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# ON THE CONVERGENCE OF SOLUTIONS TO SECOND-ORDER NEUTRAL DIFFERENCE EQUATIONS

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**Abstract.** A second-order nonlinear neutral difference equation with a quasi-difference is studied. Sufficient conditions are established under which for every real constant there exists a solution of the considered equation convergent to this constant.

**Keywords:** second-order difference equation, asymptotic behavior, quasi-differences, Krasnoselskii's fixed point theorem.

**Mathematics Subject Classification:** 39A10, 39A22, 39A30.

## 1. INTRODUCTION

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of positive integers, all integers and all real numbers respectively. Let us consider the second-order nonlinear difference equation with a quasi-difference of the form

$$\Delta(r_n\Delta(x_n + p_n x_{n-\tau})) = a_n f(x_{n-\sigma}) + b_n, \tag{1.1}$$

where $\tau \in \mathbb{N}$, $\sigma \in \mathbb{Z}$ are fixed. Here $\Delta$ is the forward difference operator defined by

$$\Delta x_n = x_{n+1} - x_n,$$

$(r_n)$ is a sequence of positive real numbers, $(a_n)$, $(b_n)$ and $(p_n)$ are sequences of real numbers, and $f$ is a real function.

Note, that equation (1.1) generalizes some well known types of classical difference equations. For example the Sturm-Liouville difference equation

$$\Delta(r_n\Delta x_n) = a_n x_{n+1},$$

 

Emden-Fowler difference equation of the form

$$\Delta^2(x_n - p_n x_{n-\tau}) = a_n x_{n-\sigma}^{\alpha},$$

or Legendre's difference equation of the form

$$\Delta((n^2-1)\Delta x_n) = a_n x_n.$$

Let

$$\eta = \max(\tau, \sigma).$$

By a *solution* of equation (1.1), we mean a sequence $x$ which satisfies equality (1.1) for $n$ sufficiently large. If (1.1) is satisfied for all $n \geq \eta$ we say that $x$ is a *full solution* of (1.1). A solution $x$ is said to be *nonoscillatory* if it is eventually positive or negative. In the sequel, the space of all sequences $x : \mathbb{N} \to \mathbb{R}$ we denote by SQ. The Banach space of all bounded sequences $x \in \mathrm{SQ}$ equipped with sup norm we denote by BS. If $x, y \in \mathrm{SQ}$, then $xy$ and $|x|$ denote the sequences defined by $xy(n) = x_n y_n$ and $|x|(n) = |x_n|$, respectively.

Nonlinear difference equations are of paramount importance in applications. They are used in mathematical models in diverse areas such as electrical engineering, computer science, physics, economics, and biology. In particular, the second-order difference equation of type (1.1) and its special cases were considered by many authors. In 1987, Drozdowicz and Popenda [3] using Schauder's fixed point theorem gave necessary and sufficient conditions for the existence of an asymptotically constant solution to the following nonlinear difference equation

$$\Delta^2 x_n + p_n f(x_n) = 0.$$

The study of boundedness and convergences of solutions was continued by many authors, also for equations with deviating arguments, with quasidifferences and of neutral type.

For example, Thandapani *et al.* [17] established sufficient conditions for the asymptotic behavior of certain types of nonoscillatory solutions to the following equation

$$\Delta(r_n \Delta x_n) + a_n f(x_{n+1}) = 0.$$

The above equation has been also studied in [16]. Using the Darbo's fixed point theorem the authors obtained the existence of an asymptotically $\omega$-periodic solution and Lyapunov type stability. In [11], for the equation of the form

$$\Delta(r_n \Delta x_n) + a_n f(x_{n-k}) = b_n$$

sufficient conditions under which for an arbitrary real constant there exists a solution convergent to this constant and sufficient conditions for the existence of an asymptotically linear solution are obtained.

In [13], for a neutral type difference equation

$$\Delta^2(x_n + p x_{n-k}) + f(n, x_n) = 0$$

there were found certain conditions, under which all nonoscillatory solutions have the property $x_n = cn + d + o(1)$. Here the discrete Bihari type inequality was used. Liu *et al.* in [8] proved the existence of uncountably many bounded nonoscillatory solutions to the problem

$$\Delta(r_n \Delta(x_n + px_{n-k})) + f(n, x_{n-d_{1n}}, \dots, x_{n-d_{kn}}) = c_n$$

using Banach's fixed point theorem, under the Lipschitz continuity condition. Agarwal *et al.* [1] studied the existence of a nonoscillatory solution to the equation

$$\Delta(r_n \Delta(x_n + px_{n-k})) + F(n+1, x_{n+1-\sigma}) = 0,$$

where $p \in \mathbb{R}, |p| \neq 1$. Recently the existence of bounded solutions to neutral difference equations was also studied, for example, in [2, 4–7, 9, 12, 14] or [15].

In this paper, using Krasnoselskii's type fixed point theorem and Schaueder's fixed point theorem, we present sufficient conditions, under which for an arbitrary real constant $c$ there exists a solution to (1.1) convergent to $c$. Moreover, our technic allow us to control the degree of convergence of solutions. More precisely, we present sufficient conditions, under which there exists a solution $x$ such that $x_n = c + \mathrm{o}(n^s)$, where $s$ is a given nonpositive real number. We consider the cases, when $|p_n| < 1$, $|p_n| > 1$ and also when $p_n \equiv 1$ or $p_n \equiv -1$.

## 2. MAIN RESULTS

Let $s \in (-\infty, 0]$. In this section, we present sufficient conditions under which for every real constant $c$ there exists a solution $x$ of equation (1.1) such that $x_n = c + \mathrm{o}(n^s)$, where $s$ is a given nonpositive real number. In the proof of the main results we will need the following lemmas.

**Lemma 2.1** ([10])**.** *Assume $k \in \mathbb{N}$, $x, z, p : \mathbb{N} \longrightarrow \mathbb{R}$, $\alpha \in (0,1)$, $s \in \mathbb{R}$,*

$$|p_n| \leq \alpha, \quad z_n = x_n - p_n x_{n-k}$$

*for large $n$ and $z(n) = \mathrm{o}(n^s)$. Then $x(n) = \mathrm{o}(n^s)$.*

**Lemma 2.2.** *Assume $s \in (-\infty, 0]$, $r, x, u \in \mathrm{SQ}$,*

$$\sum_{j=1}^{\infty} \frac{1}{j^s r_j} \sum_{i=j}^{\infty} |x_i| < \infty, \quad u_n = \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |x_i|.$$

*Then $u_n = \mathrm{o}(n^s)$.*

*Proof.* Let $g \in \mathrm{SQ}$,

$$g_n = \sum_{j=n}^{\infty} \frac{1}{j^s r_j} \sum_{i=j}^{\infty} |x_i|.$$

By assumption, $g_n = \mathrm{o}(1)$. We have

$$\begin{aligned} n^{-s}|u_n| &= n^{-s}\left|\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=j}^{\infty}x_i\right| = \left|\sum_{j=n}^{\infty}\frac{1}{n^s r_j}\sum_{i=j}^{\infty}x_i\right| \\ &\le \sum_{j=n}^{\infty}\frac{1}{n^s r_j}\sum_{i=j}^{\infty}|x_i| \le \sum_{j=n}^{\infty}\frac{1}{j^s r_j}\sum_{i=j}^{\infty}|x_i| = g_n. \end{aligned}$$

Hence $n^{-s}|u_n| = \mathrm{o}(1)$ and we get $u_n = n^s\mathrm{o}(1) = \mathrm{o}(n^s)$. □

The next lemma is a version of Krasnoselskii's fixed point theorem.

**Lemma 2.3.** *Assume that $c \in \mathbb{R}$, $\gamma \in \mathrm{SQ}$, $\lim\limits_{n\to\infty}\gamma_n = 0$,*

$$G = \{x \in \mathrm{BS} : |x - c| \le |\gamma|\}, \quad T_1, T_2 : G \to \mathrm{SQ}, \quad T_1(G) + T_2(G) \subset G,$$

*$T_1$ is a contraction and $T_2$ is continuous. Then $G$ is convex and compact and there exists a point $x \in G$ such that*

$$x = T_1x + T_2x.$$

*Proof.* The assertion is a consequence of [10, Lemma 2.2 and Theorem 2.2]. □

**Remark 2.4.** Assume $c \in \mathbb{R}$, $s \in (-\infty, 0]$, $\gamma \in \mathrm{SQ}$, $\gamma_n = \mathrm{o}(n^s)$, and

$$K = \{x \in \mathrm{BS} : |x - c| \le |\gamma|\}.$$

If $x \in K$, then $|x_n - c| \le |\gamma_n| = \mathrm{o}(n^s)$. Hence $x_n - c = \mathrm{o}(n^s)$ and we get

$$x_n = c + \mathrm{o}(n^s)$$

for any $x \in K$.

First, we consider equation (1.1) with $\lim\limits_{n\to\infty} p_n \in (-1, 1)$.

**Theorem 2.5.** *Assume $s \in (-\infty, 0]$ and*

(h1) $p_n = p^* + \mathrm{o}(n^s)$, $\;p^* \in (-1, 1)$,
(h2) *$f$ is continuous,*
(h3) $\sum\limits_{j=1}^{\infty}\frac{1}{j^s r_j}\sum\limits_{i=j}^{\infty}|a_i| < \infty$ *and* $\sum\limits_{j=1}^{\infty}\frac{1}{j^s r_j}\sum\limits_{i=j}^{\infty}|b_i| < \infty$.

*Then for any $c \in \mathbb{R}$ there exists a solution $x$ of (1.1) such that $x_n = c + \mathrm{o}(n^s)$.*

*Proof.* Let $c \in \mathbb{R}$ and let us choose a positive real number $d$. There exists a constant $M > 0$ such that

$$|f(t)| \le M \quad \text{for any } t \in [c - d, c + d].$$

From (h1) it follows that there exists $\beta \in \mathbb{R}$ such that $|p_n| < \beta < 1$ for sufficiently large $n$. By (h3), the series

$$\sum_{n=1}^{\infty}\frac{1}{r_n}\sum_{j=n}^{\infty}(M\,|a_j| + |b_j|)$$

is convergent.

Let us denote

$$\rho_n := |c(p^* - p_n)| + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} (M\,|a_j| + |b_j|)$$

and

$$\gamma_n := \begin{cases} 0 & \text{for } n < \eta, \\ \rho_n + \beta\gamma_{n-\tau} & \text{for } n \geq \eta. \end{cases}$$

By (h3) and Lemma 2.2, we have $\rho_n = \mathrm{o}(n^s)$. Hence, by Lemma 2.1, we get $\gamma_n = \mathrm{o}(n^s)$. Choose an index $n_1 > \eta$ such that

$$\gamma_n < d, \quad |p_n| < \beta < 1 \tag{2.1}$$

for every $n \geq n_1$. Let

$$G = \{x \in \mathrm{BS} : \ |x - c| \leq \gamma \ \text{ and } \ x_n = c \ \text{ for } n < n_1\}. \tag{2.2}$$

Since $\gamma_n < d$ for $n \geq n_1$, we have $x_n \in [c-d, c+d]$ for all $x \in G$ and $n \in \mathbb{N}$. Hence $|f(x_n)| \leq M$ for any $x \in G$, and $n \in \mathbb{N}$. If $x \in G$, then

$$|a_n f(x_{n-\sigma}) + b_n| \leq M|a_n| + |b_n|$$

for any $n \in \mathbb{N}$. Now, we define two mappings $T_1$ and $T_2 : G \to \mathrm{BS}$ as follows:

$$T_1(x)(n) = \begin{cases} 0 & \text{for } n < n_1, \\ cp_n - p_n x_{n-\tau} & \text{for } n \geq n_1 \end{cases}$$

and

$$T_2(x)(n) = \begin{cases} c & \text{for } n < n_1, \\ c - cp_n + cp^* + \sum\limits_{i=n}^{\infty} \frac{1}{r_i} \sum\limits_{j=i}^{\infty} (a_j f(x_{j-\sigma}) + b_j) & \text{for } n \geq n_1. \end{cases}$$

We will show that $T_1$ and $T_2$ satisfy the conditions of Lemma 2.3.

(i) If $x, y \in G$, then for $n \geq n_1$ we get

$$\begin{aligned} |(T_1 x + T_2 y)(n) - c| &= \left| cp_n - p_n x_{n-\tau} - cp_n + cp^* + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} (a_j f(y_{j-\sigma}) + b_j) \right| \\ &\leq |p_n|\,|x_{n-\tau} - c| + |c\,(p^* - p_n)| + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} (M\,|a_j| + |b_j|) \\ &\leq \beta\gamma_{n-\tau} + \rho_n = \gamma_n. \end{aligned}$$

For $n < n_1$ we have $|(T_1 x + T_2 y)(n) - c| = 0$. Hence $T_1 x + T_2 y \in G$.

(ii) $T_1$ is a contraction.
Let $x, y \in G$. Then, for $n \geq n_1$ using (h3), we get

$$|T_1(x)(n) - T_1(y)(n)| = |p_n||x_{n-\tau} - y_{n-\tau}| \leq \beta \sup_{n \geq 1} |x_n - y_n|$$

and for $n < n_1$ we have $|T_1(x)(n) - T_1(y)(n)| = 0$. Thus

$$\|T_1 x - T_1 y\| \leq \beta \|x - y\|.$$

(iii) $T_2$ is continuous.
Let $x \in G$, $\varepsilon > 0$. Since the function $f$ is uniformly continuous on the interval $[c-d, c+d]$, there exists $\delta > 0$ such that for $t, s \in [c-d, c+d]$ and $|t-s| < \delta$ we have

$$|f(t) - f(s)| < \varepsilon. \tag{2.3}$$

Now, let $y \in G$ be such that $\|x - y\| < \delta$. Then $|x_n - y_n| < \delta$ for every $n \in \mathbb{N}$. Hence

$$|f(x_n) - f(y_n)| < \varepsilon \quad \text{for each} \quad n \in \mathbb{N}.$$

Using (2.3) we have

$$\|T_2 x - T_2 y\| = \sup_{n \in \mathbb{N}} \left| \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} a_j \left( f(x_{j-\sigma}) - f(y_{j-\sigma}) \right) \right| \leq \varepsilon \sum_{i=1}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} |a_j| \, .$$

Thus, by (h3), $T_2$ is continuous.

Therefore, by Lemma 2.3, there exists $x \in G$ such that $x = T_1 x + T_2 x$. For $n \geq n_1$ we have

$$x_n = c + cp^* - p_n x_{n-\tau} + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} \left( a_j f(x_{j-\sigma}) + b_j \right). \tag{2.4}$$

Hence

$$x_n + p_n x_{n-\tau} = c + cp^* + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} \left( a_j f(x_{j-\sigma}) + b_j \right).$$

Applying the operator $\Delta$ to both sides of the above equation and multiplying by $r_n$, we get

$$r_n \Delta(x_n + p_n x_{n-\tau}) = -\sum_{j=n}^{\infty} (a_j f(x_{j-\sigma}) + b_j)$$

and applying $\Delta$ for the second time we get

$$\Delta(r_n \Delta(x_n + p_n x_{n-\tau})) = a_n f(x_{n-\sigma}) + b_n$$

for $n \geq n_1$. Hence $x$ is a solution of (1.1). Since $x \in G$ and $\gamma_n = \mathrm{o}(n^s)$, we have $x_n = c + \mathrm{o}(n^s)$. $\square$

Under some additional conditions, we get from Theorem 2.5, that for any real constant there exists a full solution of equation (1.1) convergent to this constant.

**Corollary 2.6.** *Suppose that the assumptions of Theorem 2.5 are satisfied, $p_n \neq 0$ for any $n$, and $\tau > \sigma$. Then for every $c \in \mathbb{R}$ there exists a full solution $\bar{x}$ of* (1.1) *such that $\bar{x}_n = c + \mathrm{o}(n^s)$.*

*Proof.* By Theorem 2.5, there exists a solution $x$ of (1.1) such that $x_n = c + \mathrm{o}(n^s)$. Then there exists an index $n_1 > \eta$ such that the equality (1.1) is satisfied for any $n \geq n_1$. Using this $x$, we can construct a solution $\bar{x}$ of (1.1), which satisfied (1.1) for all $n \geq \eta$. Let

$$\bar{x}_n = x_n \quad \text{ for } n \geq n_1.$$

We can rewrite equation (2.4), in the following form

$$x_{n-\tau} = \frac{c + cp^*}{p_n} - \frac{x_n}{p_n} + \frac{1}{p_n} \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} \left( a_j f(x_{j-\sigma}) + b_j \right).$$

Since $\tau > \sigma$, we have $\eta = \tau$. We can find the first $n_1 - \tau$ terms of $(\bar{x}_n)$ starting with putting $n := n_1 + \tau - 1$ in the above equation. □

**Corollary 2.7.** *Suppose that the assumptions of Theorem 2.5 are satisfied, $|p_n| \leq \beta < 1$ for any $n$, and $f$ is bounded. Then for every $c \in \mathbb{R}$ there exists a full solution $\bar{x}$ of* (1.1) *such that $x_n = c + \mathrm{o}(n^s)$.*

*Proof.* In the proof of Theorem 2.5 we can take $d$ so large that the condition (2.1) is satisfied for any $n \geq \eta$. □

**Theorem 2.8.** *Assume $s \in (-\infty, 0]$ and*

(h1) $p_n = p^* + \mathrm{o}(n^s)$, $p^* \in (-\infty, -1) \cup (1, \infty)$,
(h2) $f$ *is continuous,*
(h3) $\sum\limits_{j=1}^{\infty} \frac{1}{j^s r_j} \sum\limits_{i=j}^{\infty} |a_i| < \infty$ *and* $\sum\limits_{j=1}^{\infty} \frac{1}{j^s r_j} \sum\limits_{i=j}^{\infty} |b_i| < \infty$.

*Then for any $c \in \mathbb{R}$ there exists a solution $x$ of* (1.1) *such that $x_n = c + \mathrm{o}(n^s)$.*

*Proof.* Let us choose a positive real number $d$. There exists a constant $M > 0$ such that $|f(t)| \leq M$ for any $t \in [c - d, c + d]$. Moreover, let $\beta \in \mathbb{R}$ be such that $|p_n| > \beta > 1$ for sufficiently large $n$,

$$\rho_n := \frac{1}{\beta} \left| c(p^* - p_{n+\tau}) \right| + \frac{1}{\beta} \sum_{i=n+\tau}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} \left( M|a_j| + |b_j| \right)$$

and

$$\gamma_n := \frac{\beta}{\beta - 1} \rho_n.$$

There exists an index $n_1 \geq \eta$ such that

$$\gamma_n < \frac{d}{\beta}, \quad |p_n| > \beta$$

for every $n \geq n_1$. Let $G$ be given by (2.2). We define the operators $T_1$ and $T_2 : G \to \mathrm{BS}$ as follows:

$$T_1(x)(n) = \begin{cases} 0 & \text{for } \ n < n_1, \\ \frac{1}{p_{n+\tau}}\left(c - x_{n+\tau}\right) & \text{for } \ n \geq n_1 \end{cases}$$

and

$$T_2(x)(n)=\begin{cases} c & \text{for } n < n_1, \\ c + \frac{1}{p_{n+\tau}}\left(cp^* - cp_{n+\tau}\right) + \frac{1}{p_{n+\tau}} \sum\limits_{i=n+\tau}^{\infty} \frac{1}{r_i} \sum\limits_{j=i}^{\infty} \left(a_j f(x_{j-\sigma}) + b_j\right) & \text{for } n \geq n_1. \end{cases}$$

If $x, y \in G$, then for $n \geq n_1$ we have

$$\begin{aligned} |(T_1x + T_2y)(n) - c| &\leq \frac{1}{\beta}|x_{n+\tau} - c| + \frac{1}{\beta}\left|c\left(p^* - p_{n+\tau}\right)\right| \\ &\quad + \frac{1}{\beta} \sum_{i=n+\tau}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty} \left(M\left|a_j\right| + \left|b_j\right|\right) \\ &\leq \frac{1}{\beta}\gamma_{n+\tau} + \rho_{n+\tau} = \gamma_{n+\tau} \leq \gamma_n. \end{aligned}$$

Therefore $T_1G + T_2G \subset G$. Obviously $T_1$ is a contraction. Similarly as in the proof of Theorem 2.5 the map $T_2$ is continuous. By Lemma 2.3, there exists $x \in G$ such that $x = T_1x + T_2x$. For $n \geq n_1$ we have

$$x_n = \frac{c}{p_{n+\tau}} - \frac{x_{n+\tau}}{p_{n+\tau}} + \frac{cp^*}{p_{n+\tau}} + \frac{1}{p_{n+\tau}} \sum_{i=n+\tau}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty}(a_j f(x_{j-\sigma}) + b_j).$$

Multiplying by $p_{n+\tau}$ we get

$$x_{n+\tau} + p_{n+\tau}x_n = c + cp^* + \sum_{i=n+\tau}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty}(a_j f(x_{j-\sigma}) + b_j).$$

Hence, replacing $n$ by $n - \tau$, we obtain

$$x_n + p_n x_{n-\tau} = c + cp^* + \sum_{i=n}^{\infty} \frac{1}{r_i} \sum_{j=i}^{\infty}(a_j f(x_{j-\sigma}) + b_j).$$

The rest of the proof is similar to that of Theorem 2.5. □

Theorem 2.5 and Theorem 2.8 extend some results from [11].

In the next theorem, we consider the special case of (1.1), when $p_n \equiv 1$. In the proof of this theorem the Schauder's fixed point theorem will be used.

**Theorem 2.9.** *Assume $s \in (-\infty, 0]$ and*

(h1) $p_n \equiv 1$,
(h2) *$f$ is continuous,*
(h3) $\sum\limits_{j=1}^{\infty} \frac{1}{j^s r_j} \sum\limits_{i=j}^{\infty} |a_i| < \infty$ *and* $\sum\limits_{j=1}^{\infty} \frac{1}{j^s r_j} \sum\limits_{i=j}^{\infty} |b_i| < \infty$.

*Then for any $c \in \mathbb{R}$ there exists a solution $x$ of (1.1) such that $x_n = c + \mathrm{o}(n^s)$.*

*Proof.* Let $c \in \mathbb{R}$ and let us choose a positive real number $d$. There exists a constant $M > 0$ such that $|f(t)| \le M$ for $t \in [c-d, c+d]$. Let us denote

$$A_n = \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |a_i|, \quad B_n = \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |b_i|.$$

By (h3) there exists an index $n_1$ such that for $n \ge n_1$ we have

$$A_n \le \frac{d}{2M}, \quad \text{and} \quad B_n \le \frac{d}{2}. \tag{2.5}$$

We define a subset $G$ of BS by

$$G = \{x \in \mathrm{BS} : |x - c| \le MA + B \text{ and } x_n = c \text{ for } n \le n_1\}.$$

By Lemma 2.3, $G$ is a convex and compact subset of BS. Moreover, by (2.5), we have

$$x_n \in [c-d, c+d]$$

for any $x \in G$ and any $n$. Now we define a map $T : G \to \mathrm{BS}$, as follows:

$$T(x)(n) = \begin{cases} c & \text{for } n < n_1, \\ c + \sum\limits_{k=1}^{\infty} \sum\limits_{j=n+(2k-1)\tau}^{n+2k\tau-1} \frac{1}{r_j} \sum\limits_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) & \text{for } n \ge n_1. \end{cases}$$

We will show that $TG \subset G$. It is obvious that

$$\sum_{k=1}^{\infty} \sum_{j=n+(2k-1)\tau}^{n+2k\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} |a_i| \le \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |a_i|, \tag{2.6}$$

$$\sum_{k=1}^{\infty} \sum_{j=n+(2k-1)\tau}^{n+2k\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} |b_i| \le \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |b_i|. \tag{2.7}$$

Moreover, if $x \in G$, then $|x_n - c| \le d$ for all $n \in \mathbb{N}$. Hence $|f(x_n)| \le M$ for every $x \in G$, $n \in \mathbb{N}$. Therefore by (2.6) and (2.7), for $n \ge n_1$, we get

$$\begin{aligned} |T(x)(n) - c| &= \left| \sum_{k=1}^{\infty} \sum_{j=n+(2k-1)\tau}^{n+2k\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) \right| \\ &\le M \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |a_i| + \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |b_i| = MA_n + B_n. \end{aligned}$$

This gives $Tx \in G$ for every $x \in G$. Hence $TG \subset G$.

The next step is to show the continuity of $T$. Choose $\varepsilon > 0$. There exists a $\delta > 0$ such that

$$\text{if } t_1, t_2 \in [c-d, c+d] \text{ and } |t_1 - t_2| < \delta, \text{ then } |f(t_1) - f(t_2)| < \varepsilon.$$

Let $x, y \in G$, $\|x-y\| < \delta$. Then $|x_n - y_n| < \delta$ for any $n$ and we get

$$\begin{aligned}\|Tx - Ty\| &= \sup_{n \geq n_1} |T(x)(n) - T(y)(n)| \\ &\leq \sup_{n \geq n_1} \sum_{k=1}^{\infty} \sum_{j=n+(2k-1)\tau}^{n+2k\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} |a_i||f(x_{i-\sigma}) - f(y_{i-\sigma})| \leq A_1 \varepsilon.\end{aligned}$$

Hence $T$ is continuous. By Schauder's fixed point theorem there exists $x \in G$ such that $Tx = x$. For $n \geq n_1$ we have

$$x_n = c + \sum_{k=1}^{\infty} \sum_{j=n+2k\tau-\tau}^{n+2k\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right).$$

Hence

$$\begin{aligned}x_n + x_{n-\tau} = 2c &+ \sum_{k=1}^{\infty} \sum_{j=n+2k\tau-\tau}^{n+2k\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) \\ &+ \sum_{k=1}^{\infty} \sum_{j=n+2k\tau-2\tau}^{n+2k\tau-\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) \\ &= 2c + \sum_{k=1}^{\infty} \sum_{j=n+2k\tau-2\tau}^{n+2k\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) \\ &= 2c + \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right).\end{aligned}$$

Therefore

$$\Delta\left(x_n + x_{n-\tau}\right) = -\frac{1}{r_n} \sum_{i=n}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right),$$

$$r_n \Delta\left(x_n + x_{n-\tau}\right) = -\sum_{i=n}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right),$$

and finally

$$\Delta\left(r_n \Delta\left(x_n + x_{n-\tau}\right)\right) = a_n f\left(x_{n-\sigma}\right) + b_n.$$

Hence $x$ is a solution of (1.1). By Lemma 2.2, $MA_n + B_n = \mathrm{o}(n^s)$. Since $x \in G$, we have $x_n = c + \mathrm{o}(n^s)$, that is our claim. □

Note that, taking $s = 0$ in Theorems 2.5, 2.8, and 2.9 we get the following result.

**Theorem 2.10.** *Assume*

(h1) $\lim\limits_{n\to\infty} p_n = p^*$, $|p^*| \neq 1$ *or* $p_n \equiv 1$,
(h2) $f$ *is continuous,*
(h3) $\sum\limits_{j=1}^{\infty} \frac{1}{r_j} \sum\limits_{i=j}^{\infty} |a_i| < \infty$ *and* $\sum\limits_{j=1}^{\infty} \frac{1}{r_j} \sum\limits_{i=j}^{\infty} |b_i| < \infty$.

*Then for any* $c \in \mathbb{R}$ *there exists a solution* $x$ *of* (1.1) *such that* $\lim\limits_{n\to\infty} x(n) = c$.

In the case, when $p_n \equiv -1$ we need stronger summability conditions for the sequences $(a_n)$ and $(b_n)$.

**Theorem 2.11.** *Assume* $s \in (-\infty, 0]$ *and*

(h1) $p_n \equiv -1$,
(h2) $f$ *is continuous,*
(h3) $\sum\limits_{j=1}^{\infty} \frac{1}{j^{s-1} r_j} \sum\limits_{i=j}^{\infty} |a_i| < \infty$ *and* $\sum\limits_{j=1}^{\infty} \frac{1}{j^{s-1} r_j} \sum\limits_{i=j}^{\infty} |b_i| < \infty$.

*Then for any* $c \in \mathbb{R}$ *there exists a solution* $x$ *of* (1.1) *such that* $x_n = c + \mathrm{o}(n^s)$.

*Proof.* Let us choose a positive real number $d$. There exists a constant $M > 0$ such that $|f(t)| \leq M$ for $t \in [c-d, c+d]$. Let us denote

$$A_n = \sum_{j=n}^{\infty} \frac{j}{r_j} \sum_{i=j}^{\infty} |a_i|, \quad B_n = \sum_{j=n}^{\infty} \frac{j}{r_j} \sum_{i=j}^{\infty} |b_i|.$$

By $(h3)$ there exists an index $n_1$ such that for $n \geq n_1$ we have

$$A_n \leq \frac{d}{2M}, \quad \text{and} \quad B_n \leq \frac{d}{2}. \tag{2.8}$$

We define a subset $G$ of BS by

$$G = \{x \in \mathrm{BS} : |x - c| \leq MA + B \text{ and } x_n = c \text{ for } n \leq n_1\}.$$

By Lemma 2.3, $G$ is a convex and compact subset of BS. Moreover, by (2.8), we have $x_n \in [c-d, c+d]$ for any $x \in G$ and any $n$. Now, we define a map $T : G \to \mathrm{BS}$ as follows:

$$T(x)(n) = \begin{cases} c & \text{for } n < n_1, \\ c + \sum\limits_{k=1}^{\infty} \sum\limits_{j=n+k\tau}^{\infty} \frac{1}{r_j} \sum\limits_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) & \text{for } n \geq n_1. \end{cases}$$

Note that

$$\sum_{k=1}^{\infty} \sum_{j=n+k\tau}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |a_i| \leq \sum_{j=n}^{\infty} \frac{j}{r_j} \sum_{i=j}^{\infty} |a_i|, \tag{2.9}$$

$$\sum_{k=1}^{\infty} \sum_{j=n+k\tau}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} |b_i| \leq \sum_{j=n}^{\infty} \frac{j}{r_j} \sum_{i=j}^{\infty} |b_i|. \tag{2.10}$$

We will show that $TG \subset G$. If $x \in G$, then $|x_n - c| \le d$ for all $n \in \mathbb{N}$. Hence $|f(x_n)| \le M$ for every $x \in G$, $n \in \mathbb{N}$. Therefore by (2.9) and (2.10), for $n \ge n_1$, we get

$$\begin{aligned}|T(x)(n) - c| &= \left|\sum_{k=1}^{\infty} \sum_{j=n+k\tau}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right)\right| \\ &\le M \sum_{j=n}^{\infty} \frac{j}{r_j} \sum_{i=j}^{\infty} |a_i| + \sum_{j=n}^{\infty} \frac{j}{r_j} \sum_{i=j}^{\infty} |b_i| = MA_n + B_n.\end{aligned}$$

This gives $Tx \in G$ for every $x \in G$. Hence $TG \subset G$.
Similarly as in the proof of Theorem 2.9 we can show the continuity of $T$. Hence, by Schauder's fixed point theorem there exists $x \in G$ such that $Tx = x$. For $n \ge n_1$ we have

$$x_n = c + \sum_{k=1}^{\infty} \sum_{j=n+k\tau}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right).$$

Therefore

$$\begin{aligned}x_n - x_{n-\tau} &= \sum_{k=1}^{\infty} \sum_{j=n+k\tau}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) \\ &\quad - \sum_{k=1}^{\infty} \sum_{j=n+k\tau-\tau}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) \\ &= \sum_{k=1}^{\infty} \sum_{j=n+k\tau-\tau}^{n+k\tau-1} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right) \\ &= \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right)\end{aligned}$$

and

$$\Delta\left(x_n - x_{n-\tau}\right) = -\frac{1}{r_n} \sum_{i=n}^{\infty} \left(a_i f\left(x_{i-\sigma}\right) + b_i\right).$$

The rest of the proof is similar to that of Theorem 2.9 and is omitted. □

## 3. EXAMPLES

In this section we present two examples to illustrate the obtained results.

**Example 3.1.** Let $r_n = (n+1)n$, $p_n = \frac{1}{n}$, $\tau = 1$, $\sigma = 0$, $s = 0$, $p^* = 0$, $f(t) = t^2$ for any $t$,

$$a_n = -\frac{2n}{(n-1)^3} \quad \text{and} \quad b_n = 0.$$

Then equation (1.1) takes the form

$$\Delta\left((n+1)n\Delta\left(x_n+\frac{1}{n}\cdot x_{n-1}\right)\right)=-\frac{2n}{(n-1)^3}\,x_n^2. \tag{3.1}$$

We have

$$\sum_{n=1}^{\infty}\frac{1}{r_n}<\infty, \qquad \sum_{n=1}^{\infty}|a_n|<\infty.$$

By Theorem 2.5, for any $c\in\mathbb{R}$ there exists a solution $x$ of the equation (3.1) such that $x_n=c+\mathrm{o}(1)$. In fact, the sequence $x_n=1-\frac{1}{n}$ is a solution of (3.1) with such property.

**Example 3.2.** Let $r_n=2^n$, $p_n=8^{-n}$, $\tau=3$, $\sigma=-1$, $f(t)=t$ for any $t$,

$$a_n=\frac{21}{32}\cdot 4^{-n}, \quad b_n=\frac{7419}{128}\cdot 16^{-n}+\frac{3}{8}\cdot 2^{-n},$$

and let $s\in(-\infty,0]$ be arbitrary but fixed. Since

$$\sum_{k=n}^{\infty}\lambda^k=\frac{\lambda^n}{1-\lambda}\ \text{for any } \lambda\in(-1,1) \text{ and any } n\in\mathbb{N},$$

and $a_n\le 2^{-n}$ for any $n$, we have

$$\sum_{j=1}^{\infty}\frac{1}{j^s r_j}\sum_{i=j}^{\infty}|a_i|\le\sum_{j=1}^{\infty}\frac{j^{-s}}{2^j}\sum_{i=j}^{\infty}2^{-i}=\sum_{j=1}^{\infty}\frac{j^{-s}}{2^j}\frac{2}{2^j}=2\sum_{j=1}^{\infty}\frac{j^{-s}}{4^j}<\infty.$$

Analogously

$$\sum_{j=1}^{\infty}\frac{1}{j^s r_j}\sum_{i=j}^{\infty}|b_i|<\infty.$$

Hence, by Theorem 2.5, for any $c\in\mathbb{R}$ there exists a solution $x$ of the equation

$$\Delta\left(2^n\Delta\left(x_n+\frac{1}{8^n}\cdot x_{n-3}\right)\right)=\frac{21}{32}\cdot 4^{-n}x_{n+1}+\frac{7419}{128}\cdot 16^{-n}+\frac{3}{8}\cdot 2^{-n} \tag{3.2}$$

such that $x_n=c+\mathrm{o}(n^s)$. In fact, for any $c\in\mathbb{R}$ the sequence $x_n=c+4^{-n}$ is a solution of (3.2). Note that $4^{-n}=\mathrm{o}(n^s)$.

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REFERENCES


[1] R.P. Agarwal, S.R. Grace, D. O'Regan, *Nonoscillatory solutions for discrete equations*, Comput. Math. Appl. **45** (2003), 1297–1302.

[2] A. Bezubik, M. Migda, M. Nockowska-Rosiak, E. Schmeidel, *Trichotomy of nonoscillatory solutions to second-order neutral difference equation with quasi-difference*, Adv. Difference Equ. **192** (2015), 14 pp.

[3] A. Drozdowicz, J. Popenda, *Asymptotic behavior of solutions of the second order difference equation*, Proc. Amer. Math. Soc. **99** (1987) 1, 135–140.

[4] M. Galewski, E. Schmeidel, *On the well posed solutions for nonlinear second order neutral difference equations*, Math. Slovac. **66** (2016) 4, 933–944.

[5] M. Galewski, R. Jankowski, M. Nockowska–Rosiak, E. Schmeidel, *On the existence of bounded solutions for nonlinear second order neutral difference equations*, Electron. J. Qual. Theory Differ. Equ. **72** (2014), 1–12.

[6] R. Jankowski, E. Schmeidel, *Almost oscillation criteria for second order neutral difference equation with quasidifferences*, Int. J. Difference Equ. **9** (2014), 77–86.

[7] B.S. Lalli, B.G. Zhang, *On existence of positive solutions and bounded oscillations of neutral difference equations*, J. Math. Anal. Appl. **166** (1992), 272–287.

[8] Z. Liu, Y. Xu, S.M. Kang, *Global solvability for a second order nonlinear neutral delay difference equation*, Comput. Math. Appl. **57** (2009) 4, 587–595.

[9] Z. Liu, M. Jia, S.M. Kang, Y.C. Kwun, *Bounded positive solutions for a third order discrete equation*, Abst. Appl. Anal. **2012** (2012), Article ID 237036, 12 pp.

[10] J. Migda, *Approximative solutions to difference equations of neutral type*, Appl. Math. Comput. **268** (2015), 763–774.

[11] M. Migda, *Asymptotic behaviour of solutions of nonlinear difference equations*, Fasc. Math. **31** (2001), 57–63.

[12] M. Migda, J. Migda, *On a class of first order nonlinear difference equations of neutral type*, Math. Comput. Modelling **40** (2004), 297–306.

[13] M. Migda, J. Migda, *Asymptotic properties of solutions of second-order neutral difference equations*, Nonlinear Anal. **63** (2005), e789–e799.

[14] M. Migda, E. Schmeidel, M. Zdanowicz, *Bounded solutions of k-dimensional system of nonlinear difference equations of neutral type*, Electron. J. Qual. Theory Differ. Equ. **80** (2015), 17 pp.

[15] E. Schmeidel, *An application of measures of noncompactness in investigation of boundedness of solutions of second order neutral difference equations*, Adv. Difference Equ. **91** (2013), 9 pp.

[16] E. Schmeidel, Z. Zbąszyniak, *An application of Darbo's fixed point theorem in the investigation of periodicity of solutions of difference equations*, Comput. Math. Appl. **64** (2012), 2185–2191.

[17] E. Thandapani, M.M.S. Manuel, J.R. Graef, P.W. Spikes, *Monotone properties of certain classes of solutions of second-order difference equations*, Comput. Math. Appl. **36** (1998), 291–297.

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