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IMPROVED BOUNDS FOR SOLUTIONS OF ϕ-LAPLACIANS

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Abstract. In this short paper we prove a parametric version of the Harnack inequality for ϕ-Laplacian equations. In this sense, the estimates are optimal and represent an improvement of previous bounds for this kind of operators.

Keywords: Orlicz-Sobolev space, Harnack inequality, ϕ-Laplacian.

Mathematics Subject Classification: 35B50, 35J20, 35J60.

1. INTRODUCTION

Consider the following ϕ-Laplacian equation

$$-\text{div}\left(\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = \mathcal{B}(\cdot, u) \text{ in } \Omega \tag{1.1}$$

on the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$, where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain which has the segment property [5]. (Complete treatments and characterizations of the Orlicz-Sobolev spaces $W^1L_\Phi(\Omega)$ and $W_0^1L_\Phi(\Omega)$ can be found in [1, 8, 10]). The term $\phi : \mathbb{R} \to \mathbb{R}$ is an odd and increasing homeomorphism. However, in striking contrast with the classic case treated by Lieberman [11], ϕ is not required to be differentiable. It is rather assumed that

$$1 < p_\Phi \leq \frac{t\,\phi(t)}{\Phi(t)} \leq q_\Phi < +\infty, \quad t \neq 0 \tag{1.2}$$

for some numbers p_Φ and q_Φ, where Φ is the N-function

$$\Phi(t) = \int_0^t \phi(s)ds. \tag{1.3}$$

Note that integration of eq. (1.2) yields $\Phi(\kappa t) \leq \kappa^{q_\Phi}\Phi(t)$ for $\kappa > 1$ and $t > 0$ and this implies the useful inequality

$$\phi(\kappa t) \leq q_\Phi \kappa^{q_\Phi - 1}\phi(t) \quad \text{for all } t \geq 0 \text{ and } \kappa > 1. \tag{1.4}$$

The right-hand side $\mathcal{B} : \Omega \times W^1L_\Phi(\Omega) \to \mathbb{R}$ of (1.1) is a Carathéodory function such that

$$|\mathcal{B}(x,u)| \leq \mathfrak{a}\,\phi(|u(x)|) + \mathfrak{b} \quad a.e.\, x \text{ in } \Omega, \tag{1.5}$$

where $\mathfrak{a}, \mathfrak{b}$ are two nonnegative numbers. Let $B_R \subset\subset \Omega$ be a ball of radius $0 < R \leq 1$ and let $B_{R/2}$ be the concentric ball of radius $R/2$. It was proved in [2] that if u is a locally bounded and nonnegative solution of (1.1) then

$$\sup_{B_{R/2}} u \leq \mathcal{N}\left(\inf_{B_{R/2}} u + LR\right),$$

where $\mathcal{N} = \mathcal{N}(\mathfrak{a}, p_\Phi, q_\Phi, N)$ is a positive constant and $L > 0$ is any constant such that $\mathfrak{b} \leq \phi(L)$. In this short note we present a parametric version of the estimates obtained in [2]. In this sense, the bounds obtained here are optimal. These *improved* bounds allow for a more general interpretation of the behavior of solutions of ϕ-Laplacians and permit us to treat the problem on the regularity of the solutions [3]. Even though we do not address these properties here, it is of particular interest the special case of the p-Laplacian operator for which $\phi(s) = |s|^{p-2}s$ and $p > 1$. The particular case of variable exponents $p(x)$, where $p : \Omega \to (1, +\infty)$ is a bounded function, is treated in [14–16]. Special types of nonlinearities in connection with the p-Laplace operator have been considered recently in the article [9]. In a rather different context, the article [12] provides a geometric approach to the study of the p-Laplacian on a ball in $\mathbb{R}^N$ using techniques from dynamical systems. The author studies the invariant manifolds at the union of the solutions in the phase space and addresses the variational aspects of the corresponding tangent vector fields.

2. IMPROVED ESTIMATES

In this article, a *solution* of (1.1) will be any function $u \in W^1L_\Phi(\Omega)$ which satisfies estimates (1.5) and fulfills the identity

$$\int_\Omega \phi(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \nabla\theta dx = \int_\Omega \mathcal{B}(x,u)\,\theta dx \quad \text{for all } \theta \in W_0^1L_\Phi(\Omega). \tag{2.1}$$

(Hölder's inequality guarantees that the integrals above are finite [3]).

Let $B_s \subseteq B_R$ be two concentric balls of radii $0 < s < R \leq 1$ centered at a given point $x_0 \in \mathbb{R}^N$. A smooth function $\eta : \mathbb{R}^N \to [0,1]$ satisfying simultaneously

$$\eta|_{B_s} \equiv 1, \quad \eta|_{(\mathbb{R}^N \setminus B_R)} \equiv 0, \quad |\nabla\eta| \leq \frac{2}{R-s}$$

is called an s-cut-off function on B_R. Such functions do exist [7].

Lemma 2.1 ([2, Lemma 3.3]). *Let $B_R \subset\subset \Omega$ be any ball of radius $0 < R \le 1$ and let η be an $R/2$-cut-off function on B_R. Choose $v \ge 0$ such that $v^\alpha \in L^\infty(B_R)$, where $\alpha = \pm 1$. Set $w = R^{-1}\eta^\alpha v$. Then ${q_\Phi}^{-1} \sup_{B_R} w^\alpha \sup_{B_R}(\phi(w))^\alpha \le \sup_{B_R}(\Phi(w))^\alpha$.*

Proposition 2.2. *Let $B_R \subset\subset \Omega$ be any ball of radius $0 < R \le 1$. Choose a σR-cut-off function η on B_R, where $\sigma \in (0,1)$. Suppose that v is a nonnegative function such that $v^\alpha \in L^\infty(B_R)$, where $\alpha = \pm 1$. Set $w = R^{-1}\eta^\alpha v$ and assume that*

$$\left(\int_{B_R} \eta^{-\beta} \left(\Phi(w)\right)^{\alpha q Q} dx \right)^{1/Q} \le \frac{C\,q}{R(1-\sigma)^{q_\Phi}} \int_{B_R} \eta^{-\beta} \left(\Phi(w)\right)^{\alpha q} dx, \tag{2.2}$$

where the numbers $q \ge \beta > 0$, $Q = N/(N-1)$ and the constant C depends neither on the ball B_R nor on the number q. Then for $p > 0$ there exists a positive constant $\mathcal{C} = \mathcal{C}(\beta, p, q_\Phi, C, N)$ such that

$$\sup_{B_{\sigma R}} v^\alpha \le \frac{\mathcal{C}}{(1-\sigma)^{Nq_\Phi/p}} \left(⨍_{B_R} v^{\alpha p} dx \right)^{1/p},$$

where $B_{\sigma R}$ is the ball of radius σR concentric with B_R.

Proof. Assume first that $\beta \le p$. The particular choice $q = pQ^\nu$ in (2.2), where ν is a nonnegative integer, produces

$$\left(\int_{B_R} \eta^{-\beta} (\Phi(w))^{\alpha p Q^{\nu+1}} dx \right)^{1/pQ^{\nu+1}} \le \frac{(CpQ^\nu)^{1/pQ^\nu}}{R^{1/pQ^\nu}(1-\sigma)^{q_\Phi/pQ^\nu}} \left(\int_{B_R} \eta^{-\beta} (\Phi(w))^{\alpha p Q^\nu} dx \right)^{1/pQ^\nu}.$$

For $m \ge 1$, consider ν large enough such that $pQ^{\nu+1} > m$. Since $(\Phi(w))^\alpha \in L^\infty(B_R)$, a Moser iteration [13] of this inequality with respect to ν and the imbedding theorem yield

$$\|(\Phi(w))^\alpha\|_{L^m(B_R)} \le \left(\frac{Cp}{R(1-\sigma)^{q_\Phi}} \right)^{N/p} Q^{N(N-1)/p} \left(\int_{B_R} \eta^{-\beta} (\Phi(w))^{\alpha p} dx \right)^{1/p},$$

where $Q = N/(N-1)$. This estimate is valid for the norm $L^\infty(\Omega)$ [1, Theorem 2.14]. Since $(\Phi(w))^\alpha \le q_\Phi w^\alpha (\phi(w))^\alpha$ and $w^\alpha = R^{-\alpha}\eta^{\alpha^2} v^\alpha$, Lemma 2.1 implies

$$\sup_{B_{\sigma R}} v^\alpha \le q_\Phi^2 \left(\frac{Cp}{(1-\sigma)^{q_\Phi}} \right)^{N/p} Q^{N(N-1)/p} \omega_N^{1/p} \left(⨍_{B_R} v^{\alpha p} dx \right)^{1/p}, \tag{2.3}$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$. This concludes the proof in the case $p \geq \beta$, with $\mathcal{C} = \mathcal{D}_1(p) := q_\Phi^2 (Cp)^{N/p} Q^{N(N-1)/p} \omega_N^{1/p}$. Note that if we write $s = \sigma R$ then the previous estimate, applied with $p = \beta$, reads

$$\sup_{B_s} v^\alpha \leq \frac{\mathcal{D}_1(\beta)}{(1 - s/R)^{Nq_\Phi/\beta}} \left(\fint_{B_R} v^{\alpha\beta} dx \right)^{1/\beta}. \tag{2.4}$$

Now, suppose that $0 < p < \beta$. Since $v^{\alpha\beta} = v^{\alpha(\beta-p)} v^{\alpha p}$, estimate (2.4) produces

$$\sup_{B_s} v^\alpha \leq \frac{\mathcal{D}_1(\beta)}{(1 - s/R)^{Nq_\Phi/\beta}} \left(\sup_{B_R} v^\alpha \right)^{1-p/\beta} \left(\fint_{B_R} v^{\alpha p} dx \right)^{1/\beta}.$$

Since $cd \leq \frac{(\beta-p)}{\beta} c^{\frac{\beta}{(\beta-p)}} + \frac{p}{\beta} d^{\frac{\beta}{p}}$ for $c, d \geq 0$, the inequality below follows:

$$\sup_{B_s} v^\alpha \leq \frac{(\beta-p)}{\beta} \sup_{B_R} v^\alpha + \frac{p}{\beta} \frac{{\mathcal{D}_1}^{\beta/p}}{(R-s)^{Nq_\Phi/p}} \left(R^{Nq_\Phi} \fint_{B_R} v^{\alpha p} dx \right)^{1/p}.$$

An application of [6, Lemma 3.1] on the interval $[0, R]$ yields the conclusion. □

Let L be any real nonnegative constant such that $\mathfrak{b} \leq \phi(L)$. Choose $0 < R \leq 1$ and write $v = u + RL$ where u is a nonnegative solution of (1.1). Then

$$\mathfrak{a}\, \phi(u) + \mathfrak{b} \leq \mathfrak{a}\, \phi(u) + \phi(L) \leq \mathfrak{a}\, \phi(u + RL) + \phi\left(\frac{RL}{R}\right) \leq \mathfrak{a}\, \phi\left(\frac{Rv}{R}\right) + \phi\left(\frac{v}{R}\right).$$

By (1.5), the right-hand side of the equation (1.1) is hence bounded as follows:

$$|\mathcal{B}(\cdot, u)| \leq M\, \phi\left(\frac{v}{R}\right) \tag{2.5}$$

with the constant $M = \mathfrak{a} + 1$.

Proposition 2.3 ([2, Proposition 2.1])**.** *Let $u \in W^1 L_\Phi(\Omega)$. If* $\operatorname{supp}(u)$ *is compact contained in Ω then $u \in W_0^1 L_\Phi(\Omega)$.*

Proposition 2.4. *Let $B_R \subset\subset \Omega$ be any ball of radius $0 < R \leq 1$. Choose a σR-cut-off function η on B_R where $\sigma \in (0,1)$. Suppose that u is a locally bounded and nonnegative solution of eq.* (1.1)*. Choose any $L \geq 0$ such that $\mathfrak{b} \leq \phi(L)$. Then for any $d > 0$ and $q \geq 2 + d$,*

$$\int_\Omega \eta^{-d} (\Phi(w))^{q-1} \phi(|\nabla u|) |\nabla u| dx \leq \frac{C_1}{(1-\sigma)^{q_\Phi}} \int_\Omega \eta^{-d-2q_\Phi} (\Phi(w))^q dx \tag{2.6}$$

where $w = \eta(u + RL)/R$ and $C_1 = C_1(\mathfrak{a}, q_\Phi)$ is a positive constant.

Proof. Define $v = u + RL$. The following standard argument will be repeatedly employed. Since $\mathrm{supp}(\eta)$ is compact in Ω the function $w = \eta\, v/R \in L^\infty(\mathbb{R}^N)$ and thus there exists a constant $\mathsf{A} > 0$ such that $|w(x)| \le \mathsf{A}$ for all $x \in \mathbb{R}^N$. Define for all $t \ge 0$ the function $f_1(t) = t^{-d}\,(\Phi(t))^{q-1}$ if $t \in (0, \mathsf{A}+1)$ and write $f_1(t) = 0$ if $t \notin (0, \mathsf{A}+1)$. Note that $t \mapsto t^{-d}\,(\Phi(t))^{q-1}$ is of class $C^1(0, +\infty)$ and the derivative of this map tends to zero as $t \to 0^+$. Hence, f_1' is uniformly bounded on $[0, +\infty)$. If F_1 denotes the odd extension of f_1 to the entire real line then $F_1 \in C^1(\mathbb{R})$ and $F_1' \in L^\infty(\mathbb{R})$. Next, define the term $\theta = R^{-d} v^{d+1} F_1(w) = \eta^{-d}(\Phi(w))^{q-1} v$. The product formula [7, eq. (7.18)] and [7, Theorem 7.8] yield

$$\nabla\theta = \frac{(d+1)}{R^d} v^d F_1(w)\nabla u + \frac{v^{d+1}}{R^{d+1}} F_1'(w) v \nabla\eta + \frac{v^{d+1}}{R^{d+1}} F_1'(w)\eta\nabla u \in L_\Phi(\Omega).$$

Since $\mathrm{supp}(\theta) \subseteq \overline{B}_R \subseteq \Omega$, Proposition 2.3 implies $\theta \in W_0^1 L_\Phi(\Omega)$ and eq. (2.1) becomes

$$\begin{aligned}
&\int_\Omega \eta^{-d}\big[(q-1)(\Phi(w))^{q-2}\phi(w)w + (\Phi(w))^{q-1}\big]\phi(|\nabla u|)|\nabla u| \\
&= \int_\Omega \mathcal{B}(x,u)\eta^{-d}(\Phi(w))^{q-1} v \\
&\quad -(q-1)R\int_\Omega \eta^{-d-2}(\Phi(w))^{q-2}\phi(w)w^2\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\eta \\
&\quad +dR\int_\Omega \eta^{-d-2}(\Phi(w))^{q-1} w\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\eta.
\end{aligned}$$

The term $\phi(w)w$ in the argument of the integral on the left-hand side of the equality is bounded from below by $\Phi(w)$. The absolute value of the integrals on the right is taken. Then the bound $\phi(w)w \le q_\Phi \Phi(w)$ is applied to the argument of the second integral on the right. Since $q \ge \max\{1, d\}$, bounds $|\nabla\eta| \le 2/R(1-\sigma)$ and (2.5) produce

$$\begin{aligned}
q\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\phi(|\nabla u|)|\nabla u|dx &\le Mq\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\phi\left(\frac{v}{R}\right) v dx \\
&+ 4qq_\Phi\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\frac{w}{\eta^2}\frac{\phi(|\nabla u|)}{(1-\sigma)}dx + 2q\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\frac{w}{\eta^2}\frac{\phi(|\nabla u|)}{(1-\sigma)}dx \qquad (2.7) \\
&= Mq\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\phi\left(\frac{v}{R}\right) v dx + \mu q\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\frac{w}{\eta^2}\frac{\phi(|\nabla u|)}{(1-\sigma)}dx
\end{aligned}$$

where $\mu = 2(2q_\Phi + 1)$. Division by q is performed on both sides of the inequality. The second integral on the right is multiplied and divided by a sufficiently small quantity $\varepsilon > 0$ so as to form the term $w/\varepsilon\eta^2$ in the argument. Young's inequality

$t\phi(s) \le t\phi(t) + s\phi(s)$, with $t = w/\varepsilon\eta^2(1-\sigma)$ and $s = |\nabla u|$, is applied to this integral. From eq. (2.7) we obtain

$$(1-\varepsilon\mu)\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\phi(|\nabla u|)|\nabla u|dx$$
$$\le M\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\phi\left(\frac{v}{R}\right)vdx + \mu\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\frac{w}{\eta^2(1-\sigma)}\phi\left(\frac{w}{\varepsilon\eta^2(1-\sigma)}\right)dx.$$

Eq. (1.4) yields the following two estimates:

$$\phi(v/R)v = \phi(w/\eta)Rw/\eta \le q_\Phi \eta^{-q_\Phi}\phi(w)w$$

and

$$\phi\big(w/\varepsilon\eta^2(1-\sigma)\big) \le q_\Phi\phi(w)/\big(\varepsilon\eta^2(1-\sigma)\big)^{q_\Phi-1}.$$

Hence

$$(1-\varepsilon\mu)\int_\Omega \eta^{-d}(\Phi(w))^{q-1}\phi(|\nabla u|)|\nabla u|dx$$
$$\le Mq_\Phi\int_\Omega \eta^{-d-q_\Phi}(\Phi(w))^{q-1}\phi(w)wdx$$
$$+\frac{\mu q_\Phi}{\varepsilon^{q_\Phi-1}(1-\sigma)^{q_\Phi}}\int_\Omega \eta^{-d-2q_\Phi}(\Phi(w))^{q-1}\phi(w)wdx.$$

For ε suitably chosen (e.g. $\varepsilon = 1/2\mu$) and since $\phi(w)w \le q_\Phi\Phi(w)$ and $\eta^{-q_\Phi} \le \eta^{-2q_\Phi}$, the conclusion follows. □

Corollary 2.5. *Let u be a locally bounded and nonnegative solution of equation* (1.1) *and let $B_R \subset\subset \Omega$ be a ball of radius $0 < R \le 1$. Then for any $p > 0$,*

$$\sup_{B_{\sigma R}} u \le \frac{2^{1+1/p}\,\mathcal{C}}{(1-\sigma)^{Nq_\Phi/p}}\left(\left(\fint_{B_R} u^p dx\right)^{1/p} + RL\right)$$

where $\mathcal{C} = \mathcal{C}(\mathfrak{a}, p, q_\Phi, N)$ is the constant in Proposition 2.2 and $\sigma \in (0,1)$. The term L is any real nonnegative constant such that $\mathfrak{b} \le \phi(L)$ and $B_{\sigma R}$ is the ball of radius σR concentric with B_R.

Proof. Let η be an σR-cut-off function on B_R. As in the proof of Proposition 2.4, if we set $v = u + RL$ then $w = \eta\, v/R = \eta\,(u + RL)/R \in L^\infty(\mathbb{R}^N)$ and thus $|w(x)| \le \mathsf{A}$ for $x \in \mathbb{R}^N$ and where A is the same constant in the proof of Proposition 2.4. Take $d > 0$, $q \ge 2 + d$ (to be fixed later) and define $f_2(t) = \Phi(t)f_1(t)$ for $t \ge 0$ and where f_1 is the function defined in the proof of Proposition 2.4. It is evident that f_2' is bounded on $[0,+\infty)$. It follows that $F_2 \in C^1(\mathbb{R})$ and $F_2' \in L^\infty(\mathbb{R})$ where F_2 is the odd extension of f_2 to the entire real line. Define $\theta = F_2(w)v^d/R^d = \eta^{-d}(\Phi(w))^q$. Theorem 7.8 in [7] yields

$$\begin{aligned}\nabla\theta = &-d\eta^{-d-1}(\Phi(w))^q\nabla\eta + \eta^{-d-1}q(\Phi(w))^{q-1}\phi(w)w\,\nabla\eta \\ &+ \frac{q}{R}\eta^{-d+1}(\Phi(w))^{q-1}\phi(w)\nabla u.\end{aligned}$$

Note that $\phi(w)w \le q_\Phi\Phi(w)$, $q \ge d$ and $|\nabla\eta| \le 2/R(1-\sigma)$. Then Gagliardo–Nirenberg inequality [4, Theorem IX], applied to θ and $Q = N/(N-1)$, yields $\mathcal{C}_N > 0$ such that

$$\begin{aligned}&\left(\int_\Omega |\eta^{-d}(\Phi(w))^q|^Q\right)^{1/Q} \\ \le&\mathcal{C}_N\left(\frac{2q(1+q_\Phi)}{R(1-\sigma)}\int_\Omega \eta^{-d-1}(\Phi(w))^q + \frac{q}{R}\int_\Omega \eta^{-d+1}(\Phi(w))^{q-1}\phi(w)|\nabla u|\right).\end{aligned}$$

Since $\eta \le 1$ the first integral on the right-hand side of the inequality above is bounded by $\int_\Omega \eta^{-d-2q_\Phi}(\Phi(w))^q dx$. The argument of the second integral on the right is bounded as follows. Estimate $t\phi(s) \le t\phi(t) + s\phi(s)$ is again used (with $t = |\nabla u|$ and $s = w$) and hence

$$\begin{aligned}\eta^{-d+1}(\Phi(w))^{q-1}\phi(w)|\nabla u| &\le \eta^{-d}(\Phi(w))^{q-1}\phi(w)w + \eta^{-d}(\Phi(w))^{q-1}\phi(|\nabla u|)|\nabla u| \\ &\le q_\Phi\eta^{-d-2q_\Phi}(\Phi(w))^q + \eta^{-d}(\Phi(w))^{q-1}\phi(|\nabla u|)|\nabla u|.\end{aligned}$$

Since $1-\sigma < 1$, eq. (2.6) and the choice $-dQ = -d-2q_\Phi$ (i.e. $d = 2(N-1)q_\Phi$) yield

$$\left(\int_{B_R} \eta^{-2Nq_\Phi}(\Phi(w))^{qQ}dx\right)^{1/Q} \le \frac{C_2 q}{R(1-\sigma)^{q_\Phi}}\int_{B_R}\eta^{-2Nq_\Phi}(\Phi(w))^q dx$$

where $C_2 = C_2(\mathfrak{a}, q_\Phi, N) = \mathcal{C}_N(5q_\Phi + C_1)$. Proposition 2.2 with $C = C_2$, $\beta = 2Nq_\Phi$, $p > 0$ and $\alpha = 1$ gives

$$\sup_{B_{\sigma R}} u \le \sup_{B_{\sigma R}} v \le \frac{\mathcal{C}}{(1-\sigma)^{Nq_\Phi/p}}\left(\fint_{B_R} v^p dx\right)^{1/p}$$

for the same constant $\mathcal{C}$ in that proposition (and which, in this case, clearly depends on $\mathfrak{a}$, p, q_ϕ and N). Note that $x \mapsto x^r$ is convex in $(0,+\infty)$ for $r \ge 1$ whereas $x \mapsto 1+x^r-(1+x)^r$ is increasing in $(0,+\infty)$ if $0 < r < 1$. Hence $(c+d)^r \le 2^r(c^r+d^r)$ for $c, d, r > 0$ and this implies the bound claimed. The corollary is proved. $\square$

Proposition 2.6. *Let $B_R \subset\subset \Omega$ be any ball of radius $0 < R \leq 1$. Choose a σR-cut-off function η on B_R where $\sigma \in (0,1)$. Suppose that u is a locally bounded and nonnegative solution of eq.* (1.1)*. Choose any $L \geq 0$ such that $\mathfrak{b} \leq \phi(L)$ and let $\chi > 0$ be arbitrary. Then for any $q \geq d \geq 1$,*

$$\int\limits_{\Omega} \eta^{-d}(\Phi(w))^{-q-1}\phi(|\nabla u|)|\nabla u|dx \leq \frac{C_3}{(1-\sigma)^{q_\Phi}} \int\limits_{\Omega} \eta^{-d}(\Phi(w))^{-q}dx \tag{2.8}$$

where $w = (u + RL + \chi)/R\eta$ and $C_3 = C_3(\mathfrak{a}, q_\Phi)$ is a positive constant.

Proof. Define $v = u + RL$ and $z = R\eta/\psi$, where $\psi = v + \chi = u + RL + \chi$. Since u is nonnegative, $z \in L^\infty(\mathbb{R}^N)$ and there exists $\mathsf{B} > 0$ such that $|z(x)| \leq \mathsf{B}$ for all $x \in \mathbb{R}^N$. Consider for all $t \geq 0$ the function $f_3(t) = t^{-d}\,(\Phi(t^{-1}))^{-(q+1)}$ if $t \in (0, \mathsf{B}+1)$ and $f_3(t) = 0$ if $t \notin (0, \mathsf{B}+1)$. In this case, the map $t \mapsto t^{-d}\,(\Phi(t^{-1}))^{-(q+1)}$ is of class $C^1(0,\infty)$ and its derivative tends to zero as $t \to 0^+$. It follows that f_3' is uniformly bounded on $[0, +\infty)$. If F_3 is the odd extension of f_3 to the entire real line then $F_3 \in C^1(\mathbb{R})$ and $F_3' \in L^\infty(\mathbb{R})$. Note that $w = z^{-1}$. Define

$$\theta = R^d F_3(z)\psi^{1-d} = R\eta^{1-d}(\Phi(w))^{-q-1}w.$$

In this case,

$$\begin{aligned}\nabla\theta = {} & R^{d+1}F_3'(z)\psi^{-d}\,\nabla\eta - R^{d+1}\,\eta\,F_3'(z)\,\psi^{-(d+1)}\nabla u \\ & + (1-d)R^d F_3(z)\psi^{-d}\nabla u \in L_\Phi(\Omega).\end{aligned}$$

Since $\operatorname{supp}(\theta) \subseteq \overline{B}_R \subseteq \Omega$, Proposition 2.3 ensures that the function $\theta \in W_0^1 L_\Phi(\Omega)$. The weak formulation (2.1) in this case is equivalent to

$$\begin{aligned}&\int\limits_{\Omega} \eta^{-d}[(q+1)(\Phi(w))^{-q-2}\phi(w)w - (\Phi(w))^{-q-1}]\phi(|\nabla u|)|\nabla u|dx \\ &= R(q{+}1)\int\limits_{\Omega} \eta^{-d}(\Phi(w))^{-q-2}\phi(w)w^2\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\eta dx \\ &\quad - Rd\int\limits_{\Omega} \eta^{-d}(\Phi(w))^{-q-1}w\phi(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla\eta dx \\ &\quad - R\int\limits_{\Omega} \mathcal{B}(x,u)\eta^{1-d}(\Phi(w))^{-q-1}wdx.\end{aligned}$$

The estimate $\Phi(w) \leq \phi(w)w$ is applied to the argument of the integral on the left-hand side of the equality. After taking absolute values on the right-hand side, the bound $\phi(w)w \leq q_\Phi\Phi(w)$ is applied to the argument of the first integral on the right-hand side.

Write $\mu = 2(2q_\Phi + 1)$. Since $q \geq d \geq 1$ and $|\nabla\eta| \leq 2/R(1-\sigma)$ the above equivalence yields

$$q \int_\Omega \eta^{-d}(\Phi(w))^{-q-1}\phi(|\nabla u|)|\nabla u| dx$$
$$\leq q\mu \int_\Omega \eta^{-d}(\Phi(w))^{-q-1}\frac{w}{(1-\sigma)}\phi(|\nabla u|)dx + Mq \int_\Omega \phi\Big(\frac{v}{R}\Big)\eta^{1-d}(\Phi(w))^{-q-1} w dx.$$

The term w in the argument of the first integral on the right is divided and multiplied $\varepsilon > 0$ small. Equation (1.4) implies

$$\phi(w/\varepsilon(1-\sigma)) \leq q_\Phi\big(\varepsilon(1-\sigma)\big)^{-q_\Phi+1}\phi(w).$$

Since $\phi(v/R) \leq \phi(\eta w) \leq \phi(w)$ and $\eta^{1-d} \leq \eta^{-d}$, the bound $t\phi(s) \leq t\phi(t) + s\phi(s)$ with $t = w/\varepsilon(1-\sigma)$ and $s = |\nabla u|$ yields

$$(1-\varepsilon\mu)\int_\Omega \eta^{-d}(\Phi(w))^{-q-1}\phi(|\nabla u|)|\nabla u|$$
$$\leq \frac{q_\Phi^2\mu}{\varepsilon^{q_\Phi-1}(1-\sigma)^{q_\Phi}}\int_\Omega \eta^{-d}(\Phi(w))^{-q} + Mq_\Phi \int_\Omega \eta^{-d}(\Phi(w))^{-q}.$$

If ε is suitably chosen then the conclusion follows. □

Corollary 2.7 (Weak Harnack inequality)**.** *Let u be a locally bounded and nonnegative solution of equation* (1.1)*. Let $B_R \subset\subset \Omega$ be a ball of radius $0 < R \leq 1$ and $\sigma \in (0,1)$. Then there exist positive constants $p_0 = p_0(\mathfrak{a}, p_\Phi, q_\Phi, N)$ and $\mathsf{C} = \mathsf{C}(\mathfrak{a}, p_\Phi, q_\Phi, N)$ such that*

$$\left(\fint_{B_R} u^{p_0} dx\right)^{1/p_0} \leq \frac{\mathsf{C}}{(1-\sigma)^{Nq_\Phi/p_0}}\left(\inf_{B_{\sigma R}} u + RL\right)$$

where L is any real nonnegative constant such that $\mathfrak{b} \leq \phi(L)$ and $B_{\sigma R}$ is the ball of radius σR concentric with B_R.

Proof. Let η be an σR-cut-off function on B_R and take $\chi > 0$, arbitrary. As in the proof of Proposition 2.6, $z := R\eta/(u + RL + \chi) \in L^\infty(\mathbb{R}^N)$ and thus $|z(x)| \leq \mathsf{B}$ for $x \in \mathbb{R}^N$ and for the same constant. Take $q \geq d \geq 1$ (to be fixed later) and for all $t \geq 0$ define $f_4(t) = tR^{d-1}\Phi(t^{-1})f_3(t)$ which is differentiable with bounded derivative. It is clear that if F_4 is the odd extension of f_4 to the entire real line then $F_4 \in C^1(\mathbb{R})$ and $F_4' \in L^\infty(\mathbb{R})$. Write $w = z^{-1}$. Then [7, Theorem 7.8] applied to $F_4(z)\psi^{1-d} = \eta^{1-d}(\Phi(w))^{-q}$ yields

$$\nabla(\eta^{1-d}(\Phi(w))^{-q})$$
$$= \big[(1-d)\eta^{-d}(\Phi(w))^{-q} + q\eta^{-d}(\Phi(w))^{-q-1}\phi(w)w\big]\nabla\eta - \frac{q}{R}\eta^{-d}(\Phi(w))^{-q-1}\phi(w)\nabla u.$$

Since $q \geq d \geq 1$, the bound $\phi(t)t \leq q_\Phi \Phi(t)$ and the Gagliardo–Nirenberg inequality yield

$$\left(\int_\Omega |\eta^{1-d}(\Phi(w))^{-q}|^Q \right)^{1/Q} \leq \frac{2q\mathcal{C}_N(q_\Phi+2)}{R(1-\sigma)} \int_\Omega \eta^{-d}(\Phi(w))^{-q} + \frac{\mathcal{C}_N q}{R} \int_\Omega \eta^{-d}(\Phi(w))^{-q-1}\phi(w)|\nabla u|$$

where $Q = N/(N-1)$. It follows from eq. (2.8) (with $d = N$) and Young's inequality that

$$\left(\int_{B_R} \eta^{-N}\,(\Phi(w))^{-qQ} dx \right)^{1/Q} \leq \frac{C_4\, q}{R(1-\sigma)^{q_\Phi}} \int_{B_R} \eta^{-N}\,(\Phi(w))^{-q} dx$$

where $C_4 = C_4(\mathfrak{a}, q_\Phi, N)$. Proposition 2.2 with $\beta = N$, $v = \psi$ and $\alpha = -1$ yields

$$\left(\fint_{B_R} \psi^{-p} dx \right)^{-1} \leq \frac{\mathcal{C}^p}{(1-\sigma)^{Nq_\Phi}} \left(\inf_{B_{R/2}} \psi^{-1} \right)^p \tag{2.9}$$

where $p > 0$ and $\mathcal{C} = \mathcal{C}(\mathfrak{a}, p, q_\Phi, N)$. Let ξ be a ρ-cut-off function on B_R where $0 < \rho \leq R/2$. Consider the function $\theta = \xi^{q_\Phi}(\Phi(\mathbf{w}))^{-1}\psi$, where $\mathbf{w} = \psi/\rho$. It is easy to prove that

$$\nabla\theta = -\frac{1}{\rho}(\Phi(\mathbf{w}))^{-2}\phi(\mathbf{w})\xi^{q_\Phi}\psi\nabla v + q_\Phi(\Phi(\mathbf{w}))^{-1}\xi^{q_\Phi-1}\psi\nabla\xi + (\Phi(\mathbf{w}))^{-1}\xi^{q_\Phi}\nabla v \in L_\Phi(\Omega)$$

with $\nabla v = \nabla u$. Since $\theta \in W_0^1 L_\Phi(\Omega)$, the bound $|\nabla\xi| \leq 4/(R-\rho) \leq 4/\rho$ along with eq. (2.1) and the obvious bound

$$(\Phi(t))^{-2}\phi(t)t - (\Phi(t))^{-1} \geq (\Phi(t))^{-1}(p_\Phi - 1) > 0$$

yield

$$(p_\Phi - 1)\int_\Omega \xi^{q_\Phi}(\Phi(\mathbf{w}))^{-1}\phi(|\nabla u|)|\nabla u| \leq 4\varepsilon q_\Phi \int_\Omega \xi^{q_\Phi}(\Phi(\mathbf{w}))^{-1}\frac{\mathbf{w}}{\varepsilon\xi}\phi(|\nabla u|) + M\int_\Omega \xi^{q_\Phi}(\Phi(\mathbf{w}))^{-1}\phi\Big(\frac{v}{R}\Big)\frac{\rho\psi}{\rho},$$

where $\varepsilon > 0$ is sufficiently small. By Young's inequality (with $t = \mathbf{w}/\varepsilon\xi$ and $s = |\nabla u|$),

$$(p_\Phi - 1 - 4\varepsilon q_\Phi)\int_\Omega \xi^{q_\Phi}(\Phi(\mathbf{w}))^{-1}\phi(|\nabla u|)|\nabla u| \leq 4q_\Phi \int_\Omega \xi^{q_\Phi-1}(\Phi(\mathbf{w}))^{-1}\mathbf{w}\phi\left(\frac{\mathbf{w}}{\varepsilon\xi}\right) + M\rho\int_\Omega \xi^{q_\Phi}(\Phi(\mathbf{w}))^{-1}\phi\Big(\frac{v}{R}\Big)\mathbf{w}.$$

By (1.4), $\phi(\mathbf{w}/\varepsilon\xi) \leq q_\Phi \phi(\mathbf{w})/(\varepsilon\xi)^{q_\Phi - 1}$ and since $\phi(v/R) \leq \phi(\psi/R) \leq \phi(\psi/\rho) = \phi(\mathbf{w})$,

$$(p_\Phi - 1 - 4\varepsilon q_\Phi) \int_\Omega \xi^{q_\Phi} (\Phi(\mathbf{w}))^{-1} \phi(|\nabla u|)|\nabla u| dx \leq \left(\frac{4q_\Phi^3}{\varepsilon^{q_\Phi - 1}} + M q_\Phi \right) \int_{B_\rho} 1 dx.$$

If ε is chosen sufficiently small such that $\varepsilon < (p_\Phi - 1)/4q_\Phi$ then

$$\int_{B_\rho} \xi^{q_\Phi} \frac{\phi(|\nabla u|)|\nabla u|}{\Phi(\mathbf{w})} dx \leq C_5 \operatorname{vol}(B_\rho),$$

where $\operatorname{vol}(B_\rho)$ is the Lebesgue measure of B_ρ and $C_5 = C_5(\mathfrak{a}, p_\Phi, q_\Phi) > 0$. Since $|\nabla\psi|\, \phi(\mathbf{w}) \leq |\nabla u|\, \phi(|\nabla u|) + \phi(\mathbf{w})\mathbf{w}$ and as $\Phi(\mathbf{w}) \leq \phi(\mathbf{w})\mathbf{w}$ the following bound holds

$$\int_{B_{\rho/2}} \frac{|\nabla\psi|}{\psi} dx \leq \frac{1}{\rho} \left(\int_{B_{\rho/2}} \frac{|\nabla u|\, \phi(|\nabla u|)}{\Phi(\mathbf{w})} dx + \int_{B_{\rho/2}} 1 dx \right) \leq 2^{N-1} \omega_N \left(C_5 + \frac{1}{2^N} \right) \left(\frac{\rho}{2} \right)^{N-1},$$

where ω_N is the Lebesgue measure of the unit ball in $\mathbb{R}^N$. The result [7, Theorem 7.21], with $z = \log(\psi)$ and $\Omega' = B_R$, implies the existence of a positive constant D_N such that

$$\left(\int_{B_R} e^{p_0\, z} dx \right) \left(\int_{B_R} e^{-p_0\, z} dx \right) \leq D_N^2\, (2R)^{2N},$$

where $p_0 = p_0(\mathfrak{a}, p_\Phi, q_\Phi, N) = 2\sigma_0/(2^{2N} C_5 + 2^N)$ and $\sigma_0 = \sigma_0(N)$ is the constant produced by [7, Theorem 7.21]. Along with estimate (2.9) (with $p = p_0$) the latter yields

$$⨍_{B_R} \psi^{p_0} dx \leq \frac{\mathcal{C}^{p_0}}{(1-\sigma)^{N q_\Phi}} \left(\frac{D_N\, 2^N}{\omega_N} \right)^2 \left(\inf_{B_{R/2}} u + RL + \chi \right)^{p_0}.$$

The conclusion follows after passing to the limit $\chi \to 0$. The proof is complete. □

A combination of Corollary 2.5 and Corollary 2.7 produces the following improved version of the Harnack inequality.

Corollary 2.8 (Harnack inequality in arbitrary balls)**.** *Let u be a locally bounded and nonnegative solution of equation* (1.1)*. Let $B_R \subset\subset \Omega$ be a ball of radius $0 < R \leq 1$ and $\sigma \in (0,1)$. Then there exist positive constants p_0 and $\mathcal{M}$, which depend only on $\mathfrak{a}, p_\Phi, q_\Phi$ and N, such that*

$$\sup_{B_{\sigma R}} u \leq \frac{\mathcal{M}}{(1-\sigma)^{2N q_\Phi / p_0}} \left(\inf_{B_{\sigma R}} u + LR \right),$$

where L is any real nonnegative constant such that $\mathfrak{b} \leq \phi(L)$ and $B_{\sigma R}$ is the ball of radius σR concentric with B_R.

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