distributional data of (7.10) as follows: if $t_j \neq 0$ in h_0 , then

$$k_n^{(j)}(u_j, \dots, u_j) = \begin{cases} t_j^4 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.11}$$

for all $n \in \mathbb{N}$, for $j \in \mathbb{Z}$.

Therefore, by Observation 7.1, we immediately get the following result.

Observation 7.2. Let $u_j = l \otimes q_j$ in \mathfrak{L}_Q , and let $U_j = t_j^{-2} u_j \in \mathfrak{L}_Q$, for some $j \in \mathbb{Z}$, where the j -th coefficients t_j are the “nonzero” in the fixed vector h_0 of (7.3) in H . Then U_j is semicircular in (\mathfrak{L}_Q, τ_j) , for such $j \in \mathbb{Z}$, by Section 5.

The above Observations 7.1 and 7.2, we obtain the weighted-semicircularity and corresponding semicircularity on \mathfrak{L}_Q , for suitable $j \in \mathbb{Z}$.

From below, let us assume $t_j \neq 0$, as the j -th entry of a fixed Hilbert-space vector h_0 of (7.3), and let

$$u_j = l \otimes q_j, \text{ and } U_j = \frac{1}{\psi(q_j)} u_j = \frac{1}{t_j^2} u_j \text{ in } \mathfrak{L}_Q,$$

for such $j \in \mathbb{Z}$.

For the fixed vector $h_0 \in H = l^2(\mathbb{Z})$ of (7.3), define a subset $Supp(h_0)$ of \mathbb{Z} by

$$Supp(h_0) \stackrel{def}{=} \{j \in \mathbb{Z} : t_j \neq 0 \text{ in } h_0\}. \tag{7.12}$$

We call this set $Supp(h_0)$ of (7.12), the *support* of h_0 .

Observation 7.3. Let $h_0 = \sum_{n \in \mathbb{Z}} t_n \xi_n \in H$ be in the sense of (7.3), and let $Supp(h_0)$ be the support (7.12) of h_0 in \mathbb{Z} . If $j \in Supp(h_0)$ in \mathbb{Z} , then one has a t_j^4 -semicircular element u_j , and a semicircular element U_j in the j -th filtered probability space $\mathfrak{L}_Q(j)$.

Now, take a finite sequence $W = (j_1, \dots, j_N)$ of integers j_1, \dots, j_N in $Supp(h_0)$, for some $N \in \mathbb{N}$. Then, by Observation 7.3, one can take corresponding weighted-semicircular elements u_{j_1}, \dots, u_{j_N} , and semicircular elements U_{j_1}, \dots, U_{j_N} (which are not necessarily distinct from each other) in \mathfrak{L}_Q . Construct the direct product Banach $*$ -algebra $\mathfrak{L}_Q^{\oplus N}$ generated by $(|W| = N)$ -copies of \mathfrak{L}_Q .

On $\mathfrak{L}_Q^{\oplus N}$, define a linear functional τ_W by a linear morphism satisfying that

$$\tau_W \left(\bigoplus_{k=1}^N T_k \right) = \frac{1}{N} \sum_{k=1}^N \tau_{j_k}(T_k), \tag{7.13}$$

for all $\bigoplus_{k=1}^N T_k \in \mathfrak{L}_Q^{\oplus N}$, as in (6.9).

Then we obtain a well-defined Banach *-probability space,

$$\mathfrak{L}_Q^{\oplus N}(W) = \left(\mathfrak{L}_Q^{\oplus N}, \tau_W \right), \tag{7.14}$$

where τ_W is in the sense of (7.13).

Observation 7.4. For a fixed finite sequence W in $\text{Supp}(h_0)$, define a free random variable T_W of the Banach *-probability space $\mathfrak{L}_Q^{\oplus |W|}(W)$ of (7.14) by

$$T_W \stackrel{\text{def}}{=} \frac{1}{|W|} \left(\bigoplus_{j \in W} U_j \right) \in \mathfrak{L}_Q^{\oplus |W|}(W), \tag{7.15}$$

where U_j are our semicircular elements of $\mathfrak{L}_Q(j)$, for $j \in W$. Then the free random variable T_W of (7.15) is $\frac{1}{|W|^2}$ -semicircular in $\mathfrak{L}_Q^{\oplus |W|}(W)$, by (6.12).

Now, for a fixed finite sequence $W = (j_1, \dots, j_N)$ of “mutually distinct” integers j_1, \dots, j_N in $\text{Supp}(h_0)$, define the tensor product Banach *-algebra $\mathfrak{L}_Q^{\otimes N}$ generated by N -copies of \mathfrak{L}_Q . On this Banach *-algebra $\mathfrak{L}_Q^{\otimes N}$, define linear functionals $\{\varphi_{W,k}\}_{k=1}^N$ by the linear morphisms satisfying

$$\varphi_{W,k} \left(\bigotimes_{k=1}^N T_k \right) = t_{j_k}^2 \left(\prod_{l=1}^N \frac{1}{t_{j_l}^2} \tau_{j_l}(T_l) \right), \tag{7.16}$$

for all $k = 1, \dots, N$, as in (6.31).

Then one can get the family

$$\left\{ \mathfrak{L}_Q^{\otimes N}(W, k) \stackrel{\text{denote}}{=} \left(\mathfrak{L}_Q^{\otimes N}, \varphi_{W,k} \right) \right\}_{k=1}^N \tag{7.17}$$

of Banach *-probability spaces, where $\varphi_{W,k}$ are in the sense of (7.16).

Observation 7.5. For a fixed finite sequence $W = (j_1, \dots, j_N)$ of mutually distinct integers in $\text{Supp}(h_0)$, one can construct the family (7.17) of Banach *-probability spaces $\mathfrak{L}_Q^{\otimes |W|}(W, k)$, for $k = 1, \dots, N$. For a fixed $k \in \{1, \dots, N\}$, define a free random variable $T_{W,k}$ of $\mathfrak{L}_Q^{\otimes |W|}(W, k)$ by

$$T_{W,k} = I_{W,j_1} \otimes \dots \otimes I_{W,j_{k-1}} \otimes \underset{k\text{-th}}{u_{j_k}} \otimes I_{W,j_{k+1}} \otimes \dots \otimes I_{W,j_N}, \tag{7.18}$$

where u_{j_k} is our $t_{j_2}^2$ -semicircular element of $(\mathfrak{L}_Q, \tau_{j_k})$, and

$$I_{W,j_k} = 1_{\mathfrak{L}} \otimes (\hat{q}_{j_k}) \in \mathfrak{L}_Q, \text{ for } k = 1, \dots, N.$$

Then the operator $T_{W,k}$ of (7.18) is $t_{j_k}^4$ -semicircular in $\mathfrak{L}_Q^{\otimes |W|}(W, k)$ by (6.33), for all $k = 1, \dots, N$.

Observation 7.6. *Under the same conditions of Observation 7.5, if we take a free random variable*

$$S_{W,k} = I_{W,j_1} \otimes \dots \otimes I_{W,j_{k-1}} \otimes \underset{k\text{-th}}{U_{j_k}} \otimes I_{W,j_{k+1}} \otimes \dots \otimes I_{W,j_N},$$

in $\mathfrak{L}_Q^{\otimes N}(W, k)$, for $k \in \{1, \dots, N\}$, where U_{j_k} is our semicircular element of $(\mathfrak{L}_Q, \tau_{j_k})$, then $S_{W,k}$ is semicircular in $\mathfrak{L}_Q^{\otimes N}(W, k)$, for $k = 1, \dots, N$.

Now, let $W = (j_1, \dots, j_N)$ be an N -tuple of integers in $\text{Supp}(h_0)$, and let

$$\mathfrak{L}_Q(j_k) = (\mathfrak{L}_Q, \tau_{j_k}), \text{ for } k = 1, \dots, N,$$

be the corresponding j_k -filtered probability spaces of Q . Construct the free product Banach $*$ -probability space,

$$\mathfrak{L}_Q^N(W) = (\mathfrak{L}_Q^N, \tau^W) = \left(\underset{k=1}{\star}^N \mathfrak{L}_Q, \underset{k=1}{\star}^N \tau_{j_k} \right). \tag{7.19}$$

Observation 7.7. *Let W be a finite sequence of integers in $\text{Supp}(h_0)$, and let $\mathfrak{L}_Q^{|W|}(W)$ be in the sense of (7.19). Then the free reduced words u_k and U_k with their length-1 in the free blocks $\mathfrak{L}_Q(j_k)$ are t_k^4 -semicircular, respectively, semicircular in $\mathfrak{L}_Q^{|W|}(W)$, for all $k = 1, \dots, |W|$, by (6.48) and (6.49).*

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Ilwoo Cho
choilwoo@sau.edu

Saint Ambrose University
Department of Mathematics and Statistics
421 Ambrose Hall, 518 W. Locust St.
Davenport, Iowa, 52803, USA

Palle E.T. Jorgensen
palle-jorgensen@uiowa.edu

The University of Iowa
Department of Mathematics
Iowa City, IA 52242-1419, USA

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