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POSITIVE SOLUTIONS FOR THE ONE-DIMENSIONAL p-LAPLACIAN WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We prove the existence of positive solutions for the p-Laplacian problem

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t)f(u), \quad t \in (0,1), \\ au(0) - H_1(u'(0)) = 0, \\ cu(1) + H_2(u'(1)) = 0, \end{cases}$$

where $\phi(s) = |s|^{p-2}s$, $p > 1$, $H_i : \mathbb{R} \to \mathbb{R}$ can be nonlinear, $i = 1,2$, $f : (0,\infty) \to \mathbb{R}$ is p-superlinear or p-sublinear at ∞ and is allowed be singular ($\pm\infty$) at 0, and λ is a positive parameter.

Keywords: p-Laplacian, semipositone, nonlinear boundary conditions, positive solutions.

Mathematics Subject Classification: 34B16, 34B18.

1. INTRODUCTION

Consider the one-dimensional p-Laplacian problem

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t)f(u), \quad t \in (0,1), \\ a_1u(0) - H_1(u'(0)) = 0, \\ a_2u(1) + H_2(u'(1)) = 0, \end{cases} \tag{1.1}$$

where $\phi(s) = |s|^{p-2}s, p > 1$, a_1, a_2 are nonnegative constants with $a_1+a_2 > 0$, and λ is a positive parameter. We shall adopt the following assumptions.

(A1) $H_i : \mathbb{R} \to \mathbb{R}$ are odd, nondecreasing functions with $a_i + |H_i| \not\equiv 0$, $i = 1,2$. Furthermore, if $a_i = 0$ then H_i is strictly increasing, $i \in \{1,2\}$.
(A2) $r : [0,1] \to (0,\infty)$ is continuous.

(A3) $f : (0,\infty) \to \mathbb{R}$ is continuous and there exists a constant $\delta \in [0,1)$ such that

$$\limsup_{z\to 0^+} z^{\delta}|f(z)| < \infty.$$

(A4) $g : (0,1) \to (0,\infty)$ is continuous and $\omega^{-\delta}(t)g(t) \in L^1(0,1)$, where $\omega(t) = \min(t, 1-t)$.

(A5) There exist $i \in \{1,2\}$ and a constant $a > 0$ such that $a_i > 0$ and $H_i(z) \le az$ for $z \ge 0$.

By a solution of (1.1), we mean a function $u \in C^1[0,1]$ with $\phi(u')$ absolutely continuous on $[0,1]$, and satisfying (1.1).

Set $f_0 = \lim\limits_{z\to 0^+} \frac{f(z)}{z^{p-1}}$, $f_\infty = \lim\limits_{z\to\infty} \frac{f(z)}{z^{p-1}}$.

Our main result is the following theorem.

Theorem 1.1.

(i) *Let* (A1)–(A4) *hold and suppose* $f_\infty = \infty$. *Then there exists a constant* $\lambda_0 > 0$ *such that for* $\lambda < \lambda_0$, (1.1) *has a positive solution* u_λ *with* $u_\lambda \to \infty$ *as* $\lambda \to 0^+$ *uniformly on compact subsets of* $(0,1)$.

(ii) *Let* (A1)–(A5) *hold. Suppose* $f_\infty = 0$ *and* $\lim_{z\to\infty} f(z) = \infty$. *Then there exists a constant* $\tilde{\lambda}_0 > 0$ *such that for* $\lambda > \tilde{\lambda}_0$, (1.1) *has a positive solution* u_λ *with* $u_\lambda \to \infty$ *as* $\lambda \to \infty$ *uniformly on compact subsets of* $(0,1)$.

(iii) *Let* (A1)–(A5) *hold. Suppose* $f \ge 0$, $f_\infty = 0$, *and* $f_0 = \infty$. *Then* (1.1) *has a positive solution for all* $\lambda > 0$.

In particular, our results when applied to the model example

$$\begin{cases} -(e^t\phi(u'))' = \dfrac{\lambda}{t^\beta}\left(\dfrac{C}{u^\delta} + u^q\right), \quad t \in (0,1), \\ a_1u(0) - (u'(0))^m = 0, \\ a_2u(1) + (u'(1))^n = 0, \end{cases}$$

where m, n are positive odd integers, $C, \beta, \delta \in \mathbb{R}$ with $\beta + \delta < 1$, gives the existence of a large positive solution when $\lambda > 0$ is small, $C < 0$ and $q > p-1$ (Theorem 1.1 (i)), or when λ is large, $C < 0$, and $0 < q < p-1$ (Theorem 1.1 (ii)), and a positive solution for all $\lambda > 0$ when $C > 0, \delta > 1-p$, and $0 < q < p-1$ (Theorem 1.1 (iii)).

Since our results hold (with obvious modifications) if $(0,1)$ is replaced by (r_1, r_2) where $0 < r_1 < r_2$, it can be applied to the study of positive radial solutions of the p-Laplacian on an annulus with nonlinear boundary conditions:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda g(|x|)f(u), \ r_1 < |x| < r_2, \\ a_iu + H_i\left(\frac{\partial u}{\partial n}\right) = 0, \ |x| = r_i, \ i \in \{1,2\}, \end{cases}$$

where n denotes the outer unit normal vector on $\Omega = \{x : r_1 < |x| < r_2\}$, which has been studied extensively over the years (see [11]).

Our results are motivated by the work in [17], in which the existence of a positive solutions to the equation

$$-(\phi(u'))' = g(t)f(u), \quad t \in (0,1),$$

i.e. (1.1) with $r \equiv 1, \lambda = 1$, with one of the following nonlinear boundary conditions

$$\begin{aligned} u(0) - H_1(u'(0)) &= 0, & u(1) + H_1(u'(1)) &= 0,\\ u(0) - H_1(u'(0)) &= 0, & u'(1) &= 0,\\ u'(0) &= 0, & u(1) + H_1(u'(1)) &= 0, \end{aligned}$$

was established when f is nonsingular, nonnegative and satisfies either $f_0 = \infty$ and $f_\infty = 0$, or $f_0 = 0$ and $f_\infty = \infty$.

Note that our nonlinearity f is allowed to be singular $(\pm\infty)$ at $u = 0$, and seeking positive solutions in the singular semipositone case i.e. $\lim_{u\to 0^+} f(u) = -\infty$ is particularly challenging due to the absence of the maximum principle (see [13]). For the literature on the equation in (1.1) with linear boundary conditions, we refer the reader to [1,2,6,7,9,10,14,18,19] for the singular/nonsingular semipositone case, and to [8,12,16] for the nonpositone case. Related results in the PDE case can be found in [3,5,15].

2. PRELIMINARY RESULTS

We shall denote the norm in $L^p(0,1)$ by $\|\cdot\|_p$.

We first recall the following fixed point of Krasnoselskii's type.

Theorem 2.1 ([4, Theorem 12.3])**.** *Let E be a Banach space and $A : E \to E$ be a completely continuous operator. Suppose there exist $h \in E, h \neq 0$ and positive constants r, R with $r \neq R$ such that*

(a) *If $y \in E$ satisfies $y = \theta Ay$ for some $\theta \in (0,1]$ then $\|y\| \neq r$,*
(b) *If $y \in E$ satisfies $y = Ay + \xi h$ for some $\xi \geq 0$ then $\|y\| \neq R$.*

Then A has a fixed point $y \in E$ with $\min(r,R) < \|y\| < \max(r,R)$.

For the rest of the paper, we let $r_0 = \inf_{t\in[0,1]} r(t)$. In the following lemmas, we suppose (A1) and (A2) hold.

Lemma 2.2. *Let $h \in L^1(0,1)$. Then the problem*

$$\begin{cases} (r(t)\phi(u'))' = h(t), \quad 0 < t < 1,\\ a_1u(0) - H_1(u'(0)) = 0,\\ a_2u(1) + H_2(u'(1)) = 0, \end{cases} \tag{2.1}$$

has a unique solution $u \equiv Sh \in C^1[0,1]$. Furthermore $S : L^1(0,1) \to C[0,1]$ is completely continuous and

$$|Sh|_{C^1} \leq G(\phi^{-1}(\|h\|_1)), \tag{2.2}$$

where $G(z) = H_i(\hat{r}_0 z)/a_i + 2\phi^{-1}(2/r_0)z,\ \hat{r}_0 = \phi^{-1}(1/r(0))$, and $i \in \{1,2\}$ is smallest with $a_i > 0$.

Proof. Without loss of generality, we suppose $a_1 > 0$. By integrating, it follows that (2.1) has a unique solution u, given by

$$u(t) = \frac{H_1(\xi)}{a_1} + \int_0^t \phi^{-1}\left(\frac{r(0)\phi(\xi) + \int_0^s h}{r(s)}\right) ds, \tag{2.3}$$

where $u'(0) = \xi \in \mathbb{R}$ is the unique solution of

$$H(\xi) \equiv a_2\left(\frac{H_1(\xi)}{a_1} + \int_0^1 \phi^{-1}\left(\frac{r(0)\phi(\xi) + \int_0^s h}{r(s)}\right) ds\right)$$
$$+ H_2\left(\phi^{-1}\left(\frac{r(0)\phi(\xi) + \int_0^1 h}{r(1)}\right)\right) = 0.$$

The fact that H has a unique solution on $\mathbb{R}$ follows from the strictly increasing of G together with $\lim_{\xi\to\infty} G(\xi) = \infty$ and $\lim_{\xi\to-\infty} G(\xi) = -\infty$.

Since $H(\xi) > 0$ if $\xi > \phi^{-1}\left(\frac{1}{r(0)}\|h\|_1\right)$ and $H(\xi) < 0$ if $\xi < -\phi^{-1}\left(\frac{1}{r(0)}\|h\|_1\right)$, it follows that

$$|\xi| \le \phi^{-1}\left(\frac{1}{r(0)}\|h\|_1\right) = \hat{r}_0\phi^{-1}(\|h\|_1). \tag{2.4}$$

Hence

$$|u(t)| + |u'(t)| \le \frac{H_1\left(\hat{r}_0(\phi^{-1}\left(\|h\|_1\right)\right)}{a_1} + 2\phi^{-1}\left(\frac{2\|h\|_1}{r_0}\right)$$

for $t \in [0,1]$, from which (2.2) follows. Hence S maps bounded sets in $L^1(0,1)$ into bounded sets in $C^1[0,1]$ and hence relatively compact subsets in $C[0,1]$. We verify next that S is continuous. To this end, let $(h_n) \subset L^1(0,1)$ be such that $h_n \to h$ in $L^1(0,1)$ and let $u_n = Sh_n, u = Sh$. Then

$$u_n(t) = \frac{H_1(\xi_n)}{a_1} + \int_0^t \phi^{-1}\left(\frac{r(0)\phi(\xi_n) + \int_0^s h_n}{r(s)}\right) ds,$$

where $\xi_n = u_n'(0)$ satisfies $H(\xi_n) = 0$. We claim that

$$|\phi(\xi_n) - \phi(\xi)| \le \frac{\|h_n - h\|_1}{r(0)}. \tag{2.5}$$

Indeed, if $\phi(\xi_n) > \phi(\xi) + \frac{\|h_n - h\|_1}{r(0)}$ then $\xi_n > \xi$ and $r(0)\phi(\xi_n) + \int_0^s h_n > r(0)\phi(\xi) + \int_0^s h$ for $s \in [0,1]$, which implies $0 = H(\xi_n) > H(\xi) = 0$, a contradiction. On the other hand, if $\phi(\xi_n) < \phi(\xi) - \frac{\|h_n - h\|_1}{r(0)}$ then $\xi_n < \xi$ and $r(0)\phi(\xi_n) + \int_0^s h_n < r(0)\phi(\xi) + \int_0^s h$ for $s \in [0,1]$, which implies $0 = H(\xi_n) < H(\xi) = 0$, a contradiction. Thus (2.5) holds. In particular, $\phi(\xi_n) \to \phi(\xi)$ and therefore $\xi_n \to \xi$ as $n \to \infty$. Since

$$u_n(t) = \frac{H_1(\xi_n)}{a_1} + \int_0^t \phi^{-1}\left(\frac{r(0)\phi(\xi_n) + \int_0^s h_n}{r(s)}\right) ds$$

for $t \in [0,1]$, and u is given by (2.3), we deduce from the uniform continuity of ϕ^{-1} on bounded intervals that (u_n) converges to u uniformly on $[0,1]$. Hence S is completely continuous by the Ascoli–Arzela theorem, which completes the proof. $\square$

We next establish a comparison principle

Lemma 2.3. *Let $h_1, h_2 \in L^1(0,1)$ with $h_1 \geq h_2$ on $(0,1)$ and let $u_1, u_2 \in C^1[0,1]$ satisfy*

$$\begin{cases} -(r(t)\phi(u_i'))' = h_i, \quad 0 < t < 1, \\ a_1 u_1(0) - H_1(u_1'(0)) \geq a_1 u_2(0) - H_1(u_2'(0)), \\ a_2 u_1(1) + H_2(u_1'(1)) \geq a_2 u_2(1) + H_2(u_2'(1)). \end{cases}$$

Then $u_1 \geq u_2$ on $[0,1]$.

Proof. Suppose on the contrary that there exists $t_0 \in (0,1)$ such that $u_1(t_0) < u_2(t_0)$. Let $(\alpha, \beta) \subset (0,1)$ be the largest open interval containing t_0 such that $u_1 < u_2$ on (α, β).

Multiplying the equation

$$-(r(t)(\phi(u_1') - \phi(u_2'))' = h_1 - h_2 \text{ on } (0,1)$$

by $u_1 - u_2$ and integrating on (α, β), we obtain

$$\begin{aligned} &- r(\beta)(\phi(u_1'(\beta) - \phi(u_2'(\beta))(u_1(\beta) - u_2(\beta)) \\ &+ r(\alpha)(\phi(u_1'(\alpha) - \phi(u_2'(\alpha))(u_1(\alpha) - u_2(\alpha)) \\ &+ \int_\alpha^\beta r(t)(\phi(u_1') - \phi(u_2'))(u_1' - u_2')dt = \int_\alpha^\beta (h_1 - h_2)(u_1 - u_2)dt \leq 0. \end{aligned} \tag{2.6}$$

We claim that $(\phi(u_1'(\beta) - \phi(u_2'(\beta))(u_1(\beta) - u_2(\beta)) \leq 0$. Clearly it is true if $u_1(\beta) = u_2(\beta)$. Suppose $u_1(\beta) < u_2(\beta)$. Then $\beta = 1$ and it follows from the boundary inequality at 1 that

$$H_2(u_1'(1)) - H_2(u_2'(1)) \geq a_2(u_2(1) - u_1(1)) \geq 0$$

with strict inequality if $a_2 > 0$. Since H_2 is nondecreasing and is strictly increasing if $a_2 = 0$, it follows that $u_1'(1) \geq u_2'(1)$, which proves the claim.

Similarly, we obtain $(\phi(u_1'(\alpha) - \phi(u_2'(\alpha))(u_1(\alpha) - u_2(\alpha)) \geq 0$. Hence (2.6) together with the increasing of ϕ gives

$$\int_\alpha^\beta r(t)(\phi(u_1') - \phi(u_2'))(u_1' - u_2')dt = 0,$$

from which it follows that $u_1' = u_2'$ on $[0,1]$. Consequently, $u_1 = u_2 + C$ on $[\alpha, \beta]$ for some constant $C \leq 0$. If $\alpha > 0$ or $\beta < 1$ then $C = 0$. Suppose $\alpha = 0$ and $\beta = 1$. Then the boundary inequalities at 0 and 1 imply $a_1 C \geq 0$ and $a_2 C \geq 0$. Since $a_1 + a_2 > 0$, we reach a contradiction if $C < 0$. Hence $C = 0$ in both cases i.e. $u_1 = u_2$ on (α, β), a contradiction. Thus $u_1 \geq u_2$ on $[0,1]$, which completes the proof. $\square$

Remark 2.4. *Lemma 2.2 holds if* 0 *and* 1 *are replaced by* a *and* b *respectively, where* $0 \le a < b \le 1$, *and the case when* $H_i \equiv 0, a_i > 0$ *where* $i \in \{1,2\}$ *is included.*

The next lemma provides an extension of [9, Lemma 3.4] to include the case when H_i are nonlinear, $i = 1, 2$.

Lemma 2.5. *Let* $h \in L^1(0,1)$ *with* $h \ge 0$ *and let* $u \in C^1[0,1]$ *satisfy*

$$\begin{cases} (r(t)\phi(u'))' \le h, \quad 0 < t < 1, \\ a_1 u(0) - H_1(u'(0)) \ge 0, \\ a_2 u(1) + H_2(u'(1)) \ge 0. \end{cases}$$

Suppose

$$\|u\|_\infty > \max\left\{2m\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right), G(\phi^{-1}(\|h\|_1)\right\},$$

where $m = 2^{\left(\frac{2-p}{p-1}\right)+}$ *and* G *is defined in Lemma 2.1. Then*

$$u(t) \ge c\|u\|_\infty \omega(t) \tag{2.7}$$

for $t \in [0,1]$, *where* $c = \min\{1/4, \phi^{-1}(r_0/\|r\|_\infty)/4m\}$.

Proof. Let $v \in C^1[0,1]$ be the solution of

$$\begin{cases} (r(t)\phi(v'))' = h, \quad 0 < t < 1, \\ a_1 v(0) - H_1(v'(0)) = 0, \\ a_2 v(1) + H_2(v'(1)) = 0. \end{cases}$$

Then $u \ge v$ on $[0,1]$ in view of Lemma 2.2. Suppose $\|u\|_\infty = |u(\tau)|$ for some $\tau \in (0,1)$. If $u(\tau) \le 0$ then it follows from (2.2) that $\|u\|_\infty = -u(\tau) \le -v(\tau) \le G(\phi^{-1}(\|h\|_1)$, a contradiction. Hence $u(\tau) > 0$.

Let $w \in C^1[0,\tau]$ be the solution of

$$\begin{cases} (r(t)\phi(w'))' = h, \quad 0 < t < \tau, \\ a_1 w(0) - H_1(w'(0)) = 0, \\ w(\tau) = \|u\|_\infty. \end{cases}$$

A calculation shows that if $a_1 > 0$ then

$$w(t) = \frac{H_1(w'(0))}{a_1} + \int_0^t \phi^{-1}\left(\frac{r(0)\phi(w'(0)) + \int_0^s h}{r(s)}\right) ds, \tag{2.8}$$

where $w'(0) = \xi$ is the unique solution of

$$\frac{H_1(\xi)}{a_1} + \int_0^\tau \phi^{-1}\left(\frac{r(0)\phi(\xi) + \int_0^s h}{r(s)}\right) ds = \|u\|_\infty, \tag{2.9}$$

while if $a_1 = 0$ then $w'(0) = 0$ and

$$w(t) = \|u\|_\infty - \int_t^\tau \phi^{-1}\left(\frac{1}{r(s)}\int_0^s h\right) ds. \tag{2.10}$$

By Remark 2.4, $u \geq w$ on $[0, \tau]$. Suppose $a_1 > 0$. Then $w'(0) > 0$ for otherwise (2.8) gives $\|u\|_\infty = w(\tau) \leq \phi^{-1}\left(\|h\|_1/r_0\right)$, a contradiction. Using the inequality

$$\phi^{-1}(x+y) \leq m(\phi^{-1}(x) + \phi^{-1}(y)) \text{ for } x, y \geq 0,$$

we obtain

$$\int_0^\tau \phi^{-1}\left(\frac{r(0)\phi(w'(0) + \int_0^s h}{r(s)}\right) ds \leq m\left(\phi^{-1}\left(\frac{r(0)}{r_0}\right) w'(0) + \phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)\right). \tag{2.11}$$

Since $w(0) = H_1(\xi)/a_1$, it follows from (2.9) and (2.11) that

$$w(0) + m_1 w'(0) \geq \|u\|_\infty - m\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right) \geq \|u\|_\infty/2,$$

where $m_1 = m\phi^{-1}(r(0)/r_0)$. If $w(0) \geq \|u\|_\infty/4$ then since $w' \geq 0$ we get $w(t) \geq \|u\|_\infty/4 \geq \|u\|_\infty t/4$ for $t \in [0, \tau]$. On the other hand, if $m_1 w'(0) \geq \|u\|_\infty/4$ then (2.8) gives

$$\begin{aligned} w(t) &\geq \phi^{-1}\left(\frac{r(0)}{\|r\|_\infty}\right) w'(0)t \geq \frac{\phi^{-1}\left(r(0)/\|r\|_\infty\right)\|u\|_\infty t}{4m_1} \\ &= \frac{\phi^{-1}\left(r(0)/\|r\|_\infty\right)\|u\|_\infty t}{4m} \end{aligned} \tag{2.12}$$

for $t \in [0, \tau]$. Suppose next that $a_1 = 0$. Then (2.10) gives

$$w(t) \geq \|u\|_\infty - \phi^{-1}(\|h\|_1/r_0) \geq \|u\|_\infty t/2 \tag{2.13}$$

for $t \in [0, \tau]$. Next, let $z \in C^1[0,1]$ be the solution of

$$\begin{cases} (r(t)\phi(z'))' = h, \quad \tau < t < 1, \\ z(\tau) = \|u\|_\infty, \\ a_2 z(1) + H_2(z'(1)) = 0. \end{cases}$$

A calculation shows that if $a_2 > 0$ then

$$z(t) = -\frac{H_2(z'(1))}{a_2} + \int_t^1 \phi^{-1}\left(\frac{-r(1)\phi(z'(1) + \int_s^1 h}{r(s)}\right) ds, \tag{2.14}$$

where $z'(1)=\psi$ is the unique solution of

$$-\frac{H_2(\psi)}{a_2}+\int_\tau^1 \phi^{-1}\left(\frac{-r(1)\phi(\psi)+\int_s^1 h}{r(s)}\right)ds=\|u\|_\infty, \tag{2.15}$$

while if $a_2=0$ then $w'(1)=0$ and

$$z(t)=\|u\|_\infty-\int_\tau^t \phi^{-1}\left(\frac{1}{r(s)}\int_s^1 h\right) \tag{2.16}$$

for $t\in[\tau,1]$. By Remark 2.4, $u\geq z$ on $[\tau,1]$. Suppose $a_2>0$. Then $z'(1)\leq 0$ for otherwise (2.14) gives $\|u\|_\infty=z(\tau)\leq\phi^{-1}\left(\|h\|_1/r_0\right)$, a contradiction. Since

$$\int_\tau^1 \phi^{-1}\left(\frac{-r(1)\phi(z'(1))+\int_s^1 h}{r(s)}\right)ds\leq m\left(-\phi^{-1}\left(\frac{r(1)}{r_0}\right)z'(1)+\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)\right)$$

and $z(1)=-\frac{H_2(\psi)}{a_2}$, it follows from (2.15) that

$$z(1)-m_2z'(1)\geq\|u\|_\infty/2,$$

where $m_2=m\phi^{-1}(r(1)/r_0)$. If $z(1)\geq\|u\|_\infty/4$ then since $z'\leq 0$ we get $z(t)\geq\|u\|_\infty/4\geq(\|u\|_\infty/4)(1-t)$ for $t\in[\tau,1]$. On the other hand, if $-m_2z'(1)\geq\|u\|_\infty/4$ then (2.14) gives

$$\begin{aligned} z(t)&\geq-\phi^{-1}\left(\frac{r(1)}{\|r\|_\infty}\right)z'(1)(1-t)\geq\frac{\phi^{-1}\left(r(1)/\|r\|_\infty\right)\|u\|_\infty(1-t)}{4m_2}\\ &=\frac{\phi^{-1}\left(r_0/\|r\|_\infty\right)\|u\|_\infty(1-t)}{4m} \end{aligned} \tag{2.17}$$

for $t\in[\tau,1]$. Finally if $a_2=0$ then (2.16) gives

$$z(t)\geq\|u\|_\infty-\phi^{-1}\left(\frac{\|h\|_1}{r_0}\right)\geq\frac{\|u\|_\infty(1-t)}{2} \tag{2.18}$$

for $t\in[\tau,1]$. Combining (2.12),(2.13), (2.17), and (2.18), we obtain (2.7), which completes the proof. □

3. PROOF OF THE MAIN RESULT

Proof of Theorem 1.1. Let $E=C[0,1]$ be equipped with $\|\cdot\|_\infty$ and $\lambda>0$. For $v\in C[0,1]$, define $S_\lambda v(t)=-\lambda g(t)f(\tilde{v})$, where $\tilde{v}=\max(v,\omega)$. Then it follows from (A3) that

$$|S_\lambda v(t)|\leq\lambda C_v\frac{g(t)}{\tilde{v}^\delta}\leq\lambda C_v k(t)$$

for $t \in (0,1)$, where $k(t) = \frac{g(t)}{\omega^\delta(t)}$ and C_v is a positive constant depending on an upper bound of $\|v\|_\infty$. Hence by (A4), $S_\lambda : E \to L^1(0,1)$ and maps bounded sets in $C[0,1]$ into bounded sets in $L^1(0,1)$. Using the Lebesgue dominated convergence theorem, we see that S_λ is continuous. By Lemma 2.1, there exists a unique solution $u = T_\lambda v$ to the problem

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t) f(\tilde{v}), \quad 0 < t < 1, \\ a_1 u(0) - H_1(u'(0)) = 0, \\ a_2 u(1) + H_2(u'(1)) = 0. \end{cases} \tag{3.1}$$

Since $T_\lambda = S \circ S_\lambda$, where S is given by Lemma 2.1, it follows that $T_\lambda : E \to E$ is completely continuous. Without loss of generality, we suppose $a_1 > 0$.

(i) Let $M > 0$ be such that

$$g(t)|f(z)| \le M g(t) z^{-\delta} \tag{3.2}$$

for $t \in (0,1)$ and $z \in (0, 1/c)$, where c is given by Lemma 2.3. Fix $\lambda \in (0,1)$ so that $G(\phi^{-1}(\lambda M \|k\|_1) < 1/c$. We claim that

(a) *If* $u \in E$ satisfies $u = \theta T_\lambda u$ *for some* $\theta \in (0,1]$ *then* $\|u\|_\infty \ne 1/c$.

Indeed, let $u \in E$ satisfy $u = \theta T_\lambda u$ for some $\theta \in (0,1)$. Suppose $\|u\|_\infty = 1/c$. Then, since $c < 1$, we get $\|\tilde{u}\|_\infty \le 1/c$ and so (3.2) gives

$$|S_\lambda u(t))| \le \lambda M k(t)$$

for $t \in (0,1)$. Hence it follows from Lemma 2.1 that

$$1/c = \|u\|_\infty = \theta \|S(S_\lambda u)\|_\infty \le G(\phi^{-1}\|S_\lambda u\|_1) \le G(\phi^{-1}(\lambda M \|k\|_1),$$

a contradiction, which proves (a).

(b) *There exists* $R_\lambda > 1/c$ *such that if* $u = T_\lambda u + \gamma$ *for some* $\gamma \ge 0$ *then* $\|u\|_\infty < R_\lambda$.

Let $u \in E$ satisfy $u = T_\lambda u + \gamma$ for some $\gamma \ge 0$. Then $u - \gamma = T_\lambda u$ and therefore

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t) f(\tilde{u}), \quad 0 < t < 1, \\ a_1 u(0) - H_1(u'(0)) = a_1 \gamma \ge 0, \\ a_2 u(1) + H_2(u'(1)) = a_2 \gamma \ge 0. \end{cases}$$

Using (A4) and the fact that $\lim_{z\to\infty} f(z) = \infty$, it follows that there exists a constant $m_0 > 0$ such that $f(z) \geq -m_0 z^{-\delta}$ for $z > 0$. Hence

$$\lambda g(t) f(\tilde{u}) \geq -\lambda m_0 g(t) \tilde{u}^{-\delta} \geq -\lambda m_0 k(t) \equiv -h_\lambda(t) \tag{3.3}$$

for $t \in (0,1)$.

Suppose

$$\|u\|_\infty = R_\lambda > \max\left\{2m\phi^{-1}\left(\frac{\|h_\lambda\|_1}{r_0}\right), G(\phi^{-1}(\|h_\lambda\|_1), \frac{4}{c}\right\}.$$

Then Lemma 2.3 gives $u \geq 0$ on $[0,1]$ and

$$u(t) \geq c\|u\|_\infty \omega(t) \geq c_0\|u\|_\infty \geq 1 \tag{3.4}$$

for $t \in [1/4, 3/4]$, where $c_0 = c/4$. Hence

$$\lambda g(t) f(\tilde{u}) = \lambda g(t) f(u) \geq \lambda g(t)\ \bar{f}(c_0\|u\|_\infty)$$

for $t \in [1/4, 3/4]$, where $\bar{f}(z) = \inf\limits_{t \geq z} f(t)$. Let $v_0 \in C^1[1/4, 3/4]$ satisfy

$$\begin{cases} -(r(t)\phi(v_0'))' = g(t), \quad 1/4 < t < 3/4, \\ v(1/4) = 0, \\ v(3/4) = 0, \end{cases} \tag{3.5}$$

and let $v_1 = (\lambda \bar{f}(c_0\|u\|_\infty))^{\frac{1}{p-1}} v_0$. Then v_1 satisfies

$$\begin{cases} -(r(t)\phi(v_1'))' = \lambda g(t)\bar{f}(c_0\|u\|_\infty), \quad 1/4 < t < 3/4, \\ v_1(1/4) = 0, \\ v_1(3/4) = 0. \end{cases}$$

By the comparison principle, $u \geq v_1$ on $[1/4, 3/4]$, which implies

$$|u\|_\infty \geq \left(\lambda \bar{f}(c_0\|u\|_\infty)\right)^{\frac{1}{p-1}} \|v_0\|_\infty, \tag{3.6}$$

i.e.

$$\frac{\bar{f}(c_0\|u\|_\infty)}{\|u\|_\infty^{p-1}} \leq \frac{1}{\lambda\|v_0\|_\infty^{p-1}}.$$

Since $\lim\limits_{z\to\infty} \frac{f(z)}{z^{p-1}} = \infty$, it follows that $\lim\limits_{z\to\infty} \frac{\bar{f}(c_0 z)}{z^{p-1}} = \infty$ and therefore we reach a contradiction if $\|u\|_\infty$ is large enough. Thus $\|u\|_\infty \neq R_\lambda$ for $R_\lambda >> 1$, i.e. (b) holds. By Theorem 2.1, T_λ has a fixed point $u_\lambda \in E$ with $\|u_\lambda\|_\infty > 1/c$. By making λ smaller if necessary so that

$$\max\left\{2m\phi^{-1}\left(\frac{\|h_\lambda\|_1}{r_0}\right), G(\phi^{-1}(\|h_\lambda\|_1)\right\} < 1,$$

where h_λ is defined in (3.3), it follows from Lemma 2.3 that $u_\lambda \geq c\|u_\lambda\|_\infty \omega \geq \omega$ on $(0,1)$. Hence $\tilde{u}_\lambda = u_\lambda$ and u_λ is a positive solution of (1.1).

We verify next that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$. Let $b > 1, M_0 > 0$ be such that $f(z) > 0$ for $z \geq b$ and

$$g(t)f(z) \leq M_0 g(t) z^{-\delta}$$

for $z \in (0,b)$. Then

$$g(t)f(u_\lambda) \leq M_0 k(t) + g(t)\hat{f}(\max(u_\lambda, b)) \tag{3.7}$$

for $t \in (0,1)$, where $\hat{f}(s) = \sup_{b \leq t \leq s} f(t)$ for $s \geq b$. Note that $\hat{f}$ is nondecreasing. Hence, since $k \geq g$ on $(0,1)$, (3.7) implies

$$-(r(t)\phi(u_\lambda'))' = \lambda g(t) f(u_\lambda) \leq \lambda \left(M_0 + \hat{f}(\max(\|u_\lambda\|_\infty, b)\right) k(t) \tag{3.8}$$

for $t \in (0,1)$. Let $w_0 \in C^1[0,1]$ satisfy

$$\begin{cases} -(r(t)\phi(w_0'))' = k(t), \quad 0 < t < 1, \\ a_1 w_0(0) - H_1(w_0'(0)) = 0, \\ a_2 w_0(1) + H_2(w_0'(1)) = 0. \end{cases}$$

Then it follows from (3.8) and Lemma 2.2 that

$$u_\lambda \leq \lambda^{\frac{1}{p-1}} \left(M_0 + \hat{f}(\max(\|u_\lambda\|_\infty, b))\right)^{\frac{1}{p-1}} w_0$$

on $(0,1)$. Consequently,

$$\frac{M_0 + \hat{f}(\max(\|u_\lambda\|_\infty, b))}{\|u_\lambda\|_\infty^{p-1}} \geq \frac{1}{\lambda \|w_0\|_\infty^{p-1}}. \tag{3.9}$$

Since $\|u_\lambda\|_\infty > 1$ and the right side of (3.9) goes to ∞ as $\lambda \to 0^+$, it follows that $\|u_\lambda\|_\infty \to \infty$ as $\lambda \to 0^+$. In view of (2.7), we see that $u_\lambda \to \infty$ as $\lambda \to 0^+$ uniformly on compact subsets of $(0,1)$.

(ii) Without loss of generality, suppose $H_1(z) \leq az$ for $z \geq 0$. Then

$$G(z) = \frac{H_1(\hat{r}_0 z)}{a_1} + 2\phi^{-1}(2/r_0) z \leq Az \tag{3.10}$$

for $z \geq 0$, where $A = a\hat{r}_0 a_1^{-1} + 2\phi^{-1}(2/r_0)$.

Choose

$$K > \max\left\{2m\phi^{-1}\left(m_0\|k\|_1/r_0\right), A\phi^{-1}(m_0\|k\|_1)\right\},$$

where m_0 is defined in (3.3). Then

$$K\phi^{-1}(\lambda) > \max\left\{2m\phi^{-1}(\|h_\lambda\|_1/r_0), G(\phi^{-1}(\|h_\lambda\|_1, 4/c)\right\},$$

where we recall that $h_\lambda = \lambda m_0 k$.

Suppose $\lambda > \lambda_0$, where $\lambda_0 > 1$ is large enough so that

$$\bar{f}(c_0 K\phi^{-1}(\lambda_0)) > (K/\|v_0\|_\infty)^{p-1},$$

where v_0 is defined in (3.4). Note that this is possible since $\lim_{z\to\infty} f(z) = \infty$. We claim what follows.

(c) *If* $u \in E$ satisfies $u = T_\lambda u + \gamma$ *for some* $\gamma \geq 0$ *then* $\|u\|_\infty \neq K\phi^{-1}(\lambda)$.

Let $u \in E$ satisfy $u = T_\lambda u + \gamma$ for some $\gamma \geq 0$. Suppose that $\|u\|_\infty = K\phi^{-1}(\lambda)$. Then Lemma 2.3 gives (3.4) above. Hence (3.6) holds, i.e.

$$\lambda K^{p-1} = \|u\|_\infty^{p-1} \geq \lambda \bar{f}(c_0 K\phi^{-1}(\lambda))\|v_0\|_\infty^{p-1},$$

which implies $\bar{f}(c_0 K\phi^{-1}(\lambda_0)) \leq (K/\|v_0\|_\infty)^{p-1}$, a contradiction. Hence $\|u\|_\infty \neq K\phi^{-1}(\lambda)$, as claimed.

(d) *There exists* $R_\lambda >> 1$ *such that if* $u \in E$ *satisfies* $u = \theta T_\lambda u$ *for some* $\theta \in (0,1]$ *then* $\|u\|_\infty \neq R_\lambda$.

Let $u \in E$ satisfy $u = \theta T_\lambda u$ for some $\theta \in (0,1)$. Suppose $\|u\|_\infty = R_\lambda > \max(1,b)$. Then $\|\tilde{u}\|_\infty \geq b$ and (3.7) gives

$$\begin{aligned} g(t)f(\tilde{u}) &\leq M_0 k(t) + g(t)\hat{f}(\max(\tilde{u}, b)) \\ &\leq M_0 k(t) + g(t)\hat{f}(\|u\|_\infty) \end{aligned}$$

for $t \in (0,1)$, from which (3.10) and Lemma 2.1 imply

$$\begin{aligned} \|u\|_\infty &\leq \theta G(\phi^{-1}(\lambda\|g(t)f(\tilde{u})\|_1) \leq G(\phi^{-1}(\lambda(M_0\|k\|_1 + \|g\|_1\hat{f}(\|u\|_\infty))) \\ &\leq A\left[\lambda(M_0\|k\|_1 + \|g\|_1\hat{f}(\|u\|_\infty))\right]^{\frac{1}{p-1}}. \end{aligned}$$

Consequently,

$$\frac{M_0\|k\|_1 + \|g\|_1\hat{f}(\|u\|_\infty)}{\|u\|_\infty^{p-1}} \geq \frac{1}{\lambda A^{p-1}}.$$

Since

$$\lim_{z\to\infty} \frac{M_0\|k\|_1 + \|g\|_1\hat{f}(z)}{z^{p-1}} = 0,$$

we reach a contradiction if R_λ is large enough, which proves the claim. By Theorem 2.1, T_λ has a fixed point u_λ with $\|u_\lambda\|_\infty > K\phi^{-1}(\lambda)$. By making λ larger if necessary so that $cK\phi^{-1}(\lambda) > 1$, it follows from Lemma 2.3 that $u_\lambda \geq c\|u_\lambda\|_\infty \omega \geq \omega$ on $(0,1)$, i.e. $u_\lambda = \tilde{u}_\lambda$ is a positive solution of (1.1). Clearly $u_\lambda \to \infty$ as $\lambda \to \infty$ uniformly on compact subsets of $(0,1)$.

(iii) Let $z_0 \in C^1[0,1]$ be the solution of

$$\begin{cases} -(r(t)\phi(z_0'))' = g(t)\omega^{p-1}(t), \quad 0 < t < 1, \\ a_1 z_0(0) - H_1(z_0'(0)) = 0, \\ a_2 z_0(1) + H_2(z_0'(1)) = 0. \end{cases}$$

Let $\lambda > 0$ and choose $M > 0$ large enough so that $(\lambda M)^{\frac{1}{p-1}} c\|z_0\|_\infty > 1$. Since $\lim\limits_{z\to 0^+} \frac{f(z)}{z^{p-1}} = \infty$, there exists a constant $\rho \in (0,1)$ such that

$$f(z) \geq M z^{p-1}$$

for $z \in (0, \rho]$. For $v \in E$, define $u = A_\lambda v$ to be the unique solution of

$$\begin{cases} -(r(t)\phi(u'))' = \lambda g(t) f(\bar{v}), \quad 0 < t < 1, \\ a_1 u(0) - H_1(u'(0)) = 0, \\ a_2 u(1) + H_2(u'(1)) = 0, \end{cases}$$

where $\bar{v} = \max(v, \rho_0 \omega), \rho_0 = c\rho$ and c is given by Lemma 2.3. Then $A_\lambda : E \to E$ is completely continuous. We claim that

(e) *If $u \in E$ satisfies $u = A_\lambda u + \gamma$ for some $\gamma \geq 0$ then $\|u\|_\infty \neq \rho$.*

Indeed, let $u \in E$ satisfy $u = A_\lambda u + \gamma$ for some $\gamma \geq 0$, and suppose that $\|u\|_\infty = \rho$. Since

$$-(r(t)\phi(u'))' = \lambda g(t) f(\bar{u}) \geq 0, \ 0 < t < 1,$$

it follows from Lemma 2.3 with $h = 0$ that $u(t) \geq \rho_0 \omega(t)$ for $t \in (0,1)$, i.e. $\bar{u} = u$. Hence

$$\lambda g(t) f(\bar{u}) \geq \lambda M g(t) u^{p-1} \geq \lambda M \rho_0^{p-1} g(t) \omega^{p-1}(t)$$

for $t \in (0,1)$. By Lemma 2.2, $u \geq (\lambda M)^{\frac{1}{p-1}} \rho_0 z_0$ on $(0,1)$, which implies

$$\rho = \|u\|_\infty \geq (\lambda M)^{\frac{1}{p-1}} \rho_0 \|z_0\|_\infty.$$

Consequently, $(\lambda M)^{\frac{1}{p-1}} c\|z_0\|_\infty \leq 1$, a contradiction with the choice of M. Hence $\|u\|_\infty \neq \rho$ as claimed. Using the same argument as in (d) of (ii) above, we see that the following holds.

(f) *There exists $R_\lambda >> 1$ such that if $u \in E$ satisfies $u = \theta A_\lambda u$ for some $\theta \in (0,1]$ then $\|u\|_\infty \neq R_\lambda$.*

Hence A_λ has a fixed point u_λ in E with $\|u_\lambda\|_\infty > \rho$. By Lemma 2.3, $u_\lambda \geq \rho_0 \omega$ on $[0,1]$, i.e. $\bar{u}_\lambda = u_\lambda$ on $[0,1]$ and therefore u_λ is a positive solution of (1.1). This completes the proof of Theorem 1.1. □

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