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OSCILLATORY CRITERIA FOR SECOND ORDER DIFFERENTIAL EQUATIONS WITH SEVERAL SUBLINEAR NEUTRAL TERMS

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Abstract. In this paper, sufficient conditions for oscillation of the second order differential equations with several sublinear neutral terms are established. The results obtained generalize and extend those reported in the literature. Several examples are included to illustrate the importance and novelty of the presented results.

Keywords: second order neutral differential equation, sub-linear neutral term, oscillation.

Mathematics Subject Classification: 34K11, 34C10.

1. INTRODUCTION

In this paper, we study the oscillatory behavior of the second order differential equations with several sublinear neutral terms of the form

$$(a(t)z'(t))' + q(t)x^{\gamma}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{E}$$

where $z(t) = x(t) + \sum\limits_{i=1}^{m} p_i(t)x^{\alpha_i}(\tau_i(t))$, $m \geq 1$ is an integer and throughout the paper we assume that:

(H_1) $0 < \alpha_i \leq 1$ for $i = 1, 2, \dots, m$ and α_i, γ are quotients of odd positive integers;
(H_2) $a, p_i, q : [t_0, \infty) \to \mathbb{R}^+$ are continuous functions, $\lim\limits_{t\to\infty} p_i(t) = 0$ for $i = 1, 2, \dots, m$;
(H_3) $\tau_i, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $\tau_i(t) < t$, $\sigma(t) \leq t$, $\sigma'(t) > 0$ and $\tau_i(t), \sigma(t) \to \infty$ as $t \to \infty$ for $i = 1, 2, \dots, m$.

We assume that (E) is in canonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\, dt = \infty. \tag{1.1}$$

By a solution of equation (E), we mean a function $x \in C([T_x,\infty),\mathbb{R})$, $T_x \geq t_0$, which has the property $az' \in C^1([T_x,\infty),\mathbb{R})$ and satisfies equation (E) on $[T_x,\infty)$. We consider only those solutions x of equation (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$, and assume that equation (E) possesses such solutions. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise, it is called to be nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

It is well-known that second-order differential equations have applications in various problems of physics, biology, chemistry, economics, etc. Therefore, there has been permanent interest in obtaining sufficient conditions for oscillation and other asymptotic properties of such equations. So, in the past decades many oscillatory results have been presented, see e.g. [1–24].

However, there are only few results dealing with the oscillation of second order differential equations with sublinear neutral terms [2, 8, 20]. The aim of this paper is to fulfil this gap in oscillation theory and to introduce new oscillatory criteria for such equations. Presented results generalize those of Tamilvanan *et al.* [20], where the second order differential equation with only one sublinear neutral term has been investigated. The results obtained are applicable also for non-neutral differential equations ($p_i(t) \equiv 0$). Our results are of high generality, we cover all possible cases, where our equation is sublinear, superlinear and linear.

2. MAIN RESULTS

In what follows, all functional inequalities considered here are assumed to hold eventually, that is, they are satisfied for all t large enough. Without loss of generality, we can deal only with eventually positive solutions of equation (E).

We begin with the following couple of auxiliary lemmas.

Lemma 2.1. *If a and b are positive, then*

$$a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \quad \textit{for} \;\; 0 < \alpha \leq 1, \tag{2.1}$$

where equality holds if and only if $a = b$.

Proof. We consider the auxiliary function $f(u) = u^{\alpha} - \alpha u - (1-\alpha)$ with $u \in (0,\infty)$. It is easy to see, that function f is increasing for $u \in (0,1)$ and decreasing for $u \in (1,\infty)$. Moreover, $f(1) = 0$, which implies that $f(u) \leq 0$ for $u \in (0,\infty)$. Setting $u = \frac{a}{b}$, we have

$$a^{\alpha}b^{1-\alpha} - \alpha a - (1-\alpha)b \leq 0.$$

The proof is completed. □

We denote

$$R(t) = \int_{t_1}^{t} \frac{1}{a(s)}\mathrm{d}s, \;\; t_1 \geq t_0 \;\text{ is a constant large enough.}$$

Lemma 2.2. *Let x be a positive solution of equation* (E). *Assume that*

$$\int_{t_0}^{\infty} \frac{1}{a(u)} \int_{u}^{\infty} q(s)ds\; du = \infty. \tag{2.2}$$

Then the corresponding function z satisfies

(i) $z(t) > 0,\ z'(t) > 0,$ *and* $(a(t)z'(t))' < 0$*,* $t \geq t_1 \geq t_0$;
(ii) $z(t) \to \infty$ *for* $t \to \infty$;
(iii) $\frac{z(t)}{R(t)}$ *is decreasing.*

Proof. Assume that equation (E) has an eventually positive solution $x(t)$. Then $z(t) > 0$ for $t \geq t_1 > t_0$. It follows from (E) and condition (1.1) that case (i) holds true, which implies

$$z(t) \geq a(t)z'(t)R(t), \quad t \geq t_1.$$

Therefore $\frac{z(t)}{R(t)}$ is decreasing function. We claim that condition (2.2) guaranties that $z(t) \to \infty$ for $t \to \infty$. Really, since $z(t)$ is positive increasing function, there exist a constant $k > 0$ such that

$$z(t) \geq 2k > 0 \tag{2.3}$$

for $t \geq t_1$. Moreover, it follows from the definition of $z(t)$ that

$$\begin{aligned} x(t) &= z(t) - \sum_{i=1}^{m} p_i(t)x^{\alpha_i}(\tau_i(t)) \\ &\geq z(t) - \sum_{i=1}^{m} p_i(t)z^{\alpha_i}(t) \\ &\geq z(t) - \sum_{i=1}^{m} p_i(t)(\alpha_i z(t) + (1-\alpha_i)) \\ &= z(t)\left(1 - \sum_{i=1}^{m} \alpha_i p_i(t) - \frac{1}{z(t)}\sum_{i=1}^{m}(1-\alpha_i)p_i(t)\right), \end{aligned} \tag{2.4}$$

where we have used the inequality (2.1) with $b = 1$. Using notation

$$p(t) = \max_{i=1,\dots,m} p_i(t),$$

we have

$$x(t) \geq z(t)\left(1 - p(t)\sum_{i=1}^{m}\alpha_i - \frac{p(t)}{z(t)}\sum_{i=1}^{m}(1-\alpha_i)\right). \tag{2.5}$$

Setting (2.3) into (2.5), we obtain

$$x(t) \geq 2k\left(1 - p(t)\sum_{i=1}^{m}\alpha_i - \frac{p(t)}{k}\sum_{i=1}^{m}(1-\alpha_i)\right).$$

Taking (H_2) into the account, we get

$$x(t) \geq k > 0, \ \ t \geq t_1. \tag{2.6}$$

Integrating (E) from t to ∞ and using (2.6) in the resulting inequality, we have

$$z'(t) \geq \frac{k^\gamma}{a(t)} \int_t^\infty q(s) \mathrm{d}s.$$

Integrating the above inequality once more from t_1 to t, we obtain

$$z(t) \geq z(t_1) + k^\gamma \int_{t_1}^t \frac{1}{a(u)} \int_u^\infty q(s) \mathrm{d}s \mathrm{d}u,$$

which in view of (2.2) implies that $z(t) \to \infty$ for $t \to \infty$ and the proof is complete. □

Now, we present our first oscillation criterion for the case when (E) is superlinear.

Theorem 2.3. *Assume that* (2.2) *holds and* $\gamma > 1$. *If*

$$\limsup_{t\to\infty} \Bigg\{ R^{-\gamma}(\sigma(t)) \int_{t_1}^{\sigma(t)} R(s) q(s) R^\gamma(\sigma(s)) \mathrm{d}s + R^{1-\gamma}(\sigma(t)) \int_{\sigma(t)}^t q(s) R^\gamma(\sigma(s)) \mathrm{d}s + R(\sigma(t)) \int_t^\infty q(s) \mathrm{d}s \Bigg\} > 0,$$

then (E) *is oscillatory.*

Proof. Assume to the contrary that equation (E) possesses an eventually positive solution $x(t)$. Taking (H_2) and properties of $z(t)$ into account, one can see that

$$p(t) \sum_{i=1}^m \alpha_i + \frac{p(t)}{z(t)} \sum_{i=1}^m (1-\alpha_i) < \varepsilon$$

for any $\varepsilon \in (0,1)$. It follows from the above inequality and (2.5) that

$$x(t) \geq \lambda z(t), \tag{2.7}$$

where $\lambda = (1-\varepsilon) \in (0,1)$. Setting (2.7) into (E), we have

$$(a(t) z'(t))' + q(t) z^\gamma(\sigma(t)) \lambda^\gamma \leq 0. \tag{2.8}$$

An integration of (2.8) yields

$$\begin{aligned}
z(t) &\geq \int_{t_1}^{t} \frac{1}{a(u)} \int_{u}^{\infty} q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s \mathrm{d}u \\
&= \int_{t_1}^{t} \frac{1}{a(u)} \int_{u}^{t} q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s \mathrm{d}u + \int_{t_1}^{t} \frac{1}{a(u)} \int_{t}^{\infty} q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s \mathrm{d}u \\
&= \int_{t_1}^{t} R(s) q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s + R(t) \int_{t}^{\infty} q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
z(\sigma(t)) &\geq \int_{t_1}^{\sigma(t)} R(s) q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s + R(\sigma(t)) \int_{\sigma(t)}^{\infty} q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s \\
&= \int_{t_1}^{\sigma(t)} R(s) q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s + R(\sigma(t)) \int_{\sigma(t)}^{t} q(s) z^{\lambda}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s \\
&\quad + R(\sigma(t)) \int_{t}^{\infty} q(s) z^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s.
\end{aligned}$$

Using that $z(t)$ is increasing and $\frac{z(t)}{R(t)}$ is decreasing, we obtain

$$\begin{aligned}
z(\sigma(t)) \geq \frac{z^{\gamma}(\sigma(t))}{R^{\gamma}(\sigma(t))} &\int_{t_1}^{\sigma(t)} R(s) q(s) R^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s \\
&+ R^{1-\gamma}(\sigma(t)) z^{\gamma}(\sigma(t)) \int_{\sigma(t)}^{t} q(s) R^{\gamma}(\sigma(s)) \lambda^{\gamma} \mathrm{d}s \\
&+ R(\sigma(t)) z^{\gamma}(\sigma(t)) \int_{t}^{\infty} q(s) \lambda^{\gamma} \mathrm{d}s.
\end{aligned} \tag{2.9}$$

That is

$$z^{1-\gamma}(\sigma(t)) \geq R^{-\gamma}(\sigma(t)) \int_{t_1}^{\sigma(t)} R(s)q(s)R^{\gamma}(\sigma(s))\lambda^{\gamma}\mathrm{d}s + R^{1-\gamma}(\sigma(t)) \int_{\sigma(t)}^{t} q(s)R^{\gamma}(\sigma(s))\lambda^{\gamma}\mathrm{d}s + R(\sigma(t)) \int_{t}^{\infty} q(s)\lambda^{\gamma}\mathrm{d}s.$$

Since $z(t) \to \infty$ as $t \to \infty$, taking $\limsup$ as $t \to \infty$ on both sides of the previous inequality, we are led to a contradiction with the assumptions of the theorem. The proof is complete. □

The following oscillatory results cover the case when equation (E) is sublinear.

Theorem 2.4. *Let* $0 < \gamma < 1$ *and*

$$\int_{t_1}^{\infty} q(s)R^{\gamma}(\sigma(s))ds = \infty. \tag{2.10}$$

If

$$\limsup_{t\to\infty} \left\{ \frac{1}{R(\sigma(t))} \int_{t_1}^{\sigma(t)} R(s)q(s)R^{\gamma}(\sigma(s))\mathrm{d}s + \int_{\sigma(t)}^{t} q(s)R^{\gamma}(\sigma(s))\mathrm{d}s + R^{\gamma}(\sigma(t))) \int_{t}^{\infty} q(s)\mathrm{d}s \right\} > 0, \tag{2.11}$$

then (E) *is oscillatory.*

Proof. Assume that (E) possesses an eventually positive solution $x(t) > 0$ for $t \geq t_0$. It is easy to see that $z(t) > 0$ for $t \geq t_1$. Moreover $z(t)$ is an increasing function and $\frac{z(t)}{R(t)}$ is decreasing. We shall prove that (2.10) implies

$$\lim_{t\to\infty} \frac{z(t)}{R(t)} = 0. \tag{2.12}$$

Assume on the contrary that

$$\lim_{t\to\infty} \frac{z(t)}{R(t)} = \ell > 0.$$

Then $\frac{z(t)}{R(t)} > \ell$ and so

$$z^\gamma(\sigma(t)) \geq \ell^\gamma R^\gamma(\sigma(t)).$$

Moreover, by integrating (2.8) from t_1 to ∞, we get

$$a(t_1)z'(t_1) \geq \lambda^\gamma \ell^\gamma \int_{t_1}^{\infty} q(s)R^\gamma(\sigma(s))ds.$$

This contradicts the assumption of the theorem and we conclude that (2.12) holds. On the other hand, setting

$$u(t) = \frac{z(\sigma(t))}{R(\sigma(t))}$$

in the inequality (2.9), we see that

$$\begin{aligned} u(t)^{1-\gamma} \geq \frac{1}{R(\sigma(t))} & \int_{t_1}^{\sigma(t)} R(s)q(s)R^\gamma(\sigma(s))\lambda^\gamma \mathrm{d}s \\ & + \int_{\sigma(t)}^{t} q(s)R^\gamma(\sigma(s))\lambda^\gamma \mathrm{d}s + R^\gamma(\sigma(t)) \int_{t}^{\infty} q(s)\lambda^\gamma \mathrm{d}s. \end{aligned}$$

Applying the lim sup as $t \to \infty$ on both sides of the previous inequality, we have a contradiction with the assumptions of our theorem. The proof is complete. □

For the linear case of (E) we provide the following oscillatory criterion.

Theorem 2.5. *Assume that*

$$\begin{aligned} \limsup_{t\to\infty} \Bigg\{ \frac{1}{R(\sigma(t))} & \int_{t_1}^{\sigma(t)} R(s)q(s)R(\sigma(s))\mathrm{d}s \\ & + \int_{\sigma(t)}^{t} q(s)R(\sigma(s))\mathrm{d}s + R(\sigma(t)) \int_{t}^{\infty} q(s)\mathrm{d}s \Bigg\} > 1, \end{aligned} \tag{2.13}$$

then

$$(a(t)z'(t))' + q(t)x(\sigma(t)) = 0 \tag{E_L}$$

is oscillatory.

Proof. We assume to the contrary that (E_L) has an eventually positive solution $x(t)$. Proceeding as in the proof of Theorem 2.2, we are lead to (2.9) with $\gamma = 1$. Consequently,

$$\begin{aligned}1 \geq \frac{1}{R(\sigma(t))} \int_{t_3}^{\sigma(t)} R(s)q(s)R(\sigma(s))\lambda \mathrm{d}s \\ + \int_{\sigma(t)}^{t} q(s)R(\sigma(s))\lambda \mathrm{d}s + R(\sigma(t)) \int_{t}^{\infty} q(s)\lambda \mathrm{d}s.\end{aligned}$$

Taking lim sup as $t \to \infty$ on both sides of above inequality, we obtain a contradiction with (2.13). The proof is complete. □

3. EXAMPLES

In this section, we provide some examples to illustrate the main results.

Example 3.1. Consider the following differential equation with a couple of sublinear neutral terms

$$\left(x(t) + \frac{1}{t}\, x^{1/3}\left(\frac{t}{2}\right) + \frac{1}{t^2}\, x^{1/5}\left(\frac{t}{3}\right)\right)'' + \frac{a}{t^{4/3}}\, x^{1/3}\left(\frac{t}{2}\right) = 0, \quad t > 0. \tag{3.1}$$

It is easy to verify that for our equation

$$R(t) = t, \quad f(u) = u^{1/3}.$$

Criterion (2.11) from Theorem 2.4 reduces to

$$\frac{a}{2^{1/3}}\,(4 + \ln 2) > 0,$$

and so we conclude that (3.1) is oscillatory for all $a > 0$.

Example 3.2. We consider

$$\left(x(t) + \sum_{i=1}^{m} \frac{p_i}{t^i} x^{\alpha_i}(\tau_i(t))\right)'' + \frac{a}{t^2}\, x\left(\frac{t}{3}\right) = 0, \quad t > 0, \tag{3.2}$$

where $0 < \alpha_i \leq 1$, $p_i \geq 0$, $\tau_i(t) < t$ and a is a positive constant. Criterion (2.13) from Theorem 2.5 yields

$$\frac{a}{3}(2 + \ln 3) > 1,$$

which guarantees oscillation of our equation.

4. SUMMARY

In this paper, we have established new integral conditions for oscillation of second order nonlinear differential equations with several sublinear neutral terms. Our criteria are of high generality, we cover all possible cases, where (E) is sublinear, superlinear and linear. Our results generalize and extend those reported in the literature. Presented results can be easily extended to more general nonlinear equation.

$$(a(t)z'(t))' + q(t)f(x(\sigma(t))) = 0,$$

employing condition

(H_4) $f \in C(-\infty,\infty)$, $f'(u) \geq 0$, $uf(u) > 0$, for $u \neq 0$, $-f(-uv) \geq f(uv) \geq f(u)f(v)$ for $uv > 0$.

We would like to mention that the function $f(u) = u^\gamma$ is not the only one that satisfies (H_4). It easy to verify that for

$$f(u) = \frac{0.9u^3 + 0.81u}{1+u^2}, \qquad f(u) = u \arctan u + (\pi/2 + 4)u$$

condition (H_4) holds true.

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