

ESSENTIAL NORM OF AN INTEGRAL-TYPE OPERATOR FROM ω -BLOCH SPACES TO μ -ZYGmund SPACES ON THE UNIT BALL

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Communicated by Stevo Stević

Abstract. In this paper, we give an estimate for the essential norm of an integral-type operator from ω -Bloch spaces to μ -Zygmund spaces on the unit ball.

Keywords: essential norm, integral-type operator, ω -Bloch space, μ -Zygmund space.

Mathematics Subject Classification: 47B33, 30H10.

1. INTRODUCTION

Let μ be a positive continuous function on $[0, 1)$. We say that μ is normal if there exist positive numbers a and b , $0 < a < b$, and $\delta \in [0, 1)$ such that (see [19]),

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\mu(r)}{(1-r)^a} = 0,$$
$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r \rightarrow 1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

Let \mathbb{B} be the open unit ball of \mathbb{C}^n , $H(\mathbb{B})$ be the space of all holomorphic functions in \mathbb{B} . When $n = 1$, \mathbb{B} is the open unit disk \mathbb{D} of the complex plane and $H(\mathbb{D})$ is the holomorphic function space on \mathbb{D} .

For $f \in H(\mathbb{B})$, the radial derivative and complex gradient of f at z will be denoted by $\Re f(z)$ and $\nabla f(z)$, respectively, that is,

$$\Re f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z) \text{ and } \nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \frac{\partial f}{\partial z_2}(z), \dots, \frac{\partial f}{\partial z_n}(z) \right).$$

Let ω be normal. An $f \in H(\mathbb{B})$ is said to belong to ω -Bloch space (or Bloch type space), denoted by $\mathcal{B}_\omega = \mathcal{B}_\omega(\mathbb{B})$, if

$$\|f\|_{\mathcal{B}_\omega} = |f(0)| + \sup_{z \in \mathbb{B}} \omega(|z|)|\Re f(z)| < \infty.$$

The little ω -Bloch space $\mathcal{B}_{\omega,0} = \mathcal{B}_{\omega,0}(\mathbb{B})$, consists of all $f \in \mathcal{B}_\omega$ such that

$$\lim_{|z| \rightarrow 1} \omega(|z|)|\Re f(z)| = 0.$$

\mathcal{B}_ω is a Banach space with the norm $\|\cdot\|_{\mathcal{B}_\omega}$. When $0 < \alpha < \infty$ and $\omega(t) = (1 - t^2)^\alpha$, we get the α -Bloch space (often also called Bloch type space), denoted by $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{B})$. In particular, when $\omega(t) = 1 - t^2$, we get the Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{B})$. See [32, 36] for more information of the Bloch space \mathcal{B} and ω -Bloch space \mathcal{B}_ω on the unit ball.

Suppose μ is normal on $[0, 1)$. We say that an $f \in H(\mathbb{B})$ belongs to $\mathcal{Z}_\mu = \mathcal{Z}_\mu(\mathbb{B})$ if

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + \sup_{z \in \mathbb{B}} \mu(|z|)|\Re^2 f(z)| < \infty,$$

where $\Re^2 f(z) = \Re(\Re f)(z)$. Under the above norm, \mathcal{Z}_μ becomes a Banach space. \mathcal{Z}_μ will be called the μ -Zygmund space or the Zygmund type space. The little μ -Zygmund space, denoted by $\mathcal{Z}_{\mu,0}$, is the space consisting of all $f \in \mathcal{Z}_\mu$ such that

$$\lim_{|z| \rightarrow 1} \mu(|z|)|\Re^2 f(z)| = 0.$$

When $\mu(r) = 1 - r^2$, the μ -Zygmund space becomes the Zygmund space \mathcal{Z} . See [16, 24, 36] for more information of the Zygmund space on the unit ball. There has been some recent interest in studying of various concrete operators, including integral-type ones, from or to the Zygmund type spaces, see, for example, [2, 11, 14, 15, 17, 18, 24, 27]. Some generalizations of the Zygmund type spaces and operators on them, can be found in [29, 30].

Assume that $g \in H(\mathbb{B}), g(0) = 0$ and φ is a holomorphic self-map of \mathbb{B} . In [20], Stević introduced the following integral-type operator P_φ^g on $H(\mathbb{B})$:

$$P_\varphi^g f(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}), z \in \mathbb{B}.$$

Some results on the operator has been got in [7, 21–23, 25, 26, 28, 31, 38].

Here we consider a particular case of P_φ^g when $\varphi(z) = z$, that is the operator

$$P_g f(z) = \int_0^1 f(tz)g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}), z \in \mathbb{B}.$$

For all $g \in H(\mathbb{B})$, $P_{\Re g}$ is just the extended Cesàro operator (or Riemann-Stieltjes operator), which was introduced in [3], and studied in [1, 3–5, 8–10, 12, 13, 16, 32, 34]. Some related integral-type operators can be found in [1, 17, 20–23, 37, 38].

In [17], Li and Stević studied the boundedness and compactness of the operator $P_g : \mathcal{B}_\omega(\mathcal{B}_{\omega,0}) \rightarrow \mathcal{Z}_\mu(\mathcal{Z}_{\mu,0})$. Motivated by [17], in this paper we investigate the essential norm of $P_g : \mathcal{B}_\omega(\mathcal{B}_{\omega,0}) \rightarrow \mathcal{Z}_\mu$.

Recall that the essential norm of $P_g : X \rightarrow Y$, denoted by $\|P_g\|_{e,X \rightarrow Y}$, is defined by

$$\|P_g\|_{e,X \rightarrow Y} = \inf\{\|P_g - K\|_{X \rightarrow Y}; K \text{ is a compact operator from } X \text{ to } Y\}.$$

In this paper, constants are denoted by C , they are positive and may differ from one occurrence to the next. We say that $A \lesssim B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means $A \lesssim B \lesssim A$.

2. AUXILIARY RESULTS

In this section, we give some auxiliary results which will be used in proving the main results of this paper. They are incorporated in the lemmas which follow.

Lemma 2.1 ([6]). *Suppose ω is normal on $[0, 1]$. Then there exists $\omega_* \in H(\mathbb{D})$ such that*

- (i) $\omega_*(t)$ is positive and increasing on $[0, 1]$,
- (ii) for all $z \in \mathbb{D}$, $|\omega_*(z)| \leq \omega_*(|z|)$,
- (iii) $0 < \inf_{0 < r < 1} \omega(r)\omega_*(r) \leq \sup_{0 < r < 1} \omega(r)\omega_*(r) < \infty$.

In the rest of the paper we will always use ω_* to denote the analytic function related to ω in Lemma 2.1.

To study the compactness, we need the following lemma, which can be get by Lemma 2.10 in [33].

Lemma 2.2. *Suppose ω and μ are normal. If $T : \mathcal{B}_\omega(\mathcal{B}_{\omega,0}) \rightarrow \mathcal{Z}_\mu$ is bounded, then T is compact if and only if whenever $\{f_k\}$ is bounded in $\mathcal{B}_\omega(\mathcal{B}_{\omega,0})$ and $f_k \rightarrow 0$ uniformly on compact subsets of \mathbb{B} , $\lim_{k \rightarrow \infty} \|Tf_k\|_{\mathcal{Z}_\mu} = 0$.*

Lemma 2.3 ([21]). *Assume $g \in H(\mathbb{B})$ and $g(0) = 0$. Then, for every $f \in H(\mathbb{B})$,*

$$\Re P_g f(z) = f(z)g(z).$$

By some calculations, we have the following lemma.

Lemma 2.4 ([2, Lemma 2]). *Suppose ω is normal. Then the following statements hold.*

- (i) *There exists a $\delta \in (0, 1)$, such that ω is decreasing on $[\delta, 1]$ and $\lim_{t \rightarrow 1} \omega(t) = 0$.*
- (ii) *Fix $\alpha > 1, \beta \in (0, 1)$. When $t \in (0, 1), s \in (\beta, 1)$,*

$$\omega(t) \approx \omega(t^\alpha) \approx \frac{1}{\omega_*(t)}, \quad \int_0^{s^\alpha} \frac{1}{\omega(t)} dt \approx \int_0^s \frac{1}{\omega(t)} dt.$$

- (iii) *For any $z \in \mathbb{D}$, $|\int_0^z \omega_*(\eta) d\eta| \lesssim \int_0^{|z|} \omega_*(t) dt$. If $|\eta| \leq |z|, \omega(|z|)|\omega_*(\eta)| < C$.*

Lemma 2.5 ([35, Lemmas 2.2 and 2.4]). *Suppose ω is normal. Then the following statements hold.*

- (i) *If $f \in \mathcal{B}_\omega$, then $|f(z)| \lesssim \left(1 + \int_0^{|z|} \frac{1}{\omega(t)} dt\right) \|f\|_{\mathcal{B}_\omega}$.*
- (ii) *If $\int_0^1 \frac{1}{\omega(t)} dt = \infty$ and $f \in \mathcal{B}_{\omega,0}$, then $\lim_{|z| \rightarrow 1} \frac{|f(z)|}{\int_0^{|z|} \frac{1}{\omega(t)} dt} = 0$.*
- (iii) *If $\int_0^1 \frac{1}{\omega(t)} dt < \infty$, $\{f_k\}$ is bounded in $\mathcal{B}_{\omega,0}$ and converges to 0 uniformly on compact subsets of \mathbb{B} , then $\lim_{k \rightarrow \infty} \sup_{z \in \mathbb{B}} |f_k(z)| = 0$.*

The following lemma gives some folklore estimates, but we will prove it for the completeness.

Lemma 2.6. *Suppose ω is normal, $0 < r, s < 1$ and $f \in H(\mathbb{B})$. Then, for all $|z| \leq s$,*

$$|\Re f(z)| \leq \frac{2n}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)| \quad \text{and} \quad |f(z) - f(rz)| \leq \frac{2n(1-r)}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)|.$$

Proof. Set $z = (z_1, z_2, \dots, z_n) \in \mathbb{B}$ such that $|z| \leq s$. For $i = 1, 2, \dots, n$, let

$$\Gamma_{z,i} = \left\{ \eta \in \mathbb{D} : |\eta - z_i| = \frac{1-s}{2} \right\},$$

and

$$\lambda(z, i, \eta) = (z_1, \dots, z_{i-1}, \eta, z_{i+1}, \dots, z_n), \quad \eta \in \Gamma_{z,i}.$$

Since $f \in H(\mathbb{B})$, $\frac{\partial f}{\partial z_i} \in H(\mathbb{B})$. Taking f as a one complex variable function about the i -th component of z , by Cauchy’s integral formula, we have

$$\begin{aligned} \left| \frac{\partial f}{\partial z_i}(z) \right| &= \frac{1}{2\pi} \left| \int_{\Gamma_{z,i}} \frac{f(\lambda(z, i, \eta))}{(\eta - z_i)^2} d\eta \right| \\ &= \frac{1}{\pi(1-s)} \left| \int_0^{2\pi} f\left(\lambda\left(z, i, z_i + \frac{1-s}{2} e^{i\theta}\right)\right) e^{-i\theta} d\theta \right| \leq \frac{2}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)|. \end{aligned}$$

Here we use the change $\eta = z_i + \frac{1-s}{2} e^{i\theta}$ ($0 \leq \theta \leq 2\pi$). Then,

$$|\Re f(z)| \leq \frac{2n}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)|.$$

When $|z| \leq s$,

$$\begin{aligned} |f(z) - f(rz)| &= \left| \int_r^1 \frac{df(tz)}{dt} dt \right| = \left| \int_r^1 \langle (\nabla f)(tz), \bar{z} \rangle dt \right| \\ &\leq (1-r) \sup_{|z| \leq s} |\nabla f(z)| \leq \frac{2n(1-r)}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)|. \end{aligned}$$

The proof is complete. □

3. MAIN RESULTS AND PROOFS

Theorem 3.1. Assume that μ, ω are normal and $g \in H(\mathbb{B})$ such that $g(0) = 0$ and $\int_0^1 \frac{1}{\omega(t)} dt = \infty$. If $P_g : \mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu$ is bounded, then

$$\begin{aligned} \|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu} &\approx \|P_g\|_{e, \mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu} \\ &\approx \limsup_{|z| \rightarrow 1} \mu(|z|) |\Re g(z)| \int_0^{|z|} \frac{1}{\omega(t)} dt + \limsup_{|z| \rightarrow 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|. \end{aligned}$$

Proof. The following inequality is obvious.

$$\|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu} \geq \|P_g\|_{e, \mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu}.$$

First, we find the lower estimate of $\|P_g\|_{e, \mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu}$.

Suppose $K : \mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu$ is compact and $\{z_k\} \subset \mathbb{B}$ such that $|z_k| \geq \frac{1}{2}$ and $\lim_{k \rightarrow \infty} |z_k| = 1$. Let

$$f_k(z) = \frac{2p_k^3(z)}{p_k^2(z_k)} - \frac{3p_k^2(z)}{p_k(z_k)} \quad \text{and} \quad h_k(z) = \frac{p_k^3(z)}{p_k^2(z_k)} - \frac{p_k^2(z)}{p_k(z_k)},$$

where $p_k(z) = \int_0^{\langle z, z_k \rangle} \omega_*(t) dt$.

By Lemmas 2.1 and 2.4, we have

$$|p_k(z)| \leq \int_0^{|\langle z, z_k \rangle|} \omega_*(t) dt \leq \int_0^{|z||z_k|} \omega_*(t) dt,$$

and

$$|\Re p_k(z)| = |\langle z, z_k \rangle \omega_*(\langle z, z_k \rangle)| \leq \omega_*(|z||z_k|).$$

When $j = 2, 3$, $\Re p_k^j(z) = j p_k^{j-1}(z) \Re p_k(z)$. Since $\int_0^1 \frac{1}{\omega(t)} dt = \infty$, by Lemmas 2.1 and 2.4, we have the following statements.

- (i) $f_k \in \mathcal{B}_{\omega,0}$, $\|f_k\|_{\mathcal{B}_\omega} \lesssim 1$, $\Re f_k(z_k) = 0$, $|f_k(z_k)| \approx \int_0^{|z_k|} \frac{1}{\omega(t)} dt$.
- (ii) $h_k \in \mathcal{B}_{\omega,0}$, $\|h_k\|_{\mathcal{B}_\omega} \lesssim 1$, $h_k(z_k) = 0$, $|\Re h_k(z_k)| \approx \frac{1}{|\omega(z_k)|}$.
- (iii) Both $\{f_k\}$ and $\{h_k\}$ converge to 0 uniformly on compact subsets of \mathbb{B} as $k \rightarrow \infty$.

By Lemma 2.3,

$$\begin{aligned} \|P_g - K\|_{\mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu} &\gtrsim \|(P_g - K)f_k\|_{\mathcal{Z}_\mu} \\ &\geq \mu(|z_k|) |\Re g(z_k)| |f_k(z_k)| - \mu(|z_k|) |g(z_k)| |\Re f_k(z_k)| - \|Kf_k\|_{\mathcal{Z}_\mu} \\ &\approx \mu(|z_k|) |\Re g(z_k)| \int_0^{|z_k|} \frac{1}{\omega(t)} dt - \|Kf_k\|_{\mathcal{Z}_\mu}. \end{aligned}$$

(3.1)

Letting $k \rightarrow \infty$, by Lemma 2.2, we have

$$\|P_g - K\|_{\mathcal{B}_\omega, 0 \rightarrow \mathcal{X}_\mu} \gtrsim \limsup_{k \rightarrow \infty} \mu(|z_k|) |\Re g(z_k)| \int_0^{|z_k|} \frac{1}{\omega(t)} dt.$$

Since K and $\{z_k\}$ are arbitrary, we obtain

$$\|P_g\|_{e, \mathcal{B}_\omega, 0 \rightarrow \mathcal{X}_\mu} \gtrsim \limsup_{|z| \rightarrow 1} \mu(|z|) |\Re g(z)| \int_0^{|z|} \frac{1}{\omega(t)} dt.$$

Replacing f_k by h_k in (3.1), we similarly obtain that

$$\|P_g\|_{e, \mathcal{B}_\omega, 0 \rightarrow \mathcal{X}_\mu} \gtrsim \limsup_{|z| \rightarrow 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|.$$

Therefore

$$\|P_g\|_{e, \mathcal{B}_\omega, 0 \rightarrow \mathcal{X}_\mu} \gtrsim \limsup_{|z| \rightarrow 1} \mu(|z|) |\Re g(z)| \int_0^{|z|} \frac{1}{\omega(t)} dt + \limsup_{|z| \rightarrow 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|.$$

Next, we find the upper estimate of $\|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{X}_\mu}$.

For $0 < r < 1$ and $f \in \mathcal{B}_\omega$, let $f_r(z) = f(rz)$ and $T_{g,r}f = P_g f_r$. Since $\|f_r\|_{\mathcal{B}_\omega} \lesssim \|f\|_{\mathcal{B}_\omega}$, we see that

$$\|T_{g,r}f\|_{\mathcal{X}_\mu} = \|P_g f_r\|_{\mathcal{X}_\mu} \lesssim \|P_g\|_{\mathcal{B}_\omega \rightarrow \mathcal{X}_\mu} \|f\|_{\mathcal{B}_\omega}.$$

So $T_{g,r} : \mathcal{B}_\omega \rightarrow \mathcal{X}_\mu$ is bounded.

Using the test functions $h(z) = 1$ and $q(z) = z_j$ ($j = 1, 2, \dots, n$), we have

$$M_1 := \sup_{z \in \mathbb{B}} \mu(|z|) |\Re g(z)| < \infty \text{ and } M_2 := \sup_{z \in \mathbb{B}} \mu(|z|) |g(z)| < \infty. \tag{3.2}$$

By Lemma 2.3,

$$\|T_{g,r}f\|_{\mathcal{X}_\mu} = \sup_{z \in \mathbb{B}} \mu(|z|) |\Re^2 T_{g,r}f(z)| \leq M_1 \sup_{z \in \mathbb{B}} |f(rz)| + M_2 \sup_{z \in \mathbb{B}} |(\Re f)(rz)|.$$

By Lemmas 2.2 and 2.6, $T_{g,r}$ is compact.

Assume $\|f\|_{\mathcal{B}_\omega} \leq 1$ and $s \in (\frac{1}{2}, 1)$. By Lemma 2.3,

$$\begin{aligned} \|P_g f - T_{g,r}f\|_{\mathcal{X}_\mu} &= \|P_g(f - f_r)\|_{\mathcal{X}_\mu} \\ &\leq \sup_{|z| \leq s} \mu(|z|) |g(z)| |\Re(f - f_r)(z)| \\ &\quad + \sup_{|z| \leq s} \mu(|z|) |\Re g(z)| |f(z) - f(rz)| \\ &\quad + \sup_{s < |z| < 1} \mu(|z|) |g(z)| |\Re(f - f_r)(z)| \\ &\quad + \sup_{s < |z| < 1} \mu(|z|) |\Re g(z)| |f(z) - f(rz)|. \end{aligned} \tag{3.3}$$

Since $\int_0^1 \frac{1}{\omega(t)} dt = \infty$, by Lemmas 2.5 and 2.6, we have

$$\sup_{|z| \leq s} |f(z) - f(rz)| \leq \frac{2n(1-r)}{1-s} \max_{|z| \leq \frac{1+s}{2}} |f(z)| \lesssim \frac{2n(1-r)}{1-s} \int_0^{\frac{1+s}{2}} \frac{1}{\omega(t)} dt, \tag{3.4}$$

and

$$\sup_{|z| \leq s} |\Re(f - f_r)(z)| = \sup_{|z| \leq s} |(\Re f)(z) - (\Re f)(rz)| \leq \frac{2n(1-r)}{1-s} \max_{z \leq \frac{1+s}{2}} \frac{1}{\omega(|z|)}. \tag{3.5}$$

Since $f \in \mathcal{B}_\omega$ and by Lemma 2.5, we get

$$\sup_{s < |z| < 1} \mu(|z|) |g(z)| |(\Re f - f_r)(z)| \lesssim \sup_{s < |z| < 1} \frac{\mu(|z|) |g(z)|}{\omega(|z|)} \tag{3.6}$$

and

$$\sup_{s < |z| < 1} \mu(|z|) |\Re g(z)| |f(z) - f(rz)| \lesssim \sup_{s < |z| < 1} \mu(|z|) |\Re g(z)| \int_0^{|z|} \frac{dt}{\omega(t)}. \tag{3.7}$$

By (3.2)–(3.7), we have

$$\begin{aligned} \|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu} &\leq \limsup_{r \rightarrow 1} \|P_g - T_{g,r}\|_{\mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu} \\ &\lesssim \sup_{s < |z| < 1} \frac{\mu(|z|) |g(z)|}{\omega(|z|)} + \sup_{s < |z| < 1} \mu(|z|) |\Re g(z)| \int_0^{|z|} \frac{dt}{\omega(t)}. \end{aligned}$$

Letting $s \rightarrow 1$, we get

$$\|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu} \lesssim \limsup_{|z| \rightarrow 1} \mu(|z|) |\Re g(z)| \int_0^{|z|} \frac{1}{\omega(t)} + \limsup_{|z| \rightarrow 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|, \tag{3.8}$$

as desired. The proof is complete. □

Theorem 3.2. *Assume that μ, ω are normal and $g \in H(\mathbb{B})$ such that $g(0) = 0$ and $\int_0^1 \frac{1}{\omega(t)} dt < \infty$. If $P_g : \mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu$ is bounded, then*

$$\|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu} \approx \|P_g\|_{e, \mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu} \approx \limsup_{|z| \rightarrow 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|. \tag{3.9}$$

Proof. We begin with finding the lower estimate of $\|P_g\|_{e, \mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu}$.

Suppose $K : \mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu$ is compact and $\{z_k\} \subset \mathbb{B}$ such that $\lim_{k \rightarrow \infty} |z_k| = 1$ and $|z_k| > \frac{1}{2}$. Let

$$q_k(z) = \omega(|z_k|) \int_0^{\langle z, z_k \rangle} \omega_*^2(\eta) d\eta.$$

By Lemmas 2.1 and 2.4, $\{q_k\}$ are bounded in $\mathcal{B}_{\omega,0}$ and converge to 0 uniformly on compact subsets of \mathbb{D} . By Lemmas 2.2 and 2.5, we see that

$$\lim_{k \rightarrow \infty} \|Kq_k\|_{\mathcal{E}_\mu} = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \sup_{z \in \mathbb{B}} |q_k(z)| = 0. \tag{3.10}$$

$$\begin{aligned} \|P_g - K\|_{\mathcal{B}_{\omega,0} \rightarrow \mathcal{E}_\mu} &\gtrsim \|(P_g - K)q_k\|_{\mathcal{E}_\mu} \\ &\geq \mu(|z_k|)|\Re^2(P_g q_k)(z_k)| - \|Kq_k\|_{\mathcal{E}_\mu} \\ &\geq \mu(|z_k|)|g(z_k)||\Re q_k(z_k)| - \mu(|z_k|)|\Re g(z_k)||q_k(z_k)| \\ &\quad - \|Kq_k\|_{\mathcal{E}_\mu}. \end{aligned} \tag{3.11}$$

Since $\mathcal{B}_{\omega,0} \rightarrow \mathcal{E}_\mu$ is bounded and $f(z) = 1 \in \mathcal{B}_{\omega,0}$, we get

$$\sup_{z \in \mathbb{D}} \mu(|z|)|\Re g(z)| < \infty. \tag{3.12}$$

Letting $k \rightarrow \infty$ in (3.11), by (3.10) and (3.12), we obtain

$$\|P_g - K\|_{\mathcal{B}_{\omega,0} \rightarrow \mathcal{E}_\mu} \gtrsim \limsup_{k \rightarrow \infty} \frac{\mu(|z_k|)}{\omega(|z_k|)} |g(z_k)|.$$

Since K and $\{z_k\}$ are arbitrary, we see that

$$\|P_g\|_{e, \mathcal{B}_{\omega,0} \rightarrow \mathcal{E}_\mu} \gtrsim \limsup_{|z| \rightarrow 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|. \tag{3.13}$$

Next, we find the upper estimate of $\|P_g\|_{e, \mathcal{B}_{\omega} \rightarrow \mathcal{E}_\mu}$.

Since $\int_0^1 \frac{1}{\omega(t)} dt < \infty$, for any given $\varepsilon > 0$, there is a $\xi \in (\frac{1}{2}, 1)$, when $\xi < r < 1$, we have $\int_r^1 \frac{1}{\omega(t)} dt < \varepsilon$.

Fix $r, s \in (\sqrt{\xi}, 1)$. For all $f \in \mathcal{B}_\omega$ with $\|f\|_{\mathcal{B}_\omega} \leq 1$, when $|z| > s$, we have

$$|f(z) - f(rz)| = \left| \int_r^1 \frac{df(tz)}{dt} dt \right| = \left| \int_r^1 \frac{(\Re f)(tz)}{t} dt \right| \leq \int_r^1 \frac{1}{\omega(t|z|)t} dt \lesssim \varepsilon.$$

Thus

$$\sup_{s < |z| < 1} \mu(|z|)|\Re g(z)||f(z) - f(rz)| \lesssim \varepsilon \sup_{z \in \mathbb{B}} \mu(|z|)|\Re g(z)|. \tag{3.14}$$

Then similarly to have (3.8), just instead (3.7) with (3.14), we have

$$\|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{E}_\mu} \lesssim \limsup_{|z| \rightarrow 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)| + \varepsilon \sup_{z \in \mathbb{B}} \mu(|z|)|\Re g(z)|.$$

Since ε is arbitrary, by (3.12),

$$\|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{E}_\mu} \lesssim \limsup_{|z| \rightarrow 1} \frac{\mu(|z|)}{\omega(|z|)} |g(z)|. \tag{3.15}$$

Obviously, we have

$$\|P_g\|_{e, \mathcal{B}_\omega \rightarrow \mathcal{Z}_\mu} \gtrsim \|P_g\|_{e, \mathcal{B}_{\omega,0} \rightarrow \mathcal{Z}_\mu}. \quad (3.16)$$

Then the result follows from (3.13), (3.15) and (3.16). The proof is complete. \square

Acknowledgements

This project was partially supported by the Educational Commission of Guangdong Province, China (No. 2017GkQNCX098).

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Received: January 10, 2018.

Revised: March 1, 2018.

Accepted: March 2, 2018.