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ON PROPERTIES OF MINIMIZERS OF A CONTROL PROBLEM WITH TIME-DISTRIBUTED FUNCTIONAL RELATED TO PARABOLIC EQUATIONS

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Abstract. We consider a control problem given by a mathematical model of the temperature control in industrial hothouses. The model is based on one-dimensional parabolic equations with variable coefficients. The optimal control is defined as a minimizer of a quadratic cost functional. We describe qualitative properties of this minimizer, study the structure of the set of accessible temperature functions, and prove the dense controllability for some set of control functions.

Keywords: parabolic equation, extremal problem, quadratic cost functional, minimizer, exact controllability, dense controllability.

Mathematics Subject Classification: 35K20, 35Q93, 35Q79, 49J20.

1. INTRODUCTION

Consider the mixed boundary value problem

$$u_t = (a(x,t)u_x)_x, \qquad (x,t) \in Q_T = (0,1) \times (0,T), \quad T > 0, \tag{1.1}$$

$$u(0,t) = \varphi(t), \; u_x(1,t) = \psi(t), \qquad 0 < t < T, \tag{1.2}$$

$$u(x,0) = \xi(x), \qquad 0 < x < 1, \tag{1.3}$$

with $\varphi \in W_2^1(0,T)$, $\psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$, and a sufficiently smooth function a such that $0 < a_0 \le a(x,t) \le a_1 < \infty$ on Q_T. We treat the functions ξ and ψ as fixed and the function φ as a control function to be found. The problem is to find a control function $\varphi = \varphi_0$ making the temperature $u(x,t)$ at some fixed point $x = c \in (0,1)$ maximally close to a given one, $z(t)$, during the whole time interval $(0,T)$.

The quality of the control is estimated by the quadratic cost functional

$$J_z[\varphi] = \int_0^T (u_\varphi(c,t) - z(t))^2 dt, \tag{1.4}$$

where the function $u_\varphi(x,t)$ is a solution to problem (1.1)–(1.3). This problem arises to the problem of the temperature control in industrial hothouses (see [3, 5]). Note that various extremum problems for partial differential equations with integral functionals were considered by different authors (see [7, 9, 10, 16]). The problem of minimization of a functional with final observation and the problem of the optimal control time were considered in [7–10, 23]. A review of early results on this problem is contained in [8], a survey of later works is contained in [17, 23], see also [3, 12].

The main difference between the problem considered in this paper and in previous works consists in the type of observation. We consider the point-wise observation contrary to the previously studied control problems with final and distributed observation (see, for example, [10, 13, 16]).

This paper develops results obtained in [3–6]. We consider a more general problem (equation with variable coefficient $a = a(x,t)$ and a non-homogeneous initial condition) and prove new results on qualitative properties of its minimizer. We prove these results by methods of qualitative theory of differential equations and, in particular, by some methods described in [1, 2].

2. NOTATION, DEFINITIONS AND PREVIOUS RESULTS

Definition 2.1. By $V_2^{1,0}(Q_T)$ we denote the Banach space of all functions $u \in W_2^{1,0}(Q_T)$ with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0 \le t \le T} \|u(x,t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)} \tag{2.1}$$

such that $t \mapsto u(\cdot,t)$ is a continuous mapping $[0,T] \to L_2(0,1)$.

The space $V_2^{1,0}(Q_T)$ was introduced in [15, p. 26]. Its norm (2.1) naturally occurs in the energy balance equation corresponding to a mixed problem for a parabolic equation (see [15, Chapter 3, (2.22)]).

Definition 2.2. By $\widetilde{W}_2^1(Q_T)$ we denote the space of all functions $\eta \in W_2^1(Q_T)$ such that $\eta(x,T) = 0$ for all $x \in (0;1)$ and $\eta(0,t) = 0$ for all $t \in (0;T)$.

Definition 2.3 ([15, Chapter 3, §2, p. 161])**.** We say that a function $u \in V_2^{1,0}(Q_T)$ is a weak solution to problem (1.1)–(1.3) if it satisfies the boundary condition $u(0,t) = \varphi(t)$ and the integral identity

$$\int_{Q_T} (a(x,t)u_x\eta_x - u\eta_t)\, dx\, dt = \int_0^1 \xi(x)\eta(x,0)\, dx + \int_0^T a(1,t)\psi(t)\,\eta(1,t)\, dt \tag{2.2}$$

for any function $\eta \in \widetilde{W}_2^1(Q_T)$.

Since $\varphi, \psi \in W_2^1(0,T)$, any weak solution from $W_2^{1,0}(Q_T)$ automatically belongs to $V_2^{1,0}(Q_T)$ (see [15, Chapter 3, §3]).

By standard technique (see [14, 15]) we can obtain the following estimate for solutions to problem (1.1)–(1.3):

Theorem 2.4 ([5]). *There exists a unique weak solution $u \in V_2^{1,0}(Q_T)$ to problem* (1.1)–(1.3) *satisfying the inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} \le C_1\big(\|\xi\|_{L_2(0,1)} + \|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)}\big),$$

where the constant C_1 is independent of φ, ψ, and ξ.

Hereafter we denote by u_φ the unique solution to the problem (1.1)–(1.3) with $\varphi, \psi \in W_2^1(0,T)$, $\xi \in L_2(0,1)$, existing according to Theorem 2.4.

Suppose $z \in L_2(0,T)$. Let $\Phi \subset W_2^1(0,T)$ be a bounded closed convex set. For some $c \in (0,1)$ consider the functional $J_z[\varphi]$ defined by (1.4) and put

$$m_z[\Phi] = \inf_{\varphi \in \Phi} J_z[\varphi]. \tag{2.3}$$

Consider the sets $\Phi \subset W_2^1(0,T)$ and $Z \subset L_2(0,T)$.

Definition 2.5. We call the problem (1.1)–(1.3), (2.3) exactly controllable on Z by Φ if for any $z \in Z$ there exists $\varphi_0 \in \Phi$ such that

$$J_z[\varphi_0] = 0. \tag{2.4}$$

Definition 2.6. A function $\varphi_0 \in W_2^1(0,T)$ satisfying (2.4) is called an exact control.

Definition 2.7. We call the problem (1.1)–(1.3), (2.3) densely controllable on Z by Φ if for any $z \in Z$ we have

$$m_z[\Phi] = 0.$$

3. MAIN RESULTS

Theorem 3.1. *There exists a unique function $\varphi_0 \in \Phi$ such that*

$$m_z[\Phi] = J_z[\varphi_0]. \tag{3.1}$$

Now we study properties of the minimizer φ_0 as an element of the set Φ.

Theorem 3.2. *Suppose the coefficient a in* (1.1) *does not depend on t and*

$$m_z[\Phi] > 0.$$

Then

$$\varphi_0 \in \partial\Phi.$$

Theorem 3.3. *Suppose the coefficient a in* (1.1) *does not depend on t and* Φ_j, $j = 1, 2$, *are bounded convex closed sets in* $W_2^1(0,T)$ *such that*

$$\Phi_2 \subset \text{Int}\Phi_1$$

and

$$m_z[\Phi_1] > 0.$$

Then

$$m_z[\Phi_1] < m_z[\Phi_2].$$

Another important question concerns exact controllability of the extremal problem.

The next theorem shows that the set Z of functions $z \in L_2(0,T)$ admitting exact controllability is a sufficiently "small" subset of $L_2(0,T)$.

Theorem 3.4. *The set Z of all functions* $z \in L_2(0,T)$ *admitting exact controllability, i.e. such that* $J_z[\varphi] = 0$ *for some* $\varphi \in W_2^1(0,T)$, *is a first Baire category subset in* $L_2(0,T)$.

The following result proves the dense controllability for $Z = L_2(0,T)$ and $\Phi = W_2^1(0,T)$.

Theorem 3.5. *Suppose the coefficient a in* (1.1) *does not depend on t. Then for any* $z \in L_2(0,T)$ *the following equality holds:*

$$m_z[W_2^1(0,T)] = 0. \tag{3.2}$$

4. PROOFS

First we prove Theorem 3.1.

Proof of Theorem 3.1. The proofs of results on the existence and uniqueness are based on the following lemma concerning the best approximation in Hilbert spaces.

Lemma 4.1 ([3]). *Let A be a convex closed set in a Hilbert space H. Then for any* $x \in H$ *there exists a unique element* $y \in A$ *such that*

$$\|x - y\| = \inf_{z \in A} \|x - z\|.$$

Denote

$$B = \{y = u_\varphi(c, \cdot) : \ \varphi \in \Phi\} \subset L_2(0,T).$$

By the convexity of Φ, the set B is a convex subset in $L_2(0,T)$. Now, by Theorem 2.4 we have the inequality

$$\|u_\varphi\|_{V_2^{1,0}(Q_T)} \leq C_1 \big(\|\xi\|_{L_2(0,1)} + \|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)}\big), \tag{4.1}$$

where the constant C_1 does not depend on ξ, φ and ψ. The set Φ is bounded and closed in $W_2^1(0,T)$ and by estimate (4.1) we obtain that B is a bounded and closed set in $L_2(0,T)$. Now we prove that B is a closed subset of $L_2(0,T)$. Let $\{y_k\}_{k=1}^{\infty} \subset B$ be a fundamental sequence in $L_2(0,T)$ having the limit $y \in L_2(0,T)$. The corresponding sequence $\{\varphi_k\} \subset \Phi$ is a weakly precompact set in $W_2^1(0,T)$, by the boundedness of Φ. Let $\varphi \in \Phi$ be the weak limit of its subsequence. Hence, by the Banach-Saks theorem ([20, Chapter 1, Sec. 3], see also [19]), there exists a new subsequence $\{\varphi_{k_j}\}$ such that

$$\begin{aligned} &\|\tilde{\varphi}_l - \varphi\|_{W_2^1(0,T)} \to 0, \quad l \to \infty, \\ &\tilde{\varphi}_l = \frac{1}{l}\sum_{j=1}^{l} \varphi_{k_j}. \end{aligned} \tag{4.2}$$

Therefore, for the corresponding sequence of solutions

$$u_{\tilde{\varphi}_l} = \frac{1}{l}\sum_{j=1}^{l} u_{\varphi_{k_j}}$$

we obtain

$$\|u_{\tilde{\varphi}_l} - u_{\tilde{\varphi}_m}\|_{V_2^{1,0}(Q_T)} \le C_1 \|\tilde{\varphi}_l - \tilde{\varphi}_m\|_{W_2^1(0,T)} \to 0, \quad l, m \to \infty. \tag{4.3}$$

This means that $u_{\tilde{\varphi}_l}(0,t) = \tilde{\varphi}_l(t)$ and the integral identity

$$\begin{aligned} &\int\limits_{Q_T} (a(x,t)(u_{\tilde{\varphi}_l})_x \eta_x - u_{\tilde{\varphi}_l}\eta_t)\, dx\, dt \\ &= \int\limits_0^1 \xi(x)\eta(x,0)\, dx + \int\limits_0^T a(1,t)\psi(t)\, \eta(1,t)\, dt \end{aligned} \tag{4.4}$$

holds for any function $\eta \in \widetilde{W}_2^1(Q_T)$. Taking into account relations (4.2), (4.3), and (4.4), we see that there exists the limit function $u \in V_2^{1,0}(Q_T)$, which is a weak solution to problem (1.1)–(1.3) with the boundary function φ and

$$\|u - u_{\tilde{\varphi}_l}\|_{V_2^{1,0}(Q_T)} \le C_1 \|\varphi - \tilde{\varphi}_l\|_{W_2^1(0,T)}.$$

So, by the embedding estimate (see [15, Chapter 1, Sec. 6, (6.15)]), we obtain

$$\|u(c,\cdot) - u_{\tilde{\varphi}_l}(c,\cdot)\|_{L_2(0,T)} \le C_2 \|u - u_{\tilde{\varphi}_l}\|_{V_2^{1,0}(Q_T)} \le C_1 C_2 \|\varphi - \tilde{\varphi}_l\|_{W_2^1(0,T)},$$

whence $y = u(c,\cdot) \in B$ and B is a closed subset in $L_2(0,T)$.

Therefore, by Lemma 4.1, there exists a unique function $y = u(c,\cdot)$, where $u \in V_2^{1,0}(Q_T)$ is a solution to problem (1.1)–(1.3) with some $\varphi_0 \in \Phi$ such that

$$m_z[\Phi] = J_z[\varphi_0].$$

Now we prove that such $\varphi_0 \in \Phi$ is unique. If not, consider a pair of such functions φ_1, φ_2 and the corresponding pair of solutions u_{φ_1}, u_{φ_2}. The function $\tilde{u} = u_{\varphi_1} - u_{\varphi_2}$ is a solution to the problem

$$\tilde{u}_t = (a(x,t)\tilde{u}_x)_x, \qquad 0 < t < T, \quad 0 < x < 1, \tag{4.5}$$
$$\tilde{u}(0,t) = \tilde{\varphi}(t), \qquad 0 < t < T, \quad \tilde{\varphi}(t) = \varphi_1(t) - \varphi_2(t), \tag{4.6}$$
$$\tilde{u}(c,t) = 0, \qquad 0 < t < T, \tag{4.7}$$
$$\tilde{u}_x(1,t) = 0, \qquad 0 < t < T, \tag{4.8}$$
$$\tilde{u}(x,0) = 0, \qquad 0 < x < 1. \tag{4.9}$$

Taking into account integral identity (2.2) with the function η equal to 0 on $[0,c] \times [0,T]$, we obtain that the function $\tilde{u}$ on the rectangle $Q_T^{(c)} = (c,1) \times (0,T)$ is equal to the solution to the problem

$$\hat{u}_t = (a(x,t)\hat{u}_x)_x, \qquad 0 < t < T, \quad c < x < 1, \tag{4.10}$$
$$\hat{u}(c,t) = 0, \qquad 0 < t < T, \tag{4.11}$$
$$\hat{u}_x(1,t) = 0, \qquad 0 < t < T, \tag{4.12}$$
$$\hat{u}(x,0) = 0, \qquad c < x < 1. \tag{4.13}$$

But the solution to problem (4.10)–(4.13) vanishes on $[c,1] \times [0,T]$, whence we have

$$\tilde{u}(x,t) = 0, \quad c < x < 1, \quad 0 < t < T. \tag{4.14}$$

Now we prove that

$$\tilde{u}(x,t) = 0, \quad 0 < x < 1, \quad 0 < t < T. \tag{4.15}$$

Applying the unique continuation theorem for parabolic equations ([21, Thm. 1.1]) to the solution $\tilde{u}$ to problem (4.5)–(4.9), we obtain that (4.15) follows from (4.14). Therefore, $\tilde{u}(x,t) = 0$ for any $x \in (0,1)$ and $t \in (0,T)$. This means that $\tilde{\varphi}(t) = \tilde{u}(0,t) = 0$. The proof of Theorem 3.1 is completed. □

For further considerations we need the following result analogous to the classical maximum principle.

Theorem 4.2. *Let $u \in V_2^{1,0}(Q_T)$ is a solution to the problem*

$$u_t = (a(x,t)u_x)_x, \qquad (x,t) \in Q_T, \tag{4.16}$$
$$u(0,t) = \varphi(t), \quad u_x(1,t) = 0, \qquad 0 < t < T, \tag{4.17}$$
$$u(x,0) = \xi(x), \qquad 0 < x < 1. \tag{4.18}$$

Then for almost all $(x,t) \in Q_T$ the following inequalities hold:

$$\begin{aligned} \min\Big\{0, \operatorname*{ess\,inf}_{t\in[0,T]} \varphi(t), \operatorname*{ess\,inf}_{x\in[0,1]} \xi(x)\Big\} &\le u(x,t) \\ \le \max\Big\{0, \operatorname*{ess\,sup}_{t\in[0,T]} \varphi(t), \operatorname*{ess\,sup}_{x\in[0,1]} \xi(x)\Big\}&. \end{aligned} \tag{4.19}$$

Proof. Let

$$k = \max\Big\{ \operatorname*{ess\,sup}_{t\in[0,T]} \varphi(t), \operatorname*{ess\,sup}_{t\in[0,1]} \xi(t) \Big\}.$$

We define the function $u^{(k)} = \max\{u - k, 0\}$. By the same way as in the proof of Theorem 7.1 ([14, Chapter 3, Sec. 7]) we can obtain for $0 < t_1 < T$ the following equality

$$\int_0^1 (u^{(k)}(x, t_1))^2 dx + 2 \int_0^{t_1} \left(\int_0^1 a(x,t)(u_x^{(k)})^2 dx \right) dt = 0.$$

Therefore, for almost all $(x,t) \in Q_{t_1}$ we have

$$u(x,t) \le \max\Big\{ 0, \operatorname*{ess\,sup}_{t\in[0,T]} \varphi(t), \operatorname*{ess\,sup}_{x\in[0,1]} \xi(x) \Big\}$$

and obtain the right inequality from (4.19). Similar considerations with the function $-u$ proves the left inequality from (4.19). □

Now we prove Theorem 3.4.

Proof of Theorem 3.4. Consider the solutions $u_{\varphi_j}(x,t) \in V_2^{1,0}(Q_T)$, $j = 1, 2$. Denote $\tilde{u} = u_{\varphi_1} - u_{\varphi_2}$. The function $\tilde{u}$ is a solution of equation (1.1) and satisfies the conditions

$$\begin{aligned} \tilde{u}(0,t) &= \tilde{\varphi}(t) = \varphi_1(t) - \varphi_2(t), \\ \tilde{u}_x(1,t) &= 0, \\ \tilde{u}(x,0) &= 0. \end{aligned}$$

By Theorem 4.2, the solution $\bar{u}$ satisfies the inequalities

$$\min\Big\{ 0, \operatorname*{ess\,inf}_{t\in[0,T]} \tilde{\varphi}(t) \Big\} \le \bar{u}(x,t) \le \max\Big\{ 0, \operatorname*{ess\,sup}_{t\in[0,T]} \tilde{\varphi}(t) \Big\}. \tag{4.20}$$

From (4.20) we obtain

$$\|\tilde{u}\|_{L_\infty(Q_T)} \le \|\varphi_1 - \varphi_2\|_{L_\infty(0,T)},$$

and, consequently, by the continuity of solution to equation (4.16),

$$\sup_{t\in[0,T]} |\tilde{u}(c,t)| \le \|\varphi_1 - \varphi_2\|_{L_\infty(0,T)}. \tag{4.21}$$

By integrating inequality (4.21), we obtain

$$\|\tilde{u}(c,t)\|_{L_2(0,T)} \le \sqrt{T} \|\varphi_1 - \varphi_2\|_{L_\infty(0,T)}. \tag{4.22}$$

Suppose the functions φ_1 and φ_2 are the exact control functions for given z_1 and z_2. This means that

$$J_{z_j}[\varphi_j] = \int_0^T (u_{\varphi_j}(c,t) - z_j(t))^2 dt = 0, \quad j = 1, 2.$$

In this situation inequality (4.22) invokes the inequality

$$\|z_1 - z_2\|_{L_2(0,T)} \le \sqrt{T}\|\varphi_1 - \varphi_2\|_{L_\infty(0,T)} \tag{4.23}$$

for arbitrary functions z_1 and z_2 admitting exact controllability.

Let $Z \subset L_2(0,T)$ be the set of exactly controllable functions. We have $Z = \cup_{M=1}^{\infty} Z_M$, where $Z_M \subset L_2(0,T)$ is the set of functions exactly controllable from

$$\Phi_M = \{\varphi \in W_2^1(0,T), \|\varphi\|_{W_2^1(0,T)} \le M\}.$$

For any $M = 1, 2, \dots$ consider an arbitrary sequence of control functions $\{\varphi_k\} \subset \Phi_M$ and the corresponding sequence $\{z_k(t)\} = \{u_{\varphi_k}(c,t)\} \subset Z_M$. By the embedding theorem for $W_2^1(0,T)$, we have

$$\|\varphi_{k_l} - \varphi_{k_j}\|_{L_\infty(0,T)} \to 0, \quad l, j \to \infty, \tag{4.24}$$

for some subsequence φ_{k_j}. Therefore, by (4.23), (4.24), we get for the sequence $\{z_{k_j}\} \subset Z_M$ the relation

$$\|z_{k_l} - z_{k_j}\|_{L_2(0,T)} \le \sqrt{T}\|\varphi_{k_l} - \varphi_{k_j}\|_{L_\infty(0,T)} \to 0, \quad j, l \to \infty. \tag{4.25}$$

It follows from (4.25) that Z_M is a pre-compact set in $L_2(0,T)$. So, Z_M is nowhere dense in $L_2(0,T)$. Thus, since $Z = \cup_{M=1}^{\infty} Z_M$, we conclude that Z is a first Baire category set in $L_2(0,T)$. Theorem 3.4 is proved. □

Now we prove Theorem 3.5.

Proof of Theorem 3.5. Let us represent the solution to the problem

$$\begin{aligned} u_t &= (a(x)u_x)_x, && (x,t) \in Q_T, \\ u(0,t) &= \varphi(t), && u_x(1,t) = \psi(t), \quad t > 0, \\ u(x,0) &= \xi(x), && 0 < x < 1, \end{aligned}$$

in the form

$$u_\varphi = v + w,$$

where v and w are solutions of the following boundary value problems

$$v_t - (a(x)v_x)_x = 0, \qquad 0 < x < 1, \quad 0 < t < T, \tag{4.26}$$

$$v(0,t) = \varphi(t), \qquad 0 < t < T, \tag{4.27}$$

$$v_x(1,t) = 0, \qquad 0 < t < T, \tag{4.28}$$

$$v(x,0) = 0, \qquad 0 < x < 1, \tag{4.29}$$

and

$$\begin{aligned} w_t - (a(x)w_x)_x &= 0, && 0 < x < 1, \quad 0 < t < T, \\ w(0,t) &= 0, && 0 < t < T, \\ w_x(1,t) &= \psi(t), && 0 < t < T, \\ w(x,0) &= \xi(x), && 0 < x < 1. \end{aligned}$$

Therefore, denoting $v = v_\varphi$, we have

$$J_z[\varphi] = \int_0^T (v_\varphi(c,t) - z_1(t))^2 dt, \quad c \in (0,1],$$

where $z_1(t) = z(t) - w(c,t) \in L_2(0,T)$. It follows from the inequality

$$\begin{aligned} m_z[W_2^1(0,T)] &\le m_z[\{\varphi \in W_2^1(0,T), \varphi(0) = 0\}] \\ &= \inf_{\substack{\varphi \in W_2^1(0,T) \\ \varphi(0)=0}} \int_0^T (v_\varphi(c,t) - z_1(t))^2 dt \end{aligned}$$

that to establish (3.2) it is sufficient to prove that

$$\inf_{\substack{\varphi \in W_2^1(0,T) \\ \varphi(0)=0}} \int_0^T (v_\varphi(c,t) - z_1(t))^2 dt = 0.$$

Let us construct the weak solution $v_\varphi \in W_2^{1,0}(Q_T)$ to problem (4.26)–(4.29) for $\varphi \in W_2^1(0,T)$, $\varphi(0) = 0$. Consider the function $y(x,t) = v_\varphi(x,t) - \varphi(t)$ which is the solution to the following problem:

$$\begin{aligned} y_t - (a(x)y_x)_x &= -\varphi'(t), && 0 < x < 1, \quad 0 < t < T, \\ y(0,t) &= 0, && 0 < t < T, \\ y_x(1,t) &= 0, && 0 < t < T, \\ y(x,0) &= 0, && 0 < x < 1. \end{aligned}$$

Denote by $\{\lambda_k\}_{k=1}^\infty$ and $\{X_k(x)\}_{k=1}^\infty$ the sequences of eigenvalues and orthogonal normalized in $L_2(0,1)$ eigenfunctions of the boundary value problem

$$\begin{aligned} &(a(x)X')' + \lambda X = 0, \quad 0 < x < 1, \\ &X(0) = 0, \quad X'(1) = 0. \end{aligned}$$

So,

$$\begin{aligned} y &= -\sum_{k=1}^\infty \left(\int_0^1 X_k(z)dz X_k(x) \int_0^t e^{-\lambda_k(t-\tau)} \varphi'(\tau) d\tau \right) \\ &= -\int_0^t a(0) \sum_{k=1}^\infty \frac{X_k'(0) X_k(x) e^{-\lambda_k(t-\tau)}}{\lambda_k} \varphi'(\tau) d\tau. \end{aligned}$$

Therefore,

$$\begin{aligned} v_\varphi(x,t) &= \varphi(t) - \int\limits_0^t a(0) \sum_{k=1}^{\infty} \frac{X_k'(0) X_k(x) e^{-\lambda_k (t-\tau)}}{\lambda_k} \varphi'(\tau) d\tau \\ &= \int\limits_0^t \varphi'(\tau) d\tau - \int\limits_0^t a(0) \sum_{k=1}^{\infty} \frac{X_k'(0) X_k(x) e^{-\lambda_k (t-\tau)}}{\lambda_k} \varphi'(\tau) d\tau \\ &= \int\limits_0^t \varphi'(\tau) \left(1 - a(0) \sum_{k=1}^{\infty} \frac{X_k'(0) X_k(x) e^{-\lambda_k (t-\tau)}}{\lambda_k} \right) d\tau \\ &= \int\limits_0^t \varphi'(\tau) P(x, t-\tau) d\tau, \end{aligned}$$

where

$$P(x,t) = 1 - a(0) \sum_{k=1}^{\infty} \frac{X_k'(0) X_k(x) e^{-\lambda_k t}}{\lambda_k}. \tag{4.30}$$

The function $P \in V_2^{1,0}(Q_T)$ is a weak solution to the mixed problem

$$P_t - (a(x) P_x)_x = 0, \qquad 0 < x < 1, \quad 0 < t < T, \tag{4.31}$$

$$P(0,t) = 1, \qquad 0 < t < T, \tag{4.32}$$

$$P_x(1,t) = 0, \qquad 0 < t < T, \tag{4.33}$$

$$P(x,0) = 0, \qquad 0 < x < 1, \tag{4.34}$$

and satisfies the integral identity

$$\int\limits_{Q_T} \left(a(x) P_x \eta_x - P \eta_t \right) \, dx \, dt = 0 \tag{4.35}$$

for any function $\eta \in \widetilde{W}_2^1(Q_T)$. We can define the trace $P(c,\cdot) \in L_2(0,T)$, $c \in (0,1)$. From the structure of series (4.30) we obtain that P is the Green function for problem (4.31)–(4.34) and satisfies the integral identity (4.35).

We use the following property of linear manifolds in the Hilbert space ([18, Chapter 2, §4, Lemma 2]):

Lemma 4.3. *The linear manifold G is dense in Hilbert space H if and only if there are no non-zero element which is orthogonal to any element of G.*

Now we apply these lemma to $H = L_2(0,T)$ and the linear manifold

$$G = \{ v_\varphi(c,t), \varphi(t) \in D(0,T) = C_0^\infty(0,T) \}.$$

To prove (3.2) it is sufficient to prove that if for any $\varphi(t) \in D(0,T)$ we have

$$\int_0^T z_1(t) v_\varphi(c,t)dt = \int_0^T z_1(t)\left(\int_0^t P(c,t-\tau)\varphi'(\tau)d\tau\right)dt = 0, \tag{4.36}$$

then $z_1(t) = 0$. We can transform (4.36) as

$$\begin{aligned} &\int_0^T z_1(t) \int_0^t P(c,t-\tau)\varphi'(\tau)d\tau dt \\ &= \int_0^T \varphi'(\tau) \int_\tau^T z_1(t)P(c,t-\tau)dtd\tau = 0. \end{aligned} \tag{4.37}$$

By (4.37),

$$\int_\tau^T z_1(t)P(c,t-\tau)dt = \text{const}, \quad \tau \in [0,T],$$

but

$$\int_T^T z_1(t)P(c,t-T)dt = 0,$$

so

$$\int_\tau^T z_1(t)P(c,t-\tau)dt = 0, \quad \tau \in [0,T].$$

After the transformation of variables we have

$$\begin{aligned} \{t \to \tau, \tau \to t\} \int_\tau^T z_1(t)P(c,t-\tau)dt &= \int_t^T z_1(\tau)P(c,\tau-t)d\tau \\ \{s = T-\tau\} &= \int_0^{T-t} z_1(T-s)P(c,T-s-t)ds \\ \{q = T-t\} &= \int_0^q z_1(T-s)P(c,q-s)ds \\ \{z_2(s) = z_1(T-s)\} \quad &= \int_0^q z_2(s)P(c,q-s)ds = 0, \end{aligned} \tag{4.38}$$

for almost all $q \in (0,T)$, here $z_2(t) = z_1(T-t) \in L_2(0,T) \subset L_1(0,T)$.

Now we apply the Titchmarsh convolution theorem ([22, Thm. 7]):

Theorem 4.4. *Let* $\xi(t) \in L_1(0,T)$, $\zeta(t) \in L_1(0,T)$ *are functions, such that*

$$\int_0^t \xi(\tau)\zeta(t-\tau)d\tau = 0$$

almost everywhere in the interval $0 < t < T$, *then* $\xi(t) = 0$ *almost everywhere in* $(0,\alpha)$ *and* $\zeta(t) = 0$ *almost everywhere in* $(0,\beta)$, *where* $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \geq T$.

We use Theorem 4.4 to the functions $P(c,\cdot)$ and $z_2(\cdot)$. By equality (4.38) we obtain that there exist $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta \geq T$ such that $z_2(s) = 0$ almost everywhere in $(0,\alpha)$ and

$$P(c,s) = 0$$

almost everywhere in $(0,\beta)$.

Now we prove that $\beta = 0$. In the contrary let $\beta > 0$. Applying maximum principle (4.19) from Theorem 4.2 to problem (4.31)–(4.34) we obtain that $0 \leq P(x,t) \leq 1$ almost everywhere in Q_T. It follows from equality (4.30) that P is a smooth function in $[0,1] \times [\varepsilon, T]$ for any $\varepsilon \in (0,T)$ and it is a classical solution of equation (4.31) in Q_T. Then

$$0 \leq P(x,t) \leq 1, \quad 0 \leq x \leq 1, \quad \varepsilon < t \leq T. \tag{4.39}$$

Let us suppose that

$$P(c,t) = 0, \quad 0 < c < 1, \quad 0 < t < \beta \leq T, \tag{4.40}$$

and consider the function P in the domain $Q_{\beta/2,T} = (0,1) \times (\beta/2, T)$. Note that by (4.39), (4.40),

$$P(c,\beta) = 0 = \inf_{(x,t)\in Q_{\beta/2,T}} P(x,t)$$

and $(c,\beta) \in Q_T$. By the strong maximum principle ([11, Chapter 7, §7.1, Thm. 11]) we obtain that $P = 0$ in $(0,1) \times (\beta/2, \beta)$. It is impossible due to boundary condition (4.32). These contradiction means that $\beta = 0$. So, by the inequality $\alpha + \beta \geq T$ we have $\alpha \geq T$ and $z_2(t) = 0$ almost everywhere in $(0,T)$. Now, $z_1(t) = 0$ almost everywhere in $(0,T)$. Therefore, by Lemma 4.3, we obtain equality (3.2). Theorem 3.5 is proved. □

Now we prove Theorem 3.2.

Proof of Theorem 3.2. Let us suppose that $\varphi_0 \in \Phi$, $J_z[\varphi_0] = m_z[\Phi]$ (see (2.3), (3.1)),

$$J_z[\varphi_0] > 0, \tag{4.41}$$

and the relation $\varphi_0 \in \partial\Phi$ is not true. Then

$$\varphi_0 \in \mathrm{Int}\Phi. \tag{4.42}$$

It follows from (4.41) and Theorem 3.5 that there exists a function $\varphi_1 \in W_2^1(0,T)$ such that

$$J_z[\varphi_1] < \frac{m_z[\Phi]}{4}.$$

Let $\varphi_2 = (1-\alpha)\varphi_0 + \alpha\varphi_1$, $0 \le \alpha \le 1$. By (4.42) for some $\alpha_0 \in (0,1)$ we have $\varphi_2 \in \Phi$. Now, we obtain

$$\begin{aligned}\sqrt{J_z[\varphi_2]} &= \|u_{\varphi_2}(c,t) - z(t)\|_{L_2(0,T)} \\ &= \|(1-\alpha_0)u_{\varphi_0}(c,t) + \alpha_0 u_{\varphi_1}(c,t) - z(t)\|_{L_2(0,T)} \\ &\le (1-\alpha_0)\|u_{\varphi_0}(c,t) - z(t)\|_{L_2(0,T)} + \alpha_0\|u_{\varphi_1}(c,t) - z(t)\|_{L_2(0,T)} \\ &< (1-\alpha_0)\sqrt{m_z[\Phi]} + \frac{\alpha_0}{2}\sqrt{m_z[\Phi]} = \left(1 - \frac{\alpha_0}{2}\right)\sqrt{m_z[\Phi]}.\end{aligned}$$

Thence,

$$J_z[\varphi_2] < \left(1 - \frac{\alpha_0}{2}\right)^2 m_z[\Phi] < m_z[\Phi],$$

and φ_0 is not a minimizer of $J_z[\varphi]$ on Φ. This contradiction proves Theorem 3.2. □

Now we prove Theorem 3.3.

Proof of Theorem 3.3. It follows from the inclusion $\Phi_2 \subset \Phi_1$ that $m_z[\Phi_1] \le m_z[\Phi_2]$. Suppose

$$m_z[\Phi_1] = m_z[\Phi_2]. \tag{4.43}$$

By Theorem 3.1 and (4.43), we have the unique minimizer $\varphi_0 \in \Phi_1 \cap \Phi_2$ such that $J_z[\varphi_0] = m_z[\Phi_1] = m_z[\Phi_2]$. Additionally, it follows from Theorem 3.1 that

$$\varphi_0 \in \partial\Phi_1 \cap \partial\Phi_2.$$

But the relation $\Phi_2 \subset \mathrm{Int}\Phi_1$ means $\partial\Phi_1 \cap \partial\Phi_2 = \emptyset$. This proves Theorem 3.3. □

5. CONCLUSIONS

Note, that this article extends previous authors results (see [3–6]). We consider more general mathematical model and prove new results on qualitative properties of minimizing function for the functional (1.4) connected with this model. It would be interesting to obtain the results of Theorems 3.2, 3.3 and 3.5 for time-dependent coefficient a. Now it is one of open problems to this model.

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