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INFINITELY MANY SOLUTIONS FOR SOME NONLINEAR SUPERCRITICAL PROBLEMS WITH BREAK OF SYMMETRY

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Abstract. In this paper, we prove the existence of infinitely many weak bounded solutions of the nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = g(x,u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain, $N \geq 3$, and $A(x,t,\xi)$, $g(x,t)$, $h(x)$ are given functions, with $A_t = \frac{\partial A}{\partial t}$, $a = \nabla_\xi A$, such that $A(x,\cdot,\cdot)$ is even and $g(x,\cdot)$ is odd. To this aim, we use variational arguments and the Rabinowitz's perturbation method which is adapted to our setting and exploits a weak version of the Cerami–Palais–Smale condition. Furthermore, if $A(x,t,\xi)$ grows fast enough with respect to t, then the nonlinear term related to $g(x,t)$ may have also a supercritical growth.

Keywords: quasilinear elliptic equation, weak Cerami–Palais–Smale condition, Ambrosetti–Rabinowitz condition, break of symmetry, perturbation method, supercritical growth.

Mathematics Subject Classification: 35J20, 35J62, 35J66, 58E05.

1. INTRODUCTION

During the past years there has been a considerable amount of research in obtaining multiple critical points of functionals such as

$$\mathcal{J}(u) = \int_\Omega A(x,u,\nabla u)dx - \int_\Omega F(x,u)dx, \qquad u \in \mathcal{D},$$

where $\mathcal{D}$ is a subset of a suitable Sobolev space, $A : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ and $F : \Omega\times\mathbb{R} \to \mathbb{R}$ are given functions with $\Omega \subset \mathbb{R}^N$ open bounded domain, $N \geq 3$.

A family of model problems is given by

$$A(x,t,\xi) = \sum_{i,j=1}^{N} a_{i,j}(x,t)\xi_i\xi_j$$

with $(a_{i,j}(x,t))_{i,j}$ elliptic matrix. In particular, if $a_{i,j}(x,t) = \frac{1}{2}\delta_i^j \bar{A}(x,t)$ for a given function $\bar{A} : \Omega \times \mathbb{R} \to \mathbb{R}$, then it is $A(x,t,\xi) = \frac{1}{2}\bar{A}(x,t)|\xi|^2$.

In the simplest case $A(x,t,\xi) = \frac{1}{2}|\xi|^2$, functional $\mathcal{J}$, defined on $\mathcal{D} = H_0^1(\Omega)$, is the standard action functional associated to the classical semilinear elliptic problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f(x,t) = \frac{\partial F}{\partial t}(x,t)$. If $F(x,t)$ has a subcritical growth with respect to t and verifies other suitable assumptions, existence and multiplicity of critical points of the C^1 functional $\mathcal{J}$ have been widely studied by many authors in the last sixty years (see [23, 25] and references therein).

On the other hand, when $A(x,t,\xi) = \frac{1}{2}\bar{A}(x,t)|\xi|^2$, with $\bar{A}(x,t)$ smooth, bounded, far away from zero but $\bar{A}_t(x,t) \not\equiv 0$, even if $F(x,t) \equiv 0$, the corresponding functional

$$\bar{\mathcal{J}}_0(u) = \frac{1}{2}\int_{\Omega} \bar{A}(x,u)|\nabla u|^2 dx$$

is defined in $H_0^1(\Omega)$ but is Gâteaux differentiable only along directions which are in $H_0^1(\Omega) \cap L^\infty(\Omega)$.

In the beginning, such a problem has been overcome by introducing suitable definitions of critical point and related existence results have been stated (see, e.g., [2, 3, 17, 21]). More recently, it has been proved that suitable assumptions assure that functional $\mathcal{J}$ is C^1 in the Banach space $X = H_0^1(\Omega) \cap L^\infty(\Omega)$ equipped with the norm $\|\cdot\|_X$ given by the sum of the classical norms $\|\cdot\|_H$ on $H_0^1(\Omega)$ and $|\cdot|_\infty$ in $L^\infty(\Omega)$ (see [7] if $A(x,t,\xi) = \frac{1}{2}\bar{A}(x,t)|\xi|^2$ and [8] in the general case). Furthermore, its critical points in X are weak bounded solutions of the quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi),\ a(x,t,\xi) = (\frac{\partial A}{\partial \xi_1}(x,t,\xi), \ldots, \frac{\partial A}{\partial \xi_N}(x,t,\xi)). \tag{1.1}$$

In order to study the set of critical points of a C^1 functional J on a Banach space $(Y, \|\cdot\|_Y)$, but avoiding global compactness assumptions, Palais and Smale introduced the following condition (see [20]).

Definition 1.1. A functional J satisfies the *Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(PS)_\beta$ condition, if any $(PS)_\beta$-sequence, i.e., any sequence $(u_n)_n \subset Y$ such that

$$\lim_{n\to+\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{Y'} = 0,$$

converges in Y, up to subsequences.

We note that if J satisfies $(PS)_\beta$ condition, the set of the critical points of J at level β is compact.

Later on, in [18] Cerami weakened such a definition by allowing a sequence to go to infinity but only if the gradient of the functional goes to zero “not too slowly”.

Definition 1.2. A functional J satisfies the *Cerami's variant of Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(CPS)_\beta$ condition, if any $(CPS)_\beta$-sequence, i.e., any sequence $(u_n)_n \subset Y$ such that

$$\lim_{n\to+\infty} J(u_n) = \beta \quad \text{and} \quad \lim_{n\to+\infty} \|dJ(u_n)\|_{Y'}(1 + \|u_n\|_Y) = 0,$$

converges in Y, up to subsequences.

Unfortunately, our functional $\mathcal{J}$ in X may have unbounded Palais–Smale sequences (see [11, Example 4.3]). Anyway, since X is equipped with two different norms, namely $\|\cdot\|_X$ and $\|\cdot\|_H$, according to the ideas already developed in previous papers (see, e.g., [7, 9, 11]) a weaker version of (CPS) condition can be introduced when the Banach space Y is equipped with a second norm $\|\cdot\|_*$ such that $(Y, \|\cdot\|_Y)$ is continuously imbedded in $(Y, \|\cdot\|_*)$.

Definition 1.3. A functional J satisfies *a weak version of the Cerami's variant of Palais–Smale condition at level* β ($\beta \in \mathbb{R}$), briefly $(wCPS)_\beta$ *condition*, if for every $(CPS)_\beta$-sequence $(u_n)_n$ a point $u \in Y$ exists such that

(i) $\lim_{n\to\infty} \|u_n - u\|_* = 0$ (up to subsequences),
(ii) $J(u) = \beta$, $dJ(u) = 0$.

If J satisfies the $(wCPS)_\beta$ condition at each level $\beta \in I$, I real interval, we say that J satisfies the $(wCPS)$ condition in I.

We note that if $\beta \in \mathbb{R}$ is such that $(wCPS)_\beta$ condition holds, then β is a critical level if a $(CPS)_\beta$-sequence exists, furthermore the set of the critical points of J at level β is compact but with respect to the weaker norm $\|\cdot\|_*$.

Moreover, $(wCPS)_\beta$ condition is enough for proving a Deformation Lemma (see [9, Lemma 2.3]) and extending some critical point theorems (see [15]), but, contrary to the classical (CPS) condition, it it is not sufficient for finding multiple critical points if they occur at the same critical level. We remark that such a problem is avoided by replacing $(CPS)_\beta$-sequences with $(PS)_\beta$-sequences in Definition 1.3 and then a more general Deformation Lemma can be stated (see [11, Proposition 2.4]).

If $F(x,t)$ grows as $|t|^q$ with $2 < q < 2^*$ and satisfies the Ambrosetti–Rabinowitz condition, then it is possible to find at least one critical point, or infinitely many ones if $\mathcal{J}$ is even, by applying a suitable version of the Mountain Pass Theorem, or its symmetric variant (see [7, 8] and, for the abstract setting, [9]). Such results still hold if $F(x,t)$ has a suitable supercritical growth but function $A(x,t,\xi)$ satisfies “good” growth assumptions (see [15] and, for a different type of supercritical problems, see, e.g., [1]).

Furthermore, the existence of multiple critical points has been stated in [10, 11, 14] for different sets of hypotheses on $F(x,t)$.

We note that all the previous results still hold if $A(x,t,\xi)$ increases as $|\xi|^p$ for any $p>1$.

More recently, infinitely many critical points have been found in break of symmetry if $A(x,t,\xi)=\frac{1}{2}\bar{A}(x,t)|\xi|^2$ and $F(x,t)=G(x,t)+h(x)t$, with $\bar{A}(x,\cdot)$ and $G(x,\cdot)$ even (see [16]).

In order to give an idea of the difficulties which arise dealing with functional $\mathcal{J}$ in X, in this paper we extend the result in [16] to a more general term $A(x,t,\xi)$ which increases as $|\xi|^2$.

More precisely, we look for weak bounded solutions of the nonlinear elliptic problem

$$\begin{cases} -\mathrm{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) = g(x,u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded domain, $N\geq 3$, and $A:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$, $g:\Omega\times\mathbb{R}\to\mathbb{R}$, $h:\Omega\to\mathbb{R}$ are given functions, with $A(x,\cdot,\cdot)$ even and $g(x,\cdot)$ odd.

Hence, as already remarked, under suitable assumptions for $A(x,t,\xi)$, $g(x,t)$ and $h(x)$, we study the existence of infinitely many critical points of the C^1 functional

$$\mathcal{J}(u) = \int_\Omega A(x,u,\nabla u)dx - \int_\Omega G(x,u)dx - \int_\Omega hu dx, \quad u\in X, \tag{1.3}$$

with $G(x,t)=\int_0^t g(x,s)ds$.

If $h(x)\equiv 0$, functional $\mathcal{J}$ in (1.3) reduces to the even map

$$\mathcal{J}_0(u) = \int_\Omega A(x,u,\nabla u)dx - \int_\Omega G(x,u)dx, \quad u\in X. \tag{1.4}$$

If $h(x)\not\equiv 0$ the symmetry is broken. Anyway, some perturbation methods, introduced in the classical case $A(x,t,\xi)\equiv\frac{1}{2}|\xi|^2$, allow one to prove the existence of infinitely many critical points also for a not–even functional (see [4, 5, 22, 24]). Here, we prove a multiplicity result for our functional $\mathcal{J}$ by adapting to our setting the Rabinowitz's perturbation method in [22].

As our main theorem needs a list of hypotheses, we will give its complete statement in Section 2 (see Theorem 2.6). Anyway, we point out that, as in [15, 16], if function $A(x,t,\xi)$ satisfies "good" growth assumptions then the nonlinear term $G(x,t)$ can have also a supercritical growth. Moreover, in the particular case $G(x,t)=\frac{1}{q}|t|^q$, the interval of variability for q is larger than the one found by Tanaka in [26] (see Remark 2.9).

This paper is organized as follows. In Section 2, we introduce the hypotheses for $A(x,t,\xi)$, $G(x,t)$ and $h(x)$, we give the variational formulation of our problem and state our main result. Then, in Section 3 we introduce the perturbation method and in Section 4 we prove that $\mathcal{J}$ satisfies a weak version of the Cerami–Palais–Smale condition. Finally, in Section 5, we give the proof of our main theorem.

2. VARIATIONAL SETTING AND THE MAIN RESULT

From now on, let $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that, using the notations in (1.1), the following conditions hold:

(H_0) $A(x, t, \xi)$ is a C^1 Carathéodory function, i.e.,
$A(\cdot, t, \xi) : x \in \Omega \mapsto A(x, t, \xi) \in \mathbb{R}$ is measurable for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$,
$A(x, \cdot, \cdot) : (t, \xi) \in \mathbb{R} \times \mathbb{R}^N \mapsto A(x, t, \xi) \in \mathbb{R}$ is C^1 for a.e. $x \in \Omega$;

(H_1) some positive continuous functions Φ_i, $\phi_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, 2\}$, exist such that

$$|A_t(x, t, \xi)| \leq \Phi_1(t) + \phi_1(t)|\xi|^2 \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N,$$
$$|a(x, t, \xi)| \leq \Phi_2(t) + \phi_2(t)|\xi| \quad \text{a.e. in } \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;$$

(G_0) $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$;

(G_1) a_1, $a_2 > 0$ and $q \geq 1$ exist such that

$$|g(x, t)| \leq a_1 + a_2|t|^{q-1} \qquad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}.$$

Remark 2.1. From (G_1) it follows that a_1', $a_2' > 0$ exist such that

$$|G(x, t)| \leq a_1' + a_2'|t|^q \qquad \text{a.e in } \Omega, \text{ for all } t \in \mathbb{R}. \tag{2.1}$$

We note that, unlike assumption (G_1) in [8], no upper bound on q is actually required.

In order to investigate the existence of weak solutions of the nonlinear problem (1.2), we consider the Banach space $(X, \|\cdot\|_X)$ defined as

$$X := H_0^1(\Omega) \cap L^\infty(\Omega), \qquad \|u\|_X = \|u\|_H + |u|_\infty$$

(here and in the following, $|\cdot|$ will denote the standard norm on any Euclidean space as the dimension of the considered vector is clear and no ambiguity arises).

Moreover, from the Sobolev Imbedding Theorem, for any $r \in [1, 2^*[$, $2^* = \frac{2N}{N-2}$ as $N \geq 3$, a constant $\sigma_r > 0$ exists, such that

$$|u|_r \leq \sigma_r \|u\|_H \quad \text{for all } u \in H_0^1(\Omega) \tag{2.2}$$

and the imbedding $H_0^1(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact, where $(L^r(\Omega), |\cdot|_r)$ is the standard Lebesgue space.

From definition, $X \hookrightarrow H_0^1(\Omega)$ and $X \hookrightarrow L^\infty(\Omega)$ with continuous imbeddings, and thus $X \hookrightarrow L^r(\Omega)$ for any $r \geq 1$, too.

If the perturbation term $h : \Omega \to \mathbb{R}$ is such that the associated operator

$$\mathcal{L} : u \in X \mapsto \int_\Omega h(x)u(x)dx \in \mathbb{R}$$

belongs to X', then (H_0) and (G_0) allow us to consider the functional $\mathcal{J} : X \to \mathbb{R}$ defined as in (1.3) and the following regularity result holds.

Proposition 2.2. *Let us assume that $\mathcal{L} \in X'$, the functions $A(x,t,\xi)$ and $g(x,t)$ satisfy conditions (H_0)–(H_1), (G_0)–(G_1) and two positive continuous functions Φ_0, $\phi_0 : \mathbb{R} \to \mathbb{R}$ exist such that*

$$|A(x,t,\xi)| \leq \Phi_0(t) + \phi_0(t)|\xi|^2 \quad \text{a.e. in } \Omega, \text{ for all } (t,\xi) \in \mathbb{R} \times \mathbb{R}^N. \tag{2.3}$$

If $(u_n)_n \subset X$, $u \in X$ are such that

$$\|u_n - u\|_H \to 0, \quad u_n \to u \text{ a.e. in } \Omega \quad \text{if } n \to +\infty$$
$$\text{and } M > 0 \text{ exists so that } |u_n|_\infty \leq M \text{ for all } n \in \mathbb{N},$$

then

$$\mathcal{J}(u_n) \to \mathcal{J}(u) \quad \text{and} \quad \|d\mathcal{J}(u_n) - d\mathcal{J}(u)\|_{X'} \to 0 \quad \text{if } n \to +\infty,$$

with

$$\begin{aligned}\langle d\mathcal{J}(v), w\rangle = &\int_\Omega (a(x,v,\nabla v) \cdot \nabla w + A_t(x,v,\nabla v)w)dx \\ &- \int_\Omega g(x,v)wdx - \int_\Omega hwdx \qquad \text{for any } v,\ w \in X.\end{aligned} \tag{2.4}$$

Hence, $\mathcal{J}$ is a C^1 functional on X.

Proof. The proof follows by combining the arguments in [15, Proposition 3.2] with those ones in [16, Proposition 3.3]. □

In order to prove more properties of functional $\mathcal{J}$ in (1.3), we require that some constants $\alpha_i > 0$, $i \in \{1,2,3\}$, $\eta_j > 0$, $j \in \{1,2\}$, and $s \geq 0$, $\mu > 2$, $R_0 \geq 1$, exist such that the following hypotheses are satisfied:

(H_2) $A(x,t,\xi) \leq \eta_1 a(x,t,\xi) \cdot \xi$ a.e. in Ω if $|(t,\xi)| \geq R_0$;
(H_3) $|A(x,t,\xi)| \leq \eta_2$ a.e. in Ω if $|(t,\xi)| \leq R_0$;
(H_4) $a(x,t,\xi) \cdot \xi \geq \alpha_1(1 + |t|^{2s})|\xi|^2$ a.e. in Ω, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$;
(H_5) $a(x,t,\xi) \cdot \xi + A_t(x,t,\xi)t \geq \alpha_2 a(x,t,\xi) \cdot \xi$ a.e. in Ω if $|(t,\xi)| \geq R_0$;
(H_6) $\mu A(x,t,\xi) - a(x,t,\xi) \cdot \xi - A_t(x,t,\xi)t \geq \alpha_3 a(x,t,\xi) \cdot \xi$ a.e. in Ω if $|(t,\xi)| \geq R_0$;
(H_7) for all ξ, $\xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$, it is

$$[a(x,t,\xi) - a(x,t,\xi^*)] \cdot [\xi - \xi^*] > 0 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

(G_2) $g(x,t)$ satisfies the Ambrosetti–Rabinowitz condition, i.e.

$$0 < \mu G(x,t) \leq g(x,t)t \quad \text{for all } x \in \Omega \text{ if } |t| \geq R_0.$$

Remark 2.3. If (H_1)–(H_6) hold, we deduce that in (H_5) we can take $\alpha_2 \leq 1$ and suitable constants η_3, $\eta_4 > 0$ exist such that for a.e. $x \in \Omega$, all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ the following estimates are satisfied:

$$A(x,t,\xi) \geq \alpha_1 \frac{\alpha_2 + \alpha_3}{\mu} (1 + |t|^{2s}) |\xi|^2 - \eta_3, \tag{2.5}$$

$$|A(x,t,\xi)| \leq \eta_1 (\Phi_2(t) + \phi_2(t)) |\xi|^2 + \eta_1 \Phi_2(t) + \eta_2, \tag{2.6}$$

$$a(x,t,\xi) \cdot \xi \leq \frac{\eta_4 \mu}{\alpha_2 + \alpha_3} |t|^{\mu - \frac{1+\alpha_3}{\eta_1}} |\xi|^2 \quad \text{if } |t| \geq 1 \text{ and } |\xi| \geq R_0 \tag{2.7}$$

(for more details, see Remarks 3.3, 3.4 and 3.5 in [15]).

Thus, from (2.6) the growth condition (2.3) holds and Proposition 2.2 applies.
At last, we note that (H_4) and (2.7) imply that

$$0 \le 2s \le \mu - \frac{1+\alpha_3}{\eta_1} \tag{2.8}$$

and, in particular,

$$\mu > \frac{\alpha_3}{\eta_1}. \tag{2.9}$$

From $\mu > 2$ and (2.8) it follows that $\max\{2, 2s\} < \mu$. Actually, a stronger inequality on μ can be deduced from a careful estimate of $A(x,t,\xi)$.

Remark 2.4. If (H_1)–(H_6) hold, some constants α_1^*, $\alpha_2^* > 0$ exist such that

$$|A(x,t,\xi)| \le \alpha_1^*(1+|t|^{\mu-\frac{\alpha_3}{\eta_1}}) + \alpha_2^*(1+|t|^{\mu-\frac{\alpha_3}{\eta_1}-2})|\xi|^2 \tag{2.10}$$

for a.e. $x \in \Omega$, all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$ (for more details, see [8, Lemma 6.5]).
Therefore, from (2.5) and (2.10) it results

$$2(s+1) \le \mu - \frac{\alpha_3}{\eta_1}.$$

Then, since we can always choose η_1 in (H_2) large enough, it follows that

$$0 \le 2(s+1) < \mu. \tag{2.11}$$

Remark 2.5. Assumptions $(G_0) - (G_2)$ and direct computations imply that some strictly positive constants a_3, a_4 and a_5 exist such that

$$\frac{1}{\mu}\,(g(x,t)t + a_3) \ge G(x,t) + a_4 \ge a_5\ |t|^\mu \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{2.12}$$

Hence, in our setting of assumptions on $A(x,t,\xi)$ and $g(x,t)$, estimates (2.1), (2.11) and (2.12) imply that

$$2(s+1) < \mu \le q. \tag{2.13}$$

Now, we are able to state our main result.

Theorem 2.6. *Assume that $A(x,t,\xi)$, $g(x,t)$ and $h(x)$ satisfy conditions (H_0)–(H_7), (G_0)–(G_2) and*

(H_8) $A(x,-t,-\xi) = A(x,t,\xi)$ *for a.e. $x \in \Omega$, for all $(t,\xi) \in \mathbb{R} \times \mathbb{R}^N$;*
(G_3) $g(x,-t) = -g(x,t)$ *for all $(x,t) \in \Omega \times \mathbb{R}$;*
(h_0) $h \in L^\nu(\Omega) \cap L^{\mu'}(\Omega)$ *with $\nu > \frac{N}{2}$ and $\mu' = \frac{\mu}{\mu-1}$.*

If

$$q < 2^*(s+1) \qquad \text{and} \qquad \frac{\mu}{\mu-1} < \frac{2q}{N(q-2-2s)}, \tag{2.14}$$

with s as in (H_4), q as in (G_1) and μ as in (G_2) and (H_6), then functional $\mathcal{J}$ has infinitely many critical points $(u_n)_n$ in X such that $\mathcal{J}(u_n) \nearrow +\infty$; hence, problem (1.2) *has infinitely many weak (bounded) solutions.*

Remark 2.7. We note that $h \in L^{\mu'}(\Omega)$ implies $\mathcal{L} \in X'$ and, from $X \hookrightarrow L^\mu(\Omega)$ and Hölder inequality, we obtain the estimate

$$\left| \int_\Omega hu\ dx \right| \leq |h|_{\mu'} |u|_\mu \quad \text{for all } u \in X. \tag{2.15}$$

On the other hand, we need $h \in L^\nu(\Omega)$ only for proving the boundedness of the weak limit of the (CPS)–sequences in $H_0^1(\Omega)$ (see the proof of Proposition 4.5). Anyway, if $N \geq 4$ it results $L^\nu(\Omega) \cap L^{\mu'}(\Omega) = L^\nu(\Omega)$ as $\mu > 2$ implies $\mu' < \frac{N}{2}$.

Remark 2.8. For the classical problem (1.2) with $A(x,t,\xi) \equiv \frac{1}{2}|\xi|^2$, it is $s = 0$, hence Theorem 2.6 reduces to the well known result stated in [26] (see also [12, 13] where a similar result is stated for a problem with non–homogeneous boundary conditions).

Furthermore, in the quasilinear model case $A(x,t,\xi) = \frac{1}{2}\bar{A}(x,t)|\xi|^2$, conditions (H_2) and (H_7) are trivially verified and Theorem 2.6 reduces to [16, Theorem 3.4].

Remark 2.9. In the particular case $g(x,t) = |t|^{q-2}t$ we have $\mu = q$, then estimate (2.11) and condition (2.14) imply

$$2(s+1) < q < \frac{2(N-1)}{N-2} + \frac{2Ns}{N-2}.$$

We recall that, if $A(x,t,\xi) \equiv \frac{1}{2}|\xi|^2$, in [26] Tanaka proves the existence of infinitely many solutions if

$$2 < q < \frac{2(N-1)}{N-2}. \tag{2.16}$$

Therefore, if $s > 0$ the length of the allowed range of q, equal to $\frac{2}{N-2} + \frac{4s}{N-2}$, is larger than $\frac{2}{N-2}$ which comes from (2.16).

3. A PERTURBATION METHOD

From now on, assume that (H_1)–(H_6), (G_0)–(G_2) and (h_0) hold. Thus, from Proposition 2.2 and Remarks 2.3 and 2.7, $\mathcal{J}$ in (1.3) is a C^1 functional on X.

By $\mathcal{J}_0$ we denote the functional $\mathcal{J}$ corresponding to $h \equiv 0$ defined as in (1.4).

We note that, if (H_8) and (G_3) hold, then $\mathcal{J}_0$ is the even symmetrization of $\mathcal{J}$, as

$$\frac{1}{2}\ (\mathcal{J}(u) + \mathcal{J}(-u)) = \mathcal{J}_0(u) \qquad \text{for all } u \in X.$$

We know that, under the additional assumptions (H_7)–(H_8) and (G_3), the existence of infinitely many critical points for $\mathcal{J}_0$ in X has been proved in [15]. Here, we prove a multiplicity result for the complete functional $\mathcal{J}$ in spite of the loss of symmetry. To this aim, we use a suitable version of the Rabinowitz's perturbation method in [22] (see also [16, Section 4]) which requires the following technical lemmas.

Lemma 3.1. *For all $u \in X$ it results*

$$\left(\mu - \frac{\alpha_3}{\eta_1}\right)\mathcal{J}(u) - \langle d\mathcal{J}(u), u\rangle \geq \frac{\alpha_3}{\mu\eta_1}\int_\Omega (g(x,u)u + a_3)dx - \left(\mu - \frac{\alpha_3}{\eta_1} - 1\right)\int_\Omega hudx - a_6,$$

with η_1 as in (H_2), μ and α_3 as in (H_6), a_3 as in (2.12) and $a_6 > 0$ a suitable constant.

Proof. Taking $u \in X$, from (1.3), (2.4) and direct computations we have that

$$\begin{aligned}&\left(\mu - \frac{\alpha_3}{\eta_1}\right)\mathcal{J}(u) - \langle d\mathcal{J}(u), u\rangle \\ &= \int_\Omega (\mu A(x,u,\nabla u) - a(x,u,\nabla u)\cdot\nabla u - A_t(x,u,\nabla u)u)dx \\ &\quad - \frac{\alpha_3}{\eta_1}\int_\Omega A(x,u,\nabla u)dx - \left(\mu - \frac{\alpha_3}{\eta_1}\right)\int_\Omega (G(x,u)+a_4)dx \\ &\quad + a_4\left(\mu - \frac{\alpha_3}{\eta_1}\right)|\Omega| + \int_\Omega (g(x,u)u + a_3)dx - a_3|\Omega| - \left(\mu - \frac{\alpha_3}{\eta_1} - 1\right)\int_\Omega hudx.\end{aligned}$$

Then, setting

$$\Omega^u_{R_0} = \{x \in \Omega : |(u(x), \nabla u(x))| \geq R_0\},$$

from (H_1), (H_6), (2.6), (2.9) and (2.12) a constant $a_6 > 0$ exists such that

$$\begin{aligned}\left(\mu - \frac{\alpha_3}{\eta_1}\right)\mathcal{J}(u) - \langle d\mathcal{J}(u), u\rangle \geq\ & \alpha_3\int_{\Omega^u_{R_0}} a(x,u,\nabla u)\cdot\nabla udx \\ & - \frac{\alpha_3}{\eta_1}\int_{\Omega^u_{R_0}} A(x,u,\nabla u)dx + \frac{\alpha_3}{\eta_1\mu}\int_\Omega (g(x,u)+a_3)dx \\ & - \left(\mu - \frac{\alpha_3}{\eta_1} - 1\right)\int_\Omega hudx - a_6;\end{aligned}$$

hence, the thesis follows from (H_2). □

Lemma 3.2. *A constant $\alpha^* = \alpha^*(|h|_{\mu'}) > 0$ exists, such that*

$$u \in X,\ |\langle d\mathcal{J}(u), u\rangle| \leq 1 \quad \Longrightarrow \quad \frac{1}{\mu}\int_\Omega (g(x,u)u + a_3)dx \leq \alpha^*\left(\mathcal{J}^2(u) + 1\right)^{\frac{1}{2}},$$

with μ as in (H_6) and a_3 as in (2.12).

Proof. From Lemma 3.1, (2.9) and (2.15) it follows that

$$\begin{aligned}\left(\mu-\frac{\alpha_3}{\eta_1}\right)\mathcal{J}(u)-\langle d\mathcal{J}(u),u\rangle \ \geq\ &\frac{\alpha_3}{\eta_1\mu}\int_\Omega (g(x,u)u+a_3)dx\\ &-\left(\mu-\frac{\alpha_3}{\eta_1}+1\right)|h|_{\mu'}|u|_\mu - a_6\end{aligned} \tag{3.1}$$

(as useful in the following, we make the constant $\mu-\frac{\alpha_3}{\eta_1}-1$ grow to $\mu-\frac{\alpha_3}{\eta_1}+1$).

Now, from one hand, (3.1), Young inequality with $\varepsilon=\frac{\alpha_3}{2\eta_1}a_5$, and (2.12) imply the existence of a suitable constant $b_0=b_0(\alpha_3,\eta_1,\mu,a_5)>0$ such that for all $u\in X$ we have

$$\begin{aligned}&\frac{\alpha_3}{\eta_1\mu}\int_\Omega (g(x,u)u+a_3)dx-\left(\mu-\frac{\alpha_3}{\eta_1}+1\right)|h|_{\mu'}|u|_\mu-a_6\\ &\geq \frac{\alpha_3}{\eta_1\mu}\int_\Omega (g(x,u)u+a_3)dx-\frac{\alpha_3}{2\eta_1}a_5|u|_\mu^\mu\\ &\quad - b_0\left(\mu-\frac{\alpha_3}{\eta_1}+1\right)^{\mu'}|h|_{\mu'}^{\mu'}-a_6\\ &\geq \frac{\alpha_3}{2\eta_1\mu}\int_\Omega (g(x,u)u+a_3)dx-a_7,\end{aligned} \tag{3.2}$$

with $a_7=b_0\left(\mu-\frac{\alpha_3}{\eta_1}+1\right)^{\mu'}|h|_{\mu'}^{\mu'}+a_6$.

On the other hand, taking $u\in X$ such that $|\langle d\mathcal{J}(u),u\rangle|\leq 1$, we have

$$\left(\mu-\frac{\alpha_3}{\eta_1}\right)\mathcal{J}(u)-\langle d\mathcal{J}(u),u\rangle \ \leq\ \left(\mu-\frac{\alpha_3}{\eta_1}\right)\mathcal{J}(u)+1. \tag{3.3}$$

Whence, (3.1)–(3.3) imply

$$\left(\mu-\frac{\alpha_3}{\eta_1}\right)\mathcal{J}(u)+1\geq\frac{\alpha_3}{2\eta_1\mu}\int_\Omega (g(x,u)u+a_3)dx-a_7$$

and the conclusion follows with $\alpha^*=2\sqrt{2}\frac{\eta_1}{\alpha_3}\max\{\mu-\frac{\alpha_3}{\eta_1},1+a_7\}$. □

Now, modifying functional $\mathcal{J}$, we introduce the new map

$$\mathcal{J}_1(u)=\int_\Omega A(x,u,\nabla u)dx-\int_\Omega G(x,u)dx-\psi(u)\int_\Omega hu\ dx,\quad u\in X, \tag{3.4}$$

where

$$\psi(u)=\chi\left(\frac{1}{\mathcal{F}(u)}\int_\Omega (G(x,u)+a_4)dx\right),\quad \mathcal{F}(u)=2\alpha^*\left(\mathcal{J}^2(u)+1\right)^{\frac{1}{2}}, \tag{3.5}$$

with α^* as in Lemma 3.2, and $\chi \in C^\infty(\mathbb{R}, [0,1])$ is a decreasing cut–function such that

$$\chi(t) = \begin{cases} 1 & \text{if } t \leq 1, \\ 0 & \text{if } t \geq 2 \end{cases} \tag{3.6}$$

and $-2 < \chi'(t) < 0$ for all $t \in]1,2[$.

Clearly, it is

$$\mathcal{J}_1(u) = \mathcal{J}(u) - (\psi(u) - 1) \int_\Omega hu\, dx, \quad u \in X,$$

where we have

$$0 \leq \psi(u) \leq 1 \quad \text{for all } u \in X. \tag{3.7}$$

Also if the symmetric conditions (H_8) and (G_3) hold, functional $\mathcal{J}_1$ is not even. Anyway, we can control its loss of symmetry.

Lemma 3.3. *Under the further hypotheses (H_8) and (G_3), a constant $k_0 = k_0(|h|_{\mu'}) > 0$ exists, such that*

$$|\mathcal{J}_1(u) - \mathcal{J}_1(-u)| \leq k_0 \left(|\mathcal{J}_1(u)|^{\frac{1}{\mu}} + 1 \right) \quad \textit{for all } u \in X.$$

Proof. For the proof, see [16, Lemma 4.4]. □

From Proposition 2.2, direct computations imply that $\mathcal{J}_1$ is a C^1 functional on X and for all $u \in X$ we have

$$\begin{aligned}\langle d\mathcal{J}_1(u), u\rangle =& (1 + T_1(u))\langle d\mathcal{J}(u), u\rangle - (T_2(u) - T_1(u)) \int_\Omega g(x,u)u\, dx \\ & - (\psi(u) - 1) \int_\Omega hu\, dx,\end{aligned}$$

with

$$T_1(u) = \chi' \left(\frac{1}{\mathcal{F}(u)} \int_\Omega (G(x,u) + a_4) dx \right) \frac{(2\alpha^*)^2 \mathcal{J}(u)}{\mathcal{F}^3(u)} \int_\Omega (G(x,u) + a_4) dx \int_\Omega hu\, dx,$$

$$T_2(u) = T_1(u) + \chi' \left(\frac{1}{\mathcal{F}(u)} \int_\Omega (G(x,u) + a_4) dx \right) \frac{1}{\mathcal{F}(u)} \int_\Omega hu\, dx.$$

Lemma 3.4. *Functional $\mathcal{J}_1$ verifies the following conditions:*

(i) *two strictly positive constants $M_0 = M_0(|h|_{\mu'})$ and $a_0 = a_0(|h|_{\mu'})$ exist, such that for all $M \geq M_0$ we have*

$$u \in \operatorname{supp} \psi, \quad \mathcal{J}_1(u) \geq M \qquad \Longrightarrow \qquad \mathcal{J}(u) \geq a_0 M;$$

(ii) *for any* $\varepsilon > 0$ *a constant* $M_\varepsilon > 0$ *exists, such that*

$$u \in X, \quad \mathcal{J}_1(u) \geq M_\varepsilon \qquad \Longrightarrow \qquad |T_1(u)| \leq \varepsilon, \quad |T_2(u)| \leq \varepsilon;$$

(iii) *a constant* $M_1 > 0$ *exists such that* $u \in X$,

$$\mathcal{J}_1(u) \geq M_1,\ |\langle d\mathcal{J}_1(u), u\rangle| \leq \frac{1}{2} \quad \Longrightarrow \quad \mathcal{J}_1(u) = \mathcal{J}(u),\ d\mathcal{J}_1(u) = d\mathcal{J}(u).$$

Proof. For the proof, see Lemmas 4.3, 4.5 and 4.7 in [16]. □

Remark 3.5. Any critical point of $\mathcal{J}$ is also a critical point of $\mathcal{J}_1$ with the same critical level. In fact, if u is critical point of $\mathcal{J}$ in X, from (2.12), Lemma 3.2 and (3.5) it follows that

$$\int_\Omega (G(x,u) + a_4)dx \leq \frac{1}{2}\mathcal{F}(u);$$

hence, definition (3.6) implies that $\psi(u) = 1$, $\psi'(u) = 0$, and then

$$\mathcal{J}_1(u) = \mathcal{J}(u), \qquad d\mathcal{J}_1(u) = 0.$$

On the other hand, (iii) of Lemma 3.4 states that also the vice versa is true but only for large enough critical levels.

4. THE WEAK CERAMI–PALAIS–SMALE CONDITION

The aim of this section is proving that our perturbed functional $\mathcal{J}_1$ satisfies $(wCPS)_\beta$ condition (see Definition 1.3) but if β is large enough.

From now on, let $\mathbb{N} = \{1, 2, \dots\}$ and we denote by $|C|$ the usual Lebesgue measure of a measurable set C in $\mathbb{R}^N$.

Firstly, we recall the following result.

Proposition 4.1. *If* $q < 2^*(s+1)$, *then functional* $\mathcal{J}_0$ *satisfies the* $(wCPS)$ *condition in* $\mathbb{R}$.

Proof. For the proof, see [15, Proposition 3.10]. □

Our next step is proving that also $\mathcal{J}$ satisfies $(wCPS)$ condition in $\mathbb{R}$ for any $q < 2^*(s+1)$ even if we have $h \not\equiv 0$. To this aim, we need the following variants of imbedding theorems.

Lemma 4.2. *Fix* $s \geq 0$ *and let* $(u_n)_n \subset X$ *be a sequence such that*

$$\left(\int_\Omega (1 + |u_n|^{2s})\ |\nabla u_n|^2 dx\right)_n \quad \textit{is bounded.} \tag{4.1}$$

Then, $u \in H_0^1(\Omega)$ exists such that $|u|^s u \in H_0^1(\Omega)$, too, and, up to subsequences, if $n \to +\infty$ we have

$$u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \tag{4.2}$$

$$|u_n|^s u_n \rightharpoonup |u|^s u \text{ weakly in } H_0^1(\Omega), \tag{4.3}$$

$$u_n \to u \text{ a.e. in } \Omega, \tag{4.4}$$

$$u_n \to u \text{ strongly in } L^r(\Omega) \text{ for each } r \in [1, 2^*(s+1)[. \tag{4.5}$$

Proof. For the proof, see [15, Lemma 3.8]. □

Lemma 4.3. *If $q < 2^*(s+1)$, then a constant $c_s > 0$ exists such that*

$$|u|_\mu \le c_s \left(\int_\Omega (1 + |u|^{2s}) \, |\nabla u|^2 dx \right)^{\frac{1}{2(s+1)}} \qquad \text{for all } u \in X.$$

Proof. Taking $u \in X$, we note that

$$|\nabla(|u|^s u)|^2 = (s+1)^2 \, |u|^{2s} \, |\nabla u|^2 \quad \text{a.e. in } \Omega. \tag{4.6}$$

On the other hand, setting $q_s = \frac{q}{s+1}$, condition $q < 2^*(s+1)$ implies $q_s < 2^*$, then from (2.2) and (4.6) we have that

$$\begin{aligned} |u|_q = ||u|^s u|_{q_s}^{\frac{1}{s+1}} &\le (\sigma_{q_s} |\nabla(|u|^s u)|_2)^{\frac{1}{s+1}} \\ &\le \sigma_{q_s}^{\frac{1}{s+1}} (s+1)^{\frac{1}{s+1}} \left(\int_\Omega (1 + |u|^{2s}) \, |\nabla u|^2 dx \right)^{\frac{1}{2(s+1)}}. \end{aligned}$$

Hence, the thesis follows from (2.13). □

Moreover, in order to prove the boundedness of the weak limit of a (CPS)-sequence, we need also the following particular version of [19, Theorem II.5.1].

Theorem 4.4. *Taking $v \in H_0^1(\Omega)$, assume that $L_0 > 0$ and $k_0 \in \mathbb{N}$ exist such that for all $\tilde{k} \ge k_0$ it is*

$$\int_{\Omega_{\tilde{k}}^+} |\nabla v|^2 dx \le L_0 \left(\int_{\Omega_{\tilde{k}}^+} (v - \tilde{k})^l dx \right)^{\frac{2}{l}} + L_0 \sum_{i=1}^m \tilde{k}^{l_i} |\Omega_{\tilde{k}}^+|^{1 - \frac{2}{N} + \epsilon_i},$$

with $\Omega_{\tilde{k}}^+ = \{x \in \Omega : v(x) > \tilde{k}\}$, where l, m, l_i, ϵ_i are positive constants such that

$$1 \le l < 2^*, \quad \epsilon_i > 0, \quad 2 \le l_i < \epsilon_i 2^* + 2.$$

Then $\operatorname{ess\,sup}_\Omega v$ is bounded from above by a positive constant which depends only on N, $|\Omega|$, L_0, k_0, l, m, ϵ_i, l_i, $|u|_{2^*}$.

Proposition 4.5. *If $q < 2^*(s+1)$ then functional $\mathcal{J}$ satisfies the $(wCPS)$ condition in $\mathbb{R}$.*

Proof. Let $\beta \in \mathbb{R}$ be fixed and consider a $(CPS)_\beta$-sequence $(u_n)_n \subset X$, i.e.,

$$\mathcal{J}(u_n) \to \beta \quad \text{and} \quad \|d\mathcal{J}(u_n)\|_{X'}(1 + \|u_n\|_X) \to 0. \tag{4.7}$$

For simplicity, here and in the following we will use the notation $(\varepsilon_n)_n$ for any infinitesimal sequence depending only on $(u_n)_n$.

From (H_1), (H_6), (2.6), (G_0), (G_2), (2.15), direct computations, (H_4) and Lemma 4.3, we have that some constants a_8, $a_9 > 0$ exist such that

$$\begin{aligned}\mu\beta + \varepsilon_n &= \mu\mathcal{J}(u_n) - \langle d\mathcal{J}(u_n), u_n\rangle \\ &\geq \alpha_3 \int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla u_n dx - a_8 - (\mu - 1)|h|_{\mu'}|u_n|_\mu \\ &\geq \alpha_1\alpha_3 \int_\Omega (1 + |u_n|^{2s})\ |\nabla u_n|^2 dx - a_8 - a_9 \left(\int_\Omega (1 + |u_n|^{2s})\ |\nabla u_n|^2 dx\right)^{\frac{1}{2(s+1)}}\end{aligned}$$

which implies (4.1). Then, from Lemma 4.2 it follows that $u \in H_0^1(\Omega)$ exists such that $|u|^s u \in H_0^1(\Omega)$, too, and, up to subsequences, (4.2)–(4.5) hold.

Now, we want to prove that u is essentially bounded from above. Arguing by contradiction, let us assume that

$$\operatorname*{ess\,sup}_\Omega u = +\infty; \tag{4.8}$$

thus, taking any $k \in \mathbb{N}$, $k > R_0$ ($R_0 \geq 1$ as in the hypotheses), we have that

$$|\Omega_k^+| > 0 \quad \text{with} \quad \Omega_k^+ = \{x \in \Omega : u(x) > k\}. \tag{4.9}$$

Taking any $\tilde{k} > 0$, we define the new function $R_{\tilde{k}}^+ : t \in \mathbb{R} \to R_{\tilde{k}}^+ t \in \mathbb{R}$ as

$$R_{\tilde{k}}^+ t = \begin{cases} 0 & \text{if } t \leq \tilde{k}, \\ t - \tilde{k} & \text{if } t > \tilde{k}. \end{cases}$$

Then, if $\tilde{k} = k^{s+1}$, from (4.3) it follows that

$$R_{k^{s+1}}^+(|u_n|^s u_n) \rightharpoonup R_{k^{s+1}}^+(|u|^s u) \quad \text{weakly in } H_0^1(\Omega);$$

thus, the sequentially weakly lower semicontinuity of $\|\cdot\|_H$ implies

$$\int_{\Omega_k^+} |\nabla(u^{s+1})|^2 dx \leq \liminf_{n\to+\infty} \int_{\Omega_{n,k}^+} |\nabla(u_n^{s+1})|^2 dx \tag{4.10}$$

with $\Omega_{n,k}^+ = \{x \in \Omega : u_n(x) > k\}$, as $|t|^s t > k^{s+1}$ if and only if $t > k$.

On the other hand, from $\|R_k^+ u_n\|_X \leq \|u_n\|_X$, (4.7) and (4.9) it follows that $n_k \in \mathbb{N}$ exists so that

$$\langle d\mathcal{J}(u_n), R_k^+ u_n \rangle < |\Omega_k^+| \qquad \text{for all } n \geq n_k. \tag{4.11}$$

Then, from (H_5) (with $\alpha_2 \leq 1$), (H_4), (4.6) and direct computations we have that

$$\begin{aligned} \langle d\mathcal{J}(u_n), R_k^+ u_n \rangle \geq \alpha_2 & \int_{\Omega_{n,k}^+} a(x, u_n, \nabla u_n) \cdot \nabla u_n dx - \int_\Omega g(x, u_n) R_k^+ u_n dx \\ & - \int_\Omega h R_k^+ u_n dx \\ \geq & \ \frac{\alpha_1 \alpha_2}{(s+1)^2} \int_{\Omega_{n,k}^+} |\nabla(u_n^{s+1})|^2 dx - \int_\Omega g(x, u_n) R_k^+ u_n dx - \int_\Omega h R_k^+ u_n dx. \end{aligned}$$

Thus, from (4.11), it follows that

$$\int_{\Omega_{n,k}^+} |\nabla(u_n^{s+1})|^2 dx \leq \frac{(s+1)^2}{\alpha_1 \alpha_2} \left(|\Omega_k^+| + \int_\Omega g(x, u_n) R_k^+ u_n dx + \int_\Omega h R_k^+ u_n dx \right),$$

where, since $q < 2^*(s+1)$, from (G_1) and (4.5) it results

$$\int_\Omega g(x, u_n) R_k^+ u_n dx \ \to \int_\Omega g(x, u) R_k^+ u \ dx, \quad \int_\Omega h R_k^+ u_n dx \ \to \int_\Omega h R_k^+ u \ dx.$$

Hence, passing to the limit, (4.10) implies

$$\int_{\Omega_k^+} |\nabla(u^{s+1})|^2 dx \leq \frac{(s+1)^2}{\alpha_1 \alpha_2} \left(|\Omega_k^+| + \int_\Omega g(x, u) R_k^+ u \ dx + \int_\Omega h R_k^+ u \ dx \right).$$

Now, as $h \in L^\nu(\Omega)$ with $\nu > \frac{N}{2}$, by reasoning as in the last part of *Step 2* in the proof of [16, Proposition 4.11], we are able to apply Theorem 4.4, then $\operatorname*{ess\,sup}_\Omega u < +\infty$ in contradiction with (4.8).

Similar arguments prove also that u is essentially bounded from below; hence, $u \in L^\infty(\Omega)$.

Taking $k \geq \max\{|u|_\infty, R_0\} + 1$ ($R_0 \geq 1$ as in the set of hypotheses) and the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ defined as

$$T_k t = \begin{cases} t & \text{if } |t| \leq k, \\ k \frac{t}{|t|} & \text{if } |t| > k, \end{cases}$$

thanks to the linearity of the term $v \mapsto \int_\Omega h v dx$ we can reason as in Steps 3 and 4 of the proof of [7, Proposition 3.4] and can prove that $(T_k u_n)_n$ is a Palais–Smale sequence at level β, i.e. $\mathcal{J}(T_k u_n) \to \beta$ and $\|d\mathcal{J}(T_k u_n)\|_{X'} \to 0$, and $\|T_k u_n - u\|_H \to 0$. Hence, also $\|u_n - u\|_H \to 0$ and, since $|T_k u_n|_\infty \le k$ for all $n \in \mathbb{N}$, by applying Proposition 2.2 we have $\mathcal{J}(u) = \beta$ and $d\mathcal{J}(u) = 0$. □

Proposition 4.6. *Let $q < 2^*(s+1)$. Then, taking $M_1 > 0$ as in* (iii) *of Lemma 3.4, the functional $\mathcal{J}_1$ satisfies the $(wCPS)_\beta$ condition for any $\beta > M_1$.*

Proof. Let $\beta > M_1$ and $(u_n)_n$ be a $(CPS)_\beta$-sequence of $\mathcal{J}_1$ in X. Then, for n large enough it is

$$\mathcal{J}_1(u_n) \ge M_1 \quad \text{and} \quad |\langle d\mathcal{J}_1(u_n), u_n\rangle| \le \|d\mathcal{J}_1(u_n)\|_{X'}(\|u_n\|_X + 1) \le \frac{1}{2};$$

hence, from (iii) of Lemma 3.4 we obtain

$$\mathcal{J}_1(u_n) = \mathcal{J}(u_n), \quad d\mathcal{J}_1(u_n) = d\mathcal{J}(u_n),$$

which implies that $(u_n)_n$ is a $(CPS)_\beta$-sequence of $\mathcal{J}$ in X, too. Thus, from Proposition 4.5 it follows that $u \in X$ exists such that $\|u_n - u\|_H \to 0$ (up to subsequences) and u is a critical point of $\mathcal{J}$ at level β. Then, u is a critical point of $\mathcal{J}_1$ at level β, too (see Remark 3.5). □

5. PROOF OF THE MAIN THEOREM

Throughout this section, assume that $A(x,t,\xi)$, $g(x,t)$, $h(x)$ satisfy all the hypotheses of Theorem 2.6.

In order to introduce a suitable decomposition of X, let $(\lambda_j)_j$ be the sequence of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and for each $j \in \mathbb{N}$ let $\varphi_j \in H_0^1(\Omega)$ be the eigenfunction corresponding to λ_j.

We recall that $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$, with $\lambda_j \nearrow +\infty$ as $j \to +\infty$, and $(\varphi_j)_j$ is an orthonormal basis of $H_0^1(\Omega)$ such that for each $j \in \mathbb{N}$ it is $\varphi_j \in L^\infty(\Omega)$; hence, $\varphi_j \in X$ (see [6, Section 9.8]). Then, for any $k \in \mathbb{N}$, it is

$$H_0^1(\Omega) = V_k \oplus Z_k,$$

where

$$V_k = \text{span}\{\varphi_1, \dots, \varphi_k\} \quad \text{and} \quad Z_k \text{ is its orthogonal complement.}$$

Thus, setting $Z_k^X = Z_k \cap L^\infty(\Omega)$, we have

$$X = V_k + Z_k^X \quad \text{and} \quad V_k \cap Z_k^X = \{0\};$$

whence,

$$\text{codim} Z_k^X = \text{dim} V_k = k. \tag{5.1}$$

Proposition 5.1. *If V is a finite dimensional subspace of X, then*

$$\sup_{u \in S_R^H \cap V} \mathcal{J}_1(u) \;\to\; -\infty \quad \textit{if } R \to +\infty,$$

with $S_R^H = \{u \in X : \ \|u\|_H = R\}$.

Proof. Since in a finite dimensional space all the norms are equivalent, the proof follows from definition (3.4) and the estimates (2.10), (2.12), (2.15), (3.7). □

From (5.1) and Proposition 5.1 a strictly increasing sequence of positive numbers $(R_k)_k$ exists, $R_k \nearrow +\infty$, such that for any $k \in \mathbb{N}$ we have that

$$\mathcal{J}_1(u) < 0 \quad \text{for all } u \in V_k \text{ with } \|u\|_H \geq R_k.$$

Now, we can introduce the following notations:

$$\begin{aligned}
\Gamma_k &= \{\gamma \in C(V_k, X) : \ \gamma \text{ is odd}, \quad \gamma(u) = u \text{ if } \|u\|_H \geq R_k\},\\
\Gamma_k^H &= \{\gamma \in C(V_k, H_0^1(\Omega)) : \ \gamma \text{ is odd}, \quad \gamma(u) = u \text{ if } \|u\|_H \geq R_k\},\\
\Lambda_k &= \{\gamma \in C(V_{k+1}^+, X) : \ \gamma|_{V_k} \in \Gamma_k \text{ and } \gamma(u) = u \text{ if } \|u\|_H \geq R_{k+1}\},
\end{aligned}$$

with

$$V_{k+1}^+ = \{v + t\varphi_{k+1} \in X : \ v \in V_k, \ t \geq 0\},$$

and

$$b_k = \inf_{\gamma \in \Gamma_k} \sup_{u \in V_k} \mathcal{J}_1(\gamma(u)), \quad b_k^+ = \inf_{\gamma \in \Lambda_k} \sup_{u \in V_{k+1}^+} \mathcal{J}_1(\gamma(u)).$$

The following existence result can be proved.

Proposition 5.2. *Assume $q < 2^*(s+1)$ and let $k \in \mathbb{N}$ be such that*

$$b_k^+ > b_k \geq M_1, \tag{5.2}$$

with $M_1 > 0$ as in (iii) *of Lemma 3.4. Taking $0 < \delta < b_k^+ - b_k$, define*

$$\beta_k(\delta) = \inf_{\gamma \in \Lambda_k(\delta)} \sup_{u \in V_{k+1}^+} \mathcal{J}_1(\gamma(u)),$$

where

$$\Lambda_k(\delta) = \{\gamma \in \Lambda_k : \ \mathcal{J}_1(\gamma(u)) \leq b_k + \delta \ \text{if } u \in V_k\}.$$

Then, $\beta_k(\delta)$ is a critical level of $\mathcal{J}$ in X with $\beta_k(\delta) \geq b_k^+$.

Proof. The proof follows from Proposition 4.6 by reasoning as in [16, Proposition 5.4]. □

Now, we need an estimate from below for the sequence $(b_k)_k$.

Proposition 5.3. *If $q < 2^*(s+1)$, then a constant $C_1 > 0$ exists such that*

$$b_k \geq C_1 \ k^{\frac{2q}{N(q-2-2s)}} \qquad \text{for } k \text{ large enough.}$$

Proof. Firstly, we note that from (2.1), (2.5), (2.15), (3.4), (3.7) and direct computations, some constants a_{10}, a_{11}, $a_{12} > 0$ exist, such that

$$\mathcal{J}_1(u) \geq a_{10} \ \mathcal{I}(u) - a_{11} \quad \text{for all } u \in X, \tag{5.3}$$

where $\mathcal{I} : X \to \mathbb{R}$ is the C^1 functional defined as

$$\mathcal{I}(u) = \frac{1}{2} \int_\Omega (1 + |u|^{2s})|\nabla u|^2 dx - a_{12} \int_\Omega |u|^q dx.$$

Now, taking $k \in \mathbb{N}$, reasoning as in the proof of [16, Proposition 5.6], for any $\gamma_0 \in \Gamma_k$ we can define the continuous map $\tilde{\gamma}_0 : V_k \to X$,

$$\tilde{\gamma}_0(u) = \begin{cases} |\gamma_0(u)|^s \ \gamma_0(u) & \text{if } \|u\|_H \le R_k - \delta_0, \\ |\gamma_0(u)|^{\frac{s}{\delta_0}(R_k - \|u\|_H)} \ \gamma_0(u) & \text{if } R_k - \delta_0 < \|u\|_H < R_k, \\ u & \text{if } \|u\|_H \ge R_k, \end{cases}$$

for a suitable $\delta_0 \in \,]0, R_k[$, such that $\tilde{\gamma}_0 \in \Gamma_k \subset \Gamma_k^H$ and

$$\sup_{u \in V_k} \mathcal{I}(\gamma_0(u)) \ge \frac{1}{(s+1)^2} \sup_{u \in V_k} K^*(\tilde{\gamma}_0(u)) \ \ge \ \frac{1}{(s+1)^2} \inf_{\gamma \in \Gamma_k^H} \sup_{u \in V_k} K^*(\gamma(u)),$$

with

$$K^*(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - a_{12}(s+1)^2 \int_\Omega |v|^{\frac{q}{s+1}} dx.$$

Then, the thesis follows from [26, Section 2] and (5.3). □

Proof of Theorem 2.6. Since $b_k^+ \ge b_k$ for any $k \in \mathbb{N}$ and $b_k \to +\infty$ from Proposition 5.3, the thesis follows from Proposition 5.2 once we prove that (5.2) holds for infinitely many k.

Arguing by contradiction, assume that $k_1 \in \mathbb{N}$ exists such that $b_k^+ = b_k$ for any $k \ge k_1$. From Lemma 3.3 and reasoning as in the proof of [23, Proposition 10.46], a constant $C_2 = C_2(k_1) > 0$ exists such that

$$b_k \le C_2 \ k^{\frac{\mu}{\mu-1}} \quad \text{for any } k \text{ large enough},$$

which yields a contradiction from assumption (2.14) and Proposition 5.3. □

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REFERENCES

[1] V. Ambrosio, J. Mawhin, G. Molica Bisci, *(Super)Critical nonlocal equations with periodic boundary conditions*, Sel. Math. New Ser. **24** (2018), 3723–3751.

[2] D. Arcoya, L. Boccardo, *Critical points for multiple integrals of the calculus of variations*, Arch. Rational Mech. Anal. **134** (1996), 249–274.

[3] D. Arcoya, L. Boccardo, *Some remarks on critical point theory for nondifferentiable functionals*, Nonlinear Differential Equations Appl. **6** (1999), 79–100.

[4] A. Bahri, H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc. **267** (1981), 1–32.

[5] P. Bolle, N. Ghoussoub, H. Tehrani, *The multiplicity of solutions in non–homogeneous boundary value problems*, Manuscripta Math. **101** (2000), 325–350.

[6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext XIV, Springer, New York, 2011.

[7] A.M. Candela, G. Palmieri, *Multiple solutions of some nonlinear variational problems*, Adv. Nonlinear Stud. **6** (2006), 269–286.

[8] A.M. Candela, G. Palmieri, *Infinitely many solutions of some nonlinear variational equations*, Calc. Var. Partial Differential Equations **34** (2009), 495–530.

[9] A.M. Candela, G. Palmieri, *Some abstract critical point theorems and applications*, [in:] Dynamical Systems, Differential Equations and Applications, X. Hou, X. Lu, A. Miranville, J. Su and J. Zhu (eds), Discrete Contin. Dynam. Syst. Suppl. **2009** (2009), 133–142.

[10] A.M. Candela, G. Palmieri, *Multiplicity results for some quasilinear equations in lack of symmetry*, Adv. Nonlinear Anal. **1** (2012), 121–157.

[11] A.M. Candela, G. Palmieri, *Multiplicity results for some nonlinear elliptic problems with asymptotically p-linear terms*, Calc. Var. Partial Differential Equations **56**:72 (2017).

[12] A.M. Candela, A. Salvatore, *Multiplicity results of an elliptic equation with non––homogeneous boundary conditions*, Topol. Methods Nonlinear Anal. **11** (1998), 1–18.

[13] A.M. Candela, A. Salvatore, *Some applications of a perturbative method to elliptic equations with non–homogeneous boundary conditions*, Nonlinear Anal. **53** (2003), 299–317.

[14] A.M. Candela, G. Palmieri, K. Perera, *Multiple solutions for p-Laplacian type problems with asymptotically p-linear terms via a cohomological index theory*, J. Differential Equations **259** (2015), 235–263.

[15] A.M. Candela, G. Palmieri, A. Salvatore, *Some results on supercritical quasilinear elliptic problems*, preprint (2017).

[16] A.M. Candela, G. Palmieri, A. Salvatore, *Infinitely many solutions for quasilinear elliptic equations with lack of symmetry*, Nonlinear Anal. **172** (2018), 141–162.

[17] A. Canino, *Multiplicity of solutions for quasilinear elliptic equations*, Topol. Methods Nonlinear Anal. **6** (1995), 357–370.

[18] G. Cerami, *Un criterio di esistenza per i punti critici su varietà illimitate*, Istit. Lombardo Accad. Sci. Lett. Rend. A **112** (1978), 332–336.

[19] O.A. Ladyzhenskaya, N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[20] R.S. Palais, *Critical point theory and the minimax principle. In "Global Analysis"*, Proc. Sympos. Pure Math. **15**, Amer. Math. Soc., Providence R.I. (1970), 185–202.

[21] B. Pellacci, M. Squassina, *Unbounded critical points for a class of lower semicontinuous functionals*, J. Differential Equations **201** (2004), 25–62.

[22] P.H. Rabinowitz, *Multiple critical points of perturbed symmetric functionals*, Trans. Amer. Math. Soc. **272** (1982), 753–769.

[23] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conf. Ser. in Math. **65**, Amer. Math. Soc., Providence, 1986.

[24] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems*, Manuscripta Math. **32** (1980), 335–364.

[25] M. Struwe, *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems,* 4th ed., Ergeb. Math. Grenzgeb. (4) **34**, Springer-Verlag, Berlin, 2008.

[26] K. Tanaka, *Morse indices at critical points related to the Symmetric Mountain Pass Theorem and applications*, Comm. Partial Differential Equations **14** (1989), 99–128.

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