

DIRECT AND INVERSE SPECTRAL PROBLEMS FOR DIRAC SYSTEMS WITH NONLOCAL POTENTIALS

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Abstract. The main purposes of this paper are to study the direct and inverse spectral problems of the one-dimensional Dirac operators with nonlocal potentials. Based on informations about the spectrum of the operator, we find the potential and recover the form of the Dirac system. The methods used allow us to reduce the situation to the one-dimensional case. In accordance with the given assumptions and conditions we consider problems in a specific way. We describe the spectrum, the resolvent, the characteristic function etc. Illustrative examples are also given.

Keywords: inverse spectral problem, nonlocal potential, nonlocal boundary conditions, Dirac system.

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1. INTRODUCTION

Spectral theory of differential operators, both ordinary equations and elliptic partial derivatives ones, is currently well developed in [5, 10, 11, 13, 14, 19], and by others. There is an extensive literature on exactly solvable models of quantum mechanical operators, where the potentials are the Dirac δ -functions [4]. These models describe point interactions and allow one to present various spectral characteristics such as resolvent, spectrum, scattering operator etc. in explicit form. The papers [1, 2] provide a new class of exactly solvable models of quantum-mechanical operators with nonlocal potentials. These models contain the numerical parameters and functional parameters specified by the intensity of point interactions and by the nonlocal potentials, respectively. The above fact extends the applicability of such models in concrete problems. Several results have been obtained for quantum mechanical operators with nonlocal potentials. Besides [1, 2], we cite [3] and also [12, 13, 15–17]. In particular, the algorithm for solving the inverse spectral problem of reconstructing a nonlocal potential by the set

of all eigenvalues does not use the Gelfand–Levitan–Marchenko integral equation. The Fourier coefficients of the nonlocal potential are found directly from the characteristic function which, in turn, is constructed as an infinite product with using eigenvalues.

The idea of constructing models with nonlocal potentials can be illustrated by the example of the one-dimensional Schrödinger operator. Let the potential v in the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + v(x)$$

describe the interaction in a neighborhood of the point $x = x_0$ of small radius. In other words, the function v can be defined in a neighborhood of the point x_0 . Moreover, v possess a certain singularity at this point. Therefore, it can be approximated as $v(x) = \alpha\delta(x - x_0) + \beta\delta'(x - x_0)$ as $x \rightarrow x_0$, where δ is the Dirac function (this corresponds to the model with point interaction). In this case, the eigenfunctions $\psi(x, \lambda)$ by themselves or their derivatives $\psi'(x, \lambda)$ are discontinuous functions at the point $x = x_0$.

The argumentation above gives us that, in the model with nonlocal potentials, the expression $v(x)\psi(x)$ for $x \geq x_0$ is replaced by $v(x)(\psi'(x_0 + 0) + (x - x_0)\psi(x_0 + 0))$ while for $x \leq x_0$ is replaced by $v(x)(\psi(x_0 - 0) + (x - x_0)\psi'(x_0 - 0))$. This leads to the following expression for the Schrödinger operator with nonlocal potentials v_j ($j = 1, 2, 3, 4$):

$$L = -\frac{d^2\psi}{dx^2} + v_1(x)\psi_r(x_0) + v_2(x)\psi_s(x_0) + v_3(x)\psi'_r(x_0) + v_4(x)\psi'_s(x_0),$$

for a discontinuous function f at the point $x = x_0$. For this particular point, f_r, f_s can be taken as

$$f_r(x_0) = \frac{1}{2}(f(x_0 + 0) + f(x_0 - 0)) \quad \text{and} \quad f_s(x_0) = \frac{1}{2}(f(x_0 + 0) - f(x_0 - 0)),$$

respectively. The above differential expression generates a self-adjoint Schrödinger operator with nonlocal potentials [2] on suitable functions.

In this paper we consider the inverse spectral problem for the Dirac operator

$$(\mathcal{A}\Psi)(x) = B\frac{d\Psi(x)}{dx} + V(x)\Psi^+,$$

where

$$B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V(x) = \begin{pmatrix} 0 & v_1(x) \\ v_2(x) & 0 \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad \Psi^+ = \begin{pmatrix} \psi^+ \\ \psi^+ \end{pmatrix},$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in W_2^1(0, b) \oplus W_2^1(0, b) : \psi_1(b) - \psi_2(b) + i(\langle \psi_1, v_1 \rangle + \langle \psi_2, v_2 \rangle) = 0, \right. \\ \left. \psi_1(0) = \psi_2(0), \right\},$$

and

$$v_1, v_2 \in L_2(0, b), \quad \psi^+ := \frac{1}{2} (\psi_1(b) + \psi_2(b)).$$

We propose a method due to each of the direct and inverse spectral problem for the Dirac system can be reduced to the study of an ordinary differential operator. In terms of those operators there are described spectral properties of these operators and algorithm for the inverse problem is given.

The structure of this paper is as follows. Section 2 contains the study of the momentum operator id/dx perturbed by a single nonlocal potential. We give a complete description of the spectrum of such an operator on a finite interval with the help of the characteristic function. In Section 3, the inverse eigenvalue problem for the first order differential operator is considered. We provide the algorithm for solving this problem and illustrate it with an example. Until now we rely on ideas borrowed from [16] and present the detailed arguments for the sake of completeness. Our main results are to be found in Section 4, where we describe the method which reduces the spectral problem for the Dirac system to the one-dimensional case discussed in Section 2. Here we propose the algorithm, which enables us to solve the inverse problem for the Dirac system, together with a corresponding example.

2. DIFFERENTIAL OPERATORS OF THE FIRST ORDER WITH NONLOCAL POTENTIALS. SPECTRAL ANALYSIS

In this section, we are concerned with the study of direct spectral problems for the first order differential operators with nonlocal potentials on the finite interval. Related results were obtained in [16]. We shall prove self-adjointness of such operators, describe their spectra in terms of the characteristic function. Formulas for the resolvents will be also given. All these facts will be necessary for the study of the Dirac operator undertaken in Section 4 below.

2.1. SELF-ADJOINTNESS

Consider the following eigenvalue problem

$$L\psi := i \frac{d\psi}{dx} + v(x)\psi_+ = \lambda\psi, \quad 0 \leq x \leq l, \quad (0 < l < \infty), \quad (2.1)$$

with the boundary condition

$$\psi_- + i \int_0^l \psi(x) \overline{v(x)} dx = 0, \quad (2.2)$$

where

$$\psi_+ := \frac{1}{2} (\psi(l) + \psi(0)), \quad \psi_- := \psi(l) - \psi(0),$$

and (the nonlocal potential) v is a given function from the class $L_2(0, l)$ and (the spectral parameter) λ is a fixed complex number.

In what follows, A denotes the operator generated in the space $L_2(0, l)$ by the problem (2.1)–(2.2), that is, $A\psi = \lambda\psi$. The operator A is defined by

$$(A\psi)(x) = i \frac{d\psi}{dx} + v(x)\psi_+, \quad 0 \leq x \leq l, \tag{2.3}$$

on the domain

$$\mathcal{D}(A) = \left\{ \psi \in W_2^1(0, l) : \psi_- + i \int_0^l \psi(x)\overline{v(x)}dx = 0 \right\}, \tag{2.4}$$

where $W_2^1(0, l)$ is the Sobolev space.

Next, let A_{max} denote the maximal operator associated with the differential expression L , that is, A_{max} is the linear operator defined on the Sobolev space $W_2^1(0, l)$ by

$$A_{max}\psi = L\psi, \quad \psi \in W_2^1(0, l).$$

Note that the range of the operator A_{max} is equal to the whole space $L_2(0, l)$. Indeed, for every function $h \in L_2(0, l)$ there exists a function $\eta \in W_2^1(0, l)$, namely,

$$\eta(x) = i \int_x^l h(s)ds - \frac{i}{2} \int_0^l h(s)ds, \tag{2.5}$$

such that $A_{max}\eta = h$.

The formula for integration by parts yields

$$\langle A_{max}\psi, \varphi \rangle - \langle \psi, A_{max}\varphi \rangle = i \left((\psi_- + i\langle \psi, v \rangle) \overline{\varphi}_+ + \psi_+ \overline{(\varphi_- + i\langle \varphi, v \rangle)} \right), \tag{2.6}$$

for $\psi, \varphi \in \mathcal{D}(A_{max})$. In (2.6) the symbol $\langle \cdot, \cdot \rangle$ designates the scalar product of the Hilbert space $L_2(0, l)$. Noting by Γ_0 and Γ_1 the functionals defined on $W_2^1(0, l)$ by

$$\Gamma_0\psi = \psi_- + i\langle \psi, v \rangle, \quad \Gamma_1\psi = i\psi_+,$$

the formula (2.6) is rewritten as

$$\langle A_{max}\psi, \varphi \rangle - \langle \psi, A_{max}\varphi \rangle = \Gamma_1\psi \cdot \overline{\Gamma_0\varphi} - \Gamma_0\psi \cdot \overline{\Gamma_1\varphi}. \tag{2.7}$$

The minimal operator $A_0 := A_{min}$ is the restriction of A to the domain

$$\mathcal{D}(A_0) = \left\{ \psi \in W_2^1(0, l) : \Gamma_0\psi = \Gamma_1\psi = 0 \right\}.$$

It is given by

$$A_0\psi = i \frac{d\psi}{dx}, \quad \psi \in \mathcal{D}(A_0).$$

Lemma 2.1. A_0 is a densely defined symmetric operator on $L_2(0, l)$.

Proof. We shall prove that for every function $h \in L_2(0, l)$ and for $\epsilon > 0$ there exists a function $\psi_\epsilon \in \mathcal{D}(A_0)$ such that $\|h - \psi_\epsilon\| < \epsilon$.

Since the set $C_0^\infty(0, l)$ of all infinitely differentiable (complex-valued) functions with compact support is everywhere dense in $L_2(0, l)$, for a given function h in $L_2(0, l)$ and $\epsilon > 0$ there exists φ in $C_0^\infty(0, l)$ such that

$$\|h - \varphi\| < \frac{\epsilon}{2}.$$

The support of φ is contained in $(0, l)$, then there exists $\delta > 0$ such that $\text{supp } \varphi \subset [\delta, l - \delta]$. Now we want to extend the function φ , for which $\varphi(x) = 0$, $x \in [0, \delta] \cup [l - \delta, l] =: I_\delta$, to the whole interval $[0, l]$ so as to an extension ψ_ϵ belongs to $D(A_0)$ and

$$\int_0^l |\psi_\epsilon(x) - \varphi(x)|^2 dx = \int_{I_\delta} |\psi_\epsilon(x)|^2 dx \leq \left(\frac{\epsilon}{2}\right)^2. \tag{2.8}$$

Choosing ψ_ϵ in $\mathcal{D}(A_0)$ satisfying

$$\|\psi_\epsilon - \varphi\| < \frac{\epsilon}{2},$$

we get

$$\|h - \psi_\epsilon\| \leq \|h - \varphi\| + \|\varphi - \psi_\epsilon\| < \epsilon.$$

Consider a function $e_{\delta,d} \in W_2^1(0, l)$ such that $e_{\delta,d}(0) = 1$, $e_{\delta,d}(l) = -1$ and $\text{supp } e_{\delta,d} \subset I_\delta$ also $\|e_{\delta,d}\| = d$. A function $e_{\delta,d}$ exists for all $\delta > 0$ and $d > 0$. As an extension $\psi_\epsilon \in \mathcal{D}(A_0)$ we can take the function in the following form

$$\psi_\epsilon(x) = C e_{\delta,d}(x) + \varphi(x).$$

The boundary condition $\Gamma_1 \psi_\epsilon = 0$ holds with any value of constant C . The boundary condition $\Gamma_0 \psi_\epsilon = 0$ holds with

$$C = \frac{i\langle \varphi, v \rangle}{2 - i\langle e_{\delta,d}, v \rangle} \quad \text{and} \quad d \leq \frac{\epsilon}{2} \frac{1}{1 + \|\varphi\| \|v\| + \|v\|}. \tag{2.9}$$

Since $|C| \leq \|\varphi\| \|v\|$, then the extension $\psi_\epsilon \in \mathcal{D}(A_0)$ satisfies inequality (2.8). In this way $\mathcal{D}(A_0)$ is dense in $L_2(0, l)$.

Now, take $\psi, \varphi \in \mathcal{D}(A_0)$, i.e., $\psi, \varphi \in W_2^1(0, l)$ such that $\Gamma_0 \psi = \Gamma_1 \psi = 0$ and $\Gamma_0 \varphi = \Gamma_1 \varphi = 0$. Then the right hand side of (2.7) vanishes and, thus, the operator A_0 is symmetric. \square

Indeed, A_0 is the minimal operator associated with L , because A_0 and A_{max} satisfy the following relation.

Lemma 2.2. $A_0^* = A_{max}$.

Proof. Assume that $\psi \in \mathcal{D}(A_0)$ and $\varphi \in \mathcal{D}(A_{max})$. From (2.7) we obtain the equality $\langle A_0\psi, \varphi \rangle = \langle \psi, A_{max}\varphi \rangle$. Therefore, $A_{max} \subseteq A_0^*$.

To prove the converse inclusion $A_0^* \subseteq A_{max}$, let $f \in \mathcal{D}(A_0^*)$ and set $h := A_0^*f$. Then for the function η given by (2.5) we have

$$\langle A_0\psi, f - \eta \rangle = 0 \quad \text{for all } \psi \in \mathcal{D}(A_0).$$

Hence $f - \eta \perp \mathcal{R}(A_0)$ which is equivalent to

$$(f - \eta)(x) = C \left(-1 - \frac{i}{2} \int_0^l v(s)ds + i \int_x^l v(s)ds \right),$$

where C is an arbitrary constant. In this way we obtain that

$$f - \eta \in W_2^1(0, l)$$

and $A_{max}(f - \eta) = 0$. Then $A_{max}f = h$ and $f \in \mathcal{D}(A_{max})$. Therefore $A_0^* \subseteq A_{max}$. We conclude that $A_0^* = A_{max}$. \square

Lemma 2.3. *The mapping*

$$(\Gamma_0, \Gamma_1): W_2^1(0, l) \ni \psi \mapsto (\Gamma_0\psi, \Gamma_1\psi) \in \mathbb{C}^2 \tag{2.10}$$

is surjective.

Proof. For any pair $(a, b) \in \mathbb{C}^2$, there exists a function $\hat{\psi}$ in $W_2^1(0, l)$, which can be taken to be linear, such that

$$\hat{\psi}_- := \hat{\psi}(l) - \hat{\psi}(0) = a, \quad \hat{\psi}_+ := \frac{1}{2} \left(\hat{\psi}(l) + \hat{\psi}(0) \right) = -ib.$$

In the trivial case when $v = 0$, we have $\Gamma_0\hat{\psi} = \hat{\psi}_-$ and, since $\Gamma_1\hat{\psi} = i\hat{\psi}_+ = b$, it follows that $(\Gamma_0\hat{\psi}, \Gamma_1\hat{\psi}) = (a, b)$. In the case when $v \neq 0$, for any $\varepsilon > 0$ it can be chosen a function φ in $C_0^\infty(0, l)$, such that

$$\|\hat{\psi} - \varphi\| < \varepsilon \|v\|^{-1}.$$

Letting

$$\psi = \hat{\psi} - \varphi,$$

we have $\psi_- = \hat{\psi}_-$ and $\psi_+ = \hat{\psi}_+$. Thus

$$\Gamma_0\psi = \psi_- + i\langle \psi, v \rangle = a + i\langle \psi, v \rangle,$$

so that

$$|\Gamma_0\psi - a| = |\langle \psi, v \rangle| \leq \|\psi\| \|v\| < \varepsilon,$$

and, since $\Gamma_1\psi = b$, we conclude the surjectivity property for the mapping given by (2.10). \square

Proposition 2.4. *The operator A is self-adjoint.*

Proof. Since, by Lemma 2.1, A_0 is a densely defined symmetric operator and, by Lemma 2.2, we have $A_0^* = A_{max}$ it follows from (2.6) that $(\mathbb{C}, \Gamma_0, \Gamma_1)$ is a boundary triplet for A_0^* (see, for instance, [18, Def. 14.2]). Secondly, the mapping (2.10) is surjective due to Lemma 2.3. According to the general theory of boundary triplets [7–9, 18], the restriction of A_0^* to

$$\mathcal{D}(A^\alpha) := \{ \psi \in W_2^1(0, l) : e^{i\alpha}(\Gamma_1 + i\Gamma_0)\psi = (\Gamma_1 - i\Gamma_0)\psi \}, \quad 0 \leq \alpha < 2\pi,$$

is a self-adjoint operator. In particular, for $\alpha = 0$, we have

$$\begin{aligned} \mathcal{D}(A^0) &= \{ \psi \in W_2^1(0, l) : (\Gamma_1 + i\Gamma_0)\psi = (\Gamma_1 - i\Gamma_0)\psi \} \\ &= \{ \psi \in W_2^1(0, l) : \Gamma_0\psi = 0 \} = \mathcal{D}(A). \end{aligned}$$

As a consequence, A is a self-adjoint operator on the space $L_2(0, l)$. □

2.2. THE RESOLVENT

In this subsection, a formula for the resolvent of the operator A will be given. To this end, it will be convenient to introduce two operators A_- and A_+ related to the problem (2.1)–(2.2). In fact, the operators A_- and A_+ are nothing than the differentiation operator id/dx defined on $L_2(0, l)$ on the domains

$$\mathcal{D}(A_-) = \{ \psi \in W_2^1(0, l) : \psi_- = 0 \}, \quad \mathcal{D}(A_+) = \{ \psi \in W_2^1(0, l) : \psi_+ = 0 \},$$

respectively. Clearly, both these operators are self-adjoint and have only discrete spectra. The eigenvalues of A_- are

$$\lambda_n^- = 2n\frac{\pi}{l}, \quad n \in \mathbb{Z}$$

and of A_+ are

$$\lambda_n^+ = (2n - 1)\frac{\pi}{l}, \quad n \in \mathbb{Z}$$

with the corresponding eigenfunctions $\psi_n^-(x) = e^{-i\lambda_n^-x}$ and $\psi_n^+(x) = e^{-i\lambda_n^+x}$, respectively. Since the set of eigenfunctions $\{ \psi_n^\pm : n \in \mathbb{Z} \}$ forms a complete orthogonal system in $L_2(0, l)$ and it is assumed that $v \in L_2(0, l)$, we can be represent v by the Fourier series expansion

$$v(x) = \sum_n v_n e^{-i\lambda_n^+x}, \tag{2.11}$$

where

$$v_n = \frac{1}{l} \int_0^l v(x) e^{i\lambda_n^+x} dx, \quad n \in \mathbb{Z}. \tag{2.12}$$

The summation in (2.11), and also in the further part of the paper, is taken over all the integers n .

In turns out that the operators A , A_- and A_+ are related to each other, according to the following statement.

Proposition 2.5. *A is a rank two perturbation of A_- and a rank one perturbation of A_+ .*

Proof. First of all, we observe that the resolvents $(A_- - zI)^{-1}$, $(A_+ - zI)^{-1}$ of the operators A_- , A_+ are bounded integral operators with kernels

$$\mathcal{G}_-(x, s; z) = i \frac{e^{-iz(x-s)}}{e^{-izl} - 1} \cdot \begin{cases} 1 & \text{for } s < x, \\ e^{-izl} & \text{for } s > x, \end{cases}$$

$$\mathcal{G}_+(x, s; z) = i \frac{e^{-iz(x-s)}}{e^{-izl} + 1} \cdot \begin{cases} -1 & \text{for } s < x, \\ e^{-izl} & \text{for } s > x, \end{cases}$$

respectively, i.e.,

$$(A_- - zI)^{-1}\varphi(x) = \int_0^l \mathcal{G}_-(x, s; z)\varphi(s)ds,$$

$$(A_+ - zI)^{-1}\varphi(x) = \int_0^l \mathcal{G}_+(x, s; z)\varphi(s)ds,$$

where $\varphi \in L_2(0, l)$.

By straightforward computation, it can be stated that the resolvent $(A - zI)^{-1}$ of the operator A is also an integral operator, the kernel $\mathcal{G}(x, s; z)$ of which is expressed by the formula

$$\mathcal{G}(x, s; z) - \mathcal{G}_+(x, s; z) = \frac{\cos \frac{zl}{2}\varphi(x; z)\overline{\varphi(s; \bar{z})}}{2\chi(z)},$$

where

$$\varphi(x; z) = \frac{2e^{-izx}}{e^{-izl} + 1} - \int_0^l \mathcal{G}_+(x, s; z)v(s)ds,$$

and

$$\chi(z) = \frac{i}{2} \frac{e^{izl} - e^{-izl}}{e^{izl/2} + e^{-izl/2}} - \frac{i}{2} \left(e^{izl/2} + e^{-izl/2} \right) \left[\int_0^l G_+(0, s; z)v(s)ds - \int_0^l G_+(s, 0; z)\overline{v(s)}ds \right] - \frac{1}{4} \left(e^{izl/2} + e^{-izl/2} \right) \int_0^l \int_0^l G_+(x, s; z)v(s)\overline{v(x)}dsdx.$$

As is easily seen, the kernel $\mathcal{G}(x, s; z)$ differs from $\mathcal{G}_+(x, s; z)$ by a degenerate kernel of rank 1, and since

$$\mathcal{G}_-(x, s; z) - \mathcal{G}_+(x, s; z) = \frac{2ie^{-izl}}{e^{-2izl} - 1} e^{-izx} e^{izs},$$

it follows that the operator A is a rank one perturbation of A_+ and a rank two perturbation of A_- . \square

Remark 2.6. The function χ , as will be shown in Subsection 2.4, is equal to the characteristic function of the operator A .

2.3. THE SPECTRUM OF THE OPERATOR A

Our next aim is to describe the spectrum of the operator A . It was already noted that the operator A is self-adjoint and, due to Proposition 2.5, its spectrum is discrete, hence it consists of isolated eigenvalues that are located on the real axis. If a point is an eigenvalue of A_+ , then in its sufficiently small neighborhood there are at most two distinct simple eigenvalues or one double eigenvalue ([6, Thm. 9.3.3]). However, it may happen that one of the eigenvalues of A_+ is an eigenvalue of A as well. Actually, we have the following result.

Theorem 2.7.

- 1) The eigenvalues of the operator A are of multiplicity at most equal to 2.
- 2) An eigenvalue $\lambda_n^+ = (2n - 1)\frac{\pi}{l}$ of the operator A_+ is also an eigenvalue of the operator A if and only if

$$v_n := \frac{1}{l} \int_0^l v(x)e^{i\lambda_n^+ x} dx = \frac{2i}{l}. \tag{2.13}$$

In this case λ_n^+ is of multiplicity 2 provided that

$$\sum_{k \neq n} \frac{1}{\lambda_k^+ - \lambda_n^+} \left(v_k - \bar{v}_k - \frac{i}{2} l |v_k|^2 \right) = 0. \tag{2.14}$$

- 3) All eigenvalues of the operator A different from λ_n^+ ($n \in \mathbb{Z}$) are simple.

Proof. 1) As was already mentioned, A may be viewed as a perturbation of A_+ with one-dimensional operator. Since the eigenvalues of the operator A_+ are simple, the assertion immediately follows from the general principles concerning finite-dimensional perturbations of self-adjoint operators ([6, Thm. 9.3.3]).

For the operator A , the information that its eigenvalues are of multiplicity at most equal to 2 is also due to the fact that the homogeneous problem (2.1) can have only two linearly independent solutions (for $\psi_+ = 0$ and $\psi_+ \neq 0$).

2) Let λ_n^+ be an eigenvalue of A . It is clear that the eigenfunctions $\psi \in \mathcal{D}(A)$ are solutions of the differential equation

$$i \frac{d\psi}{dx} + v(x)\psi_+ = \lambda_n \psi. \tag{2.15}$$

The general solutions of (2.15) are given by

$$\psi(x) = ce^{-i\lambda_n^+ x} + i\psi_+ \int_0^x e^{-i\lambda_n^+(x-t)} v(t) dt,$$

where c is a constant.

Since $\psi(0) = c$ and

$$\begin{aligned}\psi(l) &= ce^{-i\lambda_n^+ l} + i\psi_+ \int_0^l e^{-i\lambda_n^+(l-t)} v(t) dt \\ &= -c - i\psi_+ \int_0^l e^{i\lambda_n^+ t} v(t) dt = -c - i\psi_+ lv_n,\end{aligned}$$

it follows that

$$2\psi_+ = -i\psi_+ lv_n,$$

which yields $\psi_+ = 0$ or $v_n = \frac{2i}{l}$. If $\psi_+ = 0$, then to the eigenvalue λ_n^+ there corresponds the eigenfunction $\psi(x) = e^{-i\lambda_n^+ x}$. This function satisfies the boundary condition (2.2), so

$$\psi_- + i \int_0^l e^{-i\lambda_n^+ x} \overline{v(x)} dx = 0,$$

or, what is the same,

$$\psi_- + il\overline{v_n} = 0. \quad (2.16)$$

Since $\psi_- = \psi(l) - \psi(0) = -2$, from (2.16) we derive (2.13).

Conversely, let (2.13) be fulfilled. Then the function $\psi(x) = e^{-i\lambda_n^+ x}$ satisfies (2.2) and, since for this function $\psi_+ = 0$, it follows that ψ is an eigenfunction corresponding to the eigenvalue λ_n^+ .

Now, let λ_n^+ be a double eigenvalue for the operator A . Then $\psi(x) = e^{-i\lambda_n^+ x}$ is an eigenfunction for which $\psi_+ = 0$. We choose another eigenfunction φ , linearly independent to ψ , for which $\varphi_+ \neq 0$. Without loss of generality, we consider $\varphi_+ = 1$. We shall look for φ in the form of a sum of solutions φ_k of the differential equation

$$i \frac{d\varphi_k}{dx} - \lambda_n^+ \varphi_k = -v_k e^{-i\lambda_k^+ x}, \quad k \in \mathbb{Z}, \quad (2.17)$$

v_k being the Fourier coefficients in the expansion (2.11) of v . The solutions v_k can be chosen as

$$\varphi_k(x) = \frac{v_k}{\lambda_n^+ - \lambda_k^+} e^{-i\lambda_k^+ x} \quad \text{for } k \neq n,$$

and

$$\varphi_n(x) = -\frac{2}{l} x e^{-i\lambda_n^+ x} \quad \text{for } k = n.$$

The last function satisfies the equation (2.17) for $k = n$, due to the condition (2.13).

Thus, we can take

$$\varphi(x) = x e^{-i\lambda_n^+ x} + \frac{l}{2} \sum_{k \neq n} \frac{v_k}{\lambda_k^+ - \lambda_n^+} e^{-i\lambda_k^+ x}.$$

It is easily seen that the function φ is of the Sobolev class $W_2^1(0, l)$ and satisfies equation (2.15). Substituting φ into the boundary condition (2.2), we get

$$-l - l \sum_{k \neq n} \frac{v_k}{\lambda_k^+ - \lambda_n^+} + i \sum_{k \neq n} \frac{l \overline{v_k}}{i(\lambda_k^+ - \lambda_n^+)} + i \overline{v_n} \frac{l^2}{2} + \frac{il}{2} \sum_{k \neq n} \frac{|v_k|^2}{\lambda_k^+ - \lambda_n^+} l = 0,$$

which, taking into account the relationships (2.13) and (2.14), becomes an identity.

3) Let λ be an eigenvalue of the operator A such that $\lambda \neq \lambda_n$ ($n \in \mathbb{Z}$). Suppose, if possible, that λ is a double eigenvalue for A . Then there exist linearly independent eigenfunction ψ_j ($j = 1, 2$) corresponding to λ . Their linear combination

$$\psi = c_1 \psi_1 + c_2 \psi_2,$$

where c_1, c_2 are constants, is also an eigenfunction of A corresponding to λ . In particular, we can choose $c_1 = (\psi_2)_+$ and $c_2 = -(\psi_1)_+$. Then

$$\psi_+ = c_1 (\psi_1)_+ + c_2 (\psi_2)_+ = (\psi_2)_+ (\psi_1)_+ - (\psi_1)_+ (\psi_2)_+ = 0.$$

In this way, we find a non-zero function ψ , that is, an eigenfunction of A corresponding to λ and satisfying $\psi_+ = 0$. Then, as follows from (2.1), $\psi(x) = e^{-i\lambda x}$. Moreover, due to $\psi_+ = 0$, λ must be among λ_n^+ 's, a contradiction. \square

2.4. THE CHARACTERISTIC FUNCTION

In this subsection we introduce a function depending on the spectral parameter $\lambda \in \mathbb{C}$, which, according to its properties, can be called the characteristic function of the operator A . So, let A be the differential operator generated by the spectral problem (2.1)–(2.2) under the assumption that the potential v is an element of the space $L_2(0, l)$. Let λ be a complex number and denote $\psi(x, \lambda)$ the solution of (2.1) satisfying (2.2). In order to determine $\psi(x, \lambda)$ we shall proceed as in the previous subsection taking into account the Fourier expansion (2.11) of v with respect to the system of the eigenfunctions $e^{-i\lambda_n^+ x}$ ($n \in \mathbb{Z}$), correspondings to the eigenvalues λ_n^+ of the operator A_+ . We seek for a particular solution of the equation (2.1) in the form

$$\tilde{\psi}(x, \lambda) = \sum_n c_n e^{-i\lambda_n x},$$

where c_n are constants. Substituting this function into (2.1), we obtain, by comparison of coefficients, the following relations

$$(\lambda_n^+ - \lambda)c_n = -\psi_+ v_n, \quad n \in \mathbb{Z}. \tag{2.18}$$

Having the coefficients c_n determined from the obtained relations (2.18), the general solution $\psi(x, \lambda)$ of (2.1) can be written as follows

$$\psi(x, \lambda) = ce^{-i\lambda x} + \sum_n c_n e^{-i\lambda_n^+ x}, \tag{2.19}$$

c being an arbitrary constant.

Accordingly, the value of ψ_+ is equal to

$$\psi_+ = \frac{1}{2} (\psi(l, \lambda) + \psi(0, \lambda)) = \frac{1}{2} (ce^{-i\lambda l} + c) = \frac{1}{2} c (1 + e^{-i\lambda l}),$$

i.e.,

$$\psi_+ = \frac{1}{2} c (1 + e^{-i\lambda l}). \tag{2.20}$$

We derive from (2.18) that

$$c_n = \frac{\psi_+ v_n}{\lambda - \lambda_n^+} \quad \text{for } \lambda \neq \lambda_n^+, \tag{2.21}$$

and, since

$$\psi_+ = \frac{1}{2} ilc(\lambda - \lambda_n^+) + o(\lambda - \lambda_n^+),$$

we can take

$$c_n = \frac{1}{2} ilcv_n \quad \text{for } \lambda = \lambda_n^+. \tag{2.22}$$

Further, the function $\psi(x, \lambda)$ must satisfy the boundary condition (2.2). Substituting into (2.2) the solution $\psi(x, \lambda)$ given by (2.19) with the coefficients c_n determined by (2.21) and (2.22), we get

$$c(e^{-i\lambda l} - 1) - 2 \sum_n \frac{\psi_+ v_n}{\lambda - \lambda_n^+} + c(1 + e^{-i\lambda l}) \sum_n \frac{\bar{v}_n}{\lambda - \lambda_n^+} + il \sum_n \frac{\psi_+ |v_n|^2}{\lambda - \lambda_n^+} = 0 \tag{2.23}$$

for $\lambda \neq \lambda_n^+$, and

$$-2c - ilv_n c + il\bar{v}_n c - \frac{1}{2} cl^2 |v_n|^2 = 0 \tag{2.24}$$

for $\lambda = \lambda_n^+$. Taking into account (2.20) and that c is an arbitrary constant and multiplying (2.23), (2.24) (for the sake of symmetry) by $-\frac{1}{2} i e^{\frac{i\lambda l}{2}}$, we arrive at the equation

$$\chi(\lambda) = 0, \tag{2.25}$$

where χ is a function on λ defined by

$$\chi(\lambda) = -\sin \frac{\lambda l}{2} + \cos \frac{\lambda l}{2} \sum_n \frac{w_n}{\lambda - \lambda_n^+} \quad \text{for } \lambda \neq \lambda_n^+, \tag{2.26}$$

and

$$\chi(\lambda) = (-1)^n \left(1 + \frac{l}{2} w_n \right) \quad \text{for } \lambda = \lambda_n^+, \tag{2.27}$$

where w_n is equal to

$$w_n = iv_n - i\bar{v}_n + \frac{l}{2} |v_n|^2. \tag{2.28}$$

Remark 2.8. Along with the formula (2.27) of the characteristic function χ it follows the equality

$$\chi(\lambda_n^+) = (-1)^n \left| 1 + i \frac{l}{2} v_n \right|^2. \tag{2.29}$$

The function χ thus introduced can be called the characteristic function of the operator A or, also, of the spectral problem (2.1)–(2.2). We refer to the equation (2.25) as the characteristic equation of A or, respectively, of the problem (2.1)–(2.2). It is seen that χ is an entire function of λ .

The following theorem describes the relationship between the eigenvalues of the operator A and the characteristic function χ . It is analogous to Theorem 2.7, but the spectrum of A is described in terms of characteristic functions.

Theorem 2.9.

- 1) λ is an eigenvalue of the operator A if and only if λ is a zero of the characteristic function χ .
- 2) All zeros $\lambda \neq \lambda_n^+$ of the characteristic function are simple.
- 3) The characteristic function does not have zeros of multiplicities greater than 2.
- 4) λ is an eigenvalue of the operator A of multiplicity 2 if and only if it is a zero of multiplicity 2 of χ .

Proof. 1) From the construction of χ , presented above, it follows that if λ is an eigenvalue of the spectral problem (2.1)–(2.2), that is, the eigenvalue of A , then λ is also a zero of the characteristic function χ .

Conversely, if λ is a zero of the characteristic function χ , we construct the function $\psi(x, \lambda)$ like in (2.19), where c is an arbitrary constant and c_n ($n \in \mathbb{Z}$) are given by the formulas (2.21)–(2.22). With appropriate values of constants this function satisfies the boundary condition (2.2), because $\chi(\lambda) = 0$. It means that $\psi(x, \lambda)$ is the solution of the spectral problem (2.1)–(2.2), i.e., λ is the eigenvalue of the operator A .

2) From 1) all zeros of the characteristic function are eigenvalues of the operator A . Hence, using 3) from Theorem 2.7 we obtain the desired assertion.

3) The fact that the characteristic function χ does not have zeros of multiplicities greater than 2, based on 1), is equivalent to 1) from Theorem 2.7. It follows also immediately from vanishing of the second derivative of the characteristic function.

4) Let λ be an eigenvalue of the operator A of multiplicity 2. From 2) we have $\lambda = \lambda_n^+$. As results from Theorem 2.7, the number λ_n^+ is an eigenvalue of A of multiplicity 2 if and only if (2.13) and (2.14) hold. From (2.29) we know that $\chi(\lambda_n^+) = 0$ if and only if $v_n = \frac{2i}{l}$, which is equivalent to (2.13). Since the first derivative of χ at the point λ_n^+ has the following form

$$\frac{d\chi}{d\lambda} \Big|_{\lambda=\lambda_n^+} = (-1)^n \frac{l}{2} \sum_{k \neq n} \frac{w_k}{\lambda_n^+ - \lambda_k^+}, \tag{2.30}$$

we see that $\frac{d\chi}{d\lambda} \Big|_{\lambda=\lambda_n^+} = 0$ which is equivalent to (2.14). Moreover, from 3) we get λ is of multiplicity equal to 2.

On the other hand, if λ is a zero of multiplicity 2 of χ , then from 2) we have $\lambda = \lambda_n^+$, which ends the proof of 4). \square

Remark 2.10. $\lambda = 0$ is an eigenvalue of the operator A if and only if $\chi(0) = 0$, i.e.,

$$\sum_n \frac{w_n}{\lambda_n^+} = 0,$$

and the corresponding eigenfunction admits the extension

$$\psi(x, 0) = -\frac{i}{2} + \frac{i}{2} \sum_n \frac{v_n e^{-i\lambda_n^+ x}}{\lambda_n^+}.$$

Proposition 2.11. *There exists at least one eigenvalue of the operator A in every interval $I_n = [\lambda_n^+, \lambda_{n+1}^+]$, $n \in \mathbb{Z}$.*

Proof. If $\chi(\lambda_n^+) = 0$, then according to Theorem 2.9 λ_n^+ is an eigenvalue of A , and similarly for λ_{n+1}^+ . If $\chi(\lambda_n^+) \neq 0$ and $\chi(\lambda_{n+1}^+) \neq 0$, then according to (2.29) the function χ takes values of different signs in $\lambda = \lambda_n^+$ and $\lambda = \lambda_{n+1}^+$. Since the function χ is continuous, then it is equal to zero in some point $\mu \in I_n$. This value μ is an eigenvalue of the operator A . \square

2.5. THE DISTRIBUTION OF THE EIGENVALUES

Our next result concerns numeration and asymptotics of eigenvalues of the operator A as zeros of the characteristic function χ given by (2.26).

Proposition 2.12. *The sequence of eigenvalues of the operator A (counting multiplicities) can be numbered as*

$$\dots \leq \lambda_{-n} \leq \dots \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq \dots \tag{2.31}$$

listed in an increasing order satisfies the asymptotic distribution,

$$\lambda_n = \frac{2\pi}{l}n + \beta_n, \quad n \in \mathbb{Z}, \tag{2.32}$$

where β_n are real values such that $|\beta_n| \leq \frac{\pi}{l}$ and

$$\sum_n \beta_n^2 < \infty. \tag{2.33}$$

Proof. Assume that λ_n^+ ($n \in \mathbb{Z}$) are not eigenvalues of the operator A . Then from Theorem 2.7 all Fourier coefficients v_n of v are different from $\frac{2i}{l}$. All eigenvalues of the operator A are zeros of the characteristic function χ based on Theorem 2.9. Observe from the Rouché theorem that the entire function χ and $\sin \frac{\lambda l}{2}$ have the same number of zeros, counting multiplicities, equal to $m+n+1$, in the interval $I_{-n,m} = [\lambda_{-n}^+, \lambda_{m+1}^+]$, where m, n are large enough. We deduce from Proposition 2.11 that there exists at least one eigenvalue in each interval $\check{I}_n = (\lambda_n^+, \lambda_{n+1}^+)$. Hence there is only one eigenvalue

in each interval I_n , which we number as λ_n . Under given assumptions, we get sharp equalities in (2.31) and $|\beta_n| < \frac{\pi}{l}$ in (2.32).

Now, let λ_n^+ ($n \in \mathbb{Z}$) be eigenvalues of A . Then $v_n = \frac{2i}{l}$. This case is only for the finite number of eigenvalues, because v_n are Fourier coefficients of $v \in L_2(0, l)$. Let us change the potential v to \hat{v} in order to change the coefficients v_n to $\hat{v}_n \neq \frac{2i}{l}$ and all eigenvalues which were in intervals \check{I}_n stay in them. An eigenvalue, which was at the end of interval, goes into this interval where there was no eigenvalues of A . With such a change of the potential numbering of eigenvalues does not change. If λ_n^+ is a double eigenvalue, then one goes to the left and the other to the right intervals. We number eigenvalues of A based on the information about the position of perturbed eigenvalues. If a perturbed eigenvalue is in the interval \check{I}_n , then we number the initial eigenvalue as λ_n . We get numbering in an increasing order of eigenvalues with respect to their multiplicity (2.31) and $|\beta_n| \leq \frac{\pi}{l}$.

We shall get an estimate (2.33) for β_n using the fact that λ_n given by (2.32) are zeros of the characteristic function $\chi(\lambda)$ with $|\beta_n| \leq \frac{\pi}{l}$. Let w_n ($n \in \mathbb{Z}$), given by (2.28), be the Fourier coefficients of the function $w \in L_2(0, l)$, that is,

$$w(x) = \sum_n w_n e^{-i\lambda_n^+ x},$$

where

$$\sum_n |w_n|^2 < \infty. \tag{2.34}$$

Then the characteristic function χ , according to (2.26), has the following representation

$$\chi(\lambda) = -\sin \frac{\lambda l}{2} - \frac{i}{2} \int_0^l e^{i\lambda(x-\frac{l}{2})} w(x) dx. \tag{2.35}$$

Denote by p the function

$$p(\lambda) = -\frac{i}{2} \int_0^l e^{i\lambda(x-\frac{l}{2})} w(x) dx. \tag{2.36}$$

Because $\chi(\lambda_n) = 0$, using (2.35) and (2.36), we get the following equation

$$\sin \frac{\lambda_n l}{2} = p(\lambda_n). \tag{2.37}$$

Since p in (2.36) is the Fourier transform of the function $w \in L_2(0, l)$, then p is an analytic function of λ . Thus, the derivative of p exists and we can write

$$p(\lambda_n) = p(\lambda_n^+) + \int_{\lambda_n^+}^{\lambda_n} p'(\lambda) d\lambda \quad \text{for } \lambda_n \in [\lambda_n^+, \lambda_{n+1}^+]. \tag{2.38}$$

Now, from straightforward computation, we get

$$p(\lambda_n^+) = (-1)^n \frac{l}{2} w_n \tag{2.39}$$

and

$$p'(\lambda) = -\frac{i}{2} \int_0^l e^{i\lambda(x-\frac{l}{2})} i \left(x - \frac{l}{2}\right) w(x) dx. \tag{2.40}$$

The function p' given by (2.40) is the Fourier transform of the function $(x - \frac{l}{2})w(x) \in L_2(0, l)$. Hence $p' \in L_2(\mathbb{R})$, so

$$\int_{-\infty}^{\infty} |p'(\lambda)|^2 d\lambda < \infty. \tag{2.41}$$

Let

$$p_n := \left(\int_{\lambda_n^+}^{\lambda_{n+1}^+} |p'(\lambda)|^2 d\lambda \right)^{\frac{1}{2}},$$

then, from (2.41), we have

$$\sum_n p_n^2 < \infty. \tag{2.42}$$

We get the following estimation

$$\left| \int_{\lambda_n^+}^{\lambda_n} p'(\lambda) d\lambda \right| \leq \int_{\lambda_n^+}^{\lambda_{n+1}^+} |p'(\lambda)| d\lambda \leq \left(\frac{2\pi}{l}\right)^{\frac{1}{2}} p_n. \tag{2.43}$$

From (2.38), (2.39) and (2.43) we have

$$|p(\lambda_n)| \leq \frac{l}{2} |w_n| + \left(\frac{2\pi}{l}\right)^{\frac{1}{2}} p_n.$$

Therefore, based on (2.34) and (2.42), we obtain

$$\sum_n |p(\lambda_n)|^2 < \infty. \tag{2.44}$$

Hence

$$p(\lambda_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.45}$$

For λ_n given by (2.32), from the equation (2.37) we have

$$(-1)^n \sin \frac{\beta_n l}{2} = p(\lambda_n).$$

We know that $|\beta_n| \leq \frac{\pi}{l}$. Then $\left| \frac{\beta_n l}{2} \right| \leq \frac{\pi}{2}$, and according to (2.45)

$$\sin \frac{\beta_n l}{2} = \frac{\beta_n l}{2} (1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

Further,

$$|\beta_n| \leq \frac{2}{l} |p(\lambda_n)| (1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

Then from (2.44) we have

$$\sum_n |\beta_n|^2 < \infty.$$

□

Remark 2.13. From the proof of Proposition 2.12, the asymptotic for eigenvalues $\lambda_n = \frac{2\pi n}{l} + \beta_n$ of the operator A , as $n \rightarrow \infty$, can be noticed in the following form

$$\beta_n = (-1)^n \frac{2}{l} p \left(\frac{2\pi n}{l} \right) (1 + o(1)) = \left(\frac{1}{\pi} \sum_k \frac{w_k}{n - k + \frac{1}{2}} \right) (1 + o(1)). \quad (2.46)$$

The coefficients w_k are expressed by the Fourier coefficients of the nonlocal potential by the formula (2.28).

For the proof instead of (2.38) we can take

$$p \left(\frac{2\pi n}{l} + \beta_n \right) = p \left(\frac{2\pi n}{l} \right) + \beta_n p' \left(\frac{2\pi n}{l} \right) + \int_{\frac{2\pi n}{l}}^{\frac{2\pi n}{l} + \beta_n} \left(\frac{2\pi n}{l} + \beta_n - \lambda \right) p''(\lambda) d\lambda. \quad (2.47)$$

Using estimates

$$p' \left(\frac{2\pi n}{l} \right) = o(1)$$

and

$$\int_{\lambda_n^+}^{\lambda_{n+1}^+} |p''(\lambda)| d\lambda = o(1)$$

we obtain (2.46).

We further use the numbering of eigenvalues of the operator A from Proposition 2.12.

We can express the entire function χ as an infinite product depending on its zeros λ_n (see [19]), as follows:

$$\chi(z) = C(z - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - z)(z - \lambda_{-n})}{\lambda_n \cdot |\lambda_{-n}|}. \quad (2.48)$$

Now using [13, Lemma 3.4.2] we get the following result.

Proposition 2.14. *The characteristic function χ given by (2.48) can be expressed by the product*

$$\chi(\lambda) = -\frac{l}{2}(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{2n\pi}{l}\right)^2}. \tag{2.49}$$

Proof. Using (2.48) we get the characteristic function in the following form

$$\chi(\lambda) = -\hat{C} \frac{l}{2}(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{2n\pi}{l}\right)^2}, \tag{2.50}$$

where

$$\hat{C} = -C \frac{2}{l} \prod_{n=1}^{\infty} \frac{\left(\frac{2n\pi}{l}\right)^2}{\lambda_n \cdot |\lambda_{-n}|}.$$

The above infinite product is convergent, because λ_n ($n \in \mathbb{Z}$) are given by (2.32) and (2.33). From the explicit formula (2.26) of the characteristic function it is seen that for $\lambda = iy$ we have

$$\lim_{y \rightarrow \infty} \frac{\chi(iy)}{-\sin \frac{iy l}{2}} = 1. \tag{2.51}$$

Now, using the formulas

$$\sin \frac{\lambda l}{2} = \frac{\lambda l}{2} \prod_{n=1}^{\infty} \frac{\frac{4n^2\pi^2}{l^2} - \lambda^2}{\frac{4n^2\pi^2}{l^2}}.$$

and (2.50) we obtain

$$\frac{\chi(\lambda)}{-\sin \frac{\lambda l}{2}} = \hat{C} \frac{\lambda - \lambda_0}{\lambda} \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\frac{4n^2\pi^2}{l^2} - \lambda^2}. \tag{2.52}$$

Putting $\lambda = iy$ and passing with y to the infinity in (2.52) we get

$$1 = \lim_{y \rightarrow \infty} \frac{\chi(iy)}{-\sin \frac{iy l}{2}} = \hat{C} \lim_{y \rightarrow \infty} \prod_{n=1}^{\infty} \frac{(\lambda_n - iy)(iy - \lambda_{-n})}{\frac{4n^2\pi^2}{l^2} - i^2 y^2} = \hat{C}.$$

Since $\hat{C} = 1$ and (2.50), we conclude (2.49). □

3. INVERSE EIGENVALUE PROBLEM

This section is devoted to the inverse problem for the first order differential operators with nonlocal potentials and nonlocal boundary conditions. At the beginning, algorithm for solving the inverse problem will be given. Next, appropriate example will be shown.

3.1. ALGORITHM

Inverse spectral problems involve recovering an operator from its spectrum. In general, this operator is not unique. There is a set of isospectral operators, and therefore exists a set of isospectral potentials. The proposed algorithm makes it possible to effectively determine all isospectral potentials.

Let us assume that we know all eigenvalues λ_n ($n \in \mathbb{Z}$) of the operator A . We find the nonlocal potential $v \in L_2(0, l)$.

Step 1. We construct the characteristic function χ via (2.49)

$$\chi(\lambda) = -\frac{l}{2}(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{2n\pi}{l}\right)^2}.$$

Step 2. We calculate the values $\chi(\lambda_n^+)$ for all $n \in \mathbb{Z}$, where $\lambda_n^+ = (2n - 1)\frac{\pi}{l}$.

Step 3. We solve the quadratic equation of (2.29) for v_n :

$$\chi(\lambda_n^+) = (-1)^n \left| 1 + i\frac{l}{2}v_n \right|^2.$$

This equation can have several different solutions leading to different isospectral potentials.

Step 4. We write the potential $v(x) = \sum_n v_n e^{-i\lambda_n^+ x}$.

Now, we consider an example for solving the inverse problem.

Example 3.1. Let $\lambda_1 = \frac{1}{2}$ and $\lambda_n = n$ for $n \neq 1$ be the eigenvalues of the operator (2.3) and let $l = 2\pi$. Under this assumptions we find the potential v in the problem (2.1)–(2.2). The characteristic function χ , in this case, is the following

$$\begin{aligned} \chi(\lambda) &= -\frac{l}{2}(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{2n\pi}{l}\right)^2} \\ &= -\pi(\lambda - 0) \prod_{n=2}^{\infty} \frac{(n - \lambda)(\lambda - (-n))}{n^2} \cdot \frac{(\frac{1}{2} - \lambda)(\lambda - (-1))}{1^2} \\ &= -\pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda^2}{n^2} \cdot \frac{1}{(1 - \lambda)(\lambda + 1)} \cdot \frac{(\frac{1}{2} - \lambda)(\lambda + 1)}{1} \\ &= -\pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda^2}{n^2} \cdot \frac{\frac{1}{2} - \lambda}{1 - \lambda} \\ &= -\pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda^2}{n^2} \cdot \frac{\lambda - \frac{1}{2}}{\lambda - 1}. \end{aligned}$$

It can be checked that

$$\sin(\pi\lambda) = \pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda^2}{n^2},$$

so

$$\chi(\lambda) = -\frac{\lambda - \frac{1}{2}}{\lambda - 1} \sin(\pi\lambda).$$

For $\lambda_n^+ = n - \frac{1}{2}$, we calculate the values $\chi(\lambda_n^+)$ as follows

$$\chi(\lambda_n^+) = \chi\left(n - \frac{1}{2}\right) = -\frac{n - \frac{1}{2} - \frac{1}{2}}{n - \frac{1}{2} - 1} \sin\left(\left(n - \frac{1}{2}\right)\pi\right) = (-1)^n \frac{n - 1}{n - \frac{3}{2}}.$$

We solve the quadratic equation

$$(-1)^n |1 + i\pi v_n|^2 = (-1)^n \frac{n - 1}{n - \frac{3}{2}},$$

which is equivalent to

$$|1 + i\pi v_n|^2 = \frac{n - 1}{n - \frac{3}{2}},$$

from which we compute the Fourier coefficients of the potential v :

$$v_n = -\frac{i}{2\pi} \left(|n - 3/2| + \sqrt{(n - 1)(n - 3/2)} \right)^{-1}.$$

It is worth mentioning that the asymptotic behavior of Fourier coefficients v_n being $v_n \sim -\frac{i}{2n\pi}$, which results immediately from the formula obtained above for v_n . Finally, we obtain the potential $v(x) = \sum_n v_n e^{-i\lambda_n^+ x}$.

4. DIRAC SYSTEMS

This section is devoted to the spectral theory of the one-dimensional Dirac operator. We study the direct and inverse problem for the Dirac system with nonlocal potential. The above considerations will be carried out in the case that the interval is finite. We describe the method of reducing the spectral problem for the Dirac system to the one-dimensional case from Section 2.

4.1. GENERAL CASE

In the literature Dirac operators with local potential are considered. For more details about the Dirac systems we refer to the book [12]. Now we present the Dirac systems in the usual case.

Consider the following eigenvalue problem for the Dirac system

$$B \frac{d\psi(x)}{dx} + V(x)\psi(x) = \lambda\psi(x), \quad 0 \leq x \leq b, \quad (4.1)$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix},$$

and (the potentials) p, q are given functions from the class $L_2(0, b)$ ($0 < b < \infty$) and (the spectral parameter) λ is a fixed complex number. We consider the equation (4.1) with functions ψ_1, ψ_2 which are defined on the Sobolev space $W_2^1(0, b)$. The above equation is equivalent to the system of two first order ordinary differential equations

$$\begin{cases} \frac{d\psi_2(x)}{dx} + p(x)\psi_1(x) + q(x)\psi_2(x) = \lambda\psi_1(x), \\ -\frac{d\psi_1(x)}{dx} + q(x)\psi_1(x) - p(x)\psi_2(x) = \lambda\psi_1(x). \end{cases}$$

Let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}.$$

Then

$$UBU^{-1} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

We obtain an equivalent system

$$\hat{B} \frac{d\hat{\psi}(x)}{dx} + \hat{V}(x)\hat{\psi}(x) = \lambda\hat{\psi}(x), \quad 0 \leq x \leq b, \tag{4.2}$$

where

$$\hat{B} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \hat{V}(x) = \begin{pmatrix} 0 & \hat{v}(x) \\ \hat{v}(x) & 0 \end{pmatrix}, \quad \hat{\psi}(x) = \begin{pmatrix} \hat{\psi}_1(x) \\ \hat{\psi}_2(x) \end{pmatrix}$$

and

$$\hat{v}(x) = -(q(x) + ip(x)), \quad \hat{\psi}_1(x) = \psi_1(x) - i\psi_2(x), \quad \hat{\psi}_2(x) = -\psi_2(x) + i\psi_1(x).$$

We rewrite (4.2) as the following system of the first order ordinary differential equations

$$\begin{cases} i \frac{d\hat{\psi}_1(x)}{dx} + \hat{v}(x)\hat{\psi}_2(x) = \lambda\hat{\psi}_1(x), \\ -i \frac{d\hat{\psi}_2(x)}{dx} + \overline{\hat{v}(x)}\hat{\psi}_1(x) = \lambda\hat{\psi}_2(x). \end{cases}$$

In order for the corresponding operator to be self-adjoint we impose the following boundary conditions:

$$\begin{aligned} \hat{\psi}_1(0) &= \hat{\psi}_2(0), \\ \hat{\psi}_1(b) &= \hat{\psi}_2(b). \end{aligned}$$

4.2. REDUCING PROCEDURE. SELF-ADJOINTNESS

Now we present a method how to reduce the spectral problem for the Dirac system to the one-dimensional case.

Consider the spectral problem

$$\begin{cases} i \frac{d\psi_1(x)}{dx} + v_1(x)\psi^+ = \lambda\psi_1(x), \\ -i \frac{d\psi_2(x)}{dx} + v_2(x)\psi^+ = \lambda\psi_2(x), \end{cases} \quad 0 \leq x \leq b, \quad (0 < b < \infty), \tag{4.3}$$

where

$$\psi_1, \psi_2 \in W_2^1(0, b), \quad v_1, v_2 \in L_2(0, b),$$

and

$$\psi^+ := \frac{1}{2} (\psi_1(b) + \psi_2(b))$$

with the boundary condition

$$\psi_1(0) = \psi_2(0), \tag{4.4}$$

and the nonlocal boundary condition

$$\psi_1(b) - \psi_2(b) + i(\langle \psi_1, v_1 \rangle + \langle \psi_2, v_2 \rangle) = 0. \tag{4.5}$$

The corresponding operator \mathcal{A} is defined by

$$(\mathcal{A}\Psi)(x) = B \frac{d\Psi(x)}{dx} + V(x)\Psi^+, \quad 0 \leq x \leq b, \tag{4.6}$$

where

$$B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V(x) = \begin{pmatrix} 0 & v_1(x) \\ v_2(x) & 0 \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \quad \Psi^+ = \begin{pmatrix} \psi^+ \\ \psi^+ \end{pmatrix},$$

on the domain $\mathcal{D}(\mathcal{A})$ contains the functions $\psi_1, \psi_2 \in W_2^1(0, b)$ satisfying conditions (4.4)–(4.5).

The system (4.3) is decoupled and the equations interact with each other only through the boundary conditions (4.4)–(4.5) at points $x = 0$ and $x = b$. Therefore, we consider the problem (4.3)–(4.5) on the interval $[0, 2b]$ illustrated in Figure 1.

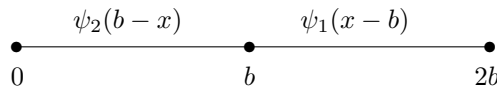


Fig. 1. Reduction procedure on the interval $[0, 2b]$.

We define the functions

$$\psi(x) = \begin{cases} \psi_1(x - b), & b \leq x \leq 2b, \\ \psi_2(b - x), & 0 \leq x \leq b, \end{cases} \tag{4.7}$$

and

$$v(x) = \begin{cases} v_1(x - b), & b \leq x \leq 2b, \\ v_2(b - x), & 0 \leq x \leq b. \end{cases} \tag{4.8}$$

Then

$$\psi^+ = \frac{1}{2} (\psi(2b) + \psi(0)),$$

which is equal to $\psi_+ = \frac{1}{2}(\psi(l) + \psi(0))$ for $l = 2b$. In fact, we write the Dirac system with nonlocal potentials (4.3)–(4.5) as the eigenvalue problem for the first order differential operator, substituting $l = 2b$, as follows

$$i \frac{d\psi(x)}{dx} + v(x)\psi_+ = \lambda\psi(x), \quad 0 \leq x \leq l \tag{4.9}$$

with the nonlocal boundary condition

$$\psi(l) - \psi(0) + i\langle \psi, v \rangle = 0. \tag{4.10}$$

We considered such a problem in Section 2. The operator \mathcal{A} given by (4.6) has the same form as the operator A in (2.3)–(2.4).

Observe that the spectral problem (4.3)–(4.5) is equivalent to the problem (2.1)–(2.2). This implies that the operator \mathcal{A} is self-adjoint.

4.3. SPECTRAL PROPERTIES

The aim of this subsection is describe the spectrum and the characteristic function of the operator \mathcal{A} . We use the fact that the problem (4.3)–(4.5) is equivalent to the problem (2.1)–(2.2). For $\psi_+ = 0$, the system of equations (4.3) has the form

$$\begin{cases} i \frac{d\psi_1(x)}{dx} = \lambda\psi_1(x), \\ -i \frac{d\psi_2(x)}{dx} = \lambda\psi_2(x), \end{cases} \quad 0 \leq x \leq b, \quad (0 < b < \infty),$$

and the corresponding operator \mathcal{A}_+ is defined by

$$(\mathcal{A}_+ \Psi)(x) = B \frac{d\Psi(x)}{dx},$$

where

$$B = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

on the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in W_2^1(0, b) \oplus W_2^1(0, b) : \begin{matrix} \psi_1(0) = \psi_2(0), \\ \psi_1(b) + \psi_2(b) = 0 \end{matrix} \right\}.$$

The numbers $\lambda_n^+ = (n - \frac{1}{2})\frac{\pi}{b}$ are eigenvalues of the operator \mathcal{A}_+ and corresponding eigenfunctions are given by $\psi_1(x) = e^{-i\lambda_n^+ x}$ and $\psi_2(x) = e^{i\lambda_n^+ x}$. Each of these families forms a complete orthogonal system in $L_2(0, b)$. Hence, the nonlocal potentials v_1 and v_2 can be represented by the Fourier series

$$v_j(x) = \sum_n v_n^{(j)} \psi_j(x), \quad 0 \leq x \leq b, \quad j = 1, 2,$$

and, respectively,

$$v_n^{(j)} = \frac{1}{b} \int_0^b v_j(x) \overline{\psi_j(x)} dx, \quad j = 1, 2.$$

In what follows, using the reducing procedure (4.7)–(4.8), we transfer the results obtained for the operator A to the case of the operator \mathcal{A} . Doing this, we substitute $l = 2b$.

We start by formulating a counterpart of Theorem 2.7 for the Dirac system.

Theorem 4.1.

- 1) The eigenvalues of the operator \mathcal{A} are of multiplicity at most equal to 2.
- 2) An eigenvalue $\lambda_n^+ = (n - \frac{1}{2})\frac{\pi}{b}$ of the operator \mathcal{A}_+ is also an eigenvalue of the operator \mathcal{A} if and only if

$$v_n^{(1)} + v_n^{(2)} = \frac{2}{b}(-1)^{(n+1)}. \tag{4.11}$$

In this case λ_n^+ is of multiplicity 2 provided that

$$\begin{aligned} & \sum_{k \neq n} \frac{1}{\lambda_k^+ - \lambda_n^+} \left((-1)^{k+1} (v_k^{(1)} + v_k^{(2)}) + (-1)^{k+1} (\bar{v}_k^{(1)} + \bar{v}_k^{(2)}) - \frac{b}{2} |v_k^{(1)} + v_k^{(2)}|^2 \right) \\ & = 0, \end{aligned} \tag{4.12}$$

- 3) All eigenvalues of the operator \mathcal{A} different from λ_n^+ are simple.

Proof. 1) and 3) follow immediately from Theorem 2.7 and (4.7)–(4.8).

2) Using (2.13) and (4.8) we get

$$\begin{aligned} v_n &= \frac{1}{2b} \int_0^{2b} v(x) e^{i\lambda_n^+ x} dx \\ &= \frac{1}{2b} \int_0^b v_2(b-x) e^{i\lambda_n^+ x} dx + \frac{1}{2b} \int_b^{2b} v_1(x-b) e^{i\lambda_n^+ x} dx \\ &= \frac{1}{2b} \int_0^b v_2(t) e^{i\lambda_n^+(l-t)} dt + \frac{1}{2b} \int_0^b v_1(t) e^{i\lambda_n^+(t+l)} dt \\ &= \frac{1}{2} (v_n^{(1)} + v_n^{(2)}) e^{i\lambda_n^+ l} = \frac{i}{2} (-1)^{n+1} (v_n^{(1)} + v_n^{(2)}) = \frac{i}{b}, \end{aligned}$$

which yields (4.11).

Substituting $v_k = \frac{i}{2}(-1)^{k+1}(v_k^{(1)} + v_k^{(2)})$ into (2.14), we obtain

$$\begin{aligned} & \sum_{k \neq n} \frac{1}{\lambda_k^+ - \lambda_n^+} \left(\frac{i}{2} (-1)^{k+1} (v_k^{(1)} + v_k^{(2)}) - \overline{\frac{i}{2} (-1)^{k+1} (v_k^{(1)} + v_k^{(2)})} \right. \\ & \left. - ib \left| \frac{i}{2} (-1)^{n+1} \right|^2 |v_k^{(1)} + v_k^{(2)}|^2 \right) = 0, \end{aligned}$$

which implies

$$\sum_{k \neq n} \frac{1}{\lambda_k^+ - \lambda_n^+} \left(\frac{i}{2} (-1)^{k+1} (v_k^{(1)} + v_k^{(2)}) + \frac{i}{2} (-1)^{k+1} (\bar{v}_k^{(1)} + \bar{v}_k^{(2)}) - \frac{ib}{4} |v_k^{(1)} + v_k^{(2)}|^2 \right) = 0.$$

Dividing this equation by $\frac{i}{2}$ we get (4.12). □

The characteristic function of the operator \mathcal{A} corresponding to the problem (4.3)–(4.5) has the form

$$\chi(\lambda) = -\sin(\lambda b) + \cos(\lambda b) \sum_n \frac{w_n}{\lambda - \lambda_n^+} \quad \text{for } \lambda \neq \lambda_n^+ \tag{4.13}$$

and

$$\chi(\lambda) = (-1)^n \left| \frac{b}{2} (-1)^{n+1} (v_n^{(1)} + v_n^{(2)}) - 1 \right|^2 \quad \text{for } \lambda = \lambda_n^+, \tag{4.14}$$

where

$$w_n = \frac{1}{2} (-1)^n (v_n^{(1)} + v_n^{(2)}) + \frac{1}{2} (-1)^n (\bar{v}_n^{(1)} + \bar{v}_n^{(2)}) + \frac{b}{4} |v_n^{(1)} + v_n^{(2)}|^2. \tag{4.15}$$

Theorem 4.2.

- 1) λ is an eigenvalue of the operator \mathcal{A} if and only if λ is a zero of the characteristic function χ .
- 2) λ is an eigenvalue of the operator \mathcal{A} of multiplicity 2 if and only if it is a zero of multiplicity 2 of χ .
- 3) All zeros $\lambda \neq \lambda_n^+$ of the characteristic function are simple.
- 4) The characteristic function does not have zeros of multiplicities greater than 2.

The proof of Theorem 4.2 is analogous to that of Theorem 2.9.

Next, repeating the same steps as in the proof of Proposition 2.14, we obtain the following result.

Proposition 4.3. *The characteristic function χ given by (4.13) can be expressed by the product*

$$\chi(\lambda) = -b(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{n\pi}{b}\right)^2}. \tag{4.16}$$

Furthermore, we can also easily obtain the ensuing counterpart of Proposition 2.14.

Proposition 4.4. *The sequence of eigenvalues of the operator \mathcal{A} (counting multiplicities) can be numbered as*

$$\dots \leq \lambda_{-n} \leq \dots \leq \lambda_{-1} \leq \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n \leq \dots$$

listed in an increasing order satisfies the asymptotic distribution,

$$\lambda_n = \frac{\pi}{b} n + \beta_n, \quad n \in \mathbb{Z},$$

where β_n are real values such that $|\beta_n| \leq \frac{\pi}{2b}$ and

$$\sum_n \beta_n^2 < \infty.$$

Now we find a solution of the direct problem for an exemplary Dirac system.

Example 4.5. Consider the problem (4.3)–(4.5) with nonlocal potentials $v_1(x) = -\sin \frac{x}{2}$ and $v_2(x) = \sin \frac{x}{2}$. Let $b = \pi$ and assume that $\psi_{\pm} = 1$. We calculate the Fourier coefficients of the potentials v_1 and v_2 as follows:

$$v_0^{(1)} = v_0^{(2)} = \frac{i}{2}, \quad v_1^{(1)} = v_1^{(2)} = -\frac{i}{2}, \quad \text{and} \quad v_n^{(1)} = v_n^{(2)} = 0, \quad n \in \mathbb{Z} \setminus \{0, 1\}.$$

From (4.13)–(4.15) we infer that

$$\chi(\lambda) = -\sin \lambda\pi + \frac{2\pi\lambda}{4\lambda^2 - 1} \cos \lambda\pi.$$

The numbers satisfying the equation

$$\tan \lambda\pi = \frac{2\pi\lambda}{4\lambda^2 - 1}$$

are the eigenvalues of the operator generated by the considered problem.

4.4. INVERSE PROBLEM

We are in a position to give the algorithm for solving the inverse eigenvalue problem (4.3)–(4.5). Let us assume that all eigenvalues of the operator generated by this problem are known. Then, we find the nonlocal potentials $v_1, v_2 \in L_2(0, l)$ proceeding as follows.

Step 1. We construct the characteristic function χ via (4.16)

$$\chi(\lambda) = -b(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{n\pi}{b}\right)^2}.$$

Step 2. We calculate the values $\chi(\lambda_n^+)$ for all $n \in \mathbb{Z}$, where $\lambda_n^+ = (n - \frac{1}{2})\frac{\pi}{b}$.

Step 3. We solve the quadratic equation for v_n :

$$\chi(\lambda_n^+) = (-1)^n |1 + ibv_n|^2.$$

Step 4. Using potential v , we find the potentials v_1, v_2 by the reduction procedure (4.8).

$$v(x) = \sum_n v_n e^{-i(n-\frac{1}{2})\frac{\pi}{b}x}, \quad 0 \leq x \leq 2b,$$

$$v_1(x) = v(x+b), \quad v_2(x) = v(b-x), \quad 0 \leq x \leq b.$$

Example 4.6. Let $\lambda_1 = \frac{1}{2}$ and $\lambda_n = n$ for $n \neq 1$ be the eigenvalues of the operator generated by the problem (4.3)–(4.5) and let $b = \pi$. The characteristic function χ , this case, is the following

$$\begin{aligned} \chi(\lambda) &= -b(\lambda - \lambda_0) \prod_{n=1}^{\infty} \frac{(\lambda_n - \lambda)(\lambda - \lambda_{-n})}{\left(\frac{n\pi}{b}\right)^2} \\ &= -\pi(\lambda - 0) \prod_{n=2}^{\infty} \frac{(n - \lambda)(\lambda - (-n))}{n^2} \cdot \frac{(\frac{1}{2} - \lambda)(\lambda - (-1))}{1^2} \\ &= -\pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda^2}{n^2} \cdot \frac{1}{(1 - \lambda)(\lambda + 1)} \cdot \frac{(\frac{1}{2} - \lambda)(\lambda + 1)}{1} \\ &= -\pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda^2}{n^2} \cdot \frac{\frac{1}{2} - \lambda}{1 - \lambda} \\ &= -\pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda^2}{n^2} \cdot \frac{\lambda - \frac{1}{2}}{\lambda - 1}. \end{aligned}$$

Since

$$\sin(\pi\lambda) = \pi\lambda \prod_{n=1}^{\infty} \frac{n^2 - \lambda^2}{n^2},$$

one has

$$\chi(\lambda) = -\sin(\pi\lambda) \frac{\lambda - \frac{1}{2}}{\lambda - 1}.$$

For $\lambda_n^+ = n - \frac{1}{2}$, we calculate the values $\chi(\lambda_n^+)$ as follows:

$$\chi(\lambda_n^+) = \chi\left(n - \frac{1}{2}\right) = -\sin\left(\left(n - \frac{1}{2}\right)\pi\right) \cdot \frac{n - \frac{1}{2} - \frac{1}{2}}{n - \frac{1}{2} - 1} = (-1)^n \frac{n - 1}{n - \frac{3}{2}}.$$

Solving the quadratic equation

$$(-1)^n |1 + i\pi v_n|^2 = (-1)^n \frac{n - 1}{n - \frac{3}{2}},$$

we get

$$v_n = -\frac{i}{2\pi} \left(|n - 3/2| + \sqrt{(n - 1)(n - 3/2)} \right)^{-1}.$$

Therefore,

$$v_n^{(j)} = \frac{(-1)^n}{2\pi} \left(|n - 3/2| + \sqrt{(n - 1)(n - 3/2)} \right)^{-1}, \quad j = 1, 2.$$

It remains to apply the above Step 4 to obtain explicit formulae for the potentials v_1 and v_2 .

$$v_j(x) = \sum_n v_n^{(j)} \psi_j(x), \quad j = 1, 2, \quad 0 \leq x \leq b.$$

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