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BOUNDARY VALUE PROBLEMS WITH SOLUTIONS IN CONVEX SETS

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Abstract. By means of the continuation method for contractions we prove the existence of solutions of Dirichlet boundary value problems in convex sets. As an application we prove the existence of concave solutions of certain boundary value problems in ordered Banach spaces.

Keywords: Dirichlet boundary value problems, solutions in convex sets, continuation method, ordered Banach spaces, concave solutions.

Mathematics Subject Classification: 34B15, 47H10.

1. INTRODUCTION

Let $(E, \|\cdot\|)$ be a real Banach space with dual space E^*, let C be a closed convex subset of E and let $f : [0,1] \times C \times E \to E$ be a function with the property

$$\left.\begin{array}{c} t \in [0,1],\ x \in C,\ p \in E,\ \varphi \in E^*, \\ \varphi(x) = \max\{\varphi(y) : y \in C\},\ \varphi(p) = 0 \end{array}\right\} \Longrightarrow\ \varphi(f(t,x,p)) \le 0. \tag{1.1}$$

It is known [9,10,14] that property (1.1), in tandem with certain compactness or Lipschitz conditions, leads to the existence of solutions $u : [0,1] \to C$ of the boundary value problem

$$u''(t) + f(t, u(t), u'(t)) = 0, \quad u(0) = x_0,\ u(1) = x_1, \tag{1.2}$$

where $x_0, x_1 \in C$.

We will deal with Lipschitz conditions, and to keep things quite general we make the following stipulations: Let $G : [0,1]^2 \to \mathbb{R}$ denote Green's function

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

corresponding to problem (1.2), and let W denote the set of all $(l, m) \in C([0,1],[0,\infty))^2$ such that there exist $\alpha, \beta \in C([0,1],(0,\infty))$ and $q < 1$ with

$$\frac{1}{\alpha(t)} \int_0^1 G(t,s)(l(s)\alpha(s) + m(s)\beta(s))ds \le q \quad (t \in [0,1]),$$

$$\frac{1}{\beta(t)} \int_0^1 |G_t(t,s)|(l(s)\alpha(s) + m(s)\beta(s))ds \le q \quad (t \in [0,1]).$$

For example, in case of constant functions l, m it is well known that $(l, m) \in W$ if $l/8 + m/2 < 1$ or if $m = 0$ and $l < \pi^2$, see [4,7].

In this paper we use a variant of Granas' continuation method for contractions [6] to prove the following theorem.

Theorem 1.1. *Let $C \subseteq E$ be closed and convex with nonempty interior C°, and let $f : [0,1] \times C \times E \to E$ be a continuous function with*

$$\|f(t,x,p) - f(t,y,q)\| \le l(t)\|x - y\| + m(t)\|p - q\| \tag{1.3}$$

for $(t,x,p),(t,y,q) \in [0,1] \times C \times E$ and with $(l,m) \in W$. Moreover let f have property (1.1)*. Then for each choice of $x_0, x_1 \in C$ problem* (1.2) *has a unique solution $u : [0,1] \to C$.*

Theorem 1.1 is related to the existence result of Lemmert and Volkmann in [9]. The main difference is that in [9] the function f is assumed to be defined and to satisfy a Lipschitz condition on $[0,1] \times E \times E$. The fact that in Theorem 1.1 the Lipschitz condition (1.3) is only assumed on $[0,1] \times C \times E$ will be essential for our applications in Sections 4 and 5. In case $f(t,x,p)$ is independent of p Lipschitz conditions on $[0,1] \times C$ are considered in [10]. In [9] and [10] convex sets C with empty interior are allowed, which requires different and more involved arguments. Moreover in [9] mixed boundary conditions are considered. Here, we consider Dirichlet boundary conditions but emphasize that general mixed boundary conditions can be treated in our setting with minor changes.

2. A CONTINUATION METHOD

We make use of the following variant of Granas' continuation method for contractions [6]. Several quite similar variants of this method are known, see [1, Chapter 3] and [3,5], for example. However, we give the proof of the version used in the sequel, for convenience of the reader.

Let $U \subseteq E$ be open and let $H : [0,1] \times \overline{U} \to E$ be a function with the following properties:

1. There exists $q < 1$ such that
$$\|H(\lambda, x) - H(\lambda, y)\| \leq q\|x - y\| \quad ((\lambda, x), (\lambda, y) \in [0,1] \times \overline{U}).$$
2. For each bounded subset $B \subseteq \overline{U}$ there exists $l_B > 0$ such that
$$\|H(\lambda, x) - H(\mu, x)\| \leq l_B|\lambda - \mu| \quad ((\lambda, x), (\mu, x) \in [0,1] \times B).$$
3. For all $(\lambda, x) \in [0,1) \times \partial U$ we have
$$H(\lambda, x) \neq x.$$

In the sequel, $\|\cdot\|_\infty$ denotes the maximum norm on $[0,1]$.

Proposition 2.1. *If $H(0,\cdot)$ has a fixed point, then $H(1,\cdot)$ has a unique fixed point.*

Proof. Let
$$S := \{\lambda \in [0,1) : \exists x \in U : \; H(\lambda, x) = x\}.$$
Note that $0 \in S$. We will show that S is connected, hence $S = [0,1)$.

First, we show that S is relatively open in $[0,1)$: Let $\lambda_0 \in S$ and $x_0 \in U$ with $H(\lambda_0, x_0) = x_0$. Let $r_1 > 0$ be such that $B_1 := \{x \in E : \|x - x_0\| \leq r_1\} \subseteq U$, and choose $\varepsilon > 0$ such that
$$\varepsilon \leq \frac{(1-q)r_1}{l_{B_1}}.$$
Now, for $\lambda \in I := (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \cap [0,1)$ and $x \in B_1$ we obtain
$$\|x_0 - H(\lambda, x)\| \leq \|H(\lambda_0, x_0) - H(\lambda_0, x)\| + \|H(\lambda_0, x) - H(\lambda, x)\|$$
$$\leq q\|x_0 - x\| + l_{B_1}|\lambda_0 - \lambda| \leq qr_1 + (1-q)r_1 = r_1.$$
Thus
$$H(\lambda, B_1) \subseteq B_1 \quad (\lambda \in I),$$
and, according to Banach's Fixed Point Theorem, $x \mapsto H(\lambda, x)$ has a fixed point in $B_1 \subseteq U$ for each $\lambda \in I$. Thus $I \subseteq S$.

Next, let $z \in U$ be fixed. Let $(\lambda_n)_{n=1}^\infty$ be a convergent sequence in S with limit $\lambda_0 \in [0,1]$, say, and let $(x_n)_{n=1}^\infty$ be a corresponding sequence in U with
$$H(\lambda_n, x_n) = x_n \quad (n \in \mathbb{N}).$$
We have
$$\|x_n\| \leq \|H(\lambda_n, x_n) - H(\lambda_n, z)\| + \|H(\lambda_n, z)\| \leq q\|x_n\| + q\|z\| + \|H(\cdot, z)\|_\infty$$
$$\Rightarrow \|x_n\| \leq \frac{q\|z\| + \|H(\cdot, z)\|_\infty}{1-q} =: r_2 \quad (n \in \mathbb{N}).$$

Hence $x_n \in B_2 := \{x \in \overline{U} : \|x\| \leq r_2\}$ $(n \in \mathbb{N})$, and therefore

$$\begin{aligned}\|x_n - x_m\| &\leq \|H(\lambda_n, x_n) - H(\lambda_n, x_m)\| + \|H(\lambda_n, x_m) - H(\lambda_m, x_m)\| \\ &\leq q\|x_n - x_m\| + l_{B_2}|\lambda_n - \lambda_m| \\ &\Rightarrow \|x_n - x_m\| \leq \frac{l_{B_2}}{1-q}|\lambda_n - \lambda_m| \quad (n, m \in \mathbb{N}).\end{aligned}$$

Thus, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, hence convergent to x_0, say, and $x_0 \in B_2$. We have

$$\|x_n - H(\lambda_0, x_0)\| = \|H(\lambda_n, x_n) - H(\lambda_0, x_0)\|$$

$$\leq q\|x_n - x_0\| + l_{B_2}|\lambda_n - \lambda_0| \to 0 \quad (n \to \infty).$$

We obtain $H(\lambda_0, x_0) = x_0$. Now, if $\lambda_0 \neq 1$ then $x_0 \in U$, and therefore $\lambda_0 \in S$. This shows that S is relatively closed in $[0, 1)$. Thus $S = [0, 1)$. Hence there exists sequences $(\lambda_n)_{n=1}^{\infty}$ in S with limit 1 and therefore $H(1, \cdot)$ has a fixed point in $\overline{U}$, which is clearly unique. □

Although we will only use Proposition 2.1 later, we note that a Leray Schauder type alternative holds in our setting, as usual:

Proposition 2.2 (Leray Schauder type alternative). *Let $U \subseteq E$ be open, $z \in U$ and let $F : \overline{U} \to E$ be a contraction with constant $q \in [0, 1)$. Then at least one of the following assertions is true:*

1. *F has a fixed point in $\overline{U}$.*
2. *There exists $(\lambda_0, x_0) \in (0, 1) \times \partial U$ such that $x_0 - z = \lambda_0(F(x_0) - z)$.*

In the first case the fixed point clearly is unique.

Proof. Assume that condition 2. is false. Let $H : [0, 1] \times \overline{U} \to E$ be defined by $H(\lambda, x) = \lambda(F(x) - z) + z$. Clearly z is a fixed point of $H(0, \cdot)$,

$$H(\lambda, x) \neq x \quad ((\lambda, x) \in [0, 1) \times \partial U),$$

and

$$\|H(\lambda, x) - H(\lambda, y)\| \leq q\|x - y\| \quad ((\lambda, x), (\lambda, y) \in [0, 1] \times \overline{U}).$$

Next, let $B \subseteq \overline{U}$ be bounded and let $r > 0$ be such that $\|x - z\| \leq r$ $(x \in B)$. We have

$$\|F(x) - z\| \leq \|F(x) - F(z)\| + \|F(z) - z\| \leq qr + \|F(z) - z\| =: l_B \quad (x \in B).$$

Hence, for all $(\lambda, x), (\mu, x) \in [0, 1] \times B$ we get

$$\|H(\lambda, x) - H(\mu, x)\| = \|F(x) - z\||\lambda - \mu| \leq l_B|\lambda - \mu|.$$

Summing up, Proposition 2.1 is applicable. Thus $F = H(1, \cdot)$ has a fixed point in $\overline{U}$. □

3. PROOF OF THEOREM 1.1

Proof. We assume without loss of generality that $l(t) > 0$ $(t \in [0,1])$. Let α, β, q be corresponding to $(l, m) \in W$. We consider the real Banach space $C^1([0,1], E)$ endowed with the norm $|||u||| = \max\{\|u/\alpha\|_\infty, \|u'/\beta\|_\infty\}$ and set

$$U := \{u \in C^1([0,1], E) : u(t) \in C^\circ\}.$$

Note that U is open and convex, and that

$$\overline{U} = \{u \in C^1([0,1], E) : u(t) \in C\}, \ \partial U = \{u \in \overline{U} : \exists t \in [0,1] : u(t) \in \partial C\}.$$

First, fix $c \in C^\circ$, let $x_0, x_1 \in C$, and let $H : [0,1] \times \overline{U} \to C^1([0,1], E)$ be defined as

$$H(\lambda, u)(t) := \int_0^1 G(t,s)(\lambda f(s, u(s), u'(s)) - (1-\lambda) l(s)(u(s) - c)) ds + \lambda h(t) + (1-\lambda) c$$

with $h(t) := t x_1 + (1-t) x_0$ $(t \in [0,1])$. A straightforward calculation by means of (1.3) gives

$$|||H(\lambda, u) - H(\lambda, v)||| \le q |||u - v||| \quad ((\lambda, u), (\lambda, v) \in [0,1] \times \overline{U}).$$

Moreover, if $B \subseteq \overline{U}$ is bounded, then by (1.3)

$$\{f(t, u(t), u'(t)) : u \in B, t \in [0,1]\}$$

is bounded in E. Thus there is some $l_B > 0$ such that

$$|||H(\lambda, u) - H(\mu, u)||| \le l_B |\lambda - \mu| \quad ((\lambda, u), (\mu, u) \in [0,1] \times B).$$

Next, let $(\lambda, u) \in [0,1) \times \partial U$ and assume that $H(\lambda, u) = u$. Then u solves the boundary value problem

$$u''(t) + \lambda f(t, u(t), u'(t)) - (1-\lambda) l(t)(u(t) - c) = 0,$$

$$u(0) = \lambda x_0 + (1-\lambda) c, \ u(1) = \lambda x_1 + (1-\lambda) c.$$

Since $u \in \partial U$, and since $u(0), u(1) \in C^\circ$ there exists $t_0 \in (0,1)$ such that $u(t_0) \in \partial C$. Hence, by Hahn–Banach's separation theorem, there exists $\varphi \in E^* \setminus \{0\}$ such that $\varphi(u(t_0)) = \max\{\varphi(y) : y \in C\}$. Thus in particular

$$\varphi(u(t_0)) = \max_{t \in [0,1]} \varphi(u(t)),$$

therefore $\varphi(u''(t_0)) \le 0$, $\varphi(u'(t_0)) = 0$, and by assumption (1.1)

$$\varphi(f(t_0, u(t_0), u'(t_0))) \le 0.$$

Since $c \in C^\circ$ we have $\varphi(u(t_0)) > \varphi(c)$. Summing up we obtain by $l(t_0) > 0$

$$0 = \varphi(u''(t_0)) + \lambda\varphi(f(t_0, u(t_0), u'(t_0))) - (1-\lambda)l(t_0)\varphi(u(t_0) - c) < 0,$$

a contradiction. Thus

$$H(\lambda, u) \neq u \quad ((\lambda, u) \in [0,1) \times \partial U).$$

Finally, the constant function $t \mapsto c$ ($t \in [0,1]$) is a fixed point of $H(0,\cdot)$. Now, according to Proposition 2.1 the function $H(1,\cdot)$ has a unique fixed point in $\overline{U}$, which is by construction the unique solution of (1.2). □

4. BVPS IN ORDERED BANACH SPACES

In this section we study, among other things, the existence of concave solutions of the Dirichlet boundary value problem

$$u''(t) + g(t, u(t)) = 0, \quad u(0) = x_0,\ u(1) = x_1, \tag{4.1}$$

in ordered Banach spaces. Inspired by the paper of Alvarez, Lasry and Lions [2], who studied the convexity (or equivalently concavity) of viscosity solutions of scalar second order elliptic equations with state constraints boundary conditions, we investigate the vector valued case for ordinary differential equations. Although we use completely different methods, the results in [2] suggest, that the combination of a comparison principle (hence a uniqueness and quasimonotonicity condition) and concavity of g will lead to the existence of concave solutions. The authors are indebted to Prof. Wolfgang Reichel for drawing their attention to [2] and the problem of finding concave solutions of boundary value problems.

Let the Banach space E be ordered by a cone K. A cone K is a nonempty closed convex subset of E such that $\lambda K \subseteq K$ ($\lambda \geq 0$), and $K \cap (-K) = \{0\}$. As usual

$$x \leq y :\Longleftrightarrow \quad y - x \in K, \text{ and } x \ll y :\Longleftrightarrow \quad y - x \in K^\circ.$$

Let K^* denote the dual wedge of K, that is the set of all $\varphi \in E^*$ with $\varphi(x) \geq 0$ ($x \geq 0$). For $D \subseteq E$ a function $g : D \to E$ is called quasimonotone increasing (qmi for short), in the sense of Volkmann [15], if

$$x, y \in D,\ x \leq y,\ \varphi \in K^*,\ \varphi(x) = \varphi(y) \Longrightarrow \varphi(f(x)) \leq \varphi(f(y)).$$

Moreover, a function $g : [0,1] \times E \to E$ is called concave, if

$$g(\lambda(\xi, x) + (1-\lambda)(\eta, y)) \geq \lambda g(\xi, x) + (1-\lambda) g(\eta, y)$$

for all $(\xi, x), (\eta, y) \in [0,1] \times E$ and $\lambda \in [0,1]$.

Theorem 4.1. *Let $g : [0,1] \times E \to E$ be continuous and concave, let $x \mapsto g(\xi, x)$ be qmi and locally Lipschitz continuous for each $\xi \in [0,1]$ and let $\gamma \in C([0,1], \mathbb{R})$. Let $g(\xi_0, z_0) \gg 0$ for some $(\xi_0, z_0) \in (0,1) \times E$, and let there exist $\tau, l \geq 0$ with $(l, |\gamma|) \in W$ and*

$$(\xi, x), (\eta, y) \in [0,1] \times E,\ g(\xi, x) \geq 0,\ g(\eta, y) \geq 0$$
$$\Longrightarrow \|g(\xi, x) - g(\eta, y)\| \leq \tau|\xi - \eta| + l\|x - y\|.$$

Moreover let $\mu : [0,1] \to \mathbb{R}$ denote the unique solution of the scalar boundary value problem

$$\mu''(t) + \gamma(t)\mu'(t) = 0, \quad \mu(0) = 0,\ \mu(1) = 1.$$

Then for each choice of $x_0, x_1 \in E$ with $g(0, x_0) \geq 0$ and $g(1, x_1) \geq 0$ the boundary value problem

$$u''(t) + g(\mu(t), u(t)) + \gamma(t)u'(t) = 0, \quad u(0) = x_0,\ u(1) = x_1, \tag{4.2}$$

has a solution $u : [0,1] \to E$ with $g(\mu(t), u(t)) \geq 0$ $(t \in [0,1])$, which is unique in the class of solutions with this property.

The function μ in Theorem 4.1 can be evaluated to

$$\mu(t) = \frac{\int_0^t \exp(-\int_0^s \gamma(\sigma)d\sigma)ds}{\int_0^1 \exp(-\int_0^s \gamma(\sigma)d\sigma)ds},$$

and $\mu : [0,1] \to [0,1]$ is strictly increasing and bijective. The function μ makes problem (4.2) seem a bit unnatural. However, it contains the following quite natural cases:

Corollary 4.2. *If $\gamma = 0$, then the boundary value problem* (4.1) *has a concave solution $u : [0,1] \to E$, which is unique in the class of concave solutions.*

Corollary 4.3. *If g is independent of ξ, then the boundary value problem*

$$u''(t) + g(u(t)) + \gamma(t)u'(t) = 0, \quad u(0) = x_0,\ u(1) = x_1,$$

has a solution $u : [0,1] \to E$ with $t \mapsto \exp(\int_0^t \gamma(s)ds)u'(t)$ monotone decreasing on $[0,1]$, which is unique in the class of solutions with this property.

In the proof of Theorem 4.1 we use the following consequence of a result of Volkmann on differential inequalities [16, Satz 2]:

Theorem 4.4. *Let E be a real Banach space ordered by a cone K and let $g : E \to E$ be qmi and locally Lipschitz continuous. Let $x \in E$ be such that $g(x) \geq 0$. Then the solution $v : [0, \omega_+) \to E$ (nonextendable to the right) of the initial value problem $v'(t) = g(v(t))$, $v(0) = x$ is monotone increasing.*

Proof of Theorem 4.1. We assume without loss of generality that $\tau, l > 0$, and we consider the Banach space $E_1 := \mathbb{R} \times E$ endowed with the norm

$$\|(\xi, x)\|_1 = \frac{\tau}{l}|\xi| + \|x\|.$$

Let $P : [0,1] \times E \to E_1$ be defined by

$$P(\xi, x) = (0, g(\xi, x)).$$

We consider the set

$$C := \{(\xi, x) \in [0,1] \times E : g(\xi, x) \geq 0\}.$$

The set C is closed, since g is continuous on $[0,1] \times E$. Moreover for $(\xi, x), (\eta, y) \in C$ and $\lambda \in [0,1]$ concavity of g implies

$$g(\lambda(\xi, x) + (1-\lambda)(\eta, y)) \geq \lambda g(\xi, x) + (1-\lambda) g(\eta, y) \geq 0.$$

Thus C is convex. Moreover $(\xi_0, z_0) \in C^\circ$, again by continuity of g. Next, if $(\xi, x), (\eta, y) \in C$ then

$$\|P(\xi, x) - P(\eta, y)\|_1 = \|g(\xi, x) - g(\eta, y)\| \leq \tau|\xi - \eta| + l\|x - y\| = l\|(\xi, x) - (\eta, y)\|_1.$$

Now, let $F : [0,1] \times C \times E_1 \to E_1$ be defined by

$$F(t, (\xi, x), (\rho, p)) = (\gamma(t)\rho, g(\xi, x) + \gamma(t)p) = P(\xi, x) + \gamma(t)(\rho, p).$$

Clearly, we have

$$\begin{aligned} &\|F(t, (\xi, x), (\rho, p)) - F(t, (\eta, y), (\sigma, q))\|_1 \\ &\leq l\|(\xi, x) - (\eta, y)\|_1 + |\gamma(t)|\|(\rho, p) - (\sigma, q))\|_1. \end{aligned}$$

To verify condition (1.1) for F let

$$\begin{cases} t_0 \in [0,1],\ (\zeta, c) \in C,\ (\rho, p) \in E_1,\ \varphi \in E_1^*, \\ \varphi((\zeta, c)) = \max\{\varphi((\xi, x)) : (\xi, x) \in C\},\ \varphi((\rho, p)) = 0. \end{cases} \tag{4.3}$$

Let $(\delta, \psi) \in \mathbb{R} \times E^*$ be the unique representation of φ, that is

$$\varphi((\xi, x)) = \delta\xi + \psi(x) \quad ((\xi, x) \in E_1).$$

Let $v : [0, \omega_+) \to E$ be the solution of the autonomous initial value problem

$$v'(t) = g(\zeta, v(t)), \quad v(0) = c.$$

Since $g(\zeta, c) \geq 0$ we know that v is increasing, according to Theorem 4.4. Thus $v'(t) \geq 0$ and therefore $(\zeta, v(t)) \in C$ $(t \in [0, \omega_+))$. Let $h : [0, \omega_+) \to E_1$ be defined as

$$h(t) = (\zeta, c) - (\zeta, v(t)) = (0, c - v(t)).$$

In view of (4.3) we have

$$0 \leq \varphi(h(t)) = \psi(c - v(t)) \ \ (t \in [0, \omega_+)), \quad \varphi(h(0)) = 0.$$

Hence

$$\begin{aligned} 0 &\leq \varphi(h'(0)) = -\psi(v'(0)) = -\psi(g(\zeta, c)) \\ &= -\varphi(P(\zeta, c)) = -\varphi(F(t_0, (\zeta, c), (\rho, p))). \end{aligned}$$

Since by assumption $(0, x_0), (1, x_1) \in C$, Theorem 1.1 proves the existence of a solution $(\mu, u) : [0,1] \to C$ of the boundary value problem

$$\begin{gathered} (\mu''(t), u''(t)) + F(t, (\mu(t), u(t)), (\mu'(t), u'(t))) = 0, \\ (\mu(0), u(0)) = (0, x_0), \quad (\mu(1), u(1)) = (1, x_1). \end{gathered}$$

Thus, $u : [0,1] \to E$ is a solution of (4.2) with $g(\mu(t), u(t)) \geq 0$ $(t \in [0,1])$. Finally, if $w : [0,1] \to E$ is any solution of (4.2) with $g(\mu(t), w(t)) \geq 0$ $(t \in [0,1])$, then $(\mu, w) : [0,1] \to C$ solves

$$(\mu''(t), w''(t)) + F(t, (\mu(t), w(t)), (\mu'(t), w'(t))) = 0,$$
$$(\mu(0), w(0)) = (0, x_0), \quad (\mu(1), w(1)) = (1, x_1),$$

and we have $w = u$ by the uniqueness part of Theorem 1.1. □

5. EXAMPLES

Let $(H, (\cdot,\cdot))$ be a complex Hilbert space. Let $E = L_s(H)$ denote the real Banach space of all self-adjoint operators $X : H \to H$, endowed with the operator norm ($\|X\| = r(X)$ for $X \in L_s(H)$, where r denotes the spectral radius), and ordered by the cone of positive semidefinite operators, that is

$$K = \{X \in L_s(H) : (Xx, x) \geq 0 \ \ (x \in H)\}.$$

Note that $K^\circ \neq \emptyset$, as for example $\mathrm{id}_H \in K^\circ$. The mapping $q : L_s(H) \to L_s(H)$, $q(X) = X^2$ is differentiable and midpoint convex (hence convex):

$$\frac{X^2 + Y^2}{2} - \left(\frac{X+Y}{2}\right)^2 = \frac{X^2 + Y^2 - XY - YX}{4} = \frac{(X-Y)^2}{4} \geq 0.$$

Alternatively [8, Lemma 4] can be used to show that q is convex.

Therefore $-q$ is a concave function on $L_s(H)$. Moreover q and $-q$ are qmi functions on $L_s(H)$. This is well known for the finite dimensional case [13], and can be seen in our setting in the following way: Since q is differentiable on $L_s(H)$ it is sufficient to show that each derivative of q and $-q$ is a linear qmi mapping. Fix $X \in L_s(H)$ and consider $T := q'(X) : L_s(H) \to L_s(H)$, that is $T(Y) = XY + YX$. Then

$$\exp(tT)(Y) = \exp(tX)Y\exp(tX) \geq 0 \quad (t \in \mathbb{R},\ Y \geq 0).$$

Thus T and $-T$ are qmi, cf. [11, Theorem 1 (A)].

Now, let $A : [0,1] \to L_s(H)$ be a Lipschitz continuous (with constant τ) and concave function, and let $A(\xi_0) \gg 0$ for some $\xi_0 \in (0,1)$. Then $g : [0,1] \times L_s(H) \to L_s(H)$ defined by $g(\xi, X) = A(\xi) - X^2$ is continuous and concave, and $X \mapsto g(\xi, X)$ is qmi and locally Lipschitz continuous for each $\xi \in [0,1]$. Moreover $g(\xi_0, 0) \gg 0$. Let

$$(\xi, X), (\eta, Y) \in [0,1] \times L_s(H),\ g(\xi, X) \geq 0,\ g(\eta, Y) \geq 0.$$

Then

$$0 \leq X^2 \leq A(\xi) \Rightarrow \|X^2\| \leq \|A(\xi)\| \Rightarrow \|X\| \leq \sqrt{\|A(\xi)\|},$$

and analogously $\|Y\| \leq \sqrt{\|A(\eta)\|}$. From

$$g(\xi, X) - g(\eta, Y) = (A(\xi) - A(\eta)) - X(X - Y) - (X - Y)Y$$

we obtain

$$\|g(\xi, X) - g(\eta, Y)\| \leq \tau|\xi - \eta| + l\|X - Y\|$$

with $l := 2\max_{t\in[0,1]} \sqrt{\|A(t)\|}$. Now, if $l < \pi^2$ then Corollary 4.2 applies and we obtain the following result on the operator valued boundary value problem

$$U''(t) + A(t) - U(t)^2 = 0, \quad U(0) = X_0, \;\; U(1) = X_1. \tag{5.1}$$

Proposition 5.1. *Let*

$$\max_{t\in[0,1]} \|A(t)\| < \frac{\pi^4}{4} \approx 24.352, \quad X_0^2 \le A(0), \;\; X_1^2 \le A(1).$$

Then (5.1) *has a concave solution* $U : [0,1] \to L_s(H)$*, which is unique in the class of concave solutions.*

Figure 1 illustrates the solutions of the boundary value problem

$$u''(t) + 24\sin(\pi t) - u(t)^2 = 0, \quad u(0) = u(1) = 0. \tag{5.2}$$

As can be seen there is a unique concave solution, and a second non-concave solution. In particular this indicates that, in general, there are many non-concave solutions of the operator valued boundary value problem (5.1) in Proposition 5.1.

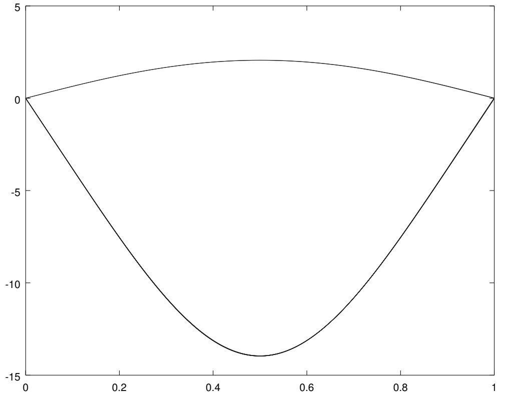

Fig. 1. Solutions of (5.2)

Now, let in addition A be constant, i.e. $A(t) = A_0$ ($t \in [0,1]$) with $A_0 \in K^\circ$, and let $\gamma \in \mathbb{R}$. We have $l = 2\sqrt{\|A_0\|}$, and if $(l, |\gamma|) \in W$ then Corollary 4.3 applies and we obtain the following result on the boundary value problem

$$U''(t) + A_0 - U(t)^2 + \gamma U'(t) = 0, \quad U(0) = X_0, \;\; U(1) = X_1. \tag{5.3}$$

Proposition 5.2. *Let*

$$\frac{\sqrt{\|A_0\|}}{4} + \frac{|\gamma|}{2} < 1, \quad X_0^2 \le A_0, \; X_1^2 \le A_0.$$

Then (5.3) *has a solution* $U : [0,1] \to L_s(H)$ *with* $t \mapsto \exp(\gamma t)U'(t)$ *monotone decreasing on* $[0,1]$*, which is unique in the class of solutions with this property.*

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