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SMALL-GAIN THEOREM FOR A CLASS OF ABSTRACT PARABOLIC SYSTEMS

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Abstract. We consider a class of abstract control system of parabolic type with observation which the state, input and output spaces are Hilbert spaces. The state space operator is assumed to generate a linear exponentially stable analytic semigroup. An observation and control action are allowed to be described by unbounded operators. It is assumed that the observation operator is admissible but the control operator may be not. Such a system is controlled in a feedback loop by a controller with static characteristic being a globally Lipschitz map from the space of outputs into the space of controls. Our main interest is to obtain a perturbation theorem of the small-gain-type which guarantees that null equilibrium of the closed-loop system will be globally asymptotically stable in Lyapunov's sense.

Keywords: control of infinite-dimensional systems, semigroups, infinite-time LQ-control problem, Lur'e feedback systems.

Mathematics Subject Classification: 49N10, 93B05, 93C25.

1. INTRODUCTION

Since G. Weiss has published his paper [15] on well-posedness of the closed-loop system treated as a feedback perturbation of the open-loop system, several papers continuing this topics appeared. A different presentation of the Weiss perturbation result has been proposed in [1] and [2].

In [7], the last results, have been reformulated in the terms of boundary control systems in factor form and completed by an original contribution. One of a novelty of that paper was that, contrary to the previous results, the author has proved that for some parabolic systems the Weiss perturbation result holds even without the admissibility of factor control operator.

In the present paper we continue this contribution by showing that under a mild assumptions the perturbation result of [7, Theorem 5.1, p. 1115] remains valid for nonlinear, globally Lipschitz perturbations. The result can be regarded as either the small-gain-type theorem or the circle criterion for an abstract parabolic system because here the perturbation describes a feedback control. To obtain this result, the standard $\mathrm{L}^2(0,\infty;\mathrm{Y})$ - admissibility of the factor control operator is replaced by the so-called Balakrishnan-Washburn estimates which represent a kind of balance between admissibility-like properties of the output and control operators.

The paper is organized as follows. In Section 2 we give an overview of the theory of boundary systems in factor form and we introduce basic concepts. L. De Simon's theorem on maximal parabolic regularity is recalled – Lemma 2.5 as an important analytic tool.

The main result of Section 3 (Theorem 3.1) links the maximal parabolic regularity with Banach's fixed point theorem. To be more precise, Theorem 3.1 provides sufficient conditions under which the perturbed feedback system has $\mathrm{L}^2(0,\infty;\mathrm{H})$ – solutions.

In the next Section 4, by adding some extra assumption, we prove a result (Theorem 4.2) on existence of weak solutions and the global asymptotic stability, in the Lyapunov's sense, of the null equilibrium point. Our proof bases on observation that though the system may not have a weak solution for an arbitrary $\mathrm{L}^2(0,\infty;\mathrm{U})$ – control, it can have weak solution for a control generated in a feedback control loop. This is due to a smoothing action of the feedback.

Section 5 brings an exhaustive example – the unloaded electric $\mathfrak{RC}$ – transmission line. It is shown that Theorem 4.2 provides sufficient conditions for existence of weak solutions to the closed loop feedback (perturbed) system though the open loop system may not have weak solution for an arbitrary control (Remark 5.3). The largest Lipschitz constant for which the origin is the globally asymptotically stable solution is identified using graphical and analytical criteria.

The final Section 6 contains a discussion of results. In particular, a link with the theory of nonlinear semigroups is indicated, which deserves some further investigations.

2. AN OVERVIEW OF CONTROL SYSTEMS IN FACTOR FORM

Consider a class of controlled systems with observation governed by the model in factor form

$$\begin{cases} \dot{x}(t) = \mathcal{A}\left[x(t) + \mathcal{D}u(t)\right], \\ x(0) = x_0, \\ y(t) = \mathcal{C}x(t), \end{cases} \tag{2.1}$$

where the *state operator* $\mathcal{A} : (D(\mathcal{A}) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates an exponentially stable C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space H with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$.

$\mathbf{L}(Z_1, Z_2)$ will be used to denote bounded everywhere defined operators acting from a Banach space Z_1 into a Banach space Z_2 and the standard abbreviation $\mathbf{L}(Z)$ will be made when $Z = Z_1 = Z_2$.

A family $\{S(t)\}_{t\geq 0} \subset \mathbf{L}(\mathrm{H})$ is a C_0-*semigroup* on H if (i) $S(0) = I$, $S(t+\tau) = S(t)S(\tau)$ for $t,\tau \geq 0$ and (ii) $S(t)z \to z$ as $t \to 0$ for every $z \in \mathrm{H}$. $\{S(t)\}_{t\geq 0}$ is *exponentially stable* (EXS) if there exist $M \geq 1$ and $\alpha > 0$ such that

$$\|S(t)z\|_{\mathrm{H}} \leq Me^{-\alpha t}\|z\|_{\mathrm{H}}, \quad t \geq 0,\ z \in \mathrm{H}. \tag{2.2}$$

We say that $\mathcal{A}$ *generates* $\{S(t)\}_{t\geq 0}$ if

$$\mathcal{A}z = \lim_{h\to 0}\frac{1}{h}[S(h)z - z], \quad D(\mathcal{A}) = \Big\{z \in \mathrm{H}: \text{ there exists } \lim_{h\to 0}\frac{1}{h}[S(h)z - z]\Big\}.$$

Such an operator is necessarily densely defined and closed.

Since $s \mapsto (sI - \mathcal{A})^{-1}z$ is the *Laplace transform* of $t \mapsto S(t)z$ then, by (2.2), the half-plane $\{s \in \mathbb{C}: \ \mathrm{Re}\, s > -\alpha\}$ is contained in $\rho(\mathcal{A})$ – the resolvent set of $\mathcal{A}$ which, in particular, implies that $\mathcal{A}$ is invertible, $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$.

Next, $\mathcal{C} : (D(\mathcal{C}) \subset \mathrm{H}) \longrightarrow \mathrm{Y}$, $\mathcal{C}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H}, \mathrm{Y})$, $\mathcal{D} \in \mathbf{L}(\mathrm{U}, \mathrm{H})$ with range $R(\mathcal{D}) \subset D(\mathcal{C})$, $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$ and Y and U are Hilbert spaces with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{U}}$, respectively.

Let us introduce

$$H := (\mathcal{C}\mathcal{A}^{-1})^* \iff H^* = \mathcal{C}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H}, \mathrm{Y})$$

to simplify future notation.

Proofs of all results appearing in this Section are given in [7, Section 2].

2.1. ADMISSIBLE OBSERVATION AND CONTROL OPERATORS

We shall use the semigroups of left-shifts on $\mathrm{L}^2(0,\infty;\mathrm{X})$, X is a Hilbert space, which will be denoted as $\{T_{\mathrm{X}}(t)\}_{t\geq 0}$,

$$(T_{\mathrm{X}}(t)f)(\tau) := f(t+\tau) \quad \text{for almost all} \quad t,\tau \geq 0.$$

It is generated by

$$\mathcal{L}f = f', \quad D(\mathcal{L}) = \mathrm{W}^{1,2}([0,\infty);\mathrm{X}), \tag{2.3}$$

$$\mathrm{W}^{1,2}([0,\infty);\mathrm{X}) := \big\{f \in \mathrm{L}^2(0,\infty;\mathrm{X}) : f' \in \mathrm{L}^2(0,\infty;\mathrm{X})\big\} \subset \mathrm{C}([0,\infty);\mathrm{X}).$$

The adjoint of $T_{\mathrm{X}}(t)$,

$$(T^*_{\mathrm{X}}(t)f)(\tau) := \begin{Bmatrix} f(\tau - t) & \text{if} & \tau \geq t \\ 0 & \text{if} & 0 \leq \tau < t \end{Bmatrix}$$

is the right-shift operator on $\mathrm{L}^2(0,\infty;\mathrm{X})$ and it is clearly generated by $\mathcal{L}^* := \mathcal{R}$,

$$\mathcal{R}f = -f', \quad D(\mathcal{R}) = \mathrm{W}^{1,2}_0([0,\infty);\mathrm{X}) \tag{2.4}$$

Define $\mathcal{Z} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$,

$$(\mathcal{Z}z)(t) := H^*S(t)z \quad \left[\iff \mathcal{Z}^*f = \int_0^\infty S^*(t)Hf(t)\mathrm{d}t\right].$$

The operator, called the *observability map*,

$$\mathbf{\Psi} := \mathcal{L}\mathcal{Z}, \quad D(\mathbf{\Psi}) = \{x \in \mathrm{H} : \mathcal{Z}x \in D(\mathcal{L})\}$$

is closed and densely defined, with $\mathbf{\Psi}|_{D(\mathcal{A})} = \mathcal{Z}\mathcal{A}$, and therefore it has closed and densely defined adjoint operator

$$\mathbf{\Psi}^* = \mathcal{A}^*\mathcal{Z}^*, \quad D(\mathbf{\Psi}^*) = \{y \in \mathrm{L}^2(0,\infty;\mathrm{Y}) : \mathcal{Z}^* y \in D(\mathcal{A}^*)\},$$

and $\mathbf{\Psi}^*|_{D(\mathcal{R})} = \mathcal{Z}^*\mathcal{R}$. Here $\mathcal{L}$, $\mathcal{R}$ are given by (2.3) and (2.4), respectively, with $\mathrm{X} = \mathrm{Y}$.

Definition 2.1. $\mathcal{C}$ is an admissible *observation* (*output*) *operator* if $\mathbf{\Psi} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$ (or, by the closed graph theorem, $R(\mathcal{Z}) \subset D(\mathcal{L})$ or $\mathbf{\Psi}$ is bounded).

Lemma 2.2. *If $\mathcal{C}$ is admissible then $\mathbf{\Psi}$ is also a linear densely defined and bounded operator from* H *into* $\mathrm{L}^1(0,\infty;\mathrm{Y})$.

Next, we define $\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{H})$ as follows:

$$\mathcal{W}f := \int_0^\infty S(t)\mathcal{D}f(t)\mathrm{d}t \quad [\iff (\mathcal{W}^* z)(t) = \mathcal{D}^* S^*(t) z].$$

The operator, called the *reachability map*,

$$\mathbf{\Phi} := \mathcal{A}\mathcal{W}, \quad D(\mathbf{\Phi}) = \{u \in \mathrm{L}^2(0,\infty;\mathrm{U}) : \mathcal{W}u \in D(\mathcal{A})\}$$

is closed and densely defined, with $\mathbf{\Phi}|_{D(\mathcal{R})} = \mathcal{W}\mathcal{R}$, and therefore it has closed and densely defined adjoint operator

$$\mathbf{\Phi}^* = \mathcal{L}\mathcal{W}^*, \quad D(\mathbf{\Phi}) = \{x \in \mathrm{H} : \ \mathcal{W}^* x \in D(\mathcal{L})\},$$

with $\mathbf{\Phi}^*|_{D(\mathcal{A}^*)} = \mathcal{W}^*\mathcal{A}^*$. Here $\mathcal{L}$, $\mathcal{R}$ are given by (2.3) and (2.4), respectively, with $\mathrm{X} = \mathrm{U}$.

Definition 2.3. $\mathcal{D}$ is an admissible *factor control operator* if $\mathbf{\Phi} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{H})$ (or, by the closed graph theorem, $R(\mathcal{W}) \subset D(\mathcal{A})$ or $\mathbf{\Phi}$ is bounded).

Using duality arguments, we can state the following result.

Lemma 2.4. *$\mathcal{D}$ is an admissible factor control operator iff $\mathcal{D}^*\mathcal{A}^*$ is an admissible observation operator with respect to the semigroup $\{S^*(t)\}_{t\geq 0}$.*

2.2. REPRESENTATION OF THE STATE UNDER PARABOLIC REGULARITY

Recall that $\{S(t)\}_{t\geq 0}$ is an *analytic* semigroup if, in addition to axioms (i), (ii) of the C_0-semigroup, there holds: (iii) for every $z \in \mathrm{H}$ the mapping $(0,\infty) \ni t \mapsto S(t)z$ is a real analytic function.

Lemma 2.5 (*maximal* $\mathrm{L}^2(0,\infty;\mathrm{H})$ - *parabolic regularity* [4]). *The following conditions are equivalent:*

(i) *$\mathcal{A}$ generates an analytic EXS semigroup $\{S(t)\}_{t\geq 0}$,*
(ii) $\mathcal{A}(sI - \mathcal{A})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H}))$,

(iii) $f \longmapsto \mathcal{A}S(\cdot) \star f \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{H}))$,
(iv) *For every* $f \in \mathrm{L}^2(0,\infty;\mathrm{H})$ *there exists a unique strong* (*absolutely continuous*) *solution of the Cauchy problem:* $\dot{z} = \mathcal{A}z + f,\ z(0) = 0$.

Thanks to Lemma 2.5, for every $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$

$$x(t) := S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau = S(t)x_0 + \int_0^t \mathcal{A}S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau, \qquad (2.5)$$

is a unique L^2-*solution* of (2.1), i.e., $x \in \mathrm{L}^2(0,\infty;\mathrm{H})$, which also satisfies (2.1) in a *weak sense*, i.e., for every $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$ and $w \in D(\mathcal{A}^*)$, the function $t \longmapsto \langle x(t), w\rangle_{\mathrm{H}}$ is in $\mathrm{W}^{1,2}(0,\infty)$, and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), w\rangle_{\mathrm{H}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left\langle S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau, w\right\rangle_{\mathrm{H}} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle x_0, S^*(t)w\rangle_{\mathrm{H}} + \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \int_0^t S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau, \mathcal{A}^*w\right\rangle_{\mathrm{H}} \\
&= \langle x_0, S^*(t)\mathcal{A}^*w\rangle_{\mathrm{H}} + \left\langle \mathcal{A}\int_0^t S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau + \mathcal{D}u(t), \mathcal{A}^*w\right\rangle_{\mathrm{H}} \\
&= \left\langle S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau + \mathcal{D}u(t), \mathcal{A}^*w\right\rangle_{\mathrm{H}} \\
&= \langle x(t) + \mathcal{D}u(t), \mathcal{A}^*w\rangle_{\mathrm{H}},
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to 0}\langle x(t), w\rangle_{\mathrm{H}} &= \lim_{t\to 0}\left\langle S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau, w\right\rangle_{\mathrm{H}} \\
&= \langle x_0, w\rangle_{\mathrm{H}} + \lim_{t\to 0}\left\langle \int_0^t S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau, \mathcal{A}^*w\right\rangle_{\mathrm{H}} = \langle x_0, w\rangle_{\mathrm{H}}.
\end{aligned}$$

However, x *is not yet a weak solution* of (2.1) as we do not know whether x is a *continuous* function satisfying $x(0) = x_0$.

Similar problem has been treated in [11] where a counterexample due to J.L. Lions was invoked to show that generally x is not continuous until u is an arbitrary $\mathrm{L}^2(0,\infty;\mathrm{U})$ function.

Lion's example is naturally obtained as a by-product of a physical problem analysed in Section 5, see especially Remark 5.3.

2.3. REPRESENTATION OF THE OUTPUT UNDER PARABOLIC REGULARITY

Under the parabolic regularity there holds

$$S(t)z \in D(\mathcal{A}^\infty), \quad t > 0, \ z \in \mathrm{H}.$$

Hence

$$(\mathbf{\Psi} z)(t) = \mathcal{C}S(t)z = H^*\mathcal{A}S(t)z, \quad t > 0, \ z \in \mathrm{H},$$

$$\mathcal{C}(sI - \mathcal{A})^{-1} = H^*\mathcal{A}(sI - \mathcal{A})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H}, \mathrm{Y}))$$

while $\mathcal{C}$ is admissible iff

$$\mathcal{C}(sI - \mathcal{A})^{-1}z \in \mathrm{H}^2(\mathbb{C}^+, \mathrm{Y}).$$

Assume that $\mathcal{C}$ is admissible. Assume also that

$$\begin{aligned} &\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}, \mathrm{Y})), \\ &\hat{G}(s) := s\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{D} - \mathcal{C}\mathcal{D} = s^2H^*(sI - \mathcal{A})^{-1}\mathcal{D} - sH^*\mathcal{D} - \mathcal{C}\mathcal{D}; \end{aligned} \tag{2.6}$$

$\widehat{G}$ is called the *transfer function* of (2.1).

Next, we introduce the *input-output operator* $\mathbb{F}$,

$$(\mathbb{F}u)(t) := \int_0^t H^*\mathcal{A}^2S(t-\tau)\mathcal{D}u(\tau)\mathrm{d}\tau, \quad (\widehat{\mathbb{F}u})(s) = \mathcal{C}\mathcal{A}(sI - \mathcal{A})^{-1}\mathcal{D}\hat{u}(s) = \hat{G}(s)\hat{u}(s),$$

so $\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0, \infty; \mathrm{U}), \mathrm{L}^2(0, \infty; \mathrm{Y}))$, provided that (2.6) holds.

Finally the output equation reads as

$$y = \mathbf{\Psi} x_0 + \mathbb{F}u, \quad x_0 \in \mathrm{H}, \quad u \in \mathrm{L}^2(0, \infty; \mathrm{U}).$$

3. A SMALL-GAIN PERTURBATION THEOREM FOR L^2-SOLUTIONS

Consider the *Lur'e system* of automatic feedback control having the structure depicted in Figure 1, where the feedback $u = -f(y)$ is given by a nonlinear mapping $f : \mathrm{Y} \ni y \mapsto f(y) \in \mathrm{U}$, called a *static characteristic* of the controller. We assume that:

$$\begin{aligned} &\text{there exists } \mu \in \mathbf{L}(\mathrm{Y}, \mathrm{U}) : \ y \longmapsto f(y) - \mu y \text{ satisfies the Lipschitz} \\ &\text{condition with Lipschitz constant } m > 0; \quad f(0) = 0. \end{aligned} \tag{3.1}$$

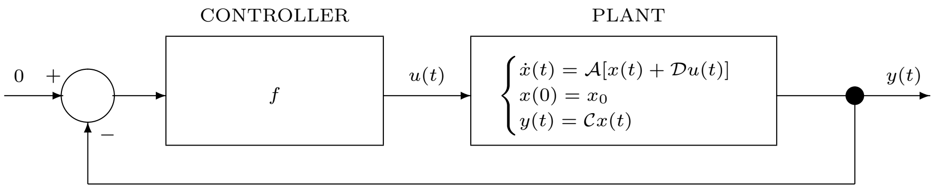

Fig. 1. The Lur'e control system with negative feedback

In what follows, $\mathcal{N}$ will denote the *Nemytskii operator of superposition* $(\mathcal{N}y)(t) := f[y(t)]$ *induced by* f. By (3.1), $\mathcal{N}_{new} := \mathcal{N} - \mu I$ is the Lipschitz mapping from $\mathrm{L}^2(0,\infty;\mathrm{Y})$ into $\mathrm{L}^2(0,\infty;\mathrm{U})$ with the same Lipschitz constant m.

The closed-loop system is governed by

$$y = \mathbf{\Psi}x_0 + \mathbb{F}u = \mathbf{\Psi}x_0 - \mathbb{F}\mathcal{N}y \iff (I + \mathbb{F}\mu)y = \mathbf{\Psi}x_0 - \mathbb{F}\mathcal{N}_{new}y. \tag{3.2}$$

Assume, in addition, that

$$I + \mathbb{F}\mu \text{ is boundedly invertible } \iff (I + \hat{G}\mu)^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{Y})). \tag{3.3}$$

Under the assumption (3.3), (3.2) takes the form

$$y = (I + \mathbb{F}\mu)^{-1}\mathbf{\Psi}x_0 - (I + \mathbb{F}\mu)^{-1}\mathbb{F}\mathcal{N}_{new}y \tag{3.4}$$

and, by Banach's fixed point theorem, (3.4) has a unique solution $y^{\mathrm{c}} \in \mathrm{L}^2(0,\infty;\mathrm{Y})$, provided that

$$\left\|(I + \hat{G}\mu)^{-1}\hat{G}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))} = \left\|\hat{G}(I + \mu\hat{G})^{-1}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))} < \frac{1}{m}, \tag{3.5}$$

where, by the Phragmén-Lindelöf principle, $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$ norm can be replaced by $\mathrm{L}^\infty(j\mathbb{R}, \mathbf{L}(\mathrm{U},\mathrm{Y}))$ norm.

Now $u^{\mathrm{c}} := -\mathcal{N}y^{\mathrm{c}} \in \mathrm{L}^2(0,\infty;\mathrm{U})$. Moreover, by (3.4) and (3.5),

$$\|y^{\mathrm{c}}\|_{\mathrm{L}^2(0,\infty;\mathrm{Y})} \le \gamma_1\|x_0\|_{\mathrm{H}}; \quad \gamma_1 := \frac{\left\|(I + \hat{G}\mu)^{-1}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{Y}))}\|\mathbf{\Psi}\|_{\mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{Y}))}}{1 - m\left\|(I + \hat{G}\mu)^{-1}\hat{G}\right\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))}}. \tag{3.6}$$

Substituting $u = u^{\mathrm{c}}$ in (2.5) and denoting the corresponding x as x^{c},

$$x^{\mathrm{c}}(t) = S(t)x_0 - \int_0^t \mathcal{A}S(t-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau, \tag{3.7}$$

we conclude that the *closed-loop state* x^{c} belongs to $\mathrm{L}^2(0,\infty;\mathrm{H})$.

The results can be gathered as the following theorem.

Theorem 3.1. *Assume that:*

(i) $\mathcal{A}$ *generates an EXS analytic semigroup,*
(ii) $\mathcal{C}$ *is admissible,*
(iii) *The transfer function* $\hat{G}$ *satisfies* (2.6)*,*
(iv) (3.1)*,* (3.3) *and* (3.5) *hold.*

Then, the closed-loop state x^c*, given by* (3.7)*, is in* $\mathrm{L}^2(0,\infty;\mathrm{H})$*.*

4. A SMALL-GAIN PERTURBATION THEOREM FOR WEAK SOLUTIONS

We begin from introducing an important definition.

Definition 4.1. Let $\mathcal{A}$ be an infinitesimal generator of an EXS *analytic* semigroup $\{S(t)\}_{t\geq 0}$ on H. We say that the pair $(H,\mathcal{D})$ satisfies the *conjugate Balakrishnan–Washburn estimates* if there exist $\alpha\in(0,\frac{1}{2})$, $\delta>0$ and two continuous increasing functions $\eta_h=\eta_h(t)$ and $\eta_d=\eta_d(t)$ defined on $[0,\infty)$ growing no faster than polynomially, such that

$$\|\mathcal{A}S(t)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}=\|\mathcal{D}^*\mathcal{A}^*S^*(t)\|_{\mathbf{L}(\mathrm{H},\mathrm{U})}\leq\eta_d(t)\frac{e^{-\delta t}}{t^{\alpha}},\quad t>0,\tag{4.1}$$

$$\|H^*\mathcal{A}S(t)\|_{\mathbf{L}(\mathrm{H},\mathrm{Y})}=\|\mathcal{A}^*S^*(t)H\|_{\mathbf{L}(\mathrm{Y},\mathrm{H})}\leq\eta_h(t)\frac{e^{-\delta t}}{t^{1-\alpha}},\quad t>0.\tag{4.2}$$

Observe that (4.2) implies that $\mathcal{C}$ is an admissible observation operator, but (4.1) does not imply that $\mathcal{D}$ is an admissible factor control operator.

If H^* is a *finite rank operator* then the observability map $\mathbf{\Psi}$ is even a Hilbert-Schmidt (HS) operator [5, Theorem 5].

In what follows $\mathrm{B(U)C}([0,\infty);\mathrm{Z})$ will denote the Banach space of *bounded* (*uniformly*) *continuous* functions defined on $[0,\infty)$ and taking values in a Hilbert space Z, equipped with standard norm

$$\|f\|_{\mathrm{B(U)C}([0,\infty);\mathrm{Z})}:=\sup_{t\geq 0}\|f(t)\|_{\mathrm{Z}},\quad f\in\mathrm{B(U)C}([0,\infty);\mathrm{Z}),$$

while $\mathrm{B(U)C}_0([0,\infty);\mathrm{Z})$ will stand for its closed subspace consisting of functions that have zero limit at infinity.

Theorem 4.2. *Assume that:*

(i) $\mathcal{A}$ *generates an EXS analytic semigroup,*
(ii) *the pair* $(H,\mathcal{D})$ *satisfies conjugate Balakrishnan–Washburn estimates* (4.2) *and* (4.1)*,*
(iii) $\hat{G}\in\mathrm{H}^\infty(\mathbb{C}^+;\mathbf{L}(\mathrm{U},\mathrm{Y}))$*,* $\infty>\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\|\hat{G}(j\omega)\|^2_{\mathbf{L}(\mathrm{U},\mathrm{Y})}d\omega:=\gamma_2^2$*,*
(iv) (3.1)*,* (3.3) *and* (3.5) *hold.*

Then, for any $x_0 \in \mathrm{H}$ *the initial-value problem for the closed-loop system*

$$\begin{cases} \dot{x}(t) = \mathcal{A}^c x(t), \\ x(0) = x_0, \end{cases} \tag{4.3}$$

$$\mathcal{A}^c x = \mathcal{A}\left[x - \mathcal{D}f(\mathcal{C}x)\right], \quad D(\mathcal{A}^c) = \{x \in \mathrm{H} : x \in D(\mathcal{C}),\, x - \mathcal{D}f(\mathcal{C}x) \in D(\mathcal{A})\} \tag{4.4}$$

has a unique weak solution in $\mathrm{BC}_0([0,\infty);\mathrm{H})$ *and the origin is globally asymptotically stable in Lyapunov's sense* (*GAS*), *i.e., it is stable:*

$$\forall \varepsilon \quad \exists \delta > 0 : \|x_0\|_{\mathrm{H}} < \delta \Longrightarrow \forall t \geq 0 : \|x(t)\|_{\mathrm{H}} < \varepsilon$$

and globally attracting: $\|x(t)\|_{\mathrm{H}} \longrightarrow$ *as* $t \to \infty$ *for any* $x_0 \in \mathrm{H}$.

Proof. Part 1. Here $\operatorname{supp} u = \operatorname{supp} G = \operatorname{supp}(\mathbb{F}u) = [0,\infty)$, and by properties of the *inverse Fourier transformation*

$$\begin{aligned} \|(\mathbb{F}u)(t)\|_{\mathrm{Y}} &\leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|(\widehat{\mathbb{F}u})(j\omega)\|_{\mathrm{Y}}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|\hat{G}(j\omega)\hat{u}(j\omega)\|_{\mathrm{Y}}\mathrm{d}\omega \\ &\leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|\hat{G}(j\omega)\|_{\mathbf{L}(\mathrm{U},\mathrm{Y})}\|\hat{u}(j\omega)\|_{\mathrm{U}}\mathrm{d}\omega \\ &\leq \gamma_2\|\hat{u}\|_{\mathrm{H}^2(\mathbb{C}^+,\mathrm{U})} = \gamma_2\|u\|_{\mathrm{L}^2(0,\infty;\mathrm{U})}, \quad t \geq 0,\ u \in \mathrm{L}^2(0,\infty;\mathrm{U}). \end{aligned}$$

This, jointly with a vector version of the Riemann–Lebesgue lemma

$$\hat{f} \in \mathrm{L}^1(j\mathbb{R};\mathrm{Y}) \Longrightarrow \lim_{t\to\infty}\int_{-\infty}^{+\infty} e^{j\omega t}\hat{f}(j\omega)\mathrm{d}\omega = 0, \tag{4.5}$$

applied to $\hat{f} = \widehat{\mathbb{F}u}$, yields

$$\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{BUC}_0([0,\infty);\mathrm{Y})), \quad \|\mathbb{F}\|_{\mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{BUC}_0([0,\infty);\mathrm{Y}))} \leq \gamma_2. \tag{4.6}$$

A classical proof of (4.5) based on the identity

$$\begin{aligned} \|tf(t)\|_{\mathrm{Y}} &= \left\| \frac{1}{2\pi j}\int_{-\infty}^{+\infty} \frac{\mathrm{d}e^{j\omega t}}{\mathrm{d}\omega}\hat{f}(j\omega) \right\|_{\mathrm{Y}} = \frac{1}{2\pi}\left\| e^{j\omega t}\hat{f}(j\omega)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{j\omega t}\hat{f}'(j\omega)\mathrm{d}\omega \right\|_{\mathrm{Y}} \\ &= \frac{1}{2\pi}\left\| \int_{-\infty}^{+\infty} e^{j\omega t}\hat{f}'(j\omega)\mathrm{d}\omega \right\|_{\mathrm{Y}} \leq \frac{1}{2\pi}\|\hat{f}'\|_{\mathrm{L}^1(j\mathbb{R};\mathrm{Y})}, \quad \hat{f} \in \mathrm{W}^{1,1}(j\mathbb{R};\mathrm{Y}) \end{aligned}$$

deserves a comment, namely: $\hat{u} \in \mathrm{W}^{1,2}(j\mathbb{R};\mathrm{Y})$ (a dense subset in $\mathrm{L}^2(j\mathbb{R};\mathrm{Y})$) implies $\hat{f} = \hat{G}\hat{u} \in \mathrm{W}^{1,1}(j\mathbb{R};\mathrm{Y})$ (a dense subset in $\mathrm{L}^1(j\mathbb{R};\mathrm{Y})$). Indeed, $\hat{f}' = \hat{G}'\hat{u} + \hat{G}\hat{u}'$.

The second component is clearly in $\mathrm{L}^1(j\mathbb{R};\mathrm{Y})$ as a product of two $\mathrm{L}^2(j\mathbb{R};\mathrm{Y})$-functions. Next observe that

$$\begin{aligned}\hat{G}'(s) &= \mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{D} - s\mathcal{C}(sI-\mathcal{A})^{-2}\mathcal{D} = \mathcal{C}(sI-\mathcal{A})^{-1}[I - s(sI-\mathcal{A})^{-1}]\mathcal{D} \\ &= -H^*\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}.\end{aligned}$$

$H^*\mathcal{A}(sI-\mathcal{A})^{-1}$ is the Laplace transform of $H^*\mathcal{A}S(\cdot)$ and, by (4.2) and the vector Paley-Wiener theorem,

$$\infty > \int_0^\infty \|H^*\mathcal{A}S(t)z\|_{\mathrm{Y}}^2 \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \|H^*\mathcal{A}(j\omega I-\mathcal{A})^{-1}z\|_{\mathrm{Y}}^2 \mathrm{d}\omega, \quad z \in \mathrm{H}.$$

Application of the principle of uniform boundedness yields: $\omega \longmapsto H^*\mathcal{A}(j\omega I-\mathcal{A})^{-1}$ is in $\mathrm{L}^2(j\mathbb{R};\mathbf{L}(\mathrm{H},\mathrm{Y}))$.

$\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}$ is the Laplace transform of $\mathcal{A}S(\cdot)\mathcal{D}$ and, by (4.1),

$$\infty > \int_0^\infty \|\mathcal{A}S(t)\mathcal{D}u\|_{\mathrm{H}}\mathrm{d}t \geq \sup_{s\in\overline{\mathbb{C}+}} \|\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D}u\|_{\mathrm{H}}\mathrm{d}\omega, \quad u \in \mathrm{U}.$$

Applying the principle of uniform boundedness yields $\omega \longmapsto \mathcal{A}(j\omega I - \mathcal{A})^{-1}\mathcal{D}$ is in $\mathrm{L}^\infty(j\mathbb{R};\mathbf{L}(\mathrm{U},\mathrm{H}))$, whence $\hat{G}' \in \mathrm{L}^2(j\mathbb{R};\mathbf{L}(\mathrm{U},\mathrm{Y}))$. Consequently $\hat{G}'\hat{u}$, $\hat{f}' \in \mathrm{L}^1(j\mathbb{R};\mathbf{L}(\mathrm{U},\mathrm{Y}))$.

Part 2. Lifting of $\mathrm{L}^2(0,\infty;\mathrm{H})$-*solutions to* $\mathrm{C}(0,\infty;\mathrm{H})$-*solutions.*

Basing on the fact that the control u^{c} is not an arbitrary $\mathrm{L}^2(0,\infty;\mathrm{U})$–function as it is generated in the feedback mode, which can smooth solutions, we shall demonstrate that $\mathrm{L}^2(0,\infty;\mathrm{H})$-solution x^{c}, given by (3.7), can be lifted to a weak solution $x^{\mathrm{c}} \in \mathrm{BC}([0,\infty);\mathrm{H})$. To achieve this goal we make the following observation. By (3.1) and because (3.2) takes the form $y^{\mathrm{c}} = \mathbf{\Psi}x_0 + \mathbb{F}u^{\mathrm{c}}$, we get for almost all $t \geq 0$:

$$\begin{aligned}\|f[y^{\mathrm{c}}(t)]\|_{\mathrm{U}} &\leq (m + \|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|y^{\mathrm{c}}(t)\|_{\mathrm{Y}} \\ &\leq (m + \|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\left[\|(\mathbf{\Psi}x_0)(t)\|_{\mathrm{Y}} + \|(\mathbb{F}u^{\mathrm{c}})(t)\|_{\mathrm{Y}}\right].\end{aligned} \tag{4.7}$$

To prove that $[0,\infty) \ni t \longmapsto \int_0^t \mathcal{A}S(t-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau \in \mathrm{H}$ is right-continuous at a fixed $t = t_1 > 0$ we take $t_2 > t_1$ and examine the expression:

$$\begin{aligned}&\int_0^{t_2} \mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau - \int_0^{t_1} \mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau \\ &= \int_0^{t_1} \mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau - \int_0^{t_1} \mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau \\ &\quad + \int_{t_1}^{t_2} \mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau.\end{aligned} \tag{4.8}$$

In the second line, the first integral tends to the second one as $t_2 \longrightarrow t_1$. Indeed, recalling that for an EXS analytic semigroup there exist C, $\epsilon > 0$ such that $\|\mathcal{A}S(t)\|_{\mathbf{L}(\mathrm{H})} \le C\frac{e^{-\epsilon t}}{t}$, we obtain

$$\begin{aligned}&\|\mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)] - \mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{H}}\\ &= \|\mathcal{A}S(t_1-\tau)\left\{[S(t_2-t_1)-I]\mathcal{D}f[y^{\mathrm{c}}(\tau)]\right\}\|_{\mathrm{H}} \le \frac{C}{t_1-\tau}\,\|[S(t_2-t_1)-I]\mathcal{D}f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{H}}\,.\end{aligned}$$

Hence, by the strong continuity of $\{S(t)\}_{t\ge 0}$, $\mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]$ strongly tends to $\mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]$ for almost all $\tau \in [0, t_1)$.

Next, $\|\mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{H}}$ is majorized by a constant multiplied by $\|f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{U}}$ because $\|\mathcal{A}S(t_2-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}$ is bounded on $[0, t_1]$ as $t_2-\tau$ is separated from 0.

The needed result follows from the Lebesgue dominated convergence theorem as $u^{\mathrm{c}} \in \mathrm{L}^2(0,\infty;\mathrm{U})$ implies $u^{\mathrm{c}} \in \mathrm{L}^1(0,t_1;\mathrm{U})$.

Employing (4.7), (4.1), (4.2), (4.6) and (3.6) we get

$$\begin{aligned}&\|\mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{H}} \le \|\mathcal{A}S(t_2-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{U}}\\ &\le (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|\mathcal{A}S(t_2-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\left[\|(\mathbf{\Psi}x_0)(\tau)\|_{\mathrm{Y}} + \|(\mathbb{F}u^{\mathrm{c}})(\tau)\|_{\mathrm{Y}}\right]\\ &\le (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|\mathcal{A}S(t_2-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|H^*\mathcal{A}S(\tau)\|_{\mathbf{L}(\mathrm{H},\mathrm{Y})}\|x_0\|_{\mathrm{H}}\\ &\quad + (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|\mathcal{A}S(t_2-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|\mathbb{F}u^{\mathrm{c}}\|_{\mathrm{BUC}_0([0,\infty);\mathrm{Y})}\\ &\le (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\left[\frac{M_d}{(t_2-\tau)^\alpha}\frac{M_h}{\tau^{1-\alpha}} + \frac{M_d}{(t_2-\tau)^\alpha}\gamma_2 m\gamma_1\right]\|x_0\|_{\mathrm{H}},\end{aligned}$$

where M_d and M_h are majorants of $\eta_d(t)e^{-\delta t}$ and $\eta_h(t)e^{-\delta t}$, respectively.

Hence

$$\begin{aligned}&\left\|\int_{t_1}^{t_2}\mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau\right\|_{\mathrm{H}}\\ &\le (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|x_0\|_{\mathrm{H}}M_dM_h\int_{t_1}^{t_2}\frac{\mathrm{d}\tau}{(t_2-\tau)^\alpha\tau^{1-\alpha}}\\ &\quad + (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|x_0\|_{\mathrm{H}}M_d\gamma_2 m\gamma_1\int_{t_1}^{t_2}\frac{\mathrm{d}\tau}{(t_2-\tau)^\alpha}\\ &= (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|x_0\|_{\mathrm{H}}M_dM_h\int_{\frac{t_1}{t_2}}^{1}\frac{\mathrm{d}r}{(1-r)^\alpha r^{1-\alpha}}\\ &\quad + (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|x_0\|_{\mathrm{H}}M_d\gamma_2 m\gamma_1\frac{1}{1-\alpha}(t_2-t_1)^{1-\alpha} \longrightarrow 0 \quad \text{as} \quad t_2 \to t_1.\end{aligned}$$

To prove the left-continuity at a fixed $t = t_1 > 0$ we take $t_2 \in (0, t_1)$ and examine the expression

$$\begin{aligned}
&\int_0^{t_1} \mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau - \int_0^{t_2} \mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau \\
&= \int_0^{t_2} \mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau - \int_0^{t_2} \mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau \\
&\quad + \int_{t_2}^{t_1} \mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau.
\end{aligned}$$

In the second line, the first integral tends to the second one as $t_2 \to t_1$. Indeed, recalling once more the estimate $\|\mathcal{A}S(t)\|_{\mathbf{L}(\mathrm{H})} \le C\frac{e^{-\epsilon t}}{t}$, one obtains

$$\begin{aligned}
&\|\mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)] - \mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{H}} \\
&= \|\mathcal{A}S(t_2-\tau)\left\{[S(t_1-t_2)-I]\mathcal{D}f[y^{\mathrm{c}}(\tau)]\right\}\|_{\mathrm{H}} \\
&\le \frac{C}{t_2-\tau}\left\|[S(t_1-t_2)-I]\mathcal{D}f[y^{\mathrm{c}}(\tau)]\right\|_{\mathrm{H}}.
\end{aligned}$$

Hence, by the strong continuity of $\{S(t)\}_{t\ge 0}$, $\mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]$ strongly tends to $\mathcal{A}S(t_2-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]$ for almost all $\tau \in [0, t_2)$.

Next, $\|\mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{H}}$ is majorized by a constant multiplied by $\|f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{U}}$ because $\|\mathcal{A}S(t_1-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}$ is bounded on $[0, t_2]$ as $t_1-\tau$ is separated from 0.

The needed result follows from the Lebesgue dominated convergence theorem as $u^{\mathrm{c}} \in \mathrm{L}^2(0,\infty;\mathrm{U})$ implies $u^{\mathrm{c}} \in \mathrm{L}^1(0, t_2;\mathrm{U})$.

Employing (4.7), (4.1), (4.2), (4.6) and (3.6) we get

$$\begin{aligned}
&\|\mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{H}} \\
&\le \|\mathcal{A}S(t_1-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|f[y^{\mathrm{c}}(\tau)]\|_{\mathrm{U}} \\
&\le (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|\mathcal{A}S(t_1-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\left[\|(\mathbf{\Psi}x_0)(\tau)\|_{\mathrm{Y}} + \|(\mathbb{F}u^{\mathrm{c}})(\tau)\|_{\mathrm{Y}}\right] \\
&\le (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|\mathcal{A}S(t_1-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|H^*\mathcal{A}S(\tau)\|_{\mathbf{L}(\mathrm{H},\mathrm{Y})}\|x_0\|_{\mathrm{H}} \\
&\quad + (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|\mathcal{A}S(t_1-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|\mathbb{F}u^{\mathrm{c}}\|_{\mathrm{BUC}_0([0,\infty);\mathrm{Y})} \\
&\le (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\left[\frac{M_d}{(t_1-\tau)^\alpha}\frac{M_h}{\tau^{1-\alpha}} + \frac{M_d}{(t_1-\tau)^\alpha}\gamma_2 m\gamma_1\right]\|x_0\|_{\mathrm{H}},
\end{aligned}$$

whence

$$\left\|\int\limits_{t_2}^{t_1} \mathcal{A}S(t_1-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau\right\|_{\mathrm{H}}$$

$$\leq (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|x_0\|_{\mathrm{H}}M_d M_h \int\limits_{t_2}^{t_1} \frac{\mathrm{d}\tau}{(t_1-\tau)^{\alpha}\tau^{1-\alpha}}$$

$$+ (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|x_0\|_{\mathrm{H}}M_d\gamma_2 m\gamma_1 \int\limits_{t_2}^{t_1} \frac{\mathrm{d}\tau}{(t_1-\tau)^{\alpha}}$$

$$= (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|x_0\|_{\mathrm{H}}M_d M_h \int\limits_{\frac{t_2}{t_1}}^{1} \frac{\mathrm{d}r}{(1-r)^{\alpha}r^{1-\alpha}}$$

$$+ (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})\|x_0\|_{\mathrm{H}}M_d\gamma_2 m\gamma_1 \frac{1}{1-\alpha}(t_1-t_2)^{1-\alpha} \longrightarrow 0 \quad \text{as} \quad t_2 \to t_1.$$

Part 3. Lifting of $\mathrm{C}(0,\infty;\mathrm{H})$*-solutions to* $\mathrm{BC}_0([0,\infty);\mathrm{H})$*-solutions.* The continuity at $t=0$ will follow from an estimate for a solution, valid on $[0,\infty)$, we are going to prove.

We recall once more the estimate

$$\left\|\int\limits_{0}^{t} \mathcal{A}S(t-\tau)\mathcal{D}f[y^{\mathrm{c}}(\tau)]\mathrm{d}\tau\right\|_{\mathrm{H}} \leq (m+\|\mu\|_{\mathbf{L}(\mathrm{Y},\mathrm{U})})[\mathfrak{E}_1(t)+\mathfrak{E}_2(t),$$

$$\mathfrak{E}_1(t) := \int\limits_{0}^{t} \|\mathcal{A}S(t-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|\left(\mathbf{\Psi}x_0\right)(\tau)\|_{\mathrm{Y}}\mathrm{d}\tau$$

$$\mathfrak{E}_2(t) := \int\limits_{0}^{t} \|\mathcal{A}S(t-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|\left(\mathbb{F}u^{\mathrm{c}}\right)(\tau)\|_{\mathrm{Y}}\mathrm{d}\tau,$$

which will be interpreted here in a slightly different manner.

Recall that, by Part 1, $\mathbb{F}u^{\mathrm{c}} \in \mathrm{BUC}_0([0,\infty);\mathrm{Y})$ with $\|\mathbb{F}u^{\mathrm{c}}\|_{\mathrm{BUC}_0([0,\infty),\mathrm{Y})} \leq \gamma_2 m\gamma_1$.

Let $\mathrm{BUC}_{00}[0,\infty)$ be a closed subspace of $\mathrm{BUC}_0[0,\infty)$ of functions vanishing at 0. By (4.1) and the standard convolution result ([3, Proposition 0.2.1], where it is shown only that $a \in \mathrm{L}^1(0,\infty)$, $b \in \mathrm{BUC}_0[0,\infty)$ implies $a \star b \in \mathrm{BUC}_0[0,\infty)$; however in this case

$$|(a\star b)(t)| \leq \|b\|_{\mathrm{BUC}_0[0,\infty)} \int\limits_{0}^{t} |a(\tau)|\mathrm{d}\tau \to 0 \quad \text{as} \quad t \to 0$$

because $a \in \mathrm{L}^1(0,\infty)$)

$$t \longmapsto \|\mathcal{A}S(t)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})} \in \mathrm{L}^1(0,\infty), \quad \mathrm{L}^1(0,\infty)\star\mathrm{BUC}_0[0,\infty) \subset \mathrm{BUC}_{00}[0,\infty) \quad (4.9)$$

and therefore

$$\mathfrak{E}_2 \in \mathrm{BUC}_{00}[0,\infty); \quad |\mathfrak{E}_2(t)| \begin{aligned}[t] &\leq \|\mathcal{A}S(\cdot)\mathcal{D}\|_{\mathrm{L}^1(0,\infty;\mathbf{L}(\mathrm{U},\mathrm{H}))} \|\mathbb{F}u^{\mathrm{c}}\|_{\mathrm{BUC}_0([0,\infty);\mathrm{Y})} \\ &\leq \|\mathcal{A}S(\cdot)\mathcal{D}\|_{\mathrm{L}^1(0,\infty;\mathbf{L}(\mathrm{U},\mathrm{H}))}\gamma_2 m\gamma_1\|x_0\|_{\mathrm{H}}. \end{aligned} \tag{4.10}$$

Passing to examination of $\mathfrak{E}_1$, we consider the sequence of scalar functions $\{\mathfrak{e}_n\}_{n\in\mathbb{N}}$,

$$\mathfrak{e}_n(t) := \int\limits_0^t \|\mathcal{A}S(t-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|\left(\mathbf{\Psi}x_n\right)(\tau)\|_{\mathrm{Y}}\mathrm{d}\tau,$$

generated by the sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(\mathcal{A})$, $\|x_n - x_0\|_{\mathrm{H}} \longrightarrow 0$ as $n \to \infty$, where the last sequence exists by density of $D(\mathcal{A})$ in H. Because

$$(\mathbf{\Psi}x_n)(t) = H^*\mathcal{A}S(t)x_n = H^*S(t)\mathcal{A}x_n, \quad t > 0,$$

then, by EXS, one has: $\mathbf{\Psi}x_n \in \mathrm{BUC}_0([0,\infty),\mathrm{Y})$ and consequently $\|\mathbf{\Psi}x_n\|_{\mathrm{Y}} \in \mathrm{BUC}_0[0,\infty)$. Employing (4.9) again, we conclude that $\{\mathfrak{e}_n\}_{n\in\mathbb{N}} \subset \mathrm{BUC}_{00}[0,\infty)$.

On the other side this sequence converges in $\mathrm{L}^\infty(0,\infty)$ to $\mathfrak{e}_\infty$,

$$\mathfrak{e}_\infty(t) := \int\limits_0^t \|\mathcal{A}S(t-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|\left(\mathbf{\Psi}x_0\right)(\tau)\|_{\mathrm{Y}}\mathrm{d}\tau,$$

where $t \geq 0$ is a Lebesgue point of

$$\mathfrak{e}_\infty \in \mathrm{L}^1(0,\infty)\cap\mathrm{L}^2(0,\infty) \quad \text{as} \quad \mathrm{L}^1(0,\infty)\star[\mathrm{L}^1(0,\infty)\cap\mathrm{L}^2(0,\infty)] \subset [\mathrm{L}^1(0,\infty)\cap\mathrm{L}^2(0,\infty)].$$

Now,

$$\begin{aligned} |\mathfrak{e}_n(t) - \mathfrak{e}_\infty(t)| &\leq \int\limits_0^t \|\mathcal{A}S(t-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})} \left|\|\left(\mathbf{\Psi}x_n\right)(\tau)\|_{\mathrm{Y}} - \|\left(\mathbf{\Psi}x_0\right)(\tau)\|_{\mathrm{Y}}\right| \mathrm{d}\tau \\ &\leq \int\limits_0^t \|\mathcal{A}S(t-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|\left(\mathbf{\Psi}x_n\right)(\tau) - \left(\mathbf{\Psi}x_0\right)(\tau)\|_{\mathrm{Y}}\mathrm{d}\tau \\ &\leq \|x_n - x_0\|_{\mathrm{H}}\mathfrak{E}_3(t), \\ \mathfrak{E}_3(t) &:= \int\limits_0^t \|\mathcal{A}S(t-\tau)\mathcal{D}\|_{\mathbf{L}(\mathrm{U},\mathrm{H})}\|H^*\mathcal{A}S(\tau)\|_{\mathbf{L}(\mathrm{H},\mathrm{Y})}\mathrm{d}\tau. \end{aligned} \tag{4.11}$$

Applying (4.2) and (4.1) to $\mathfrak{E}_3$ one obtains

$$\mathfrak{E}_3(t) \le \int_0^t \eta_d(t-\tau)\frac{e^{-\delta(t-\tau)}}{(t-\tau)^\alpha}\eta_h(\tau)\frac{e^{-\delta\tau}}{\tau^{1-\alpha}}\mathrm{d}\tau$$
$$\le \eta_d(t)\eta_h(t)e^{-\delta t}\int_0^t \frac{\mathrm{d}\tau}{(t-\tau)^\alpha\tau^{1-\alpha}}$$
$$= \eta_d(t)\eta_h(t)e^{-\delta t}\int_0^1 \frac{\mathrm{d}\xi}{(1-\xi)^\alpha\xi^{1-\alpha}} = \eta_d(t)\eta_h(t)e^{-\delta t}\frac{\pi}{\sin\pi\alpha},$$

as the last integral equals the *Beta-function*

$$B(1-\alpha,\alpha) = \Gamma(1-\alpha)\Gamma(\alpha) = \frac{\pi}{\sin\pi\alpha}.$$

Hence $\mathfrak{E}_3$ is bounded. Because $\mathrm{BUC}_{00}[0,\infty)$ is a closed subspace of $\mathrm{L}^\infty(0,\infty)$ we obtain: $\mathfrak{e}_\infty \in \mathrm{BUC}_{00}[0,\infty)$. The same arguments which lead to (4.11) show that

$$\mathfrak{E}_1(t) \le \|x_0\|_{\mathrm{H}}\eta_d(t)\eta_h(t)e^{-\delta t}\frac{\pi}{\sin\pi\alpha}. \tag{4.12}$$

Now (4.10), (4.12) jointly with EXS imply that 0 is GAS. Moreover, $\mathrm{H} \ni x_0 \longmapsto x^{\mathrm{c}} \in \mathrm{BC}_0([0,\infty);\mathrm{H})$ and $x^{\mathrm{c}}(0) = x_0$, so x^{c} is a weak solution of the open-loop system with control $u^{\mathrm{c}}(t) = -f[y^{\mathrm{c}}(t)]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x^{\mathrm{c}}(t), w\rangle_{\mathrm{H}} = \langle x^{\mathrm{c}}(t) - \mathcal{D}f[\mathcal{C}x^{\mathrm{c}}(t)], \mathcal{A}^*w\rangle_{\mathrm{H}}, \quad w \in D(\mathcal{A}^*). \tag{4.13}$$

□

5. EXAMPLE: UNLOADED ELECTRIC $\mathfrak{RC}$ TRANSMISSION LINE

Consider the system, depicted in Figure 2, consisting of a plant – an *unloaded electric* $\mathfrak{RC}$ *transmission line* and a block (controller) having the voltage to voltage static characteristic $f : \mathbb{R} \to \mathbb{R}$. The plant dynamics is governed by the equations

$$\begin{cases} 0 = -V_\theta(\theta,\tau) - \mathfrak{R}I(\theta,\tau), & \tau \ge 0, \quad 0 \le \theta \le 1, \\ \mathfrak{C}V_\tau(\theta,\tau) = -I_\theta(\theta,\tau), & \tau \ge 0, \quad 0 \le \theta \le 1, \\ I(1,\tau) = 0, & \tau \ge 0, \\ \mathfrak{U}(\tau) = V(0,\tau), & \tau \ge 0, \\ \mathfrak{Y}(\tau) = V(1,\tau), & \tau \ge 0, \end{cases}$$

where $V_\theta(\theta,\tau)$ and $I(\theta,\tau)$ denote, respectively, voltage and current at the spatial point θ and time τ, $\mathfrak{U}$ represents the control, $\mathfrak{Y}$ is the output, $\mathfrak{R}$ and $\mathfrak{C}$ stand for the resistance and capacity of the line for the unit length.

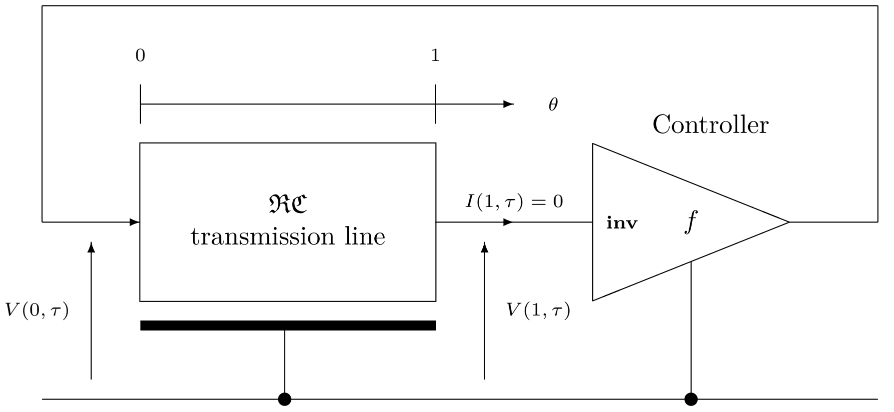

Fig. 2. Proportional feedback control of $\mathfrak{RC}$ transmission line

Time rescaling $x(\theta,t) := V(\theta,\mathfrak{RC}t)$, $u(t) := \mathfrak{U}(\mathfrak{RC}t)$ and $y(t) := \mathfrak{Y}(\mathfrak{RC}t)$ yields

$$\begin{cases} x_t(\theta,t) = x_{\theta\theta}(\theta,t) & t\geq 0, \quad 0\leq\theta\leq 1,\\ x_\theta(1,t)=0, & t\geq 0,\\ u(t)=x(0,t), & t\geq 0,\\ y(t)=x(1,t), & t\geq 0. \end{cases} \tag{5.1}$$

In the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$ with standard scalar product, the dynamics (5.1) can be written in the preliminary abstract form

$$\begin{cases} \dot{x}=\sigma x,\\ \tau x = u,\\ y = c^{\#}x, \end{cases} \tag{5.2}$$

with

$$\begin{array}{ll} \sigma x = x'', & D(\sigma)=\left\{x\in \mathrm{H}^2(0,1):\ x'(1)=0\right\};\\ \tau x = x(0), & D(\tau)=\mathrm{C}[0,1]\supset D(\sigma) \end{array}$$

and σ is a closed linear operator; the observation functional $\mathcal{C} = c^{\#}$ is given by

$$c^{\#}x = x(1), \quad D(c^{\#}) = \mathrm{C}[0,1]. \tag{5.3}$$

In order to transform (5.2), (5.3) into its abstract form (2.1) we proceed as follows. From the relationships:

$$d\in D(\sigma), \quad \sigma d = 0; \quad \tau d = -1$$

we determine a factor control vector d,

$$\left.\begin{array}{r} d''(\theta) = \ \ 0\\ d'(1) = \ \ 0\\ d(0) = -1 \end{array}\right\} \iff d = -\mathbf{1}\in \mathrm{L}^2(0,1); \quad \mathbf{1}(\theta)=1,\ 0\leq\theta\leq 1.$$

Thanks to this

$$\tau[x(t) + du(t)] = \tau x(t) + \tau du(t) = \tau x(t) - u(t) = 0,$$

i.e., $x(t) + du(t) \in \ker \tau$. Next,

$$\dot{x}(t) = \sigma x(t) = \sigma x(t) + \sigma du(t) = \sigma[x(t) + du(t)] = \mathcal{A}[x(t) + du(t)],$$

provided that $\mathcal{A} := \sigma|_{\ker \tau}$, here given by

$$\mathcal{A}x = x'', \quad D(\mathcal{A}) = \{x \in \mathrm{H}^2(0,1) : \ x'(1) = 0, \ x(0) = 0\}; \tag{5.4}$$

$\mathcal{A} = \mathcal{A}^* < 0$ with the inverse,

$$\left(\mathcal{A}^{-1}v\right)(\theta) = \int_0^1 \begin{Bmatrix} -\theta, & \text{if} \quad \theta < \vartheta \\ -\vartheta, & \text{if} \quad \theta > \vartheta \end{Bmatrix} v(\vartheta)\mathrm{d}\vartheta, \tag{5.5}$$

which is a HS operator. Thus, by discrete version of the spectral theorem, the spectrum of $\mathcal{A}$ consists of countably many eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}^*}$, $\mathbb{Z}^* := \mathbb{N} \cup \{0\}$ and there exists a system of corresponding eigenvectors $\{e_n\}_{n\in\mathbb{Z}^*}$ being an orthonormal basis of H,

$$\left.\begin{aligned} \lambda_n &= -\left(\frac{\pi}{2} + n\pi\right)^2 \\ e_n(\theta) &= \sqrt{2}\sin\left(\frac{\pi}{2} + n\pi\right)\theta, \quad 0 \le \theta \le 1 \end{aligned}\right\}, \quad n \in \mathbb{Z}^*.$$

$\mathcal{A}$ generates on H an analytic, self-adjoint semigroup $\{S(t)\}_{t\ge 0}$,

$$S(t)z = \sum_{n=0}^{\infty} e^{\lambda_n t}\langle z, e_n\rangle_{\mathrm{H}} e_n, \quad z \in \mathrm{H}, \ t \ge 0.$$

This semigroup is EXS as here, by *Parseval's identity*, (2.2) holds with $M = 1$ and $\alpha = -\lambda_0 = \frac{\pi^2}{4}$.

We have $c^{\#}\mathcal{A}^{-1}x = \langle x, h\rangle_{\mathrm{H}}$, whence $h(\theta) = -\theta$, $\theta \in [0,1]$. Similarly

$$\langle \mathcal{A}x, d\rangle_{\mathrm{H}} = -\int_0^1 x''(\theta)\mathrm{d}\theta = x'(0)$$

and therefore $d^*\mathcal{A}^* = d^*\mathcal{A}$ extends to

$$d^{\#}x = x'(0), \quad D(d^{\#}) = \mathrm{C}^1[0,1] \ni h, \quad d^{\#}h = -1 = c^{\#}d.$$

Notice that

$$c^{\#}e_n = e_n(1) = (-1)^n\sqrt{2}, \quad d^{\#}e_n = e_n'(0) = \sqrt{2}\sqrt{-\lambda_n}.$$

Since

$$x(\theta) = (sI - \mathcal{A})^{-1}d = \frac{\cosh\sqrt{s}(1-\theta) - \cosh\sqrt{s}}{s\cosh\sqrt{s}}, \tag{5.6}$$

we find using (2.6)

$$\hat{G}(s) = sc^{\#}(sI - \mathcal{A})^{-1}d - c^{\#}d = \frac{1}{\cosh\sqrt{s}}, \quad s \notin \{\lambda_n\}_{n\in\mathbb{Z}^*}. \tag{5.7}$$

Because for $s \in \overline{\mathbb{C}^+}$ there holds

$$\left|\cosh\sqrt{s}\right|^2 = \sinh^2\sqrt{\frac{|s| + \operatorname{Re} s}{2}} + 1 - \sin^2\sqrt{\frac{|s| - \operatorname{Re} s}{2}} \geq \sinh^2 z + 1 - \sin^2 z, \tag{5.8}$$

where $z = \sqrt{\frac{|s|-\operatorname{Re} s}{2}} \geq 0$, then $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+)$ with the norm $\|\hat{G}\|_{\mathrm{H}^\infty(\mathbb{C}^+)} = 1$ achieved at $s = 0$.

Boundedness of $\hat{G}$ on $j\mathbb{R}$ is confirmed by the *Nyquist curve* depicted in Figure 3, determining the *spectrum* $\sigma(\mathbb{F}) = \overline{\hat{G}(\mathbb{C}^+)}$.

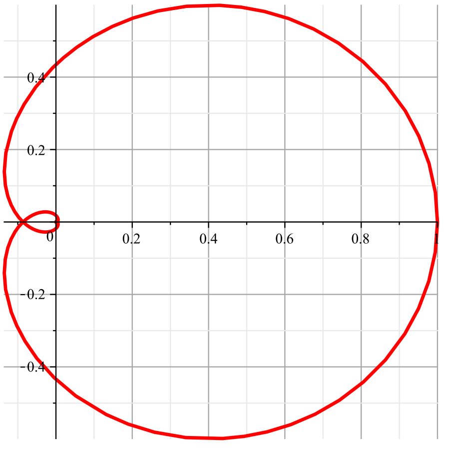

Fig. 3. The Nyquist curve $\{\hat{G}(j\omega)\}_{\omega\in\mathbb{R}}$; $\hat{G}$ given by (5.7)

To show that $\hat{G} \in \mathrm{H}^2(\mathbb{C}^+)$ we continue the estimate (5.8):

$$\left|\cosh\sqrt{s}\right|^2 \geq \sinh^2 z + 1 - \sin^2 z \geq \sinh^2 z \geq \frac{z^6}{36},$$

where $z = \sqrt{\frac{|s|-\operatorname{Re} s}{2}} \geq 0$ and $\operatorname{Re} s \geq 0$, whence

$$|\hat{G}(s)|^2 \leq \frac{288}{(|s| - \operatorname{Re} s)^3} \leq \frac{288}{(|\operatorname{Im} s| - \delta)^3}, \quad 0 \leq \operatorname{Re} s \leq \delta, \ |\operatorname{Im} s| > \delta,$$

and consequently, as $|\hat{G}(s)| \leq 1$ on $\overline{\mathbb{C}^+}$,

$$\int_{-\infty}^{\infty} |\hat{G}(s)|^2 \mathrm{d}[\operatorname{Im} s] < \infty, \quad 0 \leq \operatorname{Re} s \leq \delta.$$

From [9, Problem 2, p. 134], we get $\hat{G} \in \mathrm{H}^2(\mathbb{C}^+)$.

Lemma 5.1. *The pair* (h, d) *satisfies the conjugate Balakrishnan–Washburn estimates* (4.2) *and* (4.1) *with*

$$\alpha = \frac{1}{4}, \quad \delta = -\lambda_0, \quad \eta_h(t) := \sqrt{2}\sqrt{\sqrt{t}+1}, \quad \eta_d(t) := \frac{\pi}{\sqrt{2}}\sqrt{t\sqrt{t}+1}.$$

Proof. To prove the estimate (4.2) we use successively

$$\lambda_n - \lambda_0 \le -\pi^2 n^2, \quad n \in \mathbb{N}; \quad \int_0^\infty e^{-y^2}\mathrm{d}y = \frac{\sqrt{\pi}}{2},$$

getting

$$\begin{aligned}\sum_{n=0}^\infty |\langle \mathcal{A}S(t)h, e_n\rangle_{\mathrm{H}}|^2 &= \sum_{n=0}^\infty e^{2\lambda_n t}|c^{\#}e_n|^2 = 2\sum_{n=0}^\infty e^{2\lambda_n t} \\ &= 2e^{2\lambda_0 t}\left[1+\sum_{n=1}^\infty e^{2(\lambda_n-\lambda_0)t}\right] \le 2e^{2\lambda_0 t}\left[1+\sum_{n=1}^\infty e^{-2\pi^2 n^2 t}\right] \\ &\le 2e^{2\lambda_0 t}\left[1+\int_0^\infty e^{-2\pi^2 n^2 t}\mathrm{d}n\right] = \frac{2\sqrt{2\pi t}+1}{\sqrt{2\pi t}}\, e^{2\lambda_0 t}.\end{aligned}$$

Hence

$$\|\mathcal{A}S(t)h\|_{\mathrm{H}}^2 \le 2\frac{\sqrt{t}+1}{\sqrt{t}}\, e^{2\lambda_0 t}, \quad t \ge 0. \tag{5.9}$$

To prove the estimate (4.1) we need, in addition,

$$\frac{\lambda_n}{\lambda_0} \le 9n^2, \quad n \in \mathbb{N}; \quad xe^{-x} \le e^{-1}, \quad x \ge 0,$$

$$\begin{aligned}\|\mathcal{A}S(t)d\|_{\mathrm{H}}^2 &= \sum_{n=0}^\infty |\langle \mathcal{A}S(t)d, e_n\rangle_{\mathrm{H}}|^2 = -\sum_{n=0}^\infty 2\lambda_n e^{2\lambda_n t} \\ &= -2\lambda_0 e^{2\lambda_0 t}\left[1+\sum_{n=1}^\infty \frac{\lambda_n}{\lambda_0}\, e^{2(\lambda_n-\lambda_0)t}\right] \\ &\le -2\lambda_0 e^{2\lambda_0 t}\left[1+\frac{9}{\pi^2 t}\sum_{n=1}^\infty \pi^2 n^2 t\, e^{-2\pi^2 n^2 t}\right] \\ &\le -2\lambda_0 e^{2\lambda_0 t}\left[1+\frac{9}{\pi^2 et}\sum_{n=1}^\infty e^{-\pi^2 n^2 t}\right] \\ &\le -2\lambda_0 e^{2\lambda_0 t}\left[1+\frac{9}{\pi^2 et}\int_0^\infty e^{-\pi^2 n^2 t}\mathrm{d}n\right] = e^{2\lambda_0 t}\,\frac{2\pi^2 et\sqrt{\pi t}+9}{4et\sqrt{\pi t}}.\end{aligned}$$

Hence

$$\|\mathcal{A}S(t)d\|_{\mathrm{H}}^2 \le \frac{\pi^2}{2}\,\frac{t\sqrt{t}+1}{t\sqrt{t}}\,e^{2\lambda_0 t}, \quad t \ge 0. \tag{5.10}$$

□

The following result is borrowed from [8, Lemma 5.2, p. 27 and p. 33].

Lemma 5.2. *The factor control vector d given by* (5.3) *is not admissible.*

Remark 5.3. Taking

$$x_0 = 0 \quad \text{and} \quad u(t) = \chi_{[0,T]}(t)\frac{1}{(T-t)^\alpha}, \quad T>0,\ \alpha \in \left[\frac{1}{4},\frac{1}{2}\right),$$

we have $u \in \mathrm{L}^2(0,\infty)$ and we get

$$\begin{aligned}
\|x(T)\|_{\mathrm{H}}^2 &= \|\mathbf{\Phi} R_T u\|_{\mathrm{H}}^2 = \sum_{n=0}^{\infty} |\langle \mathbf{\Phi} R_T u, e_n\rangle_{\mathrm{H}}|^2 \\
&= \sum_{n=0}^{\infty} |\langle u, R_T \mathbf{\Phi}^* e_n\rangle_{\mathrm{L}^2(0,\infty)}|^2 = \sum_{n=0}^{\infty} \left| \int_0^T \frac{1}{(T-t)^\alpha} d^* \mathcal{A}S(T-t)e_n \mathrm{d}t \right|^2 \\
&= \sum_{n=0}^{\infty} \left| e_n'(0) \int_0^T \frac{e^{\lambda_n(T-t)}}{(T-t)^\alpha}\mathrm{d}t \right|^2 = 2\sum_{n=0}^{\infty}(-\lambda_n)\left[\int_0^T \frac{e^{\lambda_n t}}{t^\alpha}\mathrm{d}t\right]^2 = \infty
\end{aligned}$$

because [12, p. 55, last line in the proof of Proposition 2.1 with $\lambda = -\alpha$ and $x = -\lambda_n$]

$$\int_0^T \frac{e^{\lambda_n t}}{t^\alpha}\mathrm{d}t = \frac{\Gamma(1-\alpha)}{(-\lambda_n)^{(1-\alpha)}} + o\left(e^{T\lambda_n}\right) \quad \text{as} \quad n \to \infty.$$

Remark 5.3 shows that (2.1) corresponding to the electric $\mathfrak{RC}$-transmission line does not have a weak solution for an *arbitrary* $u \in \mathrm{L}^2(0,\infty)$.

Next,

$$\hat{G}(0) = 1, \quad \hat{G}(\pm 2\pi^2 j) = -\frac{1}{\cosh \pi}$$

and therefore

$$-\frac{1}{\mu} \notin \sigma(\mathbb{F}) \cap \mathbb{R} = \overline{\hat{G}(\mathbb{C}^+)} \cap \mathbb{R} \iff \mu \in (-1, \cosh \pi), \tag{5.11}$$

so, here (3.3) holds for this range of μ.

Remark 5.4. It has been proved in [5], using the *Riesz basis* approach, and in [7, Section 5.2], using a *perturbation technique*, that for this range of μ the linear closed-loop system arising by taking $f(y) = \mu y$ generates an EXS analytic semigroup on H.

The output of transmission line is plugged in the sign inverting input of a controller, whence $u = -f(y)$ is a negative feedback control, e.g., if the nonlinear block is an *operational amplifier* one has $f(y) := M\,\mathrm{sat}(\frac{\kappa}{M}y)$, where sat is the classical *saturation* function and κ denotes the *gain coefficient* of an amplifier in the linear range.

Theorem 4.2 asserts that GAS of the origin holds for f satisfying (3.1), where the Lipschitz constant m for $f(y) - \mu y$ has to be determined from (3.5).

(3.5) means geometrically that the *Nyquist plot* $\left\{\frac{\hat{G}(j\omega)}{1+\mu\hat{G}(j\omega)}\right\}_{\omega\in\mathbb{R}}$ is strictly inside the disc with centre 0 and radius $\dfrac{1}{m}$ as depicted in Figures 4–7.

Observe that magnitude of the Lipschitz constant m depends on the choice of μ: the largest $m = m_3$ is being obtained for $\mu = \mu_3$.

An equivalent analytic formulation of the above geometric condition is

$$|1 + \mu\widehat{G}(j\omega)|^2 > m^2|\widehat{G}(j\omega)|^2 > 0, \quad \omega \in \mathbb{R}. \tag{5.12}$$

Since

$$\frac{1}{\cosh(\sqrt{j\omega})} = \begin{cases} \dfrac{1}{\cosh[(1+j)\Omega]} & \text{if } \omega \geq 0, \\ \dfrac{1}{\cosh[(1-j)\Omega]} & \text{if } \omega \leq 0, \end{cases}$$

$$\Omega := \sqrt{\frac{\omega}{2}}, \quad \cosh[(1 \pm j)\Omega] = \cosh\Omega\cos\Omega \pm j\sinh\Omega\sin\Omega,$$

then (5.12) is satisfied iff

$$(0 \leq)\ l(\Omega) := (\mu + \cos\Omega\cosh\Omega)^2 + \sin^2\Omega\sinh^2\Omega > m^2.$$

We have

$$l'(\Omega) = 2(\sinh\Omega\cos\Omega - \cosh\Omega\sin\Omega)[\mu - \mathfrak{h}(\Omega)],$$

where

$$\mathfrak{h}(\Omega) := \frac{\sin 2\Omega - \sinh 2\Omega}{2(\sinh\Omega\cos\Omega - \cosh\Omega\sin\Omega)}$$

and

$$l''(\Omega) = 4[\sinh^2\Omega + \sin^2\Omega - \mu\sin\Omega\sinh\Omega].$$

Hence for $|\mu| \leq 2$, $l''(\Omega) \geq 0$, so l is convex and at $\Omega = 0$ achieves its minimum $l(0) = (\mu + 1)^2 > m^2$.

In particular, (5.12) holds for $\mu = 0$, $m < 1$ and for $\mu = 2$, $m < 3$ as confirmed by Figures 4 and Figure 5, respectively.

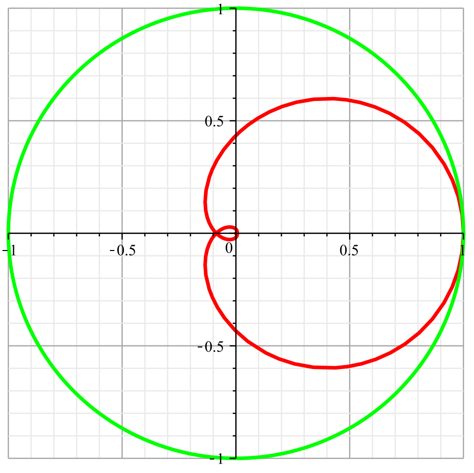

Fig. 4. Verification of (3.5) for $\mu = 0$, $m = 1$

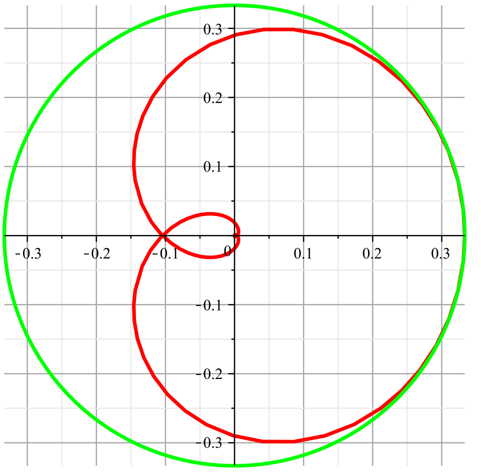

Fig. 5. Verification of (3.5) for $\mu = 2$, $m = 3$

The case of $\mu \in (2, \cosh \pi)$ involves more sophisticated analysis. We start from proving that the function $\mathfrak{h}$ is on the interval $[0, \pi]$ positive and strictly increasing from $\mathfrak{h}(0) = 2$ to $\mathfrak{h}(\pi) = \cosh \pi$; moreover $l'(\Omega) = (>)0 \iff \mu = (>)\mathfrak{h}(\Omega)$. Indeed,

$$\sinh \Omega \cos \Omega - \cosh \Omega \sin \Omega = \cosh \Omega \cos \Omega [\tanh \Omega - \tan \Omega].$$

On $[0, \pi/2)$ there holds

$$\cosh \Omega \cos \Omega > 0 \quad \text{and} \quad \tan \Omega \geq \tanh \Omega$$

with equality sign only for $\Omega = 0$.
On $(\pi/2, \pi]$ there holds

$$\cosh \Omega \cos \Omega < 0 \quad \text{and} \quad \tan \Omega < \tanh \Omega.$$

Thus

$$\sinh \Omega \cos \Omega - \cosh \Omega \sin \Omega < 0, \quad \Omega \in (0, \pi]. \tag{5.13}$$

This implies that $\mathfrak{h}(\Omega) > 0$ on $(0, \pi]$, while $\mathfrak{h}(\Omega) = 2 + \frac{1}{21}\Omega^4$ for small $|\Omega|$.
Now, $\mathfrak{h}'(\Omega) > 0$ on $(0, \pi]$ iff

$$\underbrace{[\cosh 2\Omega - \cos 2\Omega][\cosh \Omega \sin \Omega - \sinh \Omega \cos \Omega]}_{>0} \left[1 - \frac{\mathfrak{f}(\Omega)}{\mathfrak{t}(\Omega)}\right] > 0,$$

where

$$\mathfrak{f}(\Omega) := \frac{\sinh 2\Omega - \sin 2\Omega}{\cosh 2\Omega - \cos 2\Omega}, \quad \mathfrak{t}(\Omega) := \coth \Omega - \cot \Omega,$$

which reduces to proving that

$$\mathfrak{f}(\Omega) < \mathfrak{t}(\Omega) \Leftrightarrow \frac{\sinh 2\Omega - \sin 2\Omega}{\cosh 2\Omega - \cos 2\Omega} < \coth\Omega - \cot\Omega = \frac{\sinh 2\Omega}{\cosh 2\Omega - 1} - \frac{\sin 2\Omega}{1 - \cos 2\Omega},$$

or

$$\frac{\sin^4\Omega}{\sinh^4\Omega} = \left(\frac{1 - \cos 2\Omega}{\cosh 2\Omega - 1}\right)^2 > \frac{\sin 2\Omega}{\sinh 2\Omega} = \frac{\sin\Omega\cos\Omega}{\sinh\Omega\cosh\Omega}.$$

Since $\sin\Omega > 0$ on $(0, \pi)$ we have to prove that

$$\frac{\cosh\Omega}{\sinh^3\Omega} > \frac{\cos\Omega}{\sin^3\Omega} \iff \frac{\mathrm{d}}{\mathrm{d}\Omega}\left[\frac{1}{\sin^2\Omega} - \frac{1}{\sinh^2\Omega}\right] = \frac{\mathrm{d}^2\mathfrak{t}(\Omega)}{\mathrm{d}\Omega^2} > 0. \tag{5.14}$$

Recall well-known expansions

$$\coth\Omega = \frac{1}{\Omega} + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}B_n}{(2n)!}\Omega^{2n-1}, \quad \cot\Omega = \frac{1}{\Omega} - \sum_{n=1}^{\infty}\frac{2^{2n}B_n}{(2n)!}\Omega^{2n-1}, \quad |\Omega| < \pi,$$

where $B_{>}0$, $n \in \mathbb{N}$, are the Bernoulli numbers. Hence we get

$$\begin{aligned}\mathfrak{t}(\Omega) = &\sum_{n=1}^{\infty}\frac{[1 + (-1)^{n-1}]2^{2n}B_n}{(2n)!}\Omega^{2n-1} = \sum_{k=1}^{\infty}\frac{2^{4k-1}B_{2k-1}}{(4k-2)!}\Omega^{4k-3} \\ = &\ \Omega\sum_{k=1}^{\infty}\frac{2^{4k-1}B_{2k-1}}{(4k-2)!}\Omega^{4k-4}, \quad |\Omega| < \pi.\end{aligned}$$

It allows to conclude that (5.14) holds on $\Omega \in (0, \pi)$. Finally, $\mathfrak{h}'$ is positive on $(0, \pi]$.

Now, $l'(\Omega) = 0$ at some $\tilde{\Omega} \in (0, \pi]$ iff the value of $\mathfrak{h}$ exceeds μ at $\tilde{\Omega}$. Furthermore, by (5.13) and the monotonicity of $\mathfrak{h}$, l achieves minimum at $\tilde{\Omega}$ and $\sqrt{l(\tilde{\Omega})} > m$. This is the case of Figure 7, where

$$\mu = \frac{-1 + \cosh\pi}{2}, \quad \tilde{\Omega} \approx 2.652\,432\,102; \quad \sqrt{l(\tilde{\Omega})} \approx 3.463\,645\,509 > m.$$

We can also proceed in another way: since $l'(\Omega) = 0$ iff $\mu = \mathfrak{h}(\Omega)$ it is possible to eliminate μ from definition of l getting the function:

$$r(\Omega) := [\mathfrak{h}(\Omega) + \cosh\Omega\cos(\Omega)]^2 + (\sinh\Omega\sin\Omega)^2.$$

We have

$$\begin{aligned}\frac{1}{2}r'(\Omega) &= [\mathfrak{h}(\Omega) + \cosh\Omega\cos(\Omega)]\mathfrak{h}'(\Omega) + \mathfrak{h}(\Omega)[\sinh\Omega\cos\Omega - \cosh\Omega\sin\Omega] \\ &\quad + \frac{1}{2}\sinh 2\Omega - \frac{1}{2}\sin 2\Omega = [\mathfrak{h}(\Omega) + \cosh\Omega\cos(\Omega)]\mathfrak{h}'(\Omega),\end{aligned}$$

where the last equality holds due to definition of $\mathfrak{h}$. Hence, as $\mathfrak{h}'$ is positive, $r'(\Omega) = 0$ iff $\mathfrak{h}(\Omega) = -\cosh\Omega\cos(\Omega)$. Applying the definition of $\mathfrak{h}$ once more we obtain

$$\begin{aligned} &\mathfrak{h}(\Omega) = -\cosh\Omega\cos(\Omega) \\ \iff\ & -\cosh\Omega\cos(\Omega) = \frac{\sin 2\Omega - \sinh 2\Omega}{2(\sinh\Omega\cos\Omega - \cosh\Omega\sin\Omega)} \\ \iff\ & \Omega = \pi \quad \text{or} \quad \Omega \neq 0, \quad \tanh\Omega = -\tan\Omega. \end{aligned}$$

The last equation has a unique solution in $(0, \pi)$ slightly larger than $\frac{3\pi}{4} \approx 2.356\,194\,490$, namely $\bar{\Omega} \approx 2.365\,020\,372$. The corresponding value of $\sqrt{r}$ provides the maximal possible value of the Lipschitz constant m, $\sqrt{r(\bar{\Omega})} \approx 3.697\,031\,013$. This corresponds to

$$\mu = \mathfrak{h}(\bar{\Omega}) \approx 3.829\,887\,503$$

as confirmed by Figure 6.

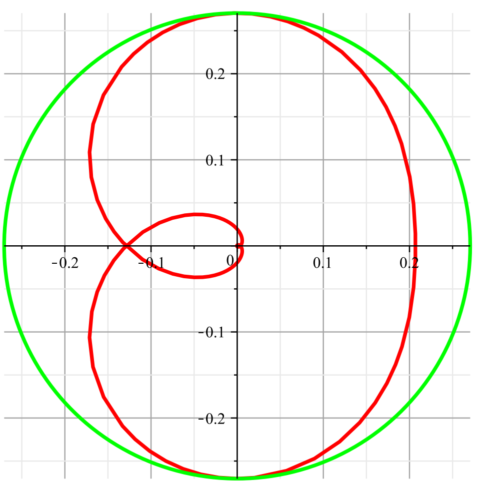

Fig. 6. Checking of (3.5) for $\mu = 3.8298875$ corresponding to $m = 3.6970310$

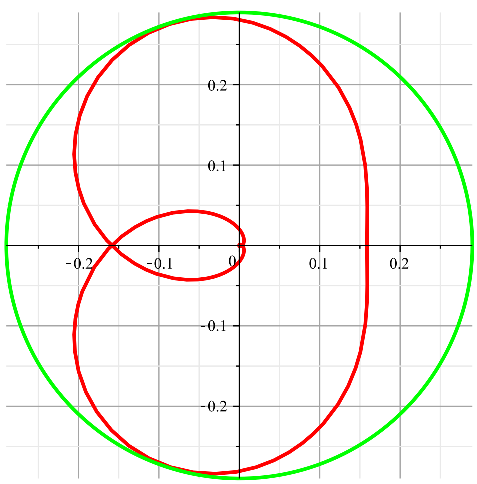

Fig. 7. Checking of (3.5) for $\mu = \frac{-1+\cosh\pi}{2}$, $m = \sqrt{l(\tilde{\Omega})} \approx 3.697031013$

6. DISCUSSION AND CONCLUSIONS

(i) In Example of Section 5 we have $h \in D[(-\mathcal{A})^{\kappa}]$ for $\kappa \in [0, \frac{3}{4})$, whilst (5.9) implies $\mathcal{A}S(\cdot)h \in \mathrm{L}^q(0,\infty;\mathrm{H})$ for $q \in [1, 4)$.

We have $d \in D[(-\mathcal{A})^{\alpha}]$ for $\alpha \in [0, \frac{1}{4})$, whilst (5.10) gives $\mathcal{A}S(\cdot)d \in \mathrm{L}^p(0,\infty;\mathrm{H})$ for $p \in [1, \frac{4}{3})$.

Therefore, $\mathfrak{E}_1$ cannot be estimated using neither Young's inequality for convolution:

$$\|a \star b\|_{\mathrm{L}^\infty(0,\infty)} \le \|a\|_{\mathrm{L}^p(0,\infty)} \|b\|_{\mathrm{L}^q(0,\infty)}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

nor the fractional powers approach.

A result on regularity of $\mathrm{L}^2(0,\infty;\mathrm{H})$-solutions was obtained in [13, Theorem 2.1/case $u_0 \in \mathrm{L}^2(\Omega)$]. It concerns continuity of the state on an interval $[h,T]$, where $h > 0$ and T is arbitrary but finite. Continuity of γ map introduced in [13, p. 348] holds on the Sobolev space $\mathrm{H}^s(\Omega)$, $s \in (0, \frac{1}{2})$, which accordingly to [13, Formula (2.8), p. 351 with $\rho = \frac{1}{4} - \frac{s}{2}$] can be identified with $D[(-\mathcal{A})^{s/2}]$. In Example of Section 5 this would hold if $h \in D[(-\mathcal{A})^{3/4+\varepsilon}]$ which is not the case. Thus our example is beyond the scope of Triggiani's results.

(ii) In Example of Section 5: the fact that $j\omega \mapsto \hat{G}(j\omega) \in \mathrm{L}^2(j\mathbb{R})$, needed while verifying the assumption (ii) of Theorem 4.2, can be proved directly. Indeed, with $\Omega := \sqrt{\omega/2}$ one has

$$\begin{aligned}
\frac{1}{8}\|\hat{G}\|^2_{\mathrm{L}^2(j\mathbb{R})} &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega}{8\left|\cosh(\sqrt{j\omega})\right|^2} = \int_0^\infty \frac{\mathrm{d}\omega}{4\left|\cosh\left[(1+j)\sqrt{\frac{\omega}{2}}\right]\right|^2} \\
&= \int_0^\infty \frac{\Omega \mathrm{d}\Omega}{\cos^2\Omega + \sinh^2\Omega} \\
&= \int_0^{\ln(1+\sqrt{2})} \frac{\Omega \mathrm{d}\Omega}{\cos^2\Omega + \sinh^2\Omega} + \int_{\ln(1+\sqrt{2})}^\infty \frac{\Omega \mathrm{d}\Omega}{\cos^2\Omega + \sinh^2\Omega} \\
&\le \int_0^{\ln(1+\sqrt{2})} \frac{\Omega \mathrm{d}\Omega}{\cos^2\Omega} + \int_{\ln(1+\sqrt{2})}^\infty \frac{\Omega \mathrm{d}\Omega}{\sinh^2\Omega} \\
&= [\Omega\tan\Omega + \ln\cos\Omega]_0^{\ln(1+\sqrt{2})} - [\Omega\coth\Omega - \ln\sinh\Omega]_{\ln(1+\sqrt{2})}^\infty < \infty.
\end{aligned}$$

(iii) Our results have to be compared with these which could be drawn from the theory of nonlinear semigroups [14, Chapter III, Section 5, especially Theorem 5.2/(iv), p. 122].

Here we limit ourselves to examining (incremental) *dissipativity* of the closed-loop operator $\mathcal{A}^{\mathrm{c}}$, given by (4.4), with respect to, generally nonequivalent, scalar product $\langle x, y\rangle_{\mathcal{H}} := \langle x, \mathcal{H}y\rangle_{\mathrm{H}}$, dictated by an operator $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* > 0$, i.e., $x^*\mathcal{H}x > 0$ for any $x \neq 0$.

Let

$$\Delta x = x_1 - x_2, \quad x_1,\ x_2 \in D(\mathcal{C}), \quad \Delta f = f(\mathcal{C}x_1) - f(\mathcal{C}x_2)$$

and

$$\Delta x - \mathcal{D}\Delta f \in D(\mathcal{A}).$$

Then

$$\begin{aligned}
&\langle \Delta\mathcal{A}^{\mathrm{c}}, \Delta x\rangle_{\mathcal{H}} + \langle \Delta x, \Delta\mathcal{A}^{\mathrm{c}}\rangle_{\mathcal{H}} = \langle \Delta\mathcal{A}^{\mathrm{c}}, \mathcal{H}\Delta x\rangle_{\mathrm{H}} + \langle \mathcal{H}\Delta x, \Delta\mathcal{A}^{\mathrm{c}}\rangle_{\mathrm{H}}\\
&= \langle \mathcal{A}(\Delta x - \mathcal{D}\Delta f), \mathcal{H}\Delta x\rangle_{\mathrm{H}} + \langle \mathcal{H}\Delta x, \mathcal{A}(\Delta x - \mathcal{D}\Delta f)\rangle_{\mathrm{H}}\\
&= \langle \mathcal{A}(\Delta x - \mathcal{D}\Delta f), \mathcal{H}(\Delta x - \mathcal{D}\Delta f)\rangle_{\mathrm{H}} + \langle \mathcal{A}(\Delta x - \mathcal{D}\Delta f), \mathcal{H}\mathcal{D}\Delta f\rangle_{\mathrm{H}}\\
&\quad + \langle \mathcal{H}(\Delta x - \mathcal{D}\Delta f), \mathcal{A}(\Delta x - \mathcal{D}\Delta f)\rangle_{\mathrm{H}} + \langle \mathcal{H}\mathcal{D}\Delta f, \mathcal{A}(\Delta x - \mathcal{D}\Delta f)\rangle_{\mathrm{H}}\\
&= \begin{bmatrix} \mathcal{A}(\Delta x - \mathcal{D}\Delta f) \\ \Delta f \end{bmatrix}^* \begin{bmatrix} \mathcal{H}\mathcal{A}^{-1} + \mathcal{A}^{-*}\mathcal{H} & \mathcal{H}\mathcal{D} \\ \mathcal{D}^*\mathcal{H} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A}(\Delta x - \mathcal{D}\Delta f) \\ \Delta f \end{bmatrix}.
\end{aligned}$$

The Lipschitz condition (3.1) can equivalently be written as

$$\begin{aligned}
0 &\geq \|\Delta f - \mu\Delta y\|_{\mathrm{U}}^2 - m^2\|\Delta y\|_{\mathrm{Y}}^2 = \|\Delta f - \mu\mathcal{C}\Delta x\|_{\mathrm{U}}^2 - m^2\|\mathcal{C}\Delta x\|_{\mathrm{Y}}^2\\
&= \|\Delta f - \mu H^*\mathcal{A}(\Delta x - \mathcal{D}\Delta f) - \mu\mathcal{C}\mathcal{D}\Delta f\|_{\mathrm{U}}^2 - m^2\|H^*\mathcal{A}(\Delta x - \mathcal{D}\Delta f) + \mathcal{C}\mathcal{D}\Delta f\|_{\mathrm{Y}}^2\\
&= \begin{bmatrix} \mathcal{A}(\Delta x - \mathcal{D}\Delta f) \\ \Delta f \end{bmatrix}^* \begin{bmatrix} H\mathbf{Q}H^* & H\mathbf{N} \\ \mathbf{N}^*H^* & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathcal{A}(\Delta x - \mathcal{D}\Delta f) \\ \Delta f \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
&\mathbf{Q} := \mu^*\mu - m^2 I \in \mathbf{L}(\mathrm{Y}), \quad \mathbf{N} := -\mu^* + \mathbf{Q}(\mathcal{C}\mathcal{D}) \in \mathbf{L}(\mathrm{U}, \mathrm{Y}),\\
&\mathbf{R} := (I - \mu\mathcal{C}\mathcal{D})^*(I - \mu\mathcal{C}\mathcal{D}) - m^2(\mathcal{C}\mathcal{D})^*(\mathcal{C}\mathcal{D}) \in \mathbf{L}(\mathrm{U}),
\end{aligned}$$

Now, adding and subtracting $\|\Delta f - \mu\Delta y\|_{\mathrm{U}}^2 - m^2\|\Delta y\|_{\mathrm{Y}}^2$, we obtain:

$$\begin{aligned}
\langle \Delta\mathcal{A}^{\mathrm{c}}, \Delta x\rangle_{\mathcal{H}} + \langle \Delta x, \Delta\mathcal{A}^{\mathrm{c}}\rangle_{\mathcal{H}} &= \|\Delta f(y) - \mu\Delta y\|_{\mathrm{U}}^2 - m^2\|\Delta y\|_{\mathrm{Y}}^2\\
&\quad + \begin{bmatrix} \mathcal{A}(\Delta x - \mathcal{D}\Delta f) \\ \Delta f \end{bmatrix}^* \mathfrak{M} \begin{bmatrix} \mathcal{A}(\Delta x - \mathcal{D}\Delta f) \\ \Delta f \end{bmatrix},
\end{aligned}$$

$$\mathfrak{M} := \begin{bmatrix} \mathcal{H}\mathcal{A}^{-1} + \mathcal{A}^{-*}\mathcal{H} - H\mathbf{Q}H^* & \mathcal{H}\mathcal{D} - H\mathbf{N} \\ \mathcal{D}^*\mathcal{H} - \mathbf{N}^*H^* & -\mathbf{R} \end{bmatrix} = \mathfrak{M}^* \in \mathbf{L}(\mathrm{H} \oplus \mathrm{U})$$

and $\mathcal{A}^{\mathrm{c}}$ is *dissipative* with respect to the scalar product $\langle x, y\rangle_{\mathcal{H}}$ if

$$\exists \mathcal{H} \in \mathbf{L}(\mathrm{H}), \mathcal{H} = \mathcal{H}^* > 0 : \quad \mathfrak{M} \leq 0. \tag{6.1}$$

It is clear that necessary conditions for (6.1) to hold are:

$$\begin{aligned}
&\exists \mathcal{H} > 0 : \ \mathcal{H}\mathcal{A}^{-1} + \mathcal{A}^{-*}\mathcal{H} - H\mathbf{Q}H^* \leq 0\\
&\iff \langle \mathcal{A}x, \mathcal{H}x\rangle_{\mathrm{H}} + \langle \mathcal{H}x, \mathcal{A}x\rangle_{\mathrm{H}} - \langle \mathcal{C}x, \mathbf{Q}\mathcal{C}x\rangle_{\mathrm{Y}} \leq 0, \quad x \in D(\mathcal{A}),
\end{aligned}$$

and, by complexifying and examining the operator matrix sign on vectors:

$$\begin{bmatrix} (j\omega I - \mathcal{A}^{-1})^{-1}\mathcal{D}u \\ u \end{bmatrix}, \quad u \in \mathrm{U},$$

we come to

$$\begin{aligned}
&\mathbf{R} + 2\,\mathrm{Re}[\mathbf{N}^*H^*(j\omega I - \mathcal{A}^{-1})^{-1}\mathcal{D}]\\
&+[H^*(j\omega I - \mathcal{A}^{-1})^{-1}\mathcal{D}]^*\mathbf{Q}[H^*(j\omega I - \mathcal{A}^{-1})^{-1}\mathcal{D}] \geq 0 \quad \forall\omega \neq 0\\
&\iff \Pi(j\omega) := I + 2\,\mathrm{Re}[\mu\widehat{G}(j\omega)] + [\widehat{G}(j\omega)]^*\mathbf{Q}\widehat{G}(j\omega) \geq 0 \ \ \forall\omega \in \mathbb{R};
\end{aligned}$$

$\Pi : j\mathbb{R} \ni j\omega \longmapsto \Pi(j\mathbb{R}) \in \mathbf{L}(\mathrm{U})$ will be called the *Popov spectral function* for $\mathfrak{M}$. Indeed, the identity

$$\begin{aligned} H^*(sI - \mathcal{A}^{-1})^{-1}\mathcal{D} &= -s^{-1}H^*\mathcal{A}(s^{-1}I - \mathcal{A})^{-1}\mathcal{D} = -s^{-1}\mathcal{C}(s^{-1}I - \mathcal{A})^{-1}\mathcal{D} \\ &= -\widehat{G}(s^{-1}) - \mathcal{C}\mathcal{D} \end{aligned} \tag{6.2}$$

yields

$$\begin{aligned} &\mathbf{R} - 2\,\mathrm{Re}\left\{\mathbf{N}^*[\widehat{G}(j\omega) + \mathcal{C}\mathcal{D}]\right\} + \left[\widehat{G}(j\omega) + \mathcal{C}\mathcal{D}\right]^* \mathbf{Q}[\widehat{G}(j\omega) + \mathcal{C}\mathcal{D}] \\ &= (I - \mu\mathcal{C}\mathcal{D})^*(I - \mu\mathcal{C}\mathcal{D}) - m^2(\mathcal{C}\mathcal{D})^*(\mathcal{C}\mathcal{D}) - 2\,\mathrm{Re}\left\{[-\mu^* + \mathbf{Q}(\mathcal{C}\mathcal{D})]^*[\widehat{G}(j\omega) + \mathcal{C}\mathcal{D}]\right\} \\ &\quad + \left[\widehat{G}(j\omega) + \mathcal{C}\mathcal{D}\right]^* \mathbf{Q}[\widehat{G}(j\omega) + \mathcal{C}\mathcal{D}] \\ &= I - (\mathcal{C}\mathcal{D})^*\mu^* - \mu\mathcal{C}\mathcal{D} + (\mathcal{C}\mathcal{D})^*\mathbf{Q}(\mathcal{C}\mathcal{D}) \\ &\quad + 2\,\mathrm{Re}[\mu\widehat{G}(j\omega)] - 2\,\mathrm{Re}[(\mathcal{C}\mathcal{D})^*\mathbf{Q}\widehat{G}(j\omega)] \\ &\quad + 2\,\mathrm{Re}[\mu\mathcal{C}\mathcal{D}] - 2(\mathcal{C}\mathcal{D})^*\mathbf{Q}(\mathcal{C}\mathcal{D}) + [\widehat{G}(j\omega)]^*\mathbf{Q}\widehat{G}(j\omega) \\ &\quad + (\mathcal{C}\mathcal{D})^*\mathbf{Q}\widehat{G}(j\omega) + [\widehat{G}(j\omega)]^*\mathbf{Q}(\mathcal{C}\mathcal{D}) + (\mathcal{C}\mathcal{D})^*\mathbf{Q}(\mathcal{C}\mathcal{D}) \\ &= I + 2\,\mathrm{Re}[\mu\widehat{G}(j\omega)] + [\widehat{G}(j\omega)]^*\mathbf{Q}\widehat{G}(j\omega). \end{aligned}$$

We have $\Pi(j0) = \mathbf{R}$ because $\widehat{G}(j0) = -\mathcal{C}\mathcal{D}$. We say that Π is *coercive* if $\Pi(j\omega) \geq \varepsilon I > 0$ for all $\omega \in \mathbb{R}$. Then $\Pi(j0) = \mathbf{R}$ is boundedly invertible.

(iv) (3.5) with $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}, \mathrm{Y}))$ norm replaced by $\mathrm{L}^\infty(j\mathbb{R}, \mathbf{L}(\mathrm{U}, \mathrm{Y}))$ norm equivalently reads as

$$\left\|\hat{G}(j\omega)(I + \mu\hat{G}(j\omega))^{-1}z\right\|_{\mathrm{Y}}^2 < \frac{1}{m^2}\,\|z\|_{\mathrm{U}}^2 \quad \forall z \in \mathrm{U} \quad \forall \omega \in \mathbb{R}.$$

In turn, the latter can be written as

$$m^2\left\|\hat{G}(j\omega)w\right\|_{\mathrm{Y}}^2 < \left\|[I + \mu\hat{G}(j\omega)]w\right\|_{\mathrm{U}}^2 \quad \forall w \in \mathrm{U} \quad \forall \omega \in \mathbb{R},$$

and finally,

$$\langle u, \Pi u\rangle_{\mathrm{U}} > 0 \quad \forall u \in \mathrm{U}, \forall \omega \in \mathbb{R} \iff \Pi(j\omega) > 0 \quad \forall \omega \in \mathbb{R}. \tag{6.3}$$

This is a frequency-domain inequality of the *circle criterion type* [10].

The question whether (6.3) is sufficient for $\mathfrak{M} \leq 0$, even without requirement that $\mathcal{H} > 0$, remains an *open problem.*

However, if $\dim \mathrm{U} < \infty$ and

$$\widehat{G}(j\omega) \xrightarrow{\ s\ } 0 \in \mathbf{L}(\mathrm{U}, \mathrm{Y}) \quad \text{as} \quad |\omega| \to \infty \tag{6.4}$$

then Π is coercive. Indeed, as $\dim \mathrm{U} < \infty$, (6.3) is equivalent to the coercivity of Π on any compact interval of the imaginary axis. Moreover, if $\widehat{G}(j\omega) \rightharpoonup 0$ then

$$\langle \mathrm{Re}[\mu\widehat{G}(j\omega)]u, u\rangle_{\mathrm{U}} = \langle \widehat{G}(j\omega)u, \mu^*u\rangle_{\mathrm{U}} + \overline{\langle \widehat{G}(j\omega)u, \mu^*u\rangle_{\mathrm{U}}} \to 0,$$

while, to ensure $\langle \widehat{G}(j\omega)u, \mathbf{Q}\widehat{G}(j\omega)u\rangle_{\mathrm{U}} \to 0$ we have then to assume that $\mathbf{Q}$ is a compact operator. But $\mathbf{Q} = \mu^*\mu - m^2 I$, where $\mu \in \mathbf{L}(\mathrm{Y}, \mathrm{U})$ is compact as U is finite dimensional, whence $m^2 I$ would be compact, which is possible only when $\dim \mathrm{Y} < \infty$. Thus we have assumed (6.4) to get $\Pi(j\omega) \stackrel{s}{\longrightarrow} I$. Thanks to this Π is coercive on $j\mathbb{R}$ and $\mathbf{R}$ is coercive, whence boundedly invertible.

Making the identifications:

$$\mathbf{A} = \mathcal{A}^{-1}, \quad \mathbf{B} = \mathcal{D}, \quad \mathbf{C} = H^*$$

one can see the following facts.

(a) $\mathcal{C}$ is admissible with respect to the semigroup generated by $\mathcal{A}$ iff H is admissible with respect to the semigroup generated by $\mathbf{A}^{-1}$ [5, p. 323].
(b) The operator-valued function $s \longmapsto \mathbf{C}(sI - \mathbf{A})^{-1}\mathbf{B}$ is in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}, \mathrm{Y}))$ which follows from (2.6) and (6.2).
(c) Let $\dim \mathrm{U} < \infty$ and (6.4) holds. Then Π is coercive, equivalently

$$\mathbf{R} + 2\,\mathrm{Re}[\mathbf{N}^*\mathbf{C}(j\omega I - \mathbf{A})^{-1}\mathbf{B}] + [\mathbf{C}(j\omega I - \mathbf{A})^{-1}\mathbf{B}]^*\mathbf{Q}[\mathbf{C}(j\omega I - \mathbf{A})^{-1}\mathbf{B}]$$

is coercive.
(d) Now, the result of [6, Theorem 2.4] yields: $\mathfrak{M} \le 0$.
Actually, the *Lur'e system of resolving equations*:

$$\begin{cases} \mathcal{H}\mathcal{A}^{-1} + \mathcal{A}^{-*}\mathcal{H} - H\mathbf{Q}H^* = -\mathbf{G}\mathbf{G}^*, \\ \qquad\qquad -\mathcal{H}\mathcal{D} + H\mathbf{N} = -\mathbf{G}\mathbf{R}^{1/2} \end{cases} \tag{6.5}$$

has a solution $(\mathcal{H}, \mathbf{G})$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^*$, $\mathbf{G} \in \mathbf{L}(\mathrm{U}, \mathrm{H})$ but without any knowledge whether $\mathcal{H} > 0$.
If, in addition, $\{e^{t\mathbf{A}}\}_{t\ge 0}$ is uniformly bounded and intersection of the spectrum of $\mathbf{A}$ with imaginary axis is contained in $\{0\}$ (the latter clearly holds in our context) then Π has the representation:

$$\Pi(j\omega) = [\Theta(j\omega)]^*\Theta(j\omega), \quad \Theta(s) := \mathbf{R}^{1/2} - \mathbf{G}^*(sI - \mathbf{A})^{-1}\mathbf{B}, \tag{6.6}$$

and $\Theta(s)$ is in $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U}))$ jointly with its inverse $[\Theta(s)]^{-1}$.
(e) If $\mathbf{Q} \le 0$ then, for all $z \in \mathrm{H}$ there holds

$$z^*\mathcal{H}z \ge \int_0^\infty \|(-\mathbf{Q})^{1/2}\mathbf{H}^*\mathcal{A}S(t)z\|^2_{\mathrm{Y}}\mathrm{d}t = \|(-\mathbf{Q})^{1/2}\mathbf{\Psi}z\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})}$$

and therefore $\mathcal{H} \ge 0$.
Now, if $\mathbf{Q} < 0$ then $\ker \mathcal{H} = \{0\}$, provided that $\ker \mathbf{\Psi} = \{0\}$, i.e., the pair $(\mathcal{A}, \mathcal{C})$ is *infinite-time approximately observable* (equivalently the pair $(\mathcal{A}^{-1}, \mathbf{H}^*)$ is infinite-time approximately observable). Therefore, if the latter holds then $\mathcal{H} > 0$. The equivalence of norms $\|\cdot\|_{\mathrm{e}}$, $\|\cdot\|_{\mathrm{H}}$ holds iff, in addition, $\mathcal{H}$ is boundedly invertible. Sufficient conditions for that are: the coercivity of $(-\mathbf{Q})$ as well as the *infinite-time exact controllability* of the pair $(\mathcal{A}, \mathcal{C})$, i.e.,

$$\exists\gamma > 0\ \forall z \in \mathrm{H}: \quad \|\mathbf{\Psi}z\|_{\mathrm{L}^2(0,\infty;\mathrm{Y})} \ge \gamma\|z\|_{\mathrm{H}}.$$

If $\dim \mathrm{Y} < \infty$ then $\mathbf{\Psi}$ is a HS operator, whence $\mathbf{\Psi}^*\mathbf{Q}\mathbf{\Psi}$ is compact and therefore $\mathcal{H}$ cannot be coercive until $\dim \mathrm{H} = \infty$.

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