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ZIG-ZAG FACIAL TOTAL-COLORING OF PLANE GRAPHS

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Abstract. In this paper we introduce the concept of zig-zag facial total-coloring of plane graphs. We obtain lower and upper bounds for the minimum number of colors which is necessary for such a coloring. Moreover, we give several sharpness examples and formulate some open problems.

Keywords: plane graph, facial coloring, total-coloring, zig-zag coloring.

Mathematics Subject Classification: 05C10, 05C15.

1. INTRODUCTION AND NOTATIONS

All graphs considered in this paper are connected and simple. We use a standard graph theory terminology according to Bondy and Murty [2]. However, we recall some important notions.

A *plane graph* is a particular drawing of a planar graph in the Euclidean plane. Let G be a plane graph with vertex set V, edge set E and face set F. The boundary of a face f is the boundary in the usual topological sense. It is a collection of all edges and vertices contained in the closure of f that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of initial vertex and direction, and is called the *boundary walk* of the face f. We denote the boundary walk of a face f by $\partial(f)$. Two distinct edges are *facially adjacent* in G if they are consecutive edges on the boundary walk of a face of G. Two distinct elements of $V \cup E$ are *facially adjacent* in G if they are incident elements, adjacent vertices or facially adjacent edges.

A *facial edge-coloring* of G is an edge-coloring such that any two facially adjacent edges receive different colors. A *facial total-coloring* of G is a total-coloring such that any two facially adjacent elements receive different colors. Facial edge-coloring was first studied for the family of cubic bridgeless plane graphs and for the family of plane triangulations. Already Tait [11] observed that the Four Color Problem is equivalent to the problem of facial 3-edge-coloring of plane triangulations and to the problem of facial 3-edge-coloring of cubic bridgeless plane graphs. It is known that every plane graph admits a facial edge-coloring with at most four colors, see [6]. Moreover, Czap and Šugerek [5] proved that every plane graph admits a facial edge-coloring with at most four colors such that at most three colors appear at each vertex. The concept of facial total-coloring of plane graphs was introduced by Fabrici, Jendroľ and Vrbjarová [6]. They showed that every bridgeless plane graph admits a facial total-coloring with at most six colors. Recently, Fabrici, Jendroľ and Voigt [7] strengthen this result. They proved that every plane graph admits a facial list total-coloring with at most six colors.

In this paper we introduce a zig-zag facial total-coloring (ZFT coloring), which strengthens the requirement for the facial total-coloring. The paper was motivated by facial colorings, see [4], and a recent book [9] by Kitaev.

A *zig-zag facial k-total-coloring* of a plane graph G is a facial total-coloring $c : V \cup E \to \{1, \dots, k\}$ such that

$$c(x_i) > \max\{c(x_{i-1}), c(x_{i+1})\} \quad \text{or} \quad c(x_i) < \min\{c(x_{i-1}), c(x_{i+1})\}$$

for any $x_{i-1}x_ix_{i+1} \subseteq \partial(f)$, $f \in F$. In other words,

$$c(x_j) > c(x_{j+1}) < c(x_{j+2}) > c(x_{j+3}) < c(x_{j+4}) > \dots$$

or

$$c(x_j) < c(x_{j+1}) > c(x_{j+2}) < c(x_{j+3}) > c(x_{j+4}) < \dots$$

holds for any $x_jx_{j+1}x_{j+2}x_{j+3}x_{j+4} \dots \subseteq \partial(f)$, $f \in F$. For an example see Figure 1.

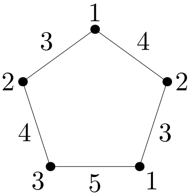


Fig. 1. A zig-zag facial 5-total-coloring of the cycle C_5

The *zig-zag facial total chromatic number* of a plane graph G, denoted by $\chi_z(G)$, is the smallest integer k such that G has a zig-zag facial k-total-coloring.

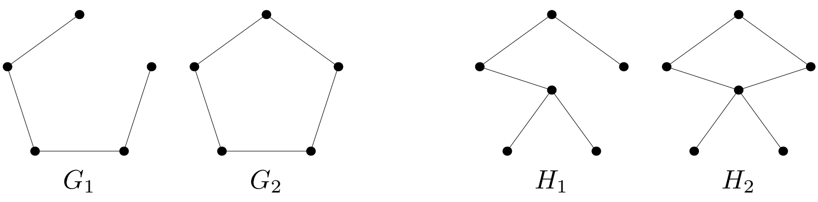
Note that this parameter is not monotone, i.e. there are graphs G_1, G_2 such that $G_1 \subseteq G_2$ and $\chi_z(G_1) < \chi_z(G_2)$ and also exist graphs H_1, H_2 such that $H_1 \subseteq H_2$ and $\chi_z(H_1) > \chi_z(H_2)$. For examples see Figure 2.

Lemma 1.1. *Let G be a connected plane graph and let c be its ZFT coloring. If $c(v) > c(e_v)$ (resp. $c(v) < c(e_v)$) for a vertex v and an indcident edge e_v, then $c(u) > c(e_u)$ (resp. ($c(u) < c(e_u)$) for every vertex u and every indcident edge e_u.*

Proof. It follows from the fact that every boundary walk is an alternating sequence of vertices and edges. □

Corollary 1.2. *Let G be a connected plane graph and let c be its ZFT coloring with colors $1, \ldots, k$. Then 1 or k appears on no vertex (edge).*

Proof. Suppose to the contrary that there is a ZFT coloring which uses both colors 1 and k on the vertices (edges) of G. If G contains a vertex (edge) of color 1, then the incident edges (vertices) have greater colors. Then, by Lemma 1.1, the edges (vertices) incident with a vertex (edge) of color k have colors greater than k, a contradiction. □



$\chi_z(G_1) = 4$ $\chi_z(G_2) = 5$ $\chi_z(H_1) = 5$ $\chi_z(H_2) = 4$

Fig. 2. Graphs which show that the parameter χ_z is not monotone

2. GENERAL BOUNDS

The *simplified medial graph* of a plane graph G is the graph $M(G)$ with vertex set $E(G)$ in which two vertices are adjacent if and only if the corresponding edges are facially adjacent in G. Clearly, the simplified medial graph is planar, moreover, it has a natural planar embedding. Observe that every proper vertex-coloring of $M(G)$ corresponds to a facial edge-coloring of G and vice versa. Let $\chi(G)$ denote the chromatic number of G. Since every planar graph G admits a proper vertex-coloring with at most four colors [1], i.e. $\chi(G) \leq 4$, we have $\chi(M(G)) \leq 4$.

Lemma 2.1. *Let G be a connected plane graph with at least three vertices and $\chi(G) = k$. Then $\chi_z(G) \geq k + 2$.*

Proof. First, let $k = 2$. Suppose to the contrary that G has a ZFT coloring c with colors $1, 2, 3$. Since $1 < 2 < 3$, there is no edge of color 2. Therefore, c uses 1 and 3 on the edges of G, which contradicts Corollary 1.2.

Now, assume that $k \in \{3, 4\}$. Corollary 1.2 implies that there is no plane graph with $\chi(G) = \chi_z(G) = k$. Suppose to the contrary that there is a plane graph H such that $\chi(H) = k$ and $\chi_z(H) = k + 1$. Let c be a ZFT coloring of H with colors $1, \ldots, k+1$. By Corollary 1.2, c uses either $1, \ldots, k$ or $2, \ldots, k+1$ on the vertices of H.

First assume that there is a vertex v of color 1. In this case the edges incident with v have greater colors than $c(v)$. Then, by Lemma 1.1, $c(u) < c(e_u)$ for every vertex u and every indcident edge e_u. Consequently, every edge incident with a vertex of color k has color $k + 1$. Therefore, every vertex of color k has degree one. This implies that the chromatic number of H is at most $k-1$ (since the leaves can be recolored), a contradiction.

If we assume that there is a vertex v of color $k+1$, then we obtain a contradiction by analogous arguments. □

Lemma 2.2. *Let G be a connected plane graph with minimum degree at least three and $\chi(G)=k$. If every vertex of G has an odd degree, then $\chi_z(G) \geq k+3$.*

Proof. Suppose to the contrary that G admits a ZFT coloring c with colors $1,\dots,k+2$. Clearly, at least three colors appear at each vertex, hence

– if $k=2$, then every vertex has color either 1 or 4. This contradicts Corollary 1.2;
– if $k=3$, then no vertex has color 3. Therefore, G has vertices u,v such that $c(u) \in \{1,2\}$ and $c(v) \in \{4,5\}$. $c(u) \in \{1,2\}$ with Lemma 1.1 implies that $c(w) < c(e_w)$ for every vertex w and every incident edge e_w, but $c(v) \in \{4,5\}$ implies $c(w) > c(e_w)$, a contradiction;
– if $k=4$, then G has vertices u,v such that $c(u) \leq 3$ and $c(v) \geq 4$. We obtain a contradiction by analogous arguments as in the previous case. □

Lemma 2.3. *Let G be a connected plane graph with at least two vertices and $\chi(M(G))=t$. Then $\chi_z(G) \geq t+2$.*

Proof. Corollary 1.2 implies that $\chi_z(G) > \chi(M(G))$. Suppose to the contrary that there is a connected plane graph H such that $\chi(M(H))=t$ and $\chi_z(H)=t+1$. Let c be a ZFT coloring of H with colors $1,\dots,t+1$. From Corollary 1.2 it follows that c uses either $1,\dots,t$ or $2,\dots,t+1$ on the edges of H.

Assume that H has an edge of color 1. Then the incident vertices have greater colors. Then, by Lemma 1.1, the endvertices of every edge of color t have the same color $t+1$, a contradiction.

If we assume that there is an edge of color $t+1$, then we obtain a contradiction by analogous arguments. □

Lemma 2.4. *Let G be a connected plane graph. Then $\chi_z(G) \leq \chi(G) + \chi(M(G))$.*

Proof. First we color the vertices of G such that adjacent vertices receive distinct colors. We use the colors $1,2,\dots,\chi(G)$. Then we color the edges of G such that facially adjacent edges receive distinct colors. We use the colors $\chi(G)+1, \chi(G)+2, \dots, \chi(G)+\chi(M(G))$. □

Corollary 2.5. *If G is a connected plane graph, then $\chi_z(G) \leq 8$. Moreover, $\chi_z(G) \leq 7$ if*

(a) *G is a connected triangle-free plane graph or*
(b) *G is a plane triangulation.*

Proof. Since $\chi(G) \leq 4$ and $\chi(M(G)) \leq 4$ hold for any plane graph G, we have $\chi_z(G) \leq 8$.

(a) follows from Grötzsch's theorem [8], which states that every triangle-free plane graph admits a proper vertex-coloring with at most three colors.

For plane triangulations the facial edge-coloring problem is equivalent to the four color problem, see e.g. the book of Saaty and Kainen [10]. From the Four Color Theorem it follows (see [10, p. 103]) that the edges of any plane triangulation G can be colored with three colors so that the edges bounding every face are colored distinctly, i.e. $\chi(M(G)) = 3$, which implies (b). □

3. SHARPNESS RESULTS

From Lemma 2.4 it follows that, if there exists a plane graph G with $\chi_z(G) = 8$, then necessarily $\chi(G) = \chi(M(G)) = 4$. In the following we determine $\chi_z(G)$ for given $\chi(G)$ and $\chi(M(G))$.

If $\chi(M(G)) = 2$, then we obtain the exact value of $\chi_z(G)$ from Lemma 2.1 and Lemma 2.4.

Theorem 3.1. *Let G be a connected plane graph such that $\chi(G) = k$ and $\chi(M(G)) = 2$. Then $\chi_z(G) = k + 2$.*

Note that there are infinitely many plane graphs such that $\chi(G) = k$ with $k \in \{2, 3, 4\}$ and $\chi(M(G)) = 2$. We can construct an infinite family in the following way. First we take a 2-connected plane graph H with chromatic number k. From H we obtain a new plane graph G such that we insert into each face f of size $d(f)$ exactly $d(f)$ vertices, thereafter we join every vertex of f with exactly one new vertex inserted to f. If we color the original edges of H with color x and the new edges with color y, then we obtain a facial edge-coloring of G with two colors. Moreover, the chromatic number of G is k, because it contains a k-chromatic subgraph.

Thus we may assume that $\chi(M(G)) \geq 3$. First, let us consider the case $\chi(G) = 4$.

Theorem 3.2. *Let G be a connected plane graph such that $\chi(G) = 4$ and $\chi(M(G)) = 3$. Then $6 \leq \chi_z(G) \leq 7$. Moreover, the bounds are tight.*

Proof. The lower bound six follows from Lemma 2.1 and the upper bound seven follows from Lemma 2.4. So it suffices to show that the bounds are tight.

Let W be a wheel on $(6n + 3) + 1$ vertices, $n \geq 0$. Since the boundary of the outer face is an odd cycle and the central vertex is adjacent with the other vertices we have $\chi(W) = 4$. A facial 3-edge-coloring of W can be obtained so that for the edges on the outer face we use the pattern $1, 2, 3, 1, 2, 3, \ldots, 1, 2, 3$, i.e. $\chi(M(W)) = 3$. Since each vertex of W has odd degree, from Lemma 2.2 it follows that $\chi_z(W) \geq 7$.

For a graph with $\chi(G) = 4$, $\chi(M(G)) = 3$ and $\chi_z(G) = 6$ see Figure 3.

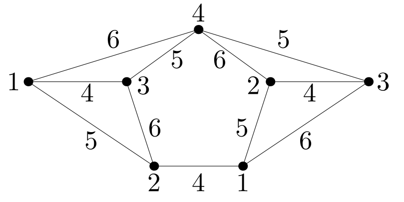


Fig. 3. A plane graph G with $\chi(G) = 4$, $\chi(M(G)) = 3$ and its ZFT 6-coloring

□

Theorem 3.3. *Let G be a connected plane graph such that $\chi(G) = 4$ and $\chi(M(G)) = 4$. Then $6 \le \chi_z(G) \le 8$. Moreover, there are graphs G_1 and G_2 with the desired properties such that $\chi_z(G_1) = 6$ and $\chi_z(G_2) = 7$.*

Proof. The lower bound six follows from Lemma 2.1 and the upper bound eight follows from Lemma 2.4.

First we show that there is a plane graph G such that $\chi(G) = 4$, $\chi(M(G)) = 4$ and $\chi_z(G) = 6$. The graph G shown in Figure 4 admits a ZFT 6-coloring. Its chromatic number is four, since it contains K_4 (the complete graph on four vertices) as a subgraph. Since it has a vertex of degree three, every facial edge-coloring uses three different colors on the incident edges. It is easy to see that no such partial coloring can be extended to a facial 3-edge-coloring of G.

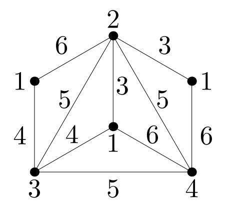


Fig. 4. A plane graph G with $\chi(G) = 4$, $\chi(M(G)) = 4$ and its ZFT 6-coloring

Now we show that there are plane graphs such that $\chi(G) = 4$, $\chi(M(G)) = 4$ and $\chi_z(G) = 7$. Let W be a wheel on $6n$ vertices, $n \ge 1$. Since the boundary of the outer face is an odd cycle and the central vertex is adjacent with the other vertices we have $\chi(W) = 4$. It is an easy exercise to show that $\chi(M(W)) = 4$. So it is sufficient to prove that $\chi_z(W) = 7$.

From Lemma 2.2 it follows that $\chi_z(W) \ge 7$. A ZFT 7-coloring of W can be defined in the following way: Color the central vertex with color 4 and the vertices on the outer face with pattern $1, 2, 3, 2, 3, \ldots, 2, 3$. Color the edge with endvertices 1 and 2 with color 4 and use the colors $5, 6, 7$ for the other edges. □

Conjecture 3.4. *There is no plane graph G with $\chi_z(G) = 8$.*

The following results are related to graphs with $\chi(G) = 3$.

Theorem 3.5. *Let G be a connected plane graph such that $\chi(G) = 3$ and $\chi(M(G)) = 3$. Then $5 \leq \chi_z(G) \leq 6$. Moreover, the bounds are tight.*

Proof. The lower bound five follows from Lemma 2.1 and the upper bound six follows from Lemma 2.4. So it suffices to show that the bounds are tight.

Let $C = v_1 v_2 \ldots v_{2k+1}$ be a cycle on $2k+1$ vertices, $k \geq 1$. Clearly, $\chi(C) = 3$ and $\chi(M(C)) = 3$. A ZFT 5-coloring c of C can be defined in the following way: $c(v_1) = 1$, $c(v_{2i}) = 2$, $c(v_{2i+1}) = 3$ for $i = 1, 2, \ldots, k$; $c(v_1 v_2) = 3$, $c(v_{2i} v_{2i+1}) = 4$, $c(v_{2i+1} v_{2i+2}) = 5$ for $i = 1, 2, \ldots, k$ where $v_{2k+2} := v_1$.

Now let H be a nonbipartite bridgeless cubic plane graph different from K_4. By Brooks' theorem [3] we have $\chi(H) = 3$. Bridgeless planar cubic graphs admit proper edge-colorings with three colors (this is an equivalent form of the Four Color Theorem, see [10]), so $\chi(M(H)) = 3$. From Lemma 2.2 it follows that $\chi_z(H) \geq 6$. □

Theorem 3.6. *Let G be a connected plane graph such that $\chi(G) = 3$ and $\chi(M(G)) = 4$. Then $6 \leq \chi_z(G) \leq 7$.*

Proof. The lower bound six follows from Lemma 2.3 and the upper bound seven follows from Lemma 2.4.

Now we show that there are infinitely many plane graphs such that $\chi(G) = 3$, $\chi(M(G)) = 4$ and $\chi_z(G) = 6$.

Let W be a wheel on $(6n+4)+1$ vertices, $n \geq 0$. It is easy to see that $\chi(G) = 3$ and $\chi(M(G)) = 4$. A ZFT 6-coloring of W can be defined in the following way: Color the central vertex with color 6 and the vertices on the outer face with colors 4 and 5 alternately. Then color the edges with endvertices 5 and 6 with color 4 and the edges with endvertices 4 and 6 with color 3. Finally, color the edges on the outer face with colors 1 and 2 alternately. □

Problem 3.7. *Is there a connected plane graph G such that $\chi(G) = 3$, $\chi(M(G)) = 4$ and $\chi_z(G) = 7$?*

For bipartite graphs we obtain the following result immediately from Lemma 2.3 and Lemma 2.4.

Theorem 3.8. *If G is a connected bipartite plane graph with $\chi(M(G)) = t$, then $\chi_z(G) = t + 2$.*

Note, that there are infinitely many plane graphs with $\chi(G) = 2$ and $\chi(M(G)) = 3$, for example, bipartite cubic plane graphs.

Problem 3.9. *Is there a connected plane graph G such that $\chi(G) = 2$ and $\chi(M(G)) = 4$?*

The cases when $\chi(G) = 1$ or $\chi(M(G)) = 1$ are trivial.

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