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CIRCULANT MATRICES: NORM, POWERS, AND POSITIVITY

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Abstract. In their recent paper "The spectral norm of a Horadam circulant matrix", Merikoski, Haukkanen, Mattila and Tossavainen study under which conditions the spectral norm of a general real circulant matrix $\mathbf{C}$ equals the modulus of its row/column sum. We improve on their sufficient condition until we have a necessary one. Our results connect the above problem to positivity of sufficiently high powers of the matrix $\mathbf{C}^\top\mathbf{C}$. We then generalize the result to complex circulant matrices.

Keywords: spectral norm, circulant matrix, eventually positive semigroups.

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1. INTRODUCTION AND PRELIMINARIES

For $n \in \mathbb{N}$ and $\mathbf{x} = (x_0, \dots, x_{n-1}) \in \mathbb{R}^n$, look at the circulant matrix

$$\mathbf{C}_\mathbf{x} := \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_{n-1} & x_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_1 \\ x_1 & \cdots & x_{n-1} & x_0 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Motivated by studies of so-called Horadam or Fibonacci circulant matrices, the authors of [2,3] ask in [2] under which conditions the spectral norm of $\mathbf{C}_\mathbf{x}$ equals $|x_0 + x_1 + \ldots + x_{n-1}|$. We give a sufficient and a necessary condition. Both have to do with the positivity of powers of $\mathbf{C}_\mathbf{x}^\top\mathbf{C}_\mathbf{x}$.

If $\mathbf{R} := \mathbf{C}_{(0,1,0,\dots,0)}$ denotes the cyclic backward shift $\mathbf{R} : (u_1, \dots, u_n) \mapsto (u_2, \dots, u_n, u_1)$, then

$$\mathbf{C}_\mathbf{x} = x_0\mathbf{R}^0 + x_1\mathbf{R}^1 + \ldots + x_{n-1}\mathbf{R}^{n-1} = c(\mathbf{R}) \quad \text{with} \quad c(t) := x_0t^0 + x_1t^1 + \ldots + x_{n-1}t^{n-1}.$$

The polynomial c is called the *symbol* of $\mathbf{C_x}$. Most of the time, we understand c as a function on

$$\mathbb{T}_n := \{t \in \mathbb{C} : t^n = 1\} = \{\omega^0, \omega^1, \dots, \omega^{n-1}\} \quad \text{with} \quad \omega := \exp(\tfrac{2\pi}{n}\mathrm{i}).$$

It is easy to see that $\mathbf{R}$ diagonalizes as $\mathbf{R} = \mathbf{FDF}^*$, where $\mathbf{D} = \operatorname{diag}(\omega^0, \dots, \omega^{n-1})$ and $\mathbf{F}$ is the so-called *Fourier matrix* $\frac{1}{\sqrt{n}}\big(\omega^{jk}\big)_{j,k=0}^{n-1}$. Note that $\mathbf{F}$ is unitary, so that $\mathbf{F}^{-1} = \mathbf{F}^*$. Consequently,

$$\mathbf{C_x} = c(\mathbf{R}) = c(\mathbf{FDF}^*) = \mathbf{F}\,c(\mathbf{D})\,\mathbf{F}^* = \mathbf{F}\operatorname{diag}(c(\omega^0), \dots, c(\omega^{n-1}))\mathbf{F}^* = \mathbf{FD_xF}^*$$

with $\mathbf{D_x} := \operatorname{diag}(c(\omega^0), \dots, c(\omega^{n-1}))$. Since $\mathbf{F}$ is an isometry of $\mathbb{C}^n$ with the Euclidean norm,

$$\|\mathbf{C_x}\| = \|\mathbf{FD_xF}^*\| = \|\mathbf{D_x}\| = \max\big(|c(\omega^0)|, |c(\omega^1)|, \dots, |c(\omega^{n-1})|\big) =: \|c\|_\infty, \quad (1.1)$$

where $\|\cdot\|$ denotes the spectral norm of a matrix; it is the matrix norm that is induced by the Euclidean norm. Of course, all of this is standard [1]. The Fourier transform $\mathbf{F}$ turns the convolution $\mathbf{C_x}$ into a multiplication $\mathbf{D_x}$. We are just fixing notations here.

The question of [2] is essentially, under which conditions

$$\|\mathbf{C_x}\| = \|c\|_\infty \quad \text{equals} \quad |x_0 + x_1 + \ldots + x_{n-1}| = |c(1)| = |c(\omega^0)|. \quad (1.2)$$

So let

$$\mathcal{C}_n := \big\{\mathbf{x} = (x_0, \dots, x_{n-1}) \in \mathbb{R}^n : \|\mathbf{C_x}\| = |x_0 + x_1 + \ldots + x_{n-1}|\big\}.$$

Looking at (1.2), we see that

$$\mathbf{x} \in \mathcal{C}_n \iff \|c\|_\infty = |c(1)|, \quad \text{i.e. } |c(\cdot)| \text{ assumes its maximum on } \mathbb{T}_n \text{ at } t = 1 = \omega^0.$$

We will work with the latter condition in what follows. We will also study the following subset of $\mathcal{C}_n$ if $n \geq 2$. Let

$$\mathcal{C}'_n := \big\{\mathbf{x} \in \mathcal{C}_n : \max_{t \in \mathbb{T}_n \setminus \{1\}} |c(t)| < |c(1)| = \|c\|_\infty\big\} \subset \mathcal{C}_n.$$

While, for $\mathbf{x} \in \mathcal{C}_n$, the maximum of $|c(\cdot)|$ in $\mathbb{T}_n$ is attained at $t = 1$, for $\mathbf{x} \in \mathcal{C}'_n$ it is **only** attained at $t = 1$, so that $\mathbf{C_x}$ has a spectral gap between the two largest (in modulus) eigenvalues. We start with a simple sufficient condition for membership in $\mathcal{C}_n$ and $\mathcal{C}'_n$, respectively. Here we write $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x} > \mathbf{0}$) or $\mathbf{M} \geq \mathbf{0}$ ($\mathbf{M} > \mathbf{0}$) if each entry of, respectively, the vector $\mathbf{x}$ or the matrix $\mathbf{M}$ is nonnegative (positive).

Lemma 1.1. *Let $n \geq 2$ and $\mathbf{x} \in \mathbb{R}^n$.*

a) *If $\mathbf{x} \geq \mathbf{0}$ or $-\mathbf{x} \geq \mathbf{0}$ (i.e. $\pm\mathbf{C_x} \geq \mathbf{0}$) then $\mathbf{x} \in \mathcal{C}_n$.* (*This is* [2, *Corollary 2*].)
b) *If $\mathbf{x} > \mathbf{0}$ or $-\mathbf{x} > \mathbf{0}$ (i.e. $\pm\mathbf{C_x} > \mathbf{0}$) then $\mathbf{x} \in \mathcal{C}'_n$.*

Proof. a) By triangle inequality, every $|c(t)|$ with $t \in \mathbb{T}_n$ is bounded as follows

$$|c(t)| = \left|x_0 + x_1 t^1 + \ldots + x_{n-1} t^{n-1}\right| \le |x_0| + |x_1| + \ldots + |x_{n-1}| \quad \text{since} \quad |t| = 1.$$

But this upper bound, and hence the maximum $\|c\|_\infty$, is attained by $|c(1)| = |x_0 + \ldots + x_{n-1}|$ as soon as all x_k have the same sign, $\mathbf{x} \ge \mathbf{0}$ or $-\mathbf{x} \ge \mathbf{0}$.

b) The statement can be derived by the Perron-Frobenius theorem but here is a more elementary proof. Let $\mathbf{x} > \mathbf{0}$. (The argument is similar for $-\mathbf{x} > \mathbf{0}$.) By a), we have $|c(1)| = \|c\|_\infty$. For every $t \in \mathbb{T}_n \setminus \{1\}$, it holds $|x_0 + x_1 t| < |x_0| + |x_1 t|$ since $x_0, x_1 > 0$ and 1 and t have different directions in $\mathbb{C}$. Consequently, noting that $|t| = 1$,

$$\begin{aligned} |c(t)| &= \left|x_0 + x_1 t^1 + \ldots + x_{n-1} t^{n-1}\right| \le \underbrace{|x_0 + x_1 t|}_{<|x_0|+|x_1 t|} + |x_2 t^2| + \ldots + |x_{n-1} t^{n-1}| \\ &< |x_0| + |x_1| + |x_2| + \ldots + |x_{n-1}| = x_0 + \ldots + x_{n-1} = c(1) = |c(1)| = \|c\|_\infty. \quad \square \end{aligned}$$

This sufficient condition for membership in $\mathcal{C}_n$ or $\mathcal{C}'_n$ seems quite generous. [2] suggests the following improvement. Put

$$\mathbf{B_x} := \mathbf{C_x^\top C_x} = \mathbf{C_x^* C_x} = (\mathbf{F D_x F^*})^* (\mathbf{F D_x F^*}) = \mathbf{F D_x^* D_x F^*} = \mathbf{F A_x F^*} \tag{1.3}$$

with

$$\mathbf{A_x} := \mathbf{D_x^* D_x} = \operatorname{diag}(b(\omega^0), \ldots, b(\omega^{n-1})),$$

where

$$b(t) := \overline{c(t)} c(t) = |c(t)|^2 \quad \text{for all} \quad t \in \mathbb{T}_n,$$

so that

$$\|b\|_\infty := \max_{t \in \mathbb{T}_n} |b(t)| = \max_{t \in \mathbb{T}_n} |c(t)|^2 = \|c\|_\infty^2.$$

Then $\mathbf{B_x}$ is again a real circulant matrix. Applying Lemma 1.1 to $\mathbf{B_x}$ (in place of $\mathbf{C_x}$), we get the following result.

Lemma 1.2. *Let $n \ge 2$, $\mathbf{x} \in \mathbb{R}^n$ and put $\mathbf{B_x} := \mathbf{C_x^\top C_x}$.*

a) *If $\mathbf{B_x} \ge \mathbf{0}$ then $\mathbf{x} \in \mathcal{C}_n$. (This is* [2, *Theorem 4*].)
b) *If $\mathbf{B_x} > \mathbf{0}$ then $\mathbf{x} \in \mathcal{C}'_n$.*

Proof. Recall that the symbol b of $\mathbf{B_x}$ is related to the symbol c of $\mathbf{C_x}$ by $b(t) = |c(t)|^2$ for all $t \in \mathbb{T}_n$. So b assumes its maximum at the same point(s) as $|c(\cdot)|$ does.

For a), by Lemma 1.1 a),

$$\mathbf{B_x} \ge \mathbf{0} \quad \Rightarrow \|b\|_\infty = |b(1)| \ \Rightarrow \ \|c\|_\infty^2 = |c(1)|^2 \ \Rightarrow \ \|c\|_\infty = |c(1)| \ \Rightarrow \ \mathbf{x} \in \mathcal{C}_n.$$

b) By Lemma 1.1 b), positivity $\mathbf{B_x} > \mathbf{0}$ implies that $|b(t)| < \|b\|_\infty$ for all $t \in \mathbb{T}_n \setminus \{1\}$. But then also $|c(t)| = |b(t)|^{1/2} < \|b\|_\infty^{1/2} = \|c\|_\infty$ for all $t \in \mathbb{T}_n \setminus \{1\}$. So $\mathbf{x} \in \mathcal{C}'_n$. $\square$

Note that the case $-\mathbf{B_x} \ge \mathbf{0}$ is impossible (unless $\mathbf{x} = \mathbf{0}$, in which case $\mathbf{B_x} = \mathbf{0}$) since the main diagonal of $\mathbf{B_x}$ carries the entry $\|\mathbf{x}\|_2^2$.

2. ITERATING THE ARGUMENT UNTIL SUFFICIENT BECOMES NECESSARY

Looking at Lemmas 1.1 and 1.2, the following questions seem natural:

(Q1) Is the new condition $\mathbf{C}_\mathbf{x}^\top \mathbf{C}_\mathbf{x} \geq \mathbf{0}$ substantially weaker than the old condition $\pm\mathbf{C}_\mathbf{x} \geq \mathbf{0}$?

(Q2) Do we get a chain of increasingly weaker sufficient conditions if we repeat the argument?

(Q3) Does that chain end in a necessary condition?

Let us address those questions, starting with (Q1): It is easy to see that for $n \in \{1,2\}$, the two conditions are equivalent but for $n \geq 3$ they differ. Table 1 below indicates that the quotient of their probabilities grows as n grows. As an example for $n = 3$, look at $\mathbf{x} = (1, -2, -3)$, where

$$\mathbf{C}_\mathbf{x} = \begin{pmatrix} 1 & -2 & -3 \\ -3 & 1 & -2 \\ -2 & -3 & 1 \end{pmatrix} \not\geq \mathbf{0}, \quad -\mathbf{C}_\mathbf{x} \not\geq \mathbf{0} \quad \text{but} \quad \mathbf{B}_\mathbf{x} := \mathbf{C}_\mathbf{x}^\top \mathbf{C}_\mathbf{x} = \begin{pmatrix} 14 & 1 & 1 \\ 1 & 14 & 1 \\ 1 & 1 & 14 \end{pmatrix} \geq \mathbf{0}.$$

So Lemma 1.1 is not strong enough to show $\mathbf{x} \in \mathcal{C}_3$, i.e. $\|\mathbf{C}_\mathbf{x}\| = |1-2-3| = 4$, but Lemma 1.2 is.

About (Q2): With $\mathbf{B}_\mathbf{x} = \mathbf{C}_\mathbf{x}^\top \mathbf{C}_\mathbf{x}$, let us now look at $\mathbf{B}_\mathbf{x}^\top \mathbf{B}_\mathbf{x}$. But since $\mathbf{B}_\mathbf{x}^\top = \mathbf{B}_\mathbf{x}$, one has $\mathbf{B}_\mathbf{x}^\top \mathbf{B}_\mathbf{x} = \mathbf{B}_\mathbf{x}^2$. This is still a circulant, to which we can apply Lemma 1.1. Then one can again multiply $\mathbf{B}_\mathbf{x}^2$ with its transpose (itself) or just with $\mathbf{B}_\mathbf{x}$ and continue like that.

Theorem 2.1. *Let $n \geq 2$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{B}_\mathbf{x} = \mathbf{C}_\mathbf{x}^\top \mathbf{C}_\mathbf{x}$.*

a) *If $\mathbf{B}_\mathbf{x}^m \geq \mathbf{0}$ for some $m \in \mathbb{N}$ then $\mathbf{x} \in \mathcal{C}_n$.*

b) *If $\mathbf{B}_\mathbf{x}^m > \mathbf{0}$ for some $m \in \mathbb{N}$ then $\mathbf{x} \in \mathcal{C}'_n$.*

Proof. For every $m \in \mathbb{N}$, we have, by (1.3),

$$\begin{aligned} \mathbf{B}_\mathbf{x}^m &= \mathbf{F}\,\mathbf{A}_\mathbf{x}^m\,\mathbf{F}^* = \mathbf{F}\,\operatorname*{diag}_{k=0}^{n-1} b(\omega^k)^m\;\mathbf{F}^*, \\ \text{so that}\quad \|\mathbf{B}_\mathbf{x}^m\| &= \max_{k=0}^{n-1} |b(\omega^k)|^m = \|b\|_\infty^m = \|c\|_\infty^{2m}. \end{aligned} \tag{2.1}$$

So $\mathbf{B}_\mathbf{x}^m$ is a circulant matrix with symbol $t \mapsto b(t)^m = |c(t)|^{2m}$. It assumes its maximum at the same point(s) of $\mathbb{T}_n$ as $|c(\cdot)|$ does. Now argue as in the proof of Lemma 1.2. □

Looking at $m = 2^0,\ 2^1,\ 2^2, \ldots$ and noting that $\mathbf{M}, \mathbf{N} \geq \mathbf{0}$ implies $\mathbf{M} \cdot \mathbf{N} \geq \mathbf{0}$, we get that

$$\begin{aligned} &\pm\mathbf{C}_\mathbf{x} \geq 0 \;\Rightarrow\; \mathbf{B}_\mathbf{x} \geq \mathbf{0} \;\Rightarrow\; \mathbf{B}_\mathbf{x}^2 \geq \mathbf{0} \;\Rightarrow\; \mathbf{B}_\mathbf{x}^4 \geq \mathbf{0} \;\Rightarrow\; \mathbf{B}_\mathbf{x}^8 \geq \mathbf{0} \;\Rightarrow\; \cdots \;\Rightarrow\; \mathbf{x} \in \mathcal{C}_n, \\ &\pm\mathbf{C}_\mathbf{x} > 0 \;\Rightarrow\; \mathbf{B}_\mathbf{x} > \mathbf{0} \;\Rightarrow\; \mathbf{B}_\mathbf{x}^2 > \mathbf{0} \;\Rightarrow\; \mathbf{B}_\mathbf{x}^4 > \mathbf{0} \;\Rightarrow\; \mathbf{B}_\mathbf{x}^8 > \mathbf{0} \;\Rightarrow\; \cdots \;\Rightarrow\; \mathbf{x} \in \mathcal{C}'_n. \end{aligned}$$

To illustrate that these are indeed chains of increasingly weaker conditions, let us approximately compute[1)] the portion of the unit ball in $\mathbb{R}^n$ that satisfies the corresponding condition (see Table 1).

[1)] using a Monte Carlo simulation with one million equally distributed points in the unit ball

Table 1. An approximate computation of the portion of points $\mathbf{x} \in \mathbb{R}^n$ of the unit ball (note that all conditions are invariant under scaling of $\mathbf{x}$) that satisfy the corresponding condition in the header. Reading from left to right, every row seems to grow – in the limit – up to the portion of the ball that belongs to $\mathcal{C}'_n$. This is a positive sign with respect to our question (Q3)

n	$\pm\mathbf{x} > \mathbf{0}$	$\mathbf{B_x} > \mathbf{0}$	$\mathbf{B_x^2} > \mathbf{0}$	$\mathbf{B_x^4} > \mathbf{0}$	$\mathbf{B_x^8} > \mathbf{0}$	$\mathbf{B_x^{16}} > \mathbf{0}$	$\mathbf{B_x^{32}} > \mathbf{0}$	$\cdots$	$\mathbf{x} \in \mathcal{C}'_n$
$n=2$	50.0%	50.0%	50.0%	50.0%	50.0%	50.0%	50.0%	$\cdots$	50.0%
$n=3$	25.0%	42.3%	42.3%	42.3%	42.3%	42.3%	42.3%	$\cdots$	42.3%
$n=4$	12.5%	25.0%	27.3%	28.9%	29.8%	30.3%	30.5%	$\cdots$	30.8%
$n=5$	6.3%	23.2%	25.4%	27.1%	28.1%	28.6%	28.9%	$\cdots$	29.2%
$n=6$	3.1%	16.7%	20.0%	21.9%	22.8%	23.1%	23.3%	$\cdots$	23.5%
$n=7$	1.6%	14.7%	18.1%	20.4%	21.7%	22.4%	22.8%	$\cdots$	23.2%
$n=8$	0.8%	10.4%	14.3%	16.8%	18.1%	18.8%	19.2%	$\cdots$	19.5%
$n=9$	0.4%	10.3%	14.4%	17.0%	18.3%	18.9%	19.2%	$\cdots$	19.5%
$n=10$	0.2%	7.5%	11.6%	14.3%	15.7%	16.3%	16.6%	$\cdots$	16.9%
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$n=20$	2^{-19}	1.9%	5.2%	7.9%	9.4%	10.1%	10.4%	$\cdots$	10.7%

Finally, we turn to our question (Q3) about necessary conditions for membership in $\mathcal{C}_n$ or $\mathcal{C}'_n$. Nonnegativity / positivity of powers of $\mathbf{B_x}$ is not necessary for membership in $\mathcal{C}_n$ (see Example 2.3 below). But, assuming a spectral gap, i.e. membership in $\mathcal{C}'_n$, we get convergence of the power method and hence positivity of large powers of $\mathbf{B_x}$ (due to the special structure of the corresponding eigenvector).

Theorem 2.2. *If $\mathbf{x} \in \mathcal{C}'_n$ then there exists an $m_0 \in \mathbb{N}$ such that $\mathbf{B_x^m} > \mathbf{0}$ for all $m \geq m_0$.*

Proof. Let $\mathbf{x} \in \mathcal{C}'_n$ and abbreviate $|c(\omega^k)| =: c_k$ for $k = 0, \dots, n-1$. Then $\|c\|_\infty = c_0 > c_1, \dots, c_{n-1} \geq 0$. From (2.1) we conclude

$$\frac{\mathbf{B_x^m}}{\|\mathbf{B_x^m}\|} = \frac{1}{c_0^{2m}} \mathbf{F} \operatorname{diag}(c_0^{2m}, c_1^{2m}, \dots, c_{n-1}^{2m}) \mathbf{F}^* = \mathbf{F} \operatorname{diag}\Big(1, \Big(\frac{c_1}{c_0}\Big)^{2m}, \dots, \Big(\frac{c_{n-1}}{c_0}\Big)^{2m}\Big) \mathbf{F}^*$$

$$\to \mathbf{F} \operatorname{diag}(1, 0, \dots, 0) \mathbf{F}^* = \frac{1}{n} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} > \mathbf{0} \quad \text{as} \quad m \to \infty, \tag{2.2}$$

so that $\mathbf{B_x^m} > \mathbf{0}$ for all sufficiently large $m \in \mathbb{N}$. □

The argument in the proof of Theorem 2.2 does not work if $|c(\cdot)|$ attains its maximum in another or in more than one point on $\mathbb{T}_n$. The following example shows that, indeed, $\mathcal{C}'_n$ cannot be replaced by $\mathcal{C}_n$ in Theorem 2.2.

Example 2.3. *Take $n = 5$ and $\mathbf{C_x} := \mathbf{F} \operatorname{diag}(1, 0, 1, 1, 0) \mathbf{F}^*$. The diagonal has its maximum in the first but also in the 3rd and 4th position, so that $\mathbf{x} \in \mathcal{C}_5 \setminus \mathcal{C}'_5$. The first row of $\mathbf{C_x}$ is $\mathbf{x} = (\frac{3}{5}, \alpha, \beta, \beta, \alpha)$ with $\alpha = \frac{1}{5}(1 + 2\cos(\frac{4\pi}{5})) < 0$ and $\beta = \frac{1}{5}(1 + 2\cos(\frac{2\pi}{5})) > 0$, so that $\mathbf{C_x} \not\geq \mathbf{0}$ and $-\mathbf{C_x} \not\geq \mathbf{0}$. But also $\mathbf{B_x^m} \not\geq \mathbf{0}$ since $\mathbf{C_x} = \mathbf{C_x^\top} = \mathbf{C_x^m} = \mathbf{B_x^m}$ for all $m \in \mathbb{N}$.*

So for membership in $\mathcal{C}'_n$, we have the following equivalence.

Corollary 2.4. *Let $n \geq 2$ and $\mathbf{x} \in \mathbb{R}^n$. Then the following are equivalent.*

(i) $\mathbf{x} \in \mathcal{C}'_n$,
(ii) $\exists m \in \mathbb{N} : \mathbf{B}_{\mathbf{x}}^m > \mathbf{0}$,
(iii) $\exists m_0 \in \mathbb{N}\, \forall\, m \geq m_0 : \mathbf{B}_{\mathbf{x}}^m > \mathbf{0}$.

Proof. (ii)⇒(i) is Theorem 2.1 b), (i)⇒(iii) is Theorem 2.2 b), and (iii)⇒(ii) is obvious. □

3. COMPLEX ENTRIES

The case $\mathbf{x} \in \mathbb{C}^n$ is only slightly different. When we refer to $\mathcal{C}_n$ or $\mathcal{C}'_n$ now, we mean the corresponding subsets of $\mathbb{C}^n$. In a complex version of Lemma 1.1 a) it would be enough to have all entries of $\mathbf{x}$ of the same phase, i.e. on the same ray $\{rz : r \geq 0\}$ with some $z \in \mathbb{C}$. But for Lemma 1.2 a), that ray would again have to be the nonnegative real axis, because the main diagonal entries of $\mathbf{B}_{\mathbf{x}} := \mathbf{C}_{\mathbf{x}}^*\mathbf{C}_{\mathbf{x}}$ are always there. The other entries of $\mathbf{B}_{\mathbf{x}}$ or $\mathbf{B}_{\mathbf{x}}^m$ need not even be real, let alone nonnegative or positive.

However, the proof of Theorem 2.2 shows that the entries of $\mathbf{B}_{\mathbf{x}}^m$ are in a certain neighborhood of the positive half axis if $\mathbf{x} \in \mathcal{C}'_n$ (also for the complex version) and m is sufficiently large. On the other hand, by the continuity of each function value $c(t)$ with respect to $\mathbf{x}$, one can generalize Lemma 1.1 to an appropriate neighborhood of the positive half axis:

Lemma 3.1. *If $n \geq 2$ and $\mathbf{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{C}^n$ is such that at least two adjacent entries of $\mathbf{x}$ are nonzero and all phases are close to zero, precisely, each*

$$\varphi_k := \arg x_k \in (-\pi, \pi] \quad \textit{is subject to} \quad |\varphi_k| < \frac{\pi}{2n}, \tag{3.1}$$

then $\mathbf{x} \in \mathcal{C}'_n$.

Proof. We start with n general complex numbers $z_0, \ldots, z_{n-1} \in \mathbb{C}$ and put $\psi_k := \arg z_k$, which we put to zero if $z_k = 0$. Then the following "generalized law of cosines" is easily verified.

$$\begin{aligned} |z_0 + \ldots + z_{n-1}|^2 &= (z_0 + \ldots + z_{n-1})\overline{(z_0 + \ldots + z_{n-1})} = \sum_{j,k=0}^{n-1} z_j \overline{z_k} \\ &= \sum_{j=0}^{n-1} |z_j|^2 + 2 \sum_{\substack{j,\,k\,=\,0 \\ j\,<\,k}}^{n-1} \operatorname{Re}(z_j \overline{z_k}) \\ &= \sum_{j=0}^{n-1} |z_j|^2 + 2 \sum_{\substack{j,\,k\,=\,0 \\ j\,<\,k}}^{n-1} |z_j||z_k| \cos(\psi_j - \psi_k). \end{aligned} \tag{3.2}$$

Putting $z_k := x_k$ from above, we have $\psi_k = \varphi_k$ and hence

$$\begin{aligned} |c(1)|^2 &= |x_0 + \ldots + x_{n-1}|^2 \\ &\overset{(3.2)}{=} \sum_{j=0}^{n-1} |x_j|^2 + 2 \sum_{\substack{j,k=0 \\ j<k}}^{n-1} |x_j||x_k| \cos(\varphi_j - \varphi_k). \end{aligned} \tag{3.3}$$

Now take $t = \omega^\ell \in \mathbb{T}_n \setminus \{1\}$ with some $\ell \in \{1, \ldots, n-1\}$ and put $z_k := x_k t^k$ in (3.2). Then $\psi_k = \arg(x_k t^k) = \arg x_k + k \arg t = \varphi_k + k\ell\vartheta$ with $\vartheta := \arg \omega = \frac{2\pi}{n}$. Plugging this into (3.2), we get

$$\begin{aligned} |c(t)|^2 &= |x_0 t^0 + \ldots + x_{n-1} t^{n-1}|^2 \\ &\overset{(3.2)}{=} \sum_{j=0}^{n-1} |x_j|^2 + 2 \sum_{\substack{j,k=0 \\ j<k}}^{n-1} |x_j||x_k| \cos\big(\varphi_j - \varphi_k + (j-k)\ell\vartheta\big). \end{aligned} \tag{3.4}$$

By our assumption (3.1), all differences $\varphi_j - \varphi_k$ are in the interval $(-\frac{\pi}{n}, \frac{\pi}{n}) =: I_n$. Since the length of I_n is $\vartheta = \frac{2\pi}{n}$,

$$\varphi_j - \varphi_k + (j-k)\ell\vartheta \quad \left\{ \begin{array}{ll} = \varphi_j - \varphi_k, & \text{if } (j-k)\ell \in n\mathbb{Z}, \\ \notin I_n, & \text{otherwise}, \end{array} \right\} \quad \text{both modulo } 2\pi.$$

Moreover, $\cos x < \cos y$ whenever $x \notin I_n$ and $y \in I_n$ (modulo 2π). Consequently, all cosines in (3.3) are larger than or equal to the corresponding cosines in (3.4). So $|c(1)| \geq |c(t)|$.

For our two adjacent j, k with x_j and x_k nonzero, we have $j - k = -1$ and hence $(j-k)\ell \notin n\mathbb{Z}$, so that the corresponding term in (3.3) is strictly larger than in (3.4). Hence, $|c(1)| > |c(t)|$. □

So it is already enough for $\mathbf{x} \in \mathcal{C}'_n$ that each entry of $\mathbf{x}$ is in a certain cone around the positive real half axis. By the same arguments as in the real case, one can look at a power of $\mathbf{B_x} := \mathbf{C_x^*}\mathbf{C_x}$, which is again a circulant matrix, and check whether the entries of its first (or any) row satisfy (3.1).

Theorem 3.2. *Let $n \geq 2$ and $\mathbf{x} \in \mathbb{C}^n$. Then the following are equivalent.*

(i) $\mathbf{x} \in \mathcal{C}'_n$,
(ii) $\exists m \in \mathbb{N}$: *at least two adjacent entries of the first row of $\mathbf{B}_\mathbf{x}^m$ are nonzero and satisfy* (3.1),
(iii) $\exists m \in \mathbb{N}$: *all entries of the first row of $\mathbf{B}_\mathbf{x}^m$ are nonzero and satisfy* (3.1),
(iv) $\exists m_0 \in \mathbb{N} \, \forall m \geq m_0$: *all entries of the first row of $\mathbf{B}_\mathbf{x}^m$ are nonzero and satisfy* (3.1).

Proof. The implications (iv)⇒(iii)⇒(ii) are obvious. It remains to check (ii)⇒(i)⇒(iv).

(ii)⇒(i) Let $m \in \mathbb{N}$ be as in (ii) and denote the circulant matrix $\mathbf{B}_{\mathbf{x}}^m$ by $\mathbf{C}_{\mathbf{y}}$. By Lemma 3.1, $\mathbf{y} \in \mathcal{C}_n'$, i.e. the symbol b of $\mathbf{B}_{\mathbf{x}}^m$ has its maximum at 1 and only there. Arguing as in the proofs of Lemma 1.2 and Theorem 2.1, the same holds for the symbol c of $\mathbf{C}_{\mathbf{x}}$, so that $\mathbf{x} \in \mathcal{C}_n'$.

(i)⇒(iv) Let $\mathbf{x} \in \mathcal{C}_n'$. Following the proof of Theorem 2.2 up to (2.2), we see that, for all entries of $\mathbf{B}_{\mathbf{x}}^m$, let us denote them by $b_{jk}^{(m)}$, we have the following limits as $m \to \infty$,

$$\frac{b_{jk}^{(m)}}{\|\mathbf{B}_{\mathbf{x}}^m\|} \to \frac{1}{n}, \quad \text{so that} \quad \frac{|b_{jk}^{(m)}|}{\|\mathbf{B}_{\mathbf{x}}^m\|} \to \left|\frac{1}{n}\right| = \frac{1}{n}$$

and hence

$$\frac{b_{jk}^{(m)}}{|b_{jk}^{(m)}|} = \frac{b_{jk}^{(m)}}{\|\mathbf{B}_{\mathbf{x}}^m\|} \frac{\|\mathbf{B}_{\mathbf{x}}^m\|}{|b_{jk}^{(m)}|} \to \frac{1}{n} \cdot n = 1,$$

showing that $\arg b_{jk}^{(m)} \to 0$. It follows that, for all sufficiently large m, all entries of $\mathbf{B}_{\mathbf{x}}^m$ are nonzero and subject to (3.1). This clearly implies (iv). □

4. CONCLUSION

Theorems 2.1 and 2.2 are clearly not meant to give efficient ways of computing the spectral norm of a generic real circulant matrix – one cannot beat formula (1.1) in terms of the computational cost. Rather than that, our theorems connect two apparently different questions to each other:

(i) whether $\|\mathbf{C}_{\mathbf{x}}\|$ equals $|x_0 + \ldots + x_{n-1}|$, and

(ii) eventual positivity of the semigroup $(\mathbf{B}_{\mathbf{x}}^m)_{m=0}^{\infty}$.

In the complex case, one has the same results but instead of being real and positive, the matrix entries of $\mathbf{B}_{\mathbf{x}}^m$ only have to belong to a certain cone (3.1) around the positive half axis.

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