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INVERSE SCATTERING PROBLEMS FOR HALF-LINE SCHRÖDINGER OPERATORS AND BANACH ALGEBRAS

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Abstract. The inverse scattering problem for half-line Schrödinger operators with potentials from the Marchenko class is shown to be closely related to some Banach algebra of functions on the line. In particular, it is proved that the topological conditions in the Marchenko theorem can be replaced by the condition that the scattering function should belong to this Banach algebra.

Keywords: inverse scattering, Schrödinger operator, Banach algebra.

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1. INTRODUCTION

Let $\mathcal{T}$ be some class of self-adjoint operators T that are perturbations of a fixed operator T_0 with purely absolutely continuous spectrum. Assume that for every pair (T, T_0) with $T \in \mathcal{T}$ there exist wave operators; we denote by $S(T, T_0)$ the corresponding scattering operator and consider the class of scattering operators $\mathcal{W} := \{S(T, T_0) \mid T \in \mathcal{T}\}$. One of the most important problems of the quantum scattering theory is the one of efficient description of the class $\mathcal{W}$.

This article starts a series of papers in which the authors plan to demonstrate that each of the classical inverse scattering problems for Schrödinger, Dirac or Jacobi operators is closely related with some Banach algebra, in terms of which the corresponding class $\mathcal{W}$ can easily be characterised.

In the present paper, we consider the class $\mathcal{T} := \{T_q \mid q \in \mathcal{Q}\}$ of self-adjoint Schrödinger operator $T_q : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ generated by the differential expression

$$\mathfrak{t}_q(f) := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q$$

and the boundary condition

$$f(0) = 0$$

with the potential q belonging to the Marchenko class

$$\mathcal{Q} := \{q \in L_1(\mathbb{R}_+, x\mathrm{d}x) \mid \operatorname{Im} q = 0\}.$$

In the class $\mathcal{T}$, the inverse scattering problem has a unique solution that was found by V. A. Marchenko [1, Ch. 3]. In particular, he proved a theorem providing a complete description of the scattering data for the operators $T \in \mathcal{T}$. Conditions of this theorem can be divided into algebraic and topological ones. The topological conditions involve only the scattering function S_q, which is an equivalent substitute of the scattering operator $S(T_q, T_0)$. Therefore, in the current setting the description of the class $\mathcal{W}$ is equivalent to the description of the class $\{S_q \mid q \in \mathcal{Q}\}$. Our aim is to show that the class $\{S_q \mid q \in \mathcal{Q}\}$ can efficiently be described in terms of some functional Banach algebra introduced below.

To formulate the main result of the paper, let us recall some definitions. The central object in inverse scattering problems is the scattering function $S = S_q$ of the operator T_q defined as

$$S(\lambda) := \frac{e(-\lambda)}{e(\lambda)}, \quad \lambda \in \mathbb{R}, \tag{1.1}$$

where $e(\lambda) := e(\lambda, 0)$ and $e(\lambda, \cdot)$ is the Jost solution of the equation

$$-y'' + qy = \lambda^2 y, \quad \lambda \in \overline{\mathbb{C}}_+. \tag{1.2}$$

Recall that $e(\lambda, \cdot)$, $\lambda \in \overline{\mathbb{C}}_+$, is called the Jost solution of the equation (1.2) if

$$e(\lambda, x) = \mathrm{e}^{\mathrm{i}\lambda x}(1 + \mathrm{o}(1)), \quad x \to +\infty.$$

The spectrum of the operator T_q with $q \in \mathcal{Q}$ consists of the absolutely continuous part filling the whole positive half-axis and the point spectrum consisting of a finite number of negative simple eigenvalues. Let us enumerate these eigenvalues in the ascending order of their moduli and denote them by $-\kappa_s^2$, $s = 1, \dots, n$, where $\kappa_s = \kappa_s(q) > 0$. To each eigenvalue $\lambda = -\kappa_s^2$, there correspond the eigenfunction $e(\mathrm{i}\kappa_s, \cdot)$ and the norming constant $m_s = m_s(q)$ defined as

$$m_s = \left(\int_0^\infty |e(\mathrm{i}\kappa_s, x)|^2 \, \mathrm{d}x \right)^{-\frac{1}{2}}.$$

The scattering data of the operator T_q are defined as the triple $\mathfrak{s}_q := (S_q, \vec{\kappa}_q, \vec{m}_q)$, where $\vec{\kappa}_q := (\kappa_s(q))_{s=1}^n$, $\vec{m}_q := (m_s(q))_{s=1}^n$. If $n = 0$, then $\mathfrak{s}_q = (S_q, 0, 0)$. Let us put

$$\Omega_n := \{(\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n \mid 0 < \kappa_1 < \dots < \kappa_n\}, \quad n \in \mathbb{N}.$$

Theorem 1.1 (The Marchenko theorem)**.** *Let $n \in \mathbb{N}$ ($n = 0$). A triple $(S, \vec{\kappa}, \vec{m})$ ($(S, 0, 0)$), where $S : \mathbb{R} \to \mathbb{C}$, $\vec{\kappa} \in \Omega_n$, $\vec{m} \in \mathbb{R}_+^n$, is the scattering data of some $T \in \mathcal{T}$ if and only if the following conditions are satisfied:*

(i) $\lim_{\xi\to\infty} S(\xi) = 1, \quad S(\lambda)S(-\lambda) = |S(\lambda)| = 1, \quad \lambda \in \mathbb{R}$;

(ii) S *is continuous on* $\mathbb{R}$ *and the function*

$$F_S(x) := \frac{1}{2\pi} \int\limits_{\mathbb{R}} (1 - S(\lambda)) e^{i\lambda x} \, d\lambda$$

is the sum of functions g_1 *and* g_2 *with* $g_1 \in L_1(\mathbb{R})$ *and* $g_2 \in L_\infty(\mathbb{R}) \cap L_2(\mathbb{R})$;

(iii) *the function* F_S *is locally absolutely continuous on* $\mathbb{R}_+$ *and* $F_S' \in L_1(\mathbb{R}_+, x\mathrm{d}x)$;

(iv) $n = \frac{\ln S(+0) - \ln S(+\infty)}{2\pi i} - \frac{1 - S(0)}{4}$.

Let X be a Banach space consisting of functions $u \in W^1_{1,\mathrm{loc}}(\mathbb{R} \setminus \{0\}) \cap L_1(\mathbb{R})$ for which the norm

$$\|u\|_X := \int\limits_{\mathbb{R}} |xu'(x)| \, dx$$

is finite. As will be shown below, X is continuously embedded in $L_1(\mathbb{R})$. Consider a Banach space

$$\mathbf{B} := \{\alpha \mathbf{1} + \widehat{\varphi} \mid \alpha \in \mathbb{C}, \varphi \in X\} \tag{1.3}$$

with the norm

$$\|\alpha \mathbf{1} + \widehat{\varphi}\|_{\mathbf{B}} := |\alpha| + 3\|\varphi\|_X. \tag{1.4}$$

Here $\mathbf{1}(x) \equiv 1$ and $\widehat{\varphi}$ is the Fourier transform of a function φ. Note that $\mathbf{B}$ consists of functions that are continuous on the one-point compactification $\widehat{\mathbb{R}}$ of the real line ($\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$). It turns out that under the standard pointwise multiplication $\mathbf{B}$ is a commutative Banach algebra with unit.

The main result of this paper is:

Theorem 1.2. *The set* $\{S_q \mid q \in \mathcal{Q}\}$ *coincides with the set*

$$\mathcal{S}_{\mathcal{Q}} = \{S \in \mathbf{B} \mid S(\infty) = 1 \text{ and } S(\lambda)S(-\lambda) = |S(\lambda)| = 1 \textit{ for all } \lambda \in \mathbb{R}\}$$

and is a multiplicative group.

By virtue of this result, the Marchenko theorem can be reformulated as follows:

Theorem 1.3. *Let* $n \in \mathbb{N}$ *(resp.* $n = 0$*). A triple* $(S, \vec{\kappa}, \vec{m})$ *(resp.* $(S, 0, 0)$*), where* $S : \mathbb{R} \to \mathbb{C}$, $\vec{\kappa} \in \Omega_n$, $\vec{m} \in \mathbb{R}^n_+$, *is the scattering data of some* $T \in \mathcal{T}$ *if and only if* S *belongs to* $\mathcal{S}_{\mathcal{Q}}$ *and*

$$[-\mathrm{ind} S/2] = n, \tag{1.5}$$

where $\mathrm{ind} S := ((\ln S)(\infty) - (\ln S)(-\infty))/2\pi i$ *and* $[x]$ *is the integer part of a number* x.

This paper is organized as follows. In Section 2, we study some properties of the algebra $\mathbf{B}$. In Section 3, we prove Theorems 1.2 and 1.3.

2. THE BANACH ALGEBRA $\mathbf{B}$

Let us consider a classical commutative Banach algebra

$$\mathbf{A} := \{\alpha\mathbf{1} + \widehat{\varphi} \mid \alpha \in \mathbb{C},\, \varphi \in L_1(\mathbb{R})\}$$

with the norm

$$\|\alpha\mathbf{1} + \widehat{\varphi}\|_{\mathbf{A}} := |\alpha| + \|\varphi\|_1. \tag{2.1}$$

Here $\|\cdot\|_1$ is the norm of $L_1(\mathbb{R})$ and $\widehat{\varphi}$ is the Fourier transform of a function φ, i.e.

$$\widehat{\varphi}(\lambda) := (\mathcal{F}\varphi)(\lambda) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\lambda t}\varphi(t)\,\mathrm{d}t, \quad \lambda \in \mathbb{R}.$$

Multiplication in $\mathbf{A}$ is the standard pointwise multiplication and

$$\|fg\|_{\mathbf{A}} \le \|f\|_{\mathbf{A}}\|g\|_{\mathbf{A}}, \quad f, g \in \mathbf{A}. \tag{2.2}$$

It is known that every function $f \in \mathbf{A}$ is continuous on $\widehat{\mathbb{R}}$.

In the algebra $\mathbf{A}$, we consider the closed subalgebras:

$$\begin{aligned} \mathbf{A}^+ &:= \{f = \alpha\mathbf{1} + \widehat{h} \mid \alpha \in \mathbb{C},\, h \in L_1(\mathbb{R}), \quad h\big|_{\mathbb{R}_-} = 0\}, \\ \mathbf{A}_0 &:= \{f = \widehat{h} \;\mid\; h \in L_1(\mathbb{R})\}. \end{aligned}$$

The main result of this section is the following theorem.

Theorem 2.1. *Equations* (1.3) *and* (1.4) *define a unital Banach algebra* $\mathbf{B}$, *that is, continuously embedded in the algebra* $\mathbf{A}$. *Moreover,* $\mathbf{B}$ *is the Wiener algebra, i.e., if* $f \in \mathbf{B}$ *and* $0 \notin f(\hat{\mathbb{R}})$, *then* f *is invertible in* $\mathbf{B}$.

First, let us make a few remarks and prove some lemmas.

Remark 2.2. *Obviously, completeness of the space* $\mathbf{B}$ *is equivalent to completeness of its subspace* $\widehat{X} := \{\widehat{u} \mid u \in X\}$. *Since* $\widehat{X}$ *and* X *are isometric* (*where the operator* $\frac{1}{3}\mathcal{F}$ *is an isometry of* X *onto* $\widehat{X}$), $\widehat{X}$ *and* $\mathbf{B}$ *are Banach spaces.*

Let us denote by X_+ and X_- the Banach spaces consisting of all those $u_+ \in W^1_{1,\mathrm{loc}}(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$ and $u_- \in W^1_{1,\mathrm{loc}}(\mathbb{R}_-) \cap L_1(\mathbb{R}_-)$ for which the norm

$$\|u_\pm\|_{X_\pm} := \int_{\mathbb{R}_\pm} |xu'_\pm(x)|\,\mathrm{d}x$$

is finite. Let us agree to identify $X_\pm$ with the subspaces $\{f \in X \mid f\big|_{\mathbb{R}_\mp} = 0\}$ in the space X. Then $X = X_+ \oplus X_-$.

Define by Λ the operator of multiplication by an independent variable acting on the space $L_{1,\mathrm{loc}}(\mathbb{R})$, i.e.

$$(\Lambda f)(x) := xf(x), \quad f \in L_{1,\mathrm{loc}}(\mathbb{R}).$$

Lemma 2.3. *The space X is continuously embedded in $L_1(\mathbb{R})$ and for all $u \in X$ it holds*

$$\|u\|_1 \le \|u\|_X, \quad \|\Lambda u\|_\infty \le \|u\|_X, \tag{2.3}$$

$$xu(x) = \mathrm{o}(1), \quad x \to +\infty \quad \text{or} \quad x \to +0. \tag{2.4}$$

Proof. Obviously, it suffices to prove (2.3) and (2.4) for $u \in X_+$. Let $u \in X_+$. Since

$$|u(x)| \le \int_x^\infty |u'(t)|\,\mathrm{d}t, \quad x \in \mathbb{R}_+,$$

we get

$$x|u(x)| \le x\int_x^\infty |u'(t)|\,\mathrm{d}t \le \int_x^\infty t|u'(t)|\,\mathrm{d}t, \quad x \in \mathbb{R}_+,$$

and

$$\int_0^\infty |u(x)|\,\mathrm{d}x \le \int_0^\infty \int_x^\infty |u'(t)|\,\mathrm{d}t\,\mathrm{d}x \le \int_{\mathbb{R}_+} t|u'(t)|\,\mathrm{d}t.$$

Also, for an arbitrary $x \in (0,1)$, it holds

$$u(x^2) = u(x) - \int_{x^2}^x u'(t)\,\mathrm{d}t$$

and, therefore,

$$x^2|u(x^2)| \le x\,x|u(x)| + \int_{x^2}^x t|u'(t)|\,\mathrm{d}t = \mathrm{o}(1), \quad x \to +0.$$

Using the above-mentioned relations we then obtain the statement of the lemma. □

Remark 2.4. *It follows from* (2.3) *that the space $\widehat{X}$ (the space $\mathbf{B}$) is continuously embedded in $\mathbf{A}_0$ ($\mathbf{A}$) and*

$$\|\varphi\|_{\mathbf{A}} \le \frac{1}{3}\|\varphi\|_{\mathbf{B}}, \quad \varphi \in \widehat{X}. \tag{2.5}$$

We denote by $\mathscr{S}$ the Schwartz space of all rapidly decreasing functions on $\mathbb{R}$, i.e.,

$$\mathscr{S} := \{f \in C^\infty(\mathbb{R}) \mid \forall k, m \in \mathbb{Z}_+ \quad f^{(k)}(x) = \mathrm{O}(x^{-m}), \quad x \to \infty\}.$$

Lemma 2.5. *$\mathscr{S}$ is dense everywhere in the space X and in $\widehat{X}$.*

Proof. Since the Fourier transform is a homeomorphism of the Schwartz class, it suffices to show that $\mathscr{S}$ is dense everywhere in the space X. Let $f \in X$. Consider the sequence $f_n := f\theta_n$ $(n \in \mathbb{N})$, where the functions $\theta_n : \mathbb{R} \to [0,1]$ are given by the formula

$$\theta_n(x) := \begin{cases} 1, & \text{if } 1/n \le |x| \le n, \\ 2 - |x|/n, & \text{if } 1 < |x|/n < 2, \\ 2n|x| - 1, & \text{if } 1 < 2n|x| < 2, \\ 0, & \text{if } 2n|x| \le 1 \quad \text{or} \quad |x|/n \ge 2. \end{cases}$$

Clearly, $f_n \in X$ and

$$\|f - f_n\|_X = \int\limits_{\mathbb{R}} |tf'(t) - tf_n'(t)|\,\mathrm{d}t \le \int\limits_{\mathbb{R}\setminus A_n} (|tf'(t)| + 2|f(t)|)\,\mathrm{d}t,$$

where $A_n := \{x \in \mathbb{R} \mid 1/n \le |x| \le n\}$. Therefore, $f_n \xrightarrow{X} f$ as $n \to +\infty$ and thus it suffices to prove that every function from the set

$$X_0 := \{f \in X \mid \exists n \in \mathbb{N} \quad \operatorname{supp} f \subset A_n\}$$

can be approximated by elements from $\mathscr{S}$ in the norm of X. Let $u \in X_0$ and $\omega \in C^\infty(\mathbb{R})$ be an arbitrary non–negative function for which

$$\operatorname{supp}\omega \subset [-1,1], \qquad \int\limits_{\mathbb{R}} \omega(t)\,\mathrm{d}t = 1.$$

Obviously, for an arbitrary $\varepsilon > 0$ the function

$$u_\varepsilon(x) := \frac{1}{\varepsilon} \int\limits_{\mathbb{R}} u(t)\omega\left(\frac{x-t}{\varepsilon}\right)\,\mathrm{d}t, \quad x \in \mathbb{R},$$

belongs to $\mathscr{S}$. Note that

$$u(x) - u_\varepsilon(x) = \int\limits_{\mathbb{R}} [u(x) - u(x - \varepsilon y)]\omega(y)\,\mathrm{d}y$$

and that both the derivative u' and the function $v(x) := xu'(x)$ belong to $L_1(\mathbb{R})$. Therefore, it holds

$$\|u - u_\varepsilon\|_X \le \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} (|v(x) - v(x - \varepsilon y)| + \varepsilon|yu'(x - \varepsilon y)|)\omega(y)\,\mathrm{d}y\,\mathrm{d}x$$

and hence $u_\varepsilon \xrightarrow{X} u$ as $\varepsilon \to +0$. The proof is complete. □

Proof of Theorem 2.1. Taking into account (1.4), (2.1) and the properties of Fourier transform, we obtain that for an arbitrary $\varphi \in \mathscr{S}$ it holds

$$\frac{1}{3}\|\varphi\|_{\mathbf{B}} = \|\mathcal{F}^{-1}\varphi\|_X = \|\Lambda(\mathcal{F}^{-1}\varphi)'\|_1 = \|\mathcal{F}\Lambda(\mathcal{F}^{-1}\varphi)'\|_{\mathbf{A}} = \|\varphi + \Lambda\varphi'\|_{\mathbf{A}}. \tag{2.6}$$

Taking into account (2.2), we obtain that for an arbitrary $\varphi, \psi \in \mathscr{S}$,

$$\|\varphi\psi + \Lambda(\varphi\psi)'\|_{\mathbf{A}} \le \|\varphi + \Lambda\varphi'\|_{\mathbf{A}}\|\psi\|_{\mathbf{A}} + \|\varphi\|_{\mathbf{A}}\|\psi + \Lambda\psi'\|_{\mathbf{A}} + \|\varphi\|_{\mathbf{A}}\|\psi\|_{\mathbf{A}}.$$

Thus, according to (2.5) and (2.6), we have

$$\|\varphi\psi\|_{\mathbf{B}} \le \|\varphi\|_{\mathbf{B}}\|\psi\|_{\mathbf{B}}, \quad \varphi, \psi \in \mathscr{S}. \tag{2.7}$$

Let $u, v \in \widehat{X}$. Taking into account Lemma 2.5, it follows that there are sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ in $\mathscr{S}$ such that $u_n \overset{\mathbf{B}}{\to} u$ and $v_n \overset{\mathbf{B}}{\to} v$ as $n \to \infty$. From (2.7) it then follows that the sequence $w_n := u_n v_n$, $n \in \mathbb{N}$, is fundamental in the space $\widehat{X}$. Since the space $\widehat{X}$ is complete (see Remark 2.2), the sequence $(w_n)_{n\in\mathbb{N}}$ converges in $\widehat{X}$ to some w. Since $\mathbf{B}$ is continuously embedded in $\mathbf{A}$ (see Remark 2.4), the sequence $(w_n)_{n\in\mathbb{N}}$ converges in $\mathbf{A}$ to the product uv and $uv = w$. Therefore,

$$\|uv\|_{\mathbf{B}} = \lim_{n\to\infty} \|u_n v_n\|_{\mathbf{B}} \le \lim_{n\to\infty} \|u_n\|_{\mathbf{B}}\|v_n\|_{\mathbf{B}} = \|u\|_{\mathbf{B}}\|v\|_{\mathbf{B}}.$$

It follows from the above that $\mathbf{B}$ is a Banach algebra with the unit $\mathbf{1}$.

Let us prove that $\mathbf{B}$ is a Wiener algebra. Let $f \in \mathbf{B}$ and $0 \notin f(\hat{\mathbb{R}})$. Then $f = \alpha\mathbf{1} + \varphi$, where $\alpha \in \mathbb{C} \setminus \{0\}$ and $\varphi \in \widehat{X}$. Taking into account Lemma 2.5, there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathscr{S}$ such that $\varphi_n \overset{\mathbf{B}}{\to} \varphi$ as $n \to \infty$. It follows from (2.6) that the sequence

$$\psi_n := \varphi_n + \Lambda\varphi_n', \quad n \in \mathbb{N},$$

is fundamental in $\mathbf{A}$. According to the classical Wiener lemma, the element f is invertible in $\mathbf{A}$. Thus, for some $n_0 \in \mathbb{N}$ the elements $f_n := \alpha\mathbf{1} + \varphi_n$, $n \ge n_0$, are invertible in $\mathbf{A}$ and the sequence $g_n := f_n^{-1}, n \ge n_0$, converges in $\mathbf{A}$. Note that the functions

$$h_n := g_n - \alpha^{-1}\mathbf{1}, \quad n \ge n_0,$$

belong to $\mathscr{S}$. To prove invertibility of f in the algebra $\mathbf{B}$, it suffices to show that the sequence $(h_n)_{n\in\mathbb{N}}$ is fundamental in $\mathbf{B}$. In view of (2.6), this is equivalent to the fact that the sequence $w_n := \Lambda h_n'$, $n \ge n_0$, is fundamental in $\mathbf{A}$. It is easily seen that

$$w_n = (\varphi_n - \psi_n)g_n^2.$$

Since the sequences $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}}$ and $(g_n)_{n=n_0}^{\infty}$ are fundamental in $\mathbf{A}$, the sequence $(w_n)_{n=n_0}^{\infty}$ is also fundamental in $\mathbf{A}$, and the proof is complete. □

Remark 2.6. *The space $\widehat{X} := \{\widehat{u} \mid u \in X\}$ is a maximal ideal in $\mathbf{B}$. If $\alpha \in \mathbb{C} \setminus \mathbb{R}$, then the function $f_\alpha(\lambda) := (\lambda + \alpha)^{-1}$ belongs to $\widehat{X}$.*

3. PROOF OF THEOREMS 1.2 AND 1.3

First we will present two auxiliary lemmas the ideas of which can be traced back to [1, Chapter 3].

Lemma 3.1. *Let* $\varphi \in L_\infty(\mathbb{R}_+)$, $\psi \in L_1(\mathbb{R}_+)$. *If the function*

$$g(x) := \varphi(x) + \int_0^\infty \varphi(t)\psi(x+t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+, \tag{3.1}$$

belongs to $L_1(\mathbb{R}_+)$, *then* $\varphi \in L_1(\mathbb{R}_+)$. *In addition, if* $g \in X_+$ *and* $\psi \in X_+$, *then* $\varphi \in X_+$.

Proof. Let us choose $\alpha > 0$ such that $\int_\alpha^\infty |\psi(x)|\,\mathrm{d}x \le 1/2$. Then the operator

$$(Vf)(x) := \int_\alpha^\infty f(t)\psi(x+t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+,$$

is continuous in $L_p(\mathbb{R}_+)$ ($p \in [1,\infty]$) and $\|V\|_{L_p \to L_p} \le 1/2$. Since the function

$$g_\alpha(x) := g(x) - \int_0^\alpha \varphi(t)\psi(x+t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+,$$

belongs to $L_1(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+)$ and $\varphi = (I+V)^{-1} g_\alpha$, we have that $\|\varphi\|_1 \le 2\|g_\alpha\|_1 < \infty$.

If $g \in X_+$ and $\psi \in X_+$, then

$$g(x) = -\int_x^\infty g'(\xi)\,\mathrm{d}\xi, \quad \psi(x) = -\int_x^\infty \psi'(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}_+.$$

In view of (3.1),

$$\varphi(x) = -\int_x^\infty g'(\xi)\,\mathrm{d}\xi + \int_0^\infty \varphi(t) \int_x^\infty \psi'(\xi+t)\,\mathrm{d}\xi\,\mathrm{d}t.$$

Using Fubini's theorem, we get

$$\varphi(x) = -\int_x^\infty \left(g'(\xi) - \int_0^\infty \varphi(t)\psi'(\xi+t)\,\mathrm{d}t \right) \mathrm{d}\xi, \quad x \in \mathbb{R}_+,$$

and thus the function φ is locally absolutely continuous on $\mathbb{R}_+$ and

$$\varphi'(x) = g'(x) - \int_0^\infty \varphi(t)\psi'(x+t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

Thus we obtain that

$$\int_0^\infty x|\varphi'(x)|\,\mathrm{d}x \le \int_0^\infty x|g'(x)|\,\mathrm{d}x + \int_0^\infty\int_0^\infty |\varphi(t)|\,|(x+t)\psi'(x+t)|\,\mathrm{d}t\,\mathrm{d}x$$

and therefore

$$\|\varphi\|_{X_+} \le \|g\|_{X_+} + \|\varphi\|_1\|\psi\|_{X_+} < \infty$$

as claimed. □

Lemma 3.2. *Let the functions $\varphi \in L_1(\mathbb{R}_+)$ and $\psi \in X_+$ be related via*

$$\varphi(x) + \psi(x) + \int_0^\infty \varphi(t)\psi(x+t)\,\mathrm{d}t = 0, \quad x \in \mathbb{R}_+, \tag{3.2}$$

and the function f be given by the formula

$$f(\lambda) = 1 + \int_0^\infty \varphi(t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t, \quad \lambda \in \mathbb{R}.$$

If $f(0) = 0$, then there exists $g \in \mathbf{B}$ such that $g(\infty) = 1$ and $f(\lambda) = \frac{\lambda}{\lambda+\mathrm{i}}g(\lambda)$.

Proof. Let the conditions of the lemma be satisfied. From Lemma 3.1, it follows that $\varphi \in X_+$ and thus $f \in \mathbf{B}$. Let us show that the function

$$h(x) := \int_x^\infty \varphi(t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+,$$

also belongs to X_+. Consider an auxiliary function

$$\Phi(x) := h(x) - \int_0^\infty h(t)\psi(x+t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

It can be easily seen that $\lim_{x\to+\infty}\Phi(x) = 0$ and $h(0) = -1$. Integrating by parts, for all $x \in \mathbb{R}_+$ we get

$$\Phi'(x) = -\varphi(x) - \int_0^\infty h(t)\psi'(x+t)\,\mathrm{d}t = -\varphi(x) - \psi(x) - \int_0^\infty \varphi(t)\psi(x+t)\,\mathrm{d}t.$$

The equality (3.2) implies that $\Phi'(x) \equiv 0$. Therefore, $\Phi(x) \equiv 0$ and, consequently,

$$h(x) = \int_0^\infty h(t)\psi(x+t)\,\mathrm{d}t, \quad x \in \mathbb{R}_+.$$

It follows from the definition of the function h that $h \in L_\infty(\mathbb{R}_+)$. Therefore, taking into account Lemma 3.1, it follows that the function h belongs to $L_1(\mathbb{R}_+)$ and to X_+ as well. Consequently, according to the formula (1.3), the function

$$g_1(\lambda) := \mathrm{i}\int_0^\infty h(t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t, \quad \lambda \in \mathbb{R},$$

belongs to the algebra $\mathbf{B}$. Integrating by parts, we get

$$\lambda g_1(\lambda) = \int_0^\infty h(t)\left(\frac{d}{dt}\mathrm{e}^{\mathrm{i}\lambda t}\right)\,\mathrm{d}t = -h(0) + \int_0^\infty \varphi(t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t = f(\lambda).$$

Let $g(\lambda) := (\lambda + \mathrm{i})g_1(\lambda)$. Since $g_1, f \in \mathbf{B}$, we get that $g \in \mathbf{B}$. Furthermore, it holds $g(\infty) = f(\infty) = 1$ and $\lambda(\lambda + \mathrm{i})^{-1}g(\lambda) = \lambda g_1(\lambda) = f(\lambda)$. □

Below we list some facts from [1, Chapter 3]. Let $q \in \mathcal{Q}$ and

$$\sigma(x) := \int_x^\infty |q(\xi)|\,\mathrm{d}\xi, \quad \sigma_1(x) := \int_x^\infty \xi|q(\xi)|\,\mathrm{d}\xi.$$

1. The solution of the Jost equation (1.2) can be represented in the form

$$e(\lambda, x) = \mathrm{e}^{\mathrm{i}\lambda x} + \int_x^\infty K(x,t)\mathrm{e}^{\mathrm{i}\lambda t}\,\mathrm{d}t, \quad \lambda \in \overline{\mathbb{C}}_+,\ x \in \mathbb{R}_+, \tag{3.3}$$

where the kernel K is continuous on the set $\Omega := \{(x,t) \in \mathbb{R}^2 \mid 0 \le x \le t\}$ and

$$|K(x,t)| \le \sigma\left(\frac{x+t}{2}\right)\exp\{\sigma_1(x)\}, \quad (x,t) \in \Omega. \tag{3.4}$$

2. The kernel K is a solution of the Marchenko equation

$$F(x+t) + K(x,t) + \int_x^\infty K(x,\xi)F(\xi+t)\,\mathrm{d}\xi = 0, \quad (x,t) \in \Omega, \tag{3.5}$$

with F given by

$$F(x) := \sum_{s=1}^n m_s^2\mathrm{e}^{-\kappa_s x} + F_S(x), \quad x \ge 0,$$

where

$$F_S(x) := \frac{1}{2\pi}\int_{\mathbb{R}} (1 - S(\lambda))\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}\lambda, \quad x \in \mathbb{R}.$$

3. For $\lambda \in \mathbb{R} \setminus \{0\}$, the following estimate for the derivative of the Jost solution holds

$$|e'(\lambda, x) - \mathrm{i}\lambda \mathrm{e}^{\mathrm{i}\lambda x}| \le \sigma(x) \exp\{\sigma_1(x)\}, \quad x \in \mathbb{R}_+, \tag{3.6}$$

and the formula

$$\omega(\lambda, x) := \frac{e(-\lambda, 0)e(\lambda, x) - e(\lambda, 0)e(-\lambda, x)}{2\mathrm{i}\lambda}, \quad x \in \mathbb{R}_+, \tag{3.7}$$

defines a solution of the equation (1.2) satisfying

$$\omega(\lambda, x) = x(1 + \mathrm{o}(1)), \quad \omega'(\lambda, x) = 1 + \mathrm{o}(1), \quad x \to +0. \tag{3.8}$$

4. The Jost function $e(\lambda) := e(\lambda, 0)$ does not vanish on $\mathbb{R} \setminus \{0\}$.

Proof of Theorem 1.2. It is clear that $\mathcal{S}_Q$ is a multiplicative group in the algebra $\mathbf{B}$.

Next we take an arbitrary $S \in \mathcal{S}_{\mathcal{Q}}$, set $n := [-\mathrm{ind} S/2]$, and fix arbitrary $\vec{\kappa} \in \Omega_n$ and $\vec{m} \in \mathbb{R}_+^n$. It follows from the definition of the set $\mathcal{S}_{\mathcal{Q}}$ that the triple $(S, \vec{\kappa}, \vec{m})$ satisfies assumptions (i)–(iii). We next show that the condition (iv) is satisfied as well.

Noting that $S(0) \in \{1; -1\}$, we denote by h a branch of the function $\frac{1}{2\pi \mathrm{i}} \ln S$ that is continuous on $\mathbb{R}$ and satisfies the conditions $h(0) = 0$ if $S(0) = 1$ and $h(0) = 1/2$ if $S(0) = -1$. Since $S(-\lambda) = \overline{S(\lambda)}$, we get

$$h(-x) = -h(x) + m, \quad x \ge 0,$$

where $m \in \{0; 1\}$. Therefore, $h(+\infty) - h(0) = h(0) - h(-\infty)$, which implies that

$$-\mathrm{ind} S = h(-\infty) - h(+\infty) = 2(h(0) - h(+\infty)).$$

Keeping in mind that $S(+\infty) = 1$, we conclude that the number $h(+\infty)$ is integer. As a result,

$$n = [-\mathrm{ind} S/2] = -h(+\infty) = \frac{\ln S(0) - \ln S(+\infty)}{2\pi \mathrm{i}} - \frac{1 - S(0)}{4},$$

i.e., (iv) holds.

Finally, by virtue of the Marchenko theorem we have $S = S_q$ for some $q \in \mathcal{Q}$ and thus

$$\mathcal{S}_{\mathcal{Q}} \subset \{S_q \mid q \in \mathcal{Q}\}. \tag{3.9}$$

Assume now that $q \in \mathcal{Q}$ and $S = S_q$. In view of (3.9) Theorem 1.2 will be proved as soon as we show that $S \in \mathcal{S}_{\mathcal{Q}}$. According to the Marchenko theorem, S satisfies conditions (i)–(iii). In view of (3.4) the function $\varphi(t) := K(0, t)$ belongs to the space $L_1(\mathbb{R}_+)$. It follows from (3.3) and (3.5) that

$$e(\lambda) = 1 + \int_0^\infty \varphi(x) e^{\mathrm{i}\lambda x} \,\mathrm{d}x, \quad \lambda \in \overline{\mathbb{C}}_+,$$

and

$$\varphi(t) + F(t) + \int_0^\infty \varphi(\xi) F(\xi + t)\, \mathrm{d}\xi = 0. \tag{3.10}$$

According to condition (iii) of the Marchenko theorem, the function F_S (and thus the function F) belongs ot the space X_+. Therefore, equation (3.10) and Lemma 3.1 imply that $\varphi \in X_+$. Based on Theorem 2.1 we conclude that the functions $\lambda \to e(\lambda)$ and $\lambda \to e(-\lambda)$ belong to the algebra $\mathbf{B}$. We next show that the function S given by formula (1.1), also belongs ot $\mathbf{B}$.

If $e(0) \neq 0$, then in view of $4°$ the function $e(\cdot)$ never vanishes on $\widehat{\mathbb{R}}$. Therefore in view of Theorem 2.1 it is an invertible element of $\mathbf{B}$, so that $S \in \mathbf{B}$.

If $e(0) = 0$, then by virtue of Lemma 3.2 there exists $g \in \mathbf{B}$ such that $g(\infty) = 1$ and

$$e(\lambda) = \frac{\lambda}{\lambda + \mathrm{i}} g(\lambda). \tag{3.11}$$

Let us show that $g(0) \neq 0$. It follows from estimate (3.6) that there is a $C > 0$ such that

$$|e'(\lambda, x)| \leq C, \quad x \in \mathbb{R}_+, \quad \lambda \in [-1, 1] \setminus \{0\}.$$

Therefore (cf. (3.7)),

$$|\omega'(\lambda, x)| \leq C(|g(-\lambda)| + |g(\lambda)|), \quad x \in \mathbb{R}_+, \quad \lambda \in [-1, 1] \setminus \{0\}.$$

It follows from (3.8) that

$$1 = \lim_{x \to +0} |\omega'(\lambda, x)| \leq C(|g(-\lambda)| + |g(\lambda)|), \quad \lambda \in [-1, 1] \setminus \{0\}.$$

Taking into account continuity of the function g, we see that $1 \leq 2C|g(0)|$, so that $g(0) \neq 0$. Combining this with $4°$ and (3.11), we conclude that g is an invertible element of the algebra $\mathbf{B}$. Since

$$S(\lambda) = \frac{\lambda + \mathrm{i}}{\lambda - \mathrm{i}} \frac{g(-\lambda)}{g(\lambda)}, \quad \lambda \in \mathbb{R},$$

in view of Remark 2.6 we arrive at the conclusion that the function S belongs to $\mathbf{B}$. □

Proof of Theorem 1.3. We first prove sufficiency. Let $\mathfrak{s} = (S, \vec{\kappa}, \vec{m})$, where $S \in \mathbf{B}$, $\vec{\kappa} \in \Omega_n$, and $\vec{m} \in \mathbb{R}^n_+$ satisfy the equality (1.5). Then the first part of the proof of Theorem 1.2 shows that the triple $(S, \vec{\kappa}, \vec{m})$ gives the scattering data for some operator $T \in \mathcal{T}$, as claimed.

Conversely, let a triple $\mathfrak{s} = (S, \vec{\kappa}, \vec{m})$ be the scattering data for some operator T_q with $q \in \mathcal{Q}$. Then $S = S_q$, and in virtue of Theorem 1.2 we have $S \in \mathcal{S}_{\mathcal{Q}}$. Moreover, the Marchenko theorem implies that (iv) is satisfied as well. However, the first part of the proof of Theorem 1.2 shows that (iv) is equivalent to the condition $[-\mathrm{ind} S/2] = n$. Therefore, the proof of necessity (and the theorem) is complete. □

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