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ON EDGE PRODUCT CORDIAL GRAPHS

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Abstract. An edge product cordial labeling is a variant of the well-known cordial labeling. In this paper we characterize graphs admitting an edge product cordial labeling. Using this characterization we investigate the edge product cordiality of broad classes of graphs, namely, dense graphs, dense bipartite graphs, connected regular graphs, unions of some graphs, direct products of some bipartite graphs, joins of some graphs, maximal k-degenerate and related graphs, product cordial graphs.

Keywords: edge product cordial labelings, dense graphs, regular graphs.

Mathematics Subject Classification: 05C78.

1. INTRODUCTION

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Cardinalities of these sets are called the *order* and *size* of G. The subgraph of a graph G induced by $U \subseteq V(G)$ is denoted by $G[U]$. The set of vertices of G adjacent to a vertex $v \in V(G)$ is denoted by $N_G(v)$.

For a graph G, a mapping $\varphi : E(G) \to \{0,1\}$ induces a vertex mapping $\varphi^* : V(G) \to \{0,1\}$ defined by

$$\varphi^*(v) = \prod_{u \in N_G(v)} \varphi(vu).$$

Set

$$\varepsilon_\varphi(i) := |\{e \in E(G) : \varphi(e) = i\}| \quad \text{and} \quad \nu_\varphi(i) := |\{v \in V(G) : \varphi^*(v) = i\}|$$

for each $i \in \{0,1\}$. A mapping $\varphi : E(G) \to \{0,1\}$ is called an *edge product cordial labeling* of G if

$$|\varepsilon_\varphi(0) - \varepsilon_\varphi(1)| \leq 1 \quad \text{and} \quad |\nu_\varphi(0) - \nu_\varphi(1)| \leq 1.$$

A graph that admits an edge product cordial labeling is called an *edge product cordial graph.*

The following claim is evident.

Observation 1.1. *A mapping* $\varphi : E(G) \to \{0,1\}$ *is an edge product cordial labeling of a graph* G *if and only if*

$$\varepsilon_\varphi(0) \in \big\{\lfloor |E(G)|/2 \rfloor, \lceil |E(G)|/2 \rceil\big\} \quad \textit{and} \quad \nu_\varphi(0) \in \big\{\lfloor |V(G)|/2 \rfloor, \lceil |V(G)|/2 \rceil\big\}.$$

An edge product cordial labeling is a version of the well-known cordial labeling defined by Cahit [2]. Vaidya and Barasara [11] introduced the concept of an edge product cordial labeling as the edge analogue of a product cordial labeling defined by Sundaram *et al.* [10]. In [11–14, 16, 17] Vaidya and Barasara presented some classes of edge product cordial graphs and also some classes of graphs admitting no edge product cordial labelings. Prajapati and co-authors [8, 9] also deal with edge product cordial graphs. Vaidya and Barasara [15] also introduced the concept of a total edge product cordial labeling, i.e., a labeling $\varphi : E(G) \to \{0,1\}$ satisfying $\big|\big(\varepsilon_\varphi(0) + \nu_\varphi(0)\big) - \big(\varepsilon_\varphi(1) + \nu_\varphi(1)\big)\big| \le 1$. The graphs admitting a total edge product cordial labeling were completely characterized in [5]. We refer the reader to [4] for comprehensive references.

2. CRUCIAL RESULTS

A *matching* in a graph is a set of pairwise nonadjacent edges. The largest number of edges in any matching of G is denoted by $\alpha(G)$. An *edge cover* of a graph G is a subset A of $E(G)$ such that every vertex of G is incident with an edge in A. The smallest number of edges in any edge cover of G is denoted by $\rho(G)$. Note that only graphs with no isolated vertices have edge covers. For such graphs Gallai [3] proved that $\alpha(G) + \rho(G) = |V(G)|$.

Now we are able to prove a crucial result of the paper.

Theorem 2.1. *Let* G *be a graph with no isolated vertex. Then* G *is an edge product cordial graph if and only if there is a set* $U \subset V(G)$ *satisfying*

(i) $|U| \in \big\{\lfloor |V(G)|/2 \rfloor, \lceil |V(G)|/2 \rceil\big\}$,
(ii) $G[U]$ *contains no isolated vertex,*
(iii) $\alpha\big(G[U]\big) \ge |U| - \lceil |E(G)|/2 \rceil$,
(iv) $\big|E(G[U])\big| \ge \big\lfloor |E(G)|/2 \big\rfloor$.

Proof. If G is an edge product cordial graph, then there is an edge product cordial labeling φ of G. Set $U := \{v \in V(G) : \varphi^*(v) = 0\}$. As $|U| = \nu_\varphi(0)$, according to Observation 1.1, condition (i) holds. Clearly, a vertex v is an element of U if and only if v is incident with an edge belonging to $A := \{e \in E(G) : \varphi(e) = 0\}$. Therefore, A is an edge cover of $G[U]$ and so (ii) holds. Moreover,

$$\lceil |E(G)|/2 \rceil \ge \varepsilon_\varphi(0) = |A| \ge \rho(G[U]) = |U| - \alpha(G[U])$$

which implies (iii).

Similarly,

$$|E(G[U])| \geq |A| = \varepsilon_\varphi(0) \geq \lfloor |E(G)|/2 \rfloor,$$

i.e., condition (iv) holds.

On the other hand, suppose that U is a set of vertices of a graph G which satisfies (i)–(iv). According to (ii), $E(G[U])$ is an edge cover of $G[U]$. Moreover, by (iii),

$$\rho(G[U]) = |U| - \alpha(G[U]) \leq \lceil |E(G)|/2 \rceil.$$

Therefore, there exists an edge cover A of $G[U]$ such that

$$\lfloor |E(G)|/2 \rfloor \leq |A| \leq \lceil |E(G)|/2 \rceil.$$

Consider the mapping $\psi : E(G) \to \{0, 1\}$ defined by

$$\psi(e) = \begin{cases} 0 & \text{when } e \in A, \\ 1 & \text{when } e \notin A. \end{cases}$$

Clearly, $\nu_\psi(0) = |U|$ and $\varepsilon_\psi(0) = |A|$. Thus, according to Observation 1.1, ψ is an edge product cordial labeling of G. □

For connected graphs we have the following result.

Corollary 2.2. *Let G be a connected graph of order at least 3. Then G is an edge product cordial graph if and only if there is a set $U \subset V(G)$ such that*

$$|U| \leq \big\lceil |V(G)|/2 \big\rceil \quad \textit{and} \quad \big|E(G[U])\big| \geq \big\lfloor |E(G)|/2 \big\rfloor.$$

Proof. Assume that G is an edge product cordial graph. According to Theorem 2.1, there is a set $U \subset V(G)$ satisfying (i)–(iv). Conditions (i) and (iv) imply $|U| \leq \lceil |V(G)|/2 \rceil$ and $\big|E(G[U])\big| \geq \lfloor |E(G)|/2 \rfloor$.

On the other hand, suppose that there exists a set $U \subset V(G)$ such that $|U| \leq \lceil |V(G)|/2 \rceil$ and $\big|E(G[U])\big| \geq \lfloor |E(G)|/2 \rfloor$. Let W be a subset of $V(G)$ such that $|W| = \lceil |V(G)|/2 \rceil$ and $G[W]$ has the largest possible size.

If w is an isolated vertex of $G[W]$, then $E(G[W]) = E(G[W - \{w\}])$. Since G is a connected graph, there is a vertex $u \in V(G) - W$ adjacent to a vertex of $W - \{w\}$. Set $W' := \big(W - \{w\}\big) \cup \{u\}$. Clearly, $|W'| = |W|$ and

$$\big|E(G[W'])\big| > \big|E(G[W - \{w\}])\big| = \big|E(G[W])\big|,$$

a contradiction. Thus, $G[W]$ contains no isolated vertex.

As G is a connected graph, $\big|E(G)\big| \geq \big|V(G)\big| - 1$. Hence,

$$\big\lceil |E(G)|/2 \big\rceil \geq \big\lceil |V(G)|/2 \big\rceil - 1.$$

Since $G[W]$ contains no isolated vertex, $\alpha\big(G[W]\big) \geq 1$. Thus,

$$\alpha\big(G[W]\big) \geq 1 \geq \big\lceil |V(G)|/2 \big\rceil - \big\lceil |E(G)|/2 \big\rceil = |W| - \big\lceil |E(G)|/2 \big\rceil.$$

Let U' be a subset of $V(G)$ such that $U \subseteq U'$ and $|U'| = \lceil |V(G)|/2 \rceil$. Clearly,

$$\big|E(G[W])\big| \geq \big|E(G[U'])\big| \geq \big|E(G[U])\big| \geq \big\lfloor |E(G)|/2 \big\rfloor.$$

Therefore, the set W satisfies conditions (i)–(iv), and by Theorem 2.1, G is an edge product cordial graph. □

Example 2.3. Let G be a connected graph of order n, $n \geq 3$. For every integer i, $1 \leq i \leq n$, define the graph G_i, vertex v_i, and integer δ_i recursively in the following way.

Set $G_1 = G$.

The minimum degree of G_i is denoted by δ_i. Let v_i be a vertex of G_i such that $\deg_{G_i}(v_i) = \delta_i$ and

$$\min \big\{ \deg_{G_i}(u) : u \in N_{G_i}(v_i) \big\} \leq \min \big\{ \deg_{G_i}(u) : u \in N_{G_i}(w) \big\}$$

for any vertex $w \in V(G_i)$ of degree δ_i. Let G_{i+1} be the subgraph of G_i induced by $V(G_i) - \{v_i\}$ (i.e., $G_{i+1} = G_i - v_i$).

Clearly, $\sum_{i=1}^{k} \delta_i + |E(G_{k+1})| = |E(G)|$, for each k, $1 \leq k < n$. Therefore, according to Corollary 2.2, we have the following sufficient condition.

$$\textit{If} \quad \sum_{i=1}^{\lfloor n/2 \rfloor} \delta_i \leq \big\lceil |E(G)|/2 \big\rceil, \textit{then } G \textit{ is an edge product cordial graph.}$$

Note that using this condition we can detect almost all known edge product cordial graphs (presented in [8, 9, 11–14]).

3. SOME CLASSES OF GRAPHS

3.1. DENSE GRAPHS

Theorem 3.1. *Let G be an edge product cordial graph of order n. Then*

$$\big|E(G)\big| \leq 1 + \lceil n/2 \rceil \big(\lceil n/2 \rceil - 1\big).$$

Proof. According to Theorem 2.1, there is a subset U of $V(G)$ such that $|U| = \lceil n/2 \rceil$ and $\big|E(G[U])\big| \geq \big\lfloor |E(G)|/2 \big\rfloor$. Since any graph of order $\lceil n/2 \rceil$ has at most $\lceil n/2 \rceil \big(\lceil n/2 \rceil - 1\big)/2$ edges, we have

$$\frac{1}{2} \lceil n/2 \rceil \big(\lceil n/2 \rceil - 1\big) \geq \big|E(G[U])\big| \geq \big\lfloor |E(G)|/2 \big\rfloor \geq \frac{|E(G)| - 1}{2},$$

which implies the desired inequality. □

It is proved in [12] that K_n, for $n \geq 4$, is not edge product cordial. By Theorem 3.1, we have immediately the following stronger result.

Corollary 3.2. *Let G be a graph of order $n \geq 4$. If the minimum degree of G is at least $\lceil n/2 \rceil$, then G is not an edge product cordial graph.*

A graph G is called *bipartite* if its vertex set can be partitioned into disjoint parts V_1, V_2 such that every edge in G joins vertices of different parts.

Theorem 3.3. *Let G be an edge product cordial bipartite graph of order n with parts V_1 and V_2 where $|V_1| \leq |V_2|$. Then the following statements hold:*

1) *If $|V_1| \leq \lfloor (1+n)/4 \rfloor$, then $\big|E(G)\big| \leq 1 + 2|V_1|\big(\lceil n/2 \rceil - |V_1|\big)$.*
2) *If $|V_1| \geq \lfloor (1+n)/4 \rfloor$, then $\big|E(G)\big| \leq 1 + 2\lfloor (1+n)/4 \rfloor \lceil n/4 \rceil$.*

Proof. According to Theorem 2.1, there is a subset U of $V(G)$ such that $|U| = \lceil n/2 \rceil$ and $\big|E(G[U])\big| \geq \big\lfloor |E(G)|/2 \big\rfloor$. Set $U_1 = U \cap V_1$ and $U_2 = U \cap V_2$. The graph $G[U]$ has at most $|U_1| \cdot |U_2| = |U_1|\big(\lceil n/2 \rceil - |U_1|\big)$ edges. Since $f(x) = x\big(\lceil n/2 \rceil - x\big)$ is a strictly increasing function on the interval $\{x : 0 \leq x \leq \lceil n/2 \rceil/2\}$, $|U_1|$ is an integer, and $\big\lfloor \lceil n/2 \rceil/2 \big\rfloor = \lfloor (1+n)/4 \rfloor$, we consider the following two cases.

If $|V_1| \leq \lfloor (1+n)/4 \rfloor$, then $G[U]$ has at most $|V_1|\big(\lceil n/2 \rceil - |V_1|\big)$ edges. Therefore

$$|V_1|\big(\lceil n/2 \rceil - |V_1|\big) \geq \big|E(G[U])\big| \geq \big\lfloor |E(G)|/2 \big\rfloor \geq \frac{|E(G)| - 1}{2},$$

which implies

$$\big|E(G)\big| \leq 1 + 2|V_1|\big(\lceil n/2 \rceil - |V_1|\big).$$

If $|V_1| \geq \lfloor (1+n)/4 \rfloor$, then the induced subgraph $G[U]$ has at most

$$\lfloor (1+n)/4 \rfloor \big(\lceil n/2 \rceil - \lfloor (1+n)/4 \rfloor\big) = \lfloor (1+n)/4 \rfloor \lceil n/4 \rceil$$

edges. Thus

$$\lfloor (1+n)/4 \rfloor \lceil n/4 \rceil \geq \big|E(G[U])\big| \geq \big\lfloor |E(G)|/2 \big\rfloor \geq \frac{|E(G)| - 1}{2},$$

which implies

$$\big|E(G)\big| \leq 1 + 2\lfloor (1+n)/4 \rfloor \lceil n/4 \rceil.$$

□

It is proved in [12] that any complete bipartite graph $K_{m,n}$, for $m \geq n \geq 2$, is not edge product cordial. By Theorem 3.3 we have the following stronger result.

Corollary 3.4. *Let G be a bipartite graph of order n with parts V_1 and V_2 where $2 \leq |V_1| \leq |V_2|$. Let $\delta_i = \min\{\deg(v) : v \in V_i\}$, $i \in \{1, 2\}$ and suppose that at least one of the following conditions is satisfied:*

1) $|V_1| \leq \lfloor (1+n)/4 \rfloor$ *and* $\delta_1 \geq 2 + |V_2| - |V_1|$,
2) $\lfloor (1+n)/4 \rfloor \leq |V_1| < n - 2\lceil n/4 \rceil$ *and* $\delta_2 \geq \lfloor (1+n)/4 \rfloor$,
3) $|V_1| \geq n - 2\lceil n/4 \rceil$ *and* $\delta_2 \geq 1 + \lfloor (1+n)/4 \rfloor$.

Then G is not an edge product cordial graph.

Proof. Suppose to the contrary that G is an edge product cordial graph.

If $|V_1| \leq \lfloor (1+n)/4 \rfloor$ and $\delta_1 \geq 2 + |V_2| - |V_1|$, then

$$\begin{aligned} |E(G)| &\geq |V_1|\delta_1 \geq |V_1|\big(2 + |V_2| - |V_1|\big) \\ &= |V_1|\big(2 + (n - |V_1|) - |V_1|\big) = 2|V_1|\big(1 + n/2 - |V_1|\big) \\ &= |V_1| + 2|V_1|\big((1+n)/2 - |V_1|\big) > 1 + 2|V_1|\big(\lceil n/2 \rceil - |V_1|\big), \end{aligned}$$

a contradiction to Theorem 3.3.

If $\lfloor (1+n)/4 \rfloor \leq |V_1| < n - 2\lceil n/4 \rceil$ and $\delta_2 \geq \lfloor (1+n)/4 \rfloor$, then the part V_2 contains $n - |V_1| > 2\lceil n/4 \rceil$ vertices, i.e., $|V_2| \geq 1 + 2\lceil n/4 \rceil$. Thus, $n \geq 7$ in this case. Consequently

$$\begin{aligned} |E(G)| \geq& |V_2|\delta_2 \geq \big(1 + 2\lceil n/4 \rceil\big)\lfloor (1+n)/4 \rfloor \\ =& \lfloor (1+n)/4 \rfloor + 2\lceil n/4 \rceil \lfloor (1+n)/4 \rfloor > 1 + 2\lceil n/4 \rceil \lfloor (1+n)/4 \rfloor, \end{aligned}$$

a contradiction.

Now assume that $|V_1| \geq n - 2\lceil n/4 \rceil$ and $\delta_2 \geq 1 + \lfloor (1+n)/4 \rfloor$. Since $|V_1| \leq |V_2|$, $|V_2| \geq \lceil n/2 \rceil \geq 2\lceil n/4 \rceil - 1$. If $|V_2| \geq 2\lceil n/4 \rceil$, then

$$|E(G)| \geq |V_2|\delta_2 \geq 2\lceil n/4 \rceil \big(1 + \lfloor (1+n)/4 \rfloor\big) > 1 + 2\lceil n/4 \rceil \lfloor (1+n)/4 \rfloor,$$

a contradiction. If $|V_2| = 2\lceil n/4 \rceil - 1$, then there is a positive integer k such that either $|V_1| = 2k$, $|V_2| = 2k+1$ or $|V_1| = |V_2| = 2k+1$. In both of these cases $\delta_2 \geq 1 + k$ and so

$$|E(G)| \geq |V_2|\delta_2 \geq (2k+1)(1+k) > 1 + 2(k+1)k = 1 + 2\lceil n/4 \rceil \lfloor (1+n)/4 \rfloor,$$

a contradiction. □

3.2. REGULAR GRAPHS

Theorem 3.5. *Let G be a connected d-regular graph of order n and size m, $m > 1$. Then the following statements hold:*

1) *If n and m are both even, then G is not an edge product cordial graph.*
2) *If n is even and m is odd, then G is an edge product cordial graph only when G contains a bridge e such that $G - e$ has two components of order $n/2$.*
3) *If n is odd, then G is an edge product cordial graph only when there is a set $U \subset V(G)$ such that $|\{uv \in E(G) : u \in U, v \notin U\}| \leq 2\lceil d/4 \rceil$ and $|U| = \lceil n/2 \rceil$.*

Proof. As G is a d-regular graph, $m = nd/2$. Moreover, G is connected and so $|E(G[U])| < |U|d/2$ for every U, $\emptyset \neq U \subsetneq V(G)$.

Consider the following cases.

Case A. $n \equiv m \equiv 0 \pmod 2$. Suppose to the contrary that G is an edge product cordial graph. According to Corollary 2.2, there is a subset U of $V(G)$ such that $|U| \leq n/2$ and $|E(G[U])| \geq m/2$. However,

$$|E(G[U])| < |U|d/2 \leq (n/2)d/2 = m/2,$$

a contradiction.

Case B. $n \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$. Suppose that G is an edge product cordial graph. According to Corollary 2.2, there is a subset U of $V(G)$ such that $|U| \leq n/2$ and $|E(G[U])| \geq (m-1)/2$. However,

$$|E(G[U])| < |U|d/2 \leq (n/2)d/2 = m/2.$$

Therefore, $|U| = n/2$ and $|E(G[U])| = (m-1)/2$ in this case. Then

$$|\{uv \in E(G) : u \in U, v \notin U\}| = |U|d - 2|E(G[U])| = 1,$$

i.e., there is a bridge e of G such that $G - e$ has components induced by U and $V(G) - U$ where $|U| = |V(G) - U| = n/2$.

Now suppose that G contains a bridge e such that $G - e$ has two components C' and C^* of order $n/2$. Then $|E(C')| \geq (m-1)/2 = \lceil m/2 \rceil$ when $|E(C')| \geq |E(C^*)|$. Thus, there is a set $U = V(C')$ such that $|E(G[U])| = |E(C')| \geq \lceil m/2 \rceil$. According to Corollary 2.2, G is an edge product cordial graph.

Case C. $n \equiv 1 \pmod 2$. Suppose that G is an edge product cordial graph. By Corollary 2.2, there is a subset U of $V(G)$ such that $|U| \leq (n+1)/2$ and $|E(G[U])| \geq \lfloor m/2 \rfloor = \lfloor nd/4 \rfloor$. If $|U| < (n+1)/2$, then

$$|E(G[U])| < |U|d/2 \leq \big((n-1)/2\big)d/2 \leq \lfloor m/2 \rfloor.$$

Therefore, $|U| = (n+1)/2$ and

$$\begin{aligned}|\{uv \in E(G) : u \in U, v \notin U\}| &= |U|d - 2|E(G[U])| \\ &\leq (n+1)d/2 - 2\lfloor nd/4 \rfloor = 2\lceil d/4 \rceil.\end{aligned}$$

Now suppose that there is a set $U \subset V(G)$ such that $|U| = \lceil n/2 \rceil$ and $|\{uv \in E(G) : u \in U, v \notin U\}| \leq 2\lceil d/4 \rceil$. Then

$$\begin{aligned}|E(G[U])| &= \big(|U|d - |\{uv \in E(G) : u \in U, v \notin U\}|\big)/2 \\ &\geq \big(\lceil n/2 \rceil d - 2\lceil d/4 \rceil\big)/2 \geq \lfloor m/2 \rfloor,\end{aligned}$$

and by Corollary 2.2, G is an edge product cordial graph. □

3.3. GRAPH OPERATIONS

The union of two vertex disjoint graphs G and H is denoted by $G \cup H$.

Theorem 3.6. *Let G_1 and G_2 be disjoint graphs without isolated vertices such that $\big||V(G_1)| - |V(G_2)|\big| \leq 1$. Then the union $G_1 \cup G_2$ is an edge product cordial graph.*

Proof. The union of graphs G_1 and G_2 is denoted by H, i.e., $H = G_1 \cup G_2$. We can assume, without loss of generality, that $|E(G_1)| \geq |E(G_2)|$. Then

$$\lceil |E(H)|/2 \rceil \leq |E(G_1)|.$$

Consider the following cases.

Case A. $\lceil |E(H)|/2 \rceil \geq \rho(G_1)$. Set $U := V(G_1)$. Then

$$\alpha(H[U]) = \alpha(G_1) = |U| - \rho(G_1) \geq |U| - \lceil |E(H)|/2 \rceil.$$

Therefore, condition (iii) holds. The other conditions of Theorem 2.1 are evident. So, H is an edge product cordial graph in this case.

Case B. $\lceil |E(H)|/2 \rceil < \rho(G_1)$. Let A_i, for $i \in \{1,2\}$, be an edge cover of G_i having $\rho(G_i)$ edges. Then $A := A_1 \cup A_2$ is an edge cover of H with $\rho(H)$ edges. Therefore, every component of $H[A]$ is a star and so

$$|V(H[F])| - 2 \leq |V(H[F - \{e\}])| \leq |V(H[F])| - 1$$

for any $e \in F \subseteq A$.

Denote the edges of A by $e_1, e_2, \ldots, e_k$ in such a way that $i < j$ whenever $e_i \in A_2$ and $e_j \in A_1$. For $t \in \{0, 1, \ldots, p\}$, where $p = k - \lceil |E(H)|/2 \rceil$, set

$$B_t = \big\{ e_{t+i} : 1 \leq i \leq \lceil |E(H)|/2 \rceil \big\}.$$

As B_t and B_{t+1} have the same cardinality and B_{t+1} contains exactly one edge not belonging to B_t,

$$|V(H[B_t])| - 1 \leq |V(H[B_{t+1}])| \leq |V(H[B_t])| + 1$$

for every t, $0 \leq t \leq p-1$. Moreover, $A_2 \subseteq B_0$, $B_p \subsetneq A_1$ and so

$$|V(H[B_0])| \geq |V(G_2)| \geq \lfloor |V(H)|/2 \rfloor, \quad |V(H[B_p])| < |V(G_1)| \leq \lceil |V(H)|/2 \rceil.$$

Therefore, there is q such that $|V(H[B_q])| = \lfloor |V(H)|/2 \rfloor$. Consider the mapping $\varphi : E(H) \to \{0,1\}$ defined by

$$\varphi(e) = \begin{cases} 0 & \text{when } e \in B_q, \\ 1 & \text{when } e \notin B_q. \end{cases}$$

Clearly, $\nu_\varphi(0) = |V(H[B_q])| = \lfloor |V(H)|/2 \rfloor$ and $\varepsilon_\varphi(0) = |B_q| = \lceil |E(H)|/2 \rceil$. Thus, by Observation 1.1, φ is an edge product cordial labeling of H. □

The *direct product* $G \times H$ of graphs G and H is a graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and two vertices (u_1, v_1), (u_2, v_2) are joined by an edge in $G \times H$ if and only if $u_1 u_2 \in E(G)$ and $v_1 v_2 \in E(H)$. It is proved in [16] that the direct product of two path is edge product cordial. By Theorem 3.6 we have the following stronger result.

Corollary 3.7. *For $i \in \{1,2\}$, let G_i be a connected bipartite graph with parts A_i, B_i. If either $|A_i| = |B_i|$ for some $i \in \{1,2\}$, or $|A_i| = |B_i| + 1$ for both $i \in \{1,2\}$, then the direct product $G_1 \times G_2$ is an edge product cordial graph.*

Proof. The graph $G_1 \times G_2$ is a disconnected graph with two components. The first component is of order $|A_1| \cdot |A_2| + |B_1| \cdot |B_2|$ and the second is of order $|A_1| \cdot |B_2| + |B_1| \cdot |A_2|$. Thus, the order of first component and the order of second component differ at most by 1. According to Theorem 3.6, $G_1 \times G_2$ is an edge product cordial graph. □

The *join* $G \oplus H$ of the disjoint graphs G and H is the graph $G \cup H$ together with all edges joining vertices of $V(G)$ and vertices of $V(H)$.

Theorem 3.8. *Let G_1 and G_2 be disjoint graphs such that*

$$3 \leq |V(G_1)| \leq |V(G_2)| \leq 3|V(G_1)| - 2.$$

Then the join $G_1 \oplus G_2$ is not an edge product cordial graph.

Proof. The join of graphs G_1 and G_2 is denoted by H, i.e., $H = G_1 \oplus G_2$. Set $n_1 := |V(G_1)|$, $n_2 := |V(G_2)|$, $n := |V(H)| = n_1 + n_2$, $k := \big\lfloor \lceil n/2 \rceil /2 \big\rfloor$. Then we have

$$k = \left\lfloor \left\lceil \frac{n}{2} \right\rceil /2 \right\rfloor = \left\lfloor \left\lceil \frac{n_1 + n_2}{2} \right\rceil /2 \right\rfloor \leq \left\lfloor \left\lceil \frac{n_1 + 3n_1 - 2}{2} \right\rceil /2 \right\rfloor = n_1 - 1.$$

Suppose to the contrary that H is an edge product cordial graph. According to Corollary 2.2, there is a subset U of $V(H)$ such that $|U| \leq \lceil n/2 \rceil$ and $|E(H[U])| \geq \lfloor |E(H)|/2 \rfloor$. The number of edges of $G_1 \cup G_2$ belonging to $H[U]$ is denoted by m, i.e., $m = \big|\big(E(G_1) \cup E(G_2)\big) \cap E(H[U])\big|$. Consider the following cases.

Case A. Suppose that $\lceil n/2 \rceil = 2k$. Then $4k - 1 \leq n \leq 4k$ and $k \geq 2$ in this case. Moreover, $|E(H[U])| \leq k^2 + m$ and

$$\begin{aligned} |E(H)| &\geq n_1 n_2 + m \geq (k+1)(n-k-1) + m \\ &\geq (k+1)(4k-1-k-1) + m = 3k^2 + k - 2 + m. \end{aligned}$$

As $|E(H[U])| \geq \lfloor |E(H)|/2 \rfloor$,

$$k^2 + m \geq \big\lfloor (3k^2 + k - 2 + m)/2 \big\rfloor.$$

Therefore, $1 + 2(k^2 + m) \geq 3k^2 + k - 2 + m$ and so $m \geq k^2 + k - 3$. Consequently,

$$|E(H)| \geq 3k^2 + k - 2 + (k^2 + k - 3) = 4k^2 + 2k - 5.$$

Then $H[U]$ has at least $2k^2 + k - 3$ edges. However, $H[U]$ has at most $2k$ vertices and so

$$|E(H[U])| \leq 2k(2k-1)/2 = 2k^2 + k - 3 + (3 - 2k),$$

a contradiction.

Case B. Suppose that $\lceil n/2 \rceil = 2k + 1 > 3$. Then $4k + 1 \leq n \leq 4k + 2$ and $k \geq 2$. Moreover, $|E(H[U])| \leq k(k+1) + m$ and

$$\begin{aligned} |E(H)| &\geq n_1 n_2 + m \geq (k+1)(n-k-1) + m \\ &\geq (k+1)(4k+1-k-1) + m = 3k^2 + 3k + m. \end{aligned}$$

As $|E(H[U])| \geq \lfloor |E(H)|/2 \rfloor$,

$$k^2 + k + m \geq \big\lfloor (3k^2 + 3k + m)/2 \big\rfloor.$$

Therefore, $1 + 2(k^2 + k + m) \geq 3k^2 + 3k + m$ and so $m \geq k^2 + k - 1$. Consequently,

$$|E(H)| \geq 3k^2 + 3k + (k^2 + k - 1) = 4k^2 + 4k - 1.$$

Then $H[U]$ has at least $2k^2 + 2k - 1$ edges. However, $H[U]$ has at most $2k + 1$ vertices and so

$$|E(H[U])| \leq (2k + 1)2k/2 = 2k^2 + 2k - 1 + (1 - k),$$

a contradiction.

Case C. Suppose that $\lceil n/2 \rceil = 3$. Then $n = 6$ and $k = 1$ in this case. Moreover, $|E(H[U])| \leq 1 \cdot 2 + m$ and $|E(H)| \geq 3 \cdot 3 + m = 9 + m$. As $|E(H[U])| \geq \lfloor |E(H)|/2 \rfloor$, $2 + m \geq \lfloor (9 + m)/2 \rfloor$. Thus, $1 + 2(2 + m) \geq 9 + m$ and so $m \geq 4$. Consequently, $|E(H)| \geq 9 + 4 = 13$. Then $H[U]$ has at least 6 edges. However, $H[U]$ has at most 3 vertices and so $|E(H[U])| \leq 3$, a contradiction. □

3.4. MAXIMAL GRAPHS

A property $\mathcal{P}$ of graphs is called *hereditary* if every subgraph of any graph with property $\mathcal{P}$ also has property $\mathcal{P}$ (see [1]). A graph G is called *$\mathcal{P}$-maximal* if it has property $\mathcal{P}$ and it looses this property after adding any edge from the complement of G. Let k, q, p ($k > 0$, $p > 0$) be integers; a hereditary property $\mathcal{P}$ is said to be *(k, q, p)-restrictive* if every $\mathcal{P}$-maximal graph of order n, $n \geq p$, has $kn + q$ edges. We are able to prove the following result.

Theorem 3.9. *Let $\mathcal{P}$ be a (k, q, p)-restrictive hereditary property. If G is a connected $\mathcal{P}$-maximal graph of order n, $n \geq 4$, then the following statements hold:*

1) *If $p \leq n$, then G is an edge product cordial graph only when it contains a subgraph of order $\lceil n/2 \rceil$ and size at least $\lfloor (kn + q)/2 \rfloor$.*
2) *If $p < n \leq 2k$, then G is not an edge product cordial graph.*
3) *If $2p \leq n \equiv 0 \pmod 2$ and $\lceil q/2 \rceil < 0$, then G is not an edge product cordial graph.*
4) *If $2p - 1 \leq n \equiv 1 \pmod 2$ and $\lceil (q + k)/2 \rceil < 0$, then G is not an edge product cordial graph.*

Proof. If $n \geq p$, then $|E(G)| = kn + q$. Therefore, the first statement is an immediate consequence of Corollary 2.2.

If $p < n \leq 2k$, then any subgraph of G on $n - 1$ vertices has at most $k(n - 1) + q = (kn + q) - k$ edges. It follows that the minimum degree of G is at least k. As $k = \lceil 2k/2 \rceil \geq \lceil n/2 \rceil$, according to Corollary 3.2, G is not edge product cordial.

If $2p \leq n \equiv 0 \pmod 2$ and $\lceil q/2 \rceil < 0$, then any subgraph of G on at most $n/2$ vertices has at most $kn/2 + q$ edges. However,

$$\lfloor |E(G)|/2 \rfloor = \lfloor (kn + q)/2 \rfloor = kn/2 + \lfloor q/2 \rfloor = kn/2 + q - \lceil q/2 \rceil > kn/2 + q.$$

Therefore, by Corollary 2.2, G is not an edge product cordial graph.

If $2p-1 \le n \equiv 1 \pmod 2$ and $\lceil (q+k)/2 \rceil < 0$, then any subgraph of G on at most $\lceil n/2 \rceil$ vertices has at most $k\lceil n/2 \rceil + q$ edges. However,

$$\begin{aligned}\big\lfloor |E(G)|/2 \big\rfloor &= \big\lfloor (kn+q)/2 \big\rfloor = \big\lfloor \big(k(n+1)+(q-k)\big)/2 \big\rfloor \\ &= k\lceil n/2 \rceil + \lfloor (q-k)/2 \rfloor = k\lceil n/2 \rceil + \big\lfloor \big(2q-(q+k)\big)/2 \big\rfloor \\ &= k\lceil n/2 \rceil + q - \lceil (q+k)/2 \rceil > k\lceil n/2 \rceil + q.\end{aligned}$$

Thus, by Corollary 2.2, G is not an edge product cordial graph. □

A graph G is called k-*degenerate* if every subgraph of G has minimum degree at most k (we also say that G has the property $\mathcal{D}_k$). The basic properties of maximal k-degenerate ($\mathcal{D}_k$-maximal) graphs have been introduced in [6, 7]. Inter alia it was proved:

1) *Let G be a maximal k-degenerate graph of order n, $n \ge k+1$. Then the minimum degree of G is equal to k and $|E(G)| = kn - k(k+1)/2$.*
2) *Let G be a graph of order n, $n \ge k+1$, and $v \in V(G)$ be a vertex of degree k. Then G is a maximal k-degenerate graph if and only if $G - v$ is maximal k-degenerate.*

Therefore, $\mathcal{D}_k$ is a $(k, -k(k+1)/2, k)$-restrictive property. Moreover, every maximal k-degenerate graph of order n contains an induced subgraph which is a maximal k-degenerate graph of order m, $1 \le m \le n$.

Note that maximal 1-degenerate graphs are trees. In [11] Vaidya and Barasara characterized edge product cordial maximal 1-degenerate graphs (trees). In [14] they also investigated the edge product cordiality of some classes of maximal 2-degenerate graphs (square and total graphs of path). By Theorem 3.9 we have immediately the following corollary.

Corollary 3.10. *Let G be a maximal k-degenerate graph of order n, $n > k$. Then G is an edge product cordial graph if and only if either $k=1$ and $n \ge 3$ or $k=2$ and $n \equiv 1 \pmod 2$.*

We say that a graph G has the property $\mathcal{D}_k^+$, if it is k-degenerate or it contains an edge e such that $G - e$ is a k-degenerate graph. Clearly, any $\mathcal{D}_k^+$-maximal graph of order at least $k+2$ is a $\mathcal{D}_k$-maximal graph with one edge added ($\mathcal{D}_1^+$-maximal graphs of order at least 3 are unicyclic graphs). Thus, $\mathcal{D}_k^+$ is a $(k, 1-k(k+1)/2, k+2)$-restrictive property.

Vaidya and Barasara [11] investigated the edge product cordiality of unicyclic graphs. Similarly, Prajapati and Patel [8] investigated the edge product cordiality of some $\mathcal{D}_2^+$-maximal graphs (sunflowers). By Theorem 3.9 we have immediately the following result.

Corollary 3.11. *Let G be a $\mathcal{D}_k^+$-maximal graph of order n, $n \ge k+2$. Then G is an edge product cordial graph if and only if one of the following conditions is satisfied:*

1) *$k=1$ and $n \equiv 1 \pmod 2$,*
2) *$k=1$, $n \equiv 0 \pmod 2$, and G contains a cycle of order at most $n/2$,*
3) *$k=2$, $n \equiv 1 \pmod 2$, and G contains an induced subgraph of order at most $\lceil n/2 \rceil$ with minimum degree 3.*

It is well-known that maximal outerplanar (maximal planar) graphs of order n, $n \geq 2$ ($n \geq 3$) have $2n-3$ ($3n-6$) edges. According to Theorem 3.9, we get the following corollary.

Corollary 3.12.

1) *A maximal outerplanar graph of order n, $n \geq 2$, is edge product cordial if and only if n is odd.*
2) *A maximal planar graph of order n, $n \geq 3$, is edge product cordial if and only if $n = 3$.*

3.5. PRODUCT CORDIAL GRAPHS

A *product cordial labeling* of a graph G is a mapping $\psi : V(G) \to \{0,1\}$ such that if each edge $uv \in E(G)$ is assigned the label $\psi(u)\psi(v)$, the number of vertices labeled with 0 and the number of vertices labeled with 1 differ by at most 1, and the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1 (see [10]). A graph with a product cordial labeling is called a *product cordial* graph.

We conclude this paper with the following assertion.

Theorem 3.13. *Let G be a connected product cordial graph of order n, $n \geq 3$. Then G is an edge product cordial graph.*

Proof. Suppose that ψ is a product cordial labeling of G. Set

$$U := \{u \in V(G) : \psi(u) = 1\}.$$

Clearly, an edge $e \in E(G)$ is labeled with 1 if and only if $e \in E\big(G[U]\big)$. Therefore,

$$\lfloor n/2 \rfloor \leq |U| \leq \lceil n/2 \rceil \quad \text{and} \quad \big\lfloor |E(G)|/2 \big\rfloor \leq \big|E(G[U])\big| \leq \big\lceil |E(G)|/2 \big\rceil.$$

According to Corollary 2.2, G is an edge product cordial graph. □

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