Opuscula Math. 39, no. 2 (2019), 297–313
https://doi.org/10.7494/OpMath.2019.39.2.297 Opuscula Mathematica

GLOBAL WELL-POSEDNESS OF A CLASS OF FOURTH-ORDER STRONGLY DAMPED NONLINEAR WAVE EQUATIONS

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Communicated by Marius Ghergu

Abstract. Global well-posedness and finite time blow up issues for some strongly damped nonlinear wave equation are investigated in the present paper. For subcritical initial energy by employing the concavity method we show a finite time blow up result of the solution. And for critical initial energy we present the global existence, asymptotic behavior and finite time blow up of the solution in the framework of the potential well. Further for supercritical initial energy we give a sufficient condition on the initial data such that the solution blows up in finite time.

Keywords: fourth-order nonlinear wave equation, strong damping, blow up, global existence.

Mathematics Subject Classification: 35B44, 35L35, 35L05.

1. INTRODUCTION

This paper investigates the IBVP (initial boundary value problem) to a class of fourth order strongly damped nonlinear wave equations

$$\begin{aligned} & u_{tt} - \Delta u + \Delta^2 u - \alpha \Delta u_t = f(u), && x \in \Omega, t > 0, \\ & u(x,0) = u_0, u_t(x,0) = u_1, && x \in \Omega, \\ & u(x,t) = \Delta u(x,t) = 0, && x \in \partial\Omega, t \geq 0, \end{aligned} \tag{1.1}$$

where α is a positive constant, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain with a smooth boundary $\partial\Omega$ and $f(u)$ satisfies

(H1) $f \in C^1$, $u(uf'(u) - f(u)) \geq 0$, where the equality holds only for $u = 0$ and $|f(u)| \leq A|u|^q, 1 < q < +\infty$ for $1 \leq n \leq 4$; $1 < q < \frac{n+4}{n-4}$ for $n \geq 5$;

(H2) there exists $p > 1$ such that for all $u \in \mathbb{R}$

$$(p+1)F(u) \le uf(u), \quad F(u) = \int_0^u f(s)ds.$$

As an example we can take $f(s) = |s|^{q-1}s$.

The original second order model of problem (1.1) was introduced by Webb [15] to consider the motion of a viscous body. Hereafter there are a lot of interesting results on the qualitative behavior of solutions for such nonlinear second order damped wave equations (see, for instance, well-posedness of solutions [2, 4, 8], decay behavior of energy [1, 11], attractors [3, 12, 22] and the papers cited therein). Up to now there are also various interesting results about the IBVP to different classes of fourth order nonlinear wave equations with certain initial energy and we refer the reader to some related papers [5–7,10,13,14,17,20,21] and the references therein. But to our knowledge up to now almost all the high-energy blow up results for nonlinear wave equations are derived in the absence of the strong damping term Δu_t or in the presence of both the strong damping term Δu_t, the weak damping term u_t and the dispersive term Δu_{tt} (see, for instance, the nonlinear fourth-order strain wave equation [14]

$$u_{tt} - \alpha\Delta u + \Delta^2 u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) = f(u),$$

the nonlinear fourth-order dispersive-dissipative wave equation

$$u_{tt} - \Delta u - \Delta u_{tt} - \Delta u_t + u_t = |u|^{p-1}u$$

in [17] and the papers cited therein). So, in the present paper we consider a fourth-order wave equation with strong damping term, i.e., problem (1.1), and solve some unsolved problems related to problem (1.1). Now let us recall some existing results on the global existence, asymptotic behavior and finite time blow up of solutions to the problem (1.1). To our knowledge Lin *et al.* [9] made the first try to consider problem (1.1) and proved the existence of global weak and strong solutions under some assumptions on the nonlinear source terms $f(u)$ and initial data in the framework of potential well. Subsequently, by introducing a family of potential wells, Xu and Yang [18] investigated the problem (1.1) and obtained the existence of global solutions as well as the asymptotic behavior of global weak solutions under some weak growth conditions on the nonlinear source terms $f(u)$, i.e., (H1) and (H2), and left some problems unsolved. These obtained results and unsolved problems are listed in Table 1 below.

The aim of the present paper is to solve some of these unsolved problems listed in Table 1 and give a comprehensive investigation on the global existence, long-time behavior and finite time blow up of solutions to the problem (1.1) at three different initial energy level (subcritical initial energy, critical initial energy and supercritical initial energy) as we did for a different model equation [19], that is

$$u_{tt} - \Delta u + \Delta^2 u - \Delta u_{tt} + \Delta^2 u_{tt} = \Delta f(u).$$

In detail, in the present paper for subcritical initial energy we obtain the finite time blow up of the solution by employing the classical concavity method in the framework of potential well [8] (Theorem 3.3). For critical initial energy by the idea [16] we obtain the global existence, asymptotic behavior and finite time blow up of the solution (Theorem 4.4, Theorem 4.6 and Theorem 4.8, respectively). For supercritical initial energy by utilizing an adapted concavity method, we give a sufficient condition on the initial data such that the solution blows up in finite time (Theorem 5.3).

Table 1

Obtained results and unsolved problems for problem (1.1)

	Global existence	Asymptotic behavior	Blow up
Subcritical initial energy $E(0) < d$	Reference [18]	Reference [18]	Theorem 3.3
Critical initial energy $E(0) = d$	Theorem 4.4	Theorem 4.6	Theorem 4.8
Supercritical initial energy $E(0) > d$	Still unsolved	Still unsolved	Theorem 5.3

2. PRELIMINARY

For simplicity, we use the notation $\|u\| := \|u\|_{L^2(\Omega)}$, $\|u\|_p := \|u\|_{L^p(\Omega)}$ and the inner product $(u, v) = \int_\Omega uvdx$ throughout the present paper. In addition we denote $H^2(\Omega) \cap H_0^1(\Omega)$ by H, the duality paring between H^{-1} and H by $\langle \cdot, \cdot \rangle$. We introduce the following functionals

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(\|\Delta u\|^2 + \|\nabla u\|^2\right) - \int_\Omega F(u)dx \tag{2.1}$$

$$= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_H^2 - \int_\Omega F(u)dx,$$

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\Delta u\|^2 - \int_\Omega F(u)dx = \frac{1}{2}\|u\|_H^2 - \int_\Omega F(u)dx, \tag{2.2}$$

$$I(u) = \|\Delta u\|^2 + \|\nabla u\|^2 - \int_\Omega uf(u)dx = \|u\|_H^2 - \int_\Omega uf(u)dx \tag{2.3}$$

and the sets

$$\mathcal{G} := \{u \in H | I(u) > 0\} \cup \{0\}, \quad \mathcal{B} := \{u \in H | I(u) < 0\}$$

as well as the definition of potential well

$$d = \inf_{u \in \mathcal{N}} J(u), \quad \mathcal{N} = \{u \in H | I(u) = 0, u \neq 0\}.$$

Definition 2.1. By solution of problem (1.1) over $[0,T_0]$ we mean a function solution

$$u(t)\in C\left([0,T_0);H\right)\cap C^1\left([0,T_0);L^2(\Omega)\right)\cap C^2\left([0,T_0];H^{-1}\right)$$

and

$$u_t\in L^2\left([0,T_0);H_0^1(\Omega)\right)$$

such that $u(x,0)=u_0$ and $u_t(x,0)=u_1$ and

$$\langle u_{tt},v\rangle+\int\limits_\Omega \Delta u\Delta v dx+\int\limits_\Omega \nabla u\nabla v dx+\alpha\int\limits_\Omega \nabla u_t\nabla v dx=\int\limits_\Omega f(u)v dx$$

for all $v\in H$ and almost every $t\in[0,T_0]$. Further there holds

$$E(t)+\alpha\int\limits_0^t \|\nabla u_\tau\|^2 d\tau=E(0). \tag{2.4}$$

By the similar arguments of [8, Theorem 3.1], we can get the following local existence of a solution to problem (1.1).

Theorem 2.2 (Local existence)**.** *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0\in H$ and $u_1\in L^2(\Omega)$. Then problem* (1.1) *admits a unique local solution over $[0,T_0]$, where T_0 is the maximal existence time of $u(x,t)$. Moreover, there holds either $T_0=+\infty$ or $T_0<+\infty$ and*

$$\lim_{t\to T_0}\|u(t)\|=+\infty.$$

Next we show that the depth of potential well is positive, which will be used in the proof of the finite time blow up of the solution to problem (1.1).

Lemma 2.3 (Depth of potential well)**.** *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*. Then there holds*

$$d=\frac{p-1}{2(p+1)}\left(\frac{1}{AC_*^{q+1}}\right)^{\frac{2}{q-1}}, \tag{2.5}$$

where $C_*=\sup_{0\neq u\in H}\frac{\|u\|_{q+1}}{\|u\|_H}$, if u is the solution to problem (1.1)*.*

Proof. From the definition of d it implies that for any $u\in\mathcal{N}$ there holds $I(u)=0$ and $\|u\|_H\neq 0$, which together with the assumption (H1) gives

$$\|u\|_H^2=\int\limits_\Omega uf(u)dx\le A\|u\|_{q+1}^{q+1}=AC_*^{q+1}\|u\|_H^{q-1}\|u\|_H^2,$$

i.e.,

$$\|u\|_H^2\ge\left(\frac{1}{AC_*^{q+1}}\right)^{\frac{2}{q-1}}. \tag{2.6}$$

In addition, by (2.2), (2.3) and the assumption (H2) we get

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_H^2 - \int_\Omega F(u)dx \\ &\geq \frac{1}{2}\|u\|_H^2 - \frac{1}{p+1}\int_\Omega uf(u)dx = \frac{p-1}{2(p+1)}\|u\|_H^2 + \frac{1}{p+1}I(u), \end{aligned} \tag{2.7}$$

which together with $I(u) = 0$ and (2.6) implies

$$J(u) \geq \frac{p-1}{2(p+1)}\|u\|_H^2 \geq \frac{p-1}{2(p+1)}\left(\frac{1}{AC_*^{q+1}}\right)^{\frac{2}{q-1}}.$$

Therefore, by the definition of potential well we get (2.5). □

3. FINITE TIME BLOW UP FOR SUBCRITICAL INITIAL ENERGY

The set $\mathcal{B}$ is invariant under the flow of problem (1.1).

Lemma 3.1. *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) < d$, then the solution to problem* (1.1) *belongs to $\mathcal{B}$, provided that $u_0(x) \in \mathcal{B}$.*

Proof. Let $u(t)$ be any weak solution to problem (1.1) with $E(0) < d$, $u_0(x) \in \mathcal{B}$ and T_0 be the maximum existence time of $u(x,t)$. Then from (2.4) it implies that $E(u(t)) = E(0) < d$ for $t \in (0, T_0)$. Next we claim $u(t) \in \mathcal{B}$ for $t \in [0, T_0)$. Arguing by contradiction we suppose that $t_* \in (0, T_0)$ is the first time such that $I(u(t_*)) = 0$ and $I(u(t)) < 0$ for $t \in [0, t_*)$. Then by the definition of depth of potential well d, we have

$$d > E(0) \geq E\left(u(t_*)\right) \geq J\left(u(t_*)\right) \geq d,$$

which is a contradiction. □

Lemma 3.2. *Under the conditions of Lemma 3.1, there holds*

$$d < \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_H^2. \tag{3.1}$$

Proof. From Lemma 3.1 we see that $I(u) < 0$, which together with the assumption (H1) implies that

$$\|u\|_H^2 < \int_\Omega uf(u)dx \leq A\|u\|_{q+1}^{q+1} = AC_*^{q+1}\|u\|_H^{q-1}\|u\|_H^2,$$

namely

$$\|u\|_H^2 > \left(\frac{1}{AC_*^{q+1}}\right)^{\frac{2}{q-1}},$$

where C_* is defined in Lemma 2.3. Recalling (2.5), it is easy to get (3.1). □

In the following, with the aid of Lemma 3.1 and Lemma 3.2 we prove a finite time blow up result of the solution when $E(0) < d$.

Theorem 3.3 (Blow up for subcritical initial energy)**.** *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) < d$ and $u_0 \in \mathcal{B}$, then the solution of problem* (1.1) *blows up in finite time.*

Proof. Let $u(t)$ be any solution of problem (1.1) with $E(0) < d$ and $u_0 \in \mathcal{B}$. Then from Lemma 3.1 it follows that $u \in \mathcal{B}$. Arguing by contradiction, we suppose that the solution $u(x,t)$ is global. Then for any $T_0 > 0$, we introduce the following auxiliary function

$$B(t) := \|u\|^2 + \alpha \int_0^t \|\nabla u(\tau)\|^2 d\tau + \alpha(T_0 - t)\|\nabla u_0\|^2. \tag{3.2}$$

Clearly $B(t) > 0$ for all $t \in [0, T_0]$. From the continuity of $B(t)$ in t we can conclude that there exists $\rho > 0$ such that

$$B(t) \geq \rho \quad \text{for all } t \in [0, T_0], \tag{3.3}$$

where ρ is independent of the choice of T_0. Further, for $t \in [0, T_0]$, one gets

$$B'(t) = 2(u, u_t) + \alpha\|\nabla u\|^2 - \alpha\|\nabla u_0\|^2 = 2(u, u_t) + 2\alpha \int_0^t (\nabla u(\tau), \nabla u_\tau(\tau))\, d\tau \tag{3.4}$$

and

$$B''(t) = 2\|u_t\|^2 + 2\langle u_{tt}, u\rangle + 2\alpha(\nabla u, \nabla u_t) = 2\|u_t\|^2 - 2I(u). \tag{3.5}$$

The equality (3.4) implies that

$$\begin{aligned}(B'(t))^2 = 4 &\left((u, u_t)^2 + 2\alpha(u, u_t) \int_0^t (\nabla u(\tau), \nabla u_\tau(\tau))\, d\tau \right) \\ &+ 4 \left(\alpha \int_0^t (\nabla u(\tau), \nabla u_\tau(\tau))\, d\tau \right)^2 .\end{aligned} \tag{3.6}$$

Then, from the Cauchy–Schwarz inequality it follows

$$(u, u_t)^2 \leq \|u\|^2\|u_t\|^2,$$

$$\left(\alpha \int_0^t (\nabla u(\tau), \nabla u_\tau(\tau))\, d\tau \right)^2 \leq \alpha \int_0^t \|\nabla u(\tau)\|^2 d\tau \alpha \int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau$$

and

$$
\begin{aligned}
&2\alpha(u,u_t)\int_0^t (\nabla u(\tau),\nabla u_\tau(\tau))\,d\tau \\
&\le 2\|u\|\|u_t\|\left(\alpha\int_0^t \|\nabla u(\tau)\|^2 d\tau\right)^{1/2}\left(\alpha\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau\right)^{1/2} \\
&\le \alpha\|u\|^2\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau + \alpha\|u_t\|^2\int_0^t \|\nabla u(\tau)\|^2 d\tau.
\end{aligned}
$$

Therefore, (3.6) becomes

$$
\begin{aligned}
\left(B'(t)\right)^2 &\le 4\left(\|u\|^2+\alpha\int_0^t \|\nabla u(\tau)\|^2 d\tau\right)\left(\|u_t\|^2+\alpha\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau\right) \\
&\le 4B(t)\left(\|u_t\|^2+\alpha\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau\right). \qquad (3.7)
\end{aligned}
$$

Hence, from (3.5) and (3.7) we have

$$
\begin{aligned}
&B''(t)B(t)-\frac{\lambda+3}{4}\left(B'(t)\right)^2 \\
&\ge B(t)\left(B''(t)-(\lambda+3)\left(\|u_t\|^2+\alpha\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau\right)\right) \\
&\ge B(t)\left(-(\lambda+1)\|u_t\|^2-2I(u)-\alpha(\lambda+3)\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau\right),
\end{aligned} \qquad (3.8)
$$

where $p>\lambda>1$ will be decided later. Now, define

$$
\xi(t) := -(\lambda+1)\|u_t\|^2-2I(u)-\alpha(\lambda+3)\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau. \qquad (3.9)
$$

Recalling (2.7), (2.1) and (2.4) we can deduce (3.9) to

$$
\begin{aligned}
\xi(t) \ge &(p-\lambda)\|u_t\|^2+(p-1)\|u\|_H^2-2(p+1)E(0) \\
&+\alpha\,(2p-1-\lambda)\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau.
\end{aligned} \qquad (3.10)
$$

Let

$$\phi(t) := (p-1)\|u(t)\|_H^2 - 2(p+1)E(0).$$

Then, from Lemma 3.2 it implies that

$$\begin{aligned}\phi(t) &= (p-1)\|u(t)\|_H^2 - 2(p+1)d + 2(p+1)d - 2(p+1)E(0)\\ &:\geq \sigma_1 > 0.\end{aligned} \tag{3.11}$$

At this point we can choose $\lambda = \frac{p+1}{2}$, which guarantees that $\lambda \in (1,p)$. Then, combining (3.8)–(3.11) yields

$$B''(t)B(t) - \frac{\lambda+3}{4}B'(t)^2 > \rho\sigma_1 > 0, \quad t \in [0,T_0]. \tag{3.12}$$

If we substitute $y(t) = B(t)^{-\frac{\lambda-1}{4}}$ into (3.12), we have

$$y''(t) < -\frac{\lambda-1}{4}\rho\sigma y(t)^{\frac{\lambda+7}{\lambda-1}}, \quad t \in [0,T_0],$$

which says $\lim\limits_{t\to T_*^-} y(t) = 0$, where $T_* < T_0$ and T_* is independent of the initial choice of T_0. Hence, we can get

$$\lim_{t\to T_*^-} B(t) = +\infty,$$

which completes the proof. □

4. GLOBAL EXISTENCE, ASYMPTOTIC BEHAVIOR AND FINITE TIME BLOW UP FOR CRITICAL INITIAL ENERGY

Let us give a preliminary Lemma 4.1 to prove the global existence, asymptotic behavior and blow up of the solution to problem (1.1) under the condition $E(0) = d$.

Lemma 4.1. *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2), *$u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $u(x,t)$ be a solution* (*not steady state solution*) *of problem* (1.1) *over* $[0,T_0]$, *where T_0 is the maximum existence time of $u(x,t)$. Then, there exists $t_0 \in (0,T_0)$ such that*

$$\int_0^{t_0} \|\nabla u_\tau\|^2 d\tau > 0. \tag{4.1}$$

Proof. Let $u(t)$ be any solution (but not steady-state solution) to problem (1.1) with $E(0) = d$ and T_0 be the maximum existence time of $u(t)$. We prove that there exists $t_0 \in (0,T_0)$ such that (4.1) holds. If it is false, then $\int_0^t \|\nabla u_\tau\|^2 d\tau \equiv 0$ for $0 \leq t < T_0$, which together with Poincaré inequality yields $\int_0^t \|u_\tau\|^2 d\tau \equiv 0$ for $0 \leq t < T_0$. Therefore we can conclude $\|u_t\| \equiv 0$ for $t \in [0,T_0)$ and $\frac{du}{dt} \equiv 0$ for $x \in \Omega$, $t \in [0,T_0)$. Then $u(x,t) \equiv u_0$ for $x \in \Omega$ and $0 \leq t < T_0$, i.e. $u(x,t)$ is a steady-state solution to problem (1.1), which is a contradiction. □

4.1. GLOBAL EXISTENCE FOR CRITICAL INITIAL ENERGY

First we present the global existence of the solution to problem (1.1) under the condition $E(0) < d$ (see [18]).

Lemma 4.2. *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) < d$ and $u_0(x) \in \mathcal{G}$, then problem* (1.1) *admits a global solution.*

The invariance of the stable set under the flow of problem (1.1) is acquired, which plays a core role in proving the existence of the global solution to problem (1.1) at critical initial energy level.

Lemma 4.3. *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) = d$, then $\mathcal{G}$ is invariant under the flow of problem* (1.1)*.*

Proof. Let $u(t)$ be a solution to problem (1.1) with $E(0) = d$, $I(u_0) > 0$ or $\|u_0\|_H = 0$, T_0 be the maximum existence time of $u(t)$. We prove that $u(t) \in \mathcal{G}$ for $0 < t < T_0$. Arguing by contradiction, we suppose that there exists $t_0 \in (0, T_0)$ such that $u(t_0) \in \partial W$, i.e. $I(u(t_0)) = 0$, $\|u(t_0)\|_H \neq 0$. Then by the definition of d we have $J(u(t_0)) \geq d$. Hence together with

$$\frac{1}{2}\|u_t\|^2 + J(u) + \alpha \int_0^t \|\nabla u_\tau\|^2 d\tau = E(0) = d,$$

we can get $\alpha \int_0^{t_0} \|\nabla u_\tau\|^2 d\tau = 0$ and $\|u_t\| = 0$ for $0 \leq t \leq t_0$, which implies $\frac{du}{dt} = 0$ for $x \in \Omega$, $0 \leq t \leq t_0$ and $u(x,t) = u_0(x)$. Hence we have $I(u(t_0)) = I(u_0) > 0$, which contradicts $I(u(t_0)) = 0$. □

In the following, we present the global existence of the solution to problem (1.1) with $E(0) = d$.

Theorem 4.4 (Global existence for critical initial energy)**.** *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) = d$ and $u_0 \in \mathcal{G}$, then problem* (1.1) *admits a global solution.*

Proof. First Theorem 2.2 gives the existence of the local solution over $[0, T_0]$, where T_0 is the maximum existence time of $u(t)$. Thus, it suffices to prove $T_0 = +\infty$. Clearly, if $u(x,t)$ is a steady-state solution of problem (1.1), then $T_0 = +\infty$. If $u(x,t)$ is a solution but not a steady-state solution of problem (1.1), then from Lemma 4.1 it follows that there exists $t_0 \in (0, T_0)$ such that

$$\int_0^{t_0} \|\nabla u_\tau\|^2 d\tau > 0.$$

From (2.4) and $E(0) = d$ we get $E(t_0) = d - \alpha \int_0^{t_0} \|\nabla u_\tau\|^2 d\tau < d$. In addition by Lemma 4.3 we can obtain $u(t_0) \in \mathcal{G}$, i.e. $I(u(t_0)) > 0$ or $\|u(t_0)\|_H = 0$. Hence let $v(t) = u(t + t_0), t \geq 0$, then $v(t)$ is a solution to problem (1.1). From Lemma 4.2 it follows that the existence time of $v(t)$ is infinite, which implies $T_0 = +\infty$. □

4.2. ASYMPTOTIC BEHAVIOR FOR CRITICAL INITIAL ENERGY

Recall the following result about exponential decay of the solution to problem (1.1) in the subcritical case [18].

Lemma 4.5 ([18]). *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2), *$u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $0 < E(0) < d$ and $u_0(x) \in \mathcal{G}$, then for the global solution u given in Lemma 4.2 there holds*

$$\|u_t(t)\|^2 + \|u(t)\|_H^2 \le Ce^{-\gamma t}, \tag{4.2}$$

for some positive constants C and γ.

In what follows, we give an asymptotic behavior of the solution to problem (1.1) for $E(0) = d$.

Theorem 4.6 (Asymptotic behavior for critical initial energy). *Under the conditions of Theorem 4.4, yield*

$$E(t) \le C_1 e^{-\gamma t}, \quad t_0 \le t < +\infty \tag{4.3}$$

and

$$\|u_t(t)\|^2 + \|u(t)\|_H^2 \le C_2 e^{-\gamma t}, \quad t_0 \le t < +\infty, \tag{4.4}$$

for some $t_0 > 0$, $C_i > 0$ $(i = 1, 2)$ and $\gamma > 0$.

Proof. First Theorem 4.4 gives the global existence of the solution. Furthermore, if $u(t)$ is not a steady-state solution to problem (1.1), then from the proof of Theorem 4.4 it follows that there exists $t_0 > 0$ such that $E(t_0) < d$, $I(u_0) > 0$ or $\|u_0\|_H = 0$. Hence, by Lemma 4.5 we have

$$E(t) \le Ce^{-\gamma(t-t_0)}, \quad t_0 \le t < +\infty$$

and (4.3), where $C_1 = Ce^{\gamma t_0}$. Furthermore, by a similar argument as that in [18, Lemma 4.5] we can obtain (4.4). □

4.3. FINITE TIME BLOW UP FOR CRITICAL INITIAL ENERGY

First by the same argument of Lemma 4.3 we can get the following invariance of the unstable set $\mathcal{B}$ under the flow of problem (1.1) at critical initial energy level, which is used to prove the finite time blow up of the solution to problem (1.1) when $E(0) = d$.

Lemma 4.7. *Let $f(u)$ the assumptions* (H1) *and* (H2), *$u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) = d$, then $\mathcal{B}$ is invariant under the flow of problem* (1.1).

Next we prove the finite time blow up of the solution to problem (1.1) with $E(0) = d$.

Theorem 4.8 (Finite time blow up for critical initial energy). *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2), *$u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) = d$ and $I(u_0) < 0$, then the solution but not the steady-state solution $u(t)$ to problem* (1.1) *blows up in finite time.*

Proof. First, Theorem 2.2 gives the existence of the local solution over $[0, T_0]$, where T_0 is the maximum existence time of $u(t)$. Let us prove that if $u(t)$ is not a steady-state solution to problem (1.1) then $T_0 < +\infty$. In fact, from Lemma 4.1 it follows that there exists a $t_0 > 0$ such that

$$\int_0^{t_0} \|\nabla u_t\|^2 dt > 0$$

and

$$E(t_0) = d - \alpha \int_0^{t_0} \|\nabla u_t\|^2 dt < d.$$

In addition by Lemma 4.7 we can obtain $I(u(t_0)) < 0$. Hence from Theorem 3.3 it follows that the maximum existence time of $u(t)$ is finite. □

5. FINITE TIME BLOW UP FOR SUPERCRITICAL INITIAL ENERGY

In this section, we consider the finite time blow up of the solution to problem (1.1) with the supercritical initial energy level $E(0) > d$. Throughout this section we let

$$1 \geq \alpha > 0. \tag{5.1}$$

First, let us prove the following lemma to aid us to obtain that the unstable set $\mathcal{B}$ is invariant under the flow of problem (1.1) with the supercritical initial energy $E(0) > d$.

Lemma 5.1 (Increasing function)**.** *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) > d$ and the initial data satisfy*

$$\|\nabla u_0\|^2 + 2(u_0, u_1) > \frac{2(p+1)(C+2)}{\alpha(p-1)C} E(0), \tag{5.2}$$

where C is the best constant of Poincaré inequality

$$\|\nabla u\|^2 \geq C\|u\|^2. \tag{5.3}$$

Then the map

$$t \mapsto \alpha\|\nabla u(t)\|^2 + 2(u, u_t)$$

is positive and strictly increasing provided that $u(t) \in \mathcal{B}$.

Proof. We introduce the following auxiliary function

$$F(t) := \alpha\|\nabla u(t)\|^2 + 2(u, u_t). \tag{5.4}$$

Then, from Equation (1.1) it follows

$$F'(t) = 2\alpha(\nabla u, \nabla u_t) + 2\langle u_{tt}, u\rangle + 2\|u_t\|^2 = 2\|u_t\|^2 - 2I(u).$$

Hence, by $u(t) \in \mathcal{B}$ we have

$$F'(t) > 0, \quad t \in [0, +\infty). \tag{5.5}$$

Moreover, from (5.2) and $E(0) > d > 0$ it implies that

$$F(0) = \alpha\|\nabla u_0\|^2 + 2(u_0, u_1) \geq \alpha\left(\|\nabla u_0\|^2 + 2(u_0, u_1)\right) > \alpha A E(0) > 0, \tag{5.6}$$

where $A = \frac{2(p+1)(C+2)}{\alpha(p-1)C}$. Therefore, from (5.5) and (5.6) we can see that $F(t) > F(0) > 0$, which tells us that the map

$$t \mapsto \alpha\|\nabla u(t)\|^2 + 2(u, u_t)$$

is positive and strictly increasing. □

In the following, we show the invariance of the unstable set $\mathcal{B}$ under the flow of problem (1.1) with the supercritical initial energy $E(0) > d$.

Lemma 5.2 (Invariant set $\mathcal{B}$)**.** *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Then the solution to problem* (1.1) *with $E(0) > d$ belongs to $\mathcal{B}$, provided that $u_0 \in \mathcal{B}$ and* (5.2) *holds.*

Proof. We prove $u(t) \in \mathcal{B}$ for $t \in [0, T_0)$. Arguing by contradiction, we suppose that $t_0 \in (0, T_0)$ is the first time such that

$$I(u(t_0)) = 0 \tag{5.7}$$

and

$$I(u(t)) < 0, \ t \in [0, t_0).$$

Hence, from Lemma 5.1 it follows that the map

$$t \mapsto \alpha\|\nabla u(t)\|^2 + 2(u, u_t)$$

is positive and strictly increasing on the interval $[0, t_0)$, which together with (5.2) gives that

$$\begin{aligned}\|\nabla u(t)\|^2 + 2(u(t), u_t(t)) &\geq \alpha\|\nabla u(t)\|^2 + 2(u(t), u_t(t)) > \alpha\|\nabla u_0\|^2 + 2(u_0, u_1) \\ &> \alpha\left(\|\nabla u_0\|^2 + 2(u_0, u_1)\right) > \alpha A E(0)\end{aligned}$$

for all $t \in (0, t_0)$, where $A = \frac{2(p+1)(C+2)}{\alpha(p-1)C}$, which means

$$\|\nabla u(t)\|^2 + 2(u(t), u_t(t)) > \frac{2(p+1)(C+2)}{(p-1)C} E(0).$$

Moreover, from the continuity of $u(t)$ and $u_t(t)$ in t, we obtain

$$\|\nabla u(t_0)\|^2 + 2(u(t_0), u_t(t_0)) > \frac{2(p+1)(C+2)}{(p-1)C} E(0). \tag{5.8}$$

Recalling (2.4), (2.1), (2.3) and (2.7), we have

$$\begin{aligned}E(0) =&E(t) + \alpha \int_0^t \|\nabla u_\tau\|^2 d\tau \\ =&\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_H^2 - \int_\Omega F(u)dx + \alpha \int_0^t \|\nabla u_\tau\|^2 d\tau \\ \geq&\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_H^2 - \frac{1}{p+1}\int_\Omega uf(u)dx + \alpha \int_0^t \|\nabla u_\tau\|^2 d\tau \\ =&\frac{1}{2}\|u_t\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_H^2 + \frac{1}{p+1}I(u) + \alpha \int_0^t \|\nabla u_\tau\|^2 d\tau,\end{aligned} \tag{5.9}$$

which together with (5.7), (5.1), $p > 1$, (5.3) and Cauchy-Schwarz inequality shows that

$$\begin{aligned}E(0) \geq&\frac{1}{2}\|u_t(t_0)\|^2 + \frac{p-1}{2(p+1)}\|\nabla u(t_0)\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)\|\Delta u(t_0)\|^2 \\ \geq&\frac{p-1}{2(p+1)}\|u_t(t_0)\|^2 + \frac{p-1}{2(p+1)}\|\nabla u(t_0)\|^2 \\ \geq&\frac{(p-1)C}{2(p+1)(C+2)}\|u_t(t_0)\|^2 + \frac{p-1}{2(p+1)}\|\nabla u(t_0)\|^2 \\ =&\frac{(p-1)C}{2(p+1)(C+2)}\left(\|u_t(t_0)\|^2 + \|\nabla u(t_0)\|^2\right) + \frac{p-1}{(p+1)(C+2)}\|\nabla u(t_0)\|^2 \\ \geq&\frac{(p-1)C}{2(p+1)(C+2)}\left(\|u_t(t_0)\|^2 + \|\nabla u(t_0)\|^2\right) + \frac{(p-1)C}{(p+1)(C+2)}\|u(t_0)\|^2 \\ =&\frac{(p-1)C}{2(p+1)(C+2)}\left(\|u_t(t_0)\|^2 + 2\|u(t_0)\|^2 + \|\nabla u(t_0)\|^2\right) \\ \geq&\frac{(p-1)C}{2(p+1)(C+2)}\left(2(u(t_0), u_t(t_0)) + \|u(t_0)\|^2 + \|\nabla u(t_0)\|^2\right) \\ \geq&\frac{(p-1)C}{2(p+1)(C+2)}\left(2(u(t_0), u_t(t_0)) + \|\nabla u(t_0)\|^2\right).\end{aligned} \tag{5.10}$$

Obviously (5.10) contradicts (5.8). So the proof is completed. □

In the end we prove the finite time blow up result of the solution to problem (1.1) with the supercritical initial energy $E(0) > d$.

Theorem 5.3 (Blow up for supercritical initial energy)**.** *Let $f(u)$ satisfy the assumptions* (H1) *and* (H2)*, $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Assume that $E(0) > d$, $u_0 \in \mathcal{B}$ and* (5.2) *holds, then the solution of problem* (1.1) *blows up in finite time.*

Proof. Recalling the auxiliary function $B(t)$ defined as (3.2) and the proof of Theorem 3.3, we have (3.8)-(3.10). Then, from (5.3) and the Cauchy–Schwarz inequality we can deduce (3.10) to

$$\begin{aligned}
\xi(t) \geq & (p-\lambda)\|u_t\|^2 + (p-1)\|\nabla u\|^2 - 2(p+1)E(0) + (p-1)\|\Delta u\|^2 \\
& + (2p-1-\lambda)\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau \\
= & (p-\lambda)\|u_t\|^2 + \frac{2(p-\lambda)}{C}\|\nabla u\|^2 - 2(p+1)E(0) \\
& + \left((p-1) - \frac{2(p-\lambda)}{C}\right)\|\nabla u\|^2 + (2p-1-\lambda)\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau \\
\geq & (p-\lambda)\|u_t\|^2 + 2(p-\lambda)\|u\|^2 - 2(p+1)E(0) \\
& + \left((p-1) - \frac{2(p-\lambda)}{C}\right)\|\nabla u\|^2 + (2p-1-\lambda)\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau \\
\geq & (p-\lambda)\left(\|u_t\|^2 + 2\|u\|^2\right) + \left((p-1) - \frac{2(p-\lambda)}{C}\right)\|\nabla u\|^2 - 2(p+1)E(0) \\
& + (2p-1-\lambda)\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau \\
\geq & (p-\lambda)\left(2(u,u_t) + \|u\|^2\right) + \left((p-1) - \frac{2(p-\lambda)}{C}\right)\|\nabla u\|^2 - 2(p+1)E(0) \\
& + (2p-1-\lambda)\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau.
\end{aligned} \tag{5.11}$$

At this point we choose $\lambda = \frac{C+2p}{C+2}$, which guarantees that $\lambda \in (1,p)$, since $p > 1$. Then, by a simple computation and $\lambda < 1 + 2p$, (5.11) becomes

$$\begin{aligned}
\xi(t) &\geq \frac{C(p-1)}{C+2}\left(2(u,u_t) + \|u\|^2 + \|\nabla u\|^2\right) - 2(p+1)E(0) \\
&> \frac{C(p-1)}{C+2}\left(2(u,u_t) + \|\nabla u\|^2\right) - 2(p+1)E(0),
\end{aligned} \tag{5.12}$$

which together with (5.1), Lemma 5.2 and Lemma 5.1 gives

$$\begin{aligned}\xi(t) &> \frac{C(p-1)}{C+2}\big(\alpha\|\nabla u(t)\|^2 + 2(u(t), u_t(t))\big) - 2(p+1)E(0),\\ &> \frac{C(p-1)}{C+2}\big(\alpha\|\nabla u_0\|^2 + 2(u_0, u_1)\big) - 2(p+1)E(0)\\ &> \frac{\alpha C(p-1)}{C+2}\big(\|\nabla u_0\|^2 + 2(u_0, u_1)\big) - 2(p+1)E(0)\\ &:= \sigma_2 > 0, \quad t \in (0, T_0).\end{aligned} \tag{5.13}$$

Therefore, by (3.8), (3.9), (5.12) and (5.13), we have

$$B''(t)B(t) - \frac{\lambda+3}{4}B'(t)^2 > \rho\sigma_2 > 0, \quad t \in [0, T_0].$$

The remainder proof of this theorem, by the concavity argument, is similar to that of Theorem 3.3. □

Acknowledgements
This work was supported by the National Natural Science Foundation of China (11471087, 11871017, 11801114), the China Postdoctoral Science Foundation (2013M540270), the Heilongjiang Postdoctoral Foundation (LBH-Z15036), the Fundamental Research Funds for the Central Universities.

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Received: March 7, 2018.
Accepted: November 3, 2018.