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DEFORMATION OF SEMICIRCULAR AND CIRCULAR LAWS VIA p-ADIC NUMBER FIELDS AND SAMPLING OF PRIMES

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Abstract. In this paper, we study semicircular elements and circular elements in a certain Banach $*$-probability space $(\mathfrak{LS}, \tau^0)$ induced by analysis on the p-adic number fields $\mathbb{Q}_p$ over primes p. In particular, by truncating the set $\mathcal{P}$ of all primes for given suitable real numbers $t < s$ in $\mathbb{R}$, two different types of truncated linear functionals $\tau_{t_1<t_2}$, and $\tau^+_{t_1<t_2}$ are constructed on the Banach $*$-algebra $\mathfrak{LS}$. We show how original free distributional data (with respect to τ^0) are distorted by the truncations on $\mathcal{P}$ (with respect to $\tau_{t<s}$, and $\tau^+_{t<s}$). As application, distorted free distributions of the semicircular law, and those of the circular law are characterized up to truncation.

Keywords: free probability, primes, p-adic number fields, Banach $*$-probability spaces, semicircular elements, circular elements, truncated linear functionals.

Mathematics Subject Classification: 11R56, 46L54, 47L30, 47L55.

1. INTRODUCTION

The main purposes of this paper are (i) to construct semicircular elements induced by analysis on the *p-adic number fields* $\mathbb{Q}_p$ over *primes* p, in a certain *Banach $*$-probability space* $(\mathfrak{LS}, \tau^0)$, (ii) to establish other types of linear functionals $\tau_{t<s}$, and $\tau^+_{t<s}$ on the Banach $*$-algebra $\mathfrak{LS}$ for suitable real numbers $t < s$ in $\mathbb{R}$, truncating the set $\mathcal{P}$ of all primes, and (iii) to study how our truncations of (ii) affect, or distort the original free-distributional data on $(\mathfrak{LS}, \tau^0)$. To do that, we restrict our interests to the Banach $*$-subalgebra $\mathbb{LS}$ of $\mathfrak{LS}$, generated by the semicircular elements of (i), and the corresponding Banach $*$-probabilistic sub-structure $(\mathbb{LS}, \tau^0)$. Our main results, in particular, characterize how *the semicircular law*, and *the circular law* are distorted by our *truncations on $\mathcal{P}$*.

In [10] and [6], we constructed and studied *weighted-semicircular elements* and *semicircular elements* induced by *p-adic number fields* $\mathbb{Q}_p$, for all $p \in \mathcal{P}$. We showed there that p-adic number theory provides *weighted-semicircular laws*, and the *semicircular law*. In this paper, certain "truncated" free-probabilistic information of the free probability of [6] is studied.

1.1. PREVIEW AND MOTIVATION

Relations between *primes* and *operators* have been studied. For instance, we considered in [5] and [4] how primes act on certain *von Neumann algebras* generated by p-adic and Adelic *measure spaces* as operators. In [3] and [9], primes are regarded as *linear functionals* acting on *arithmetic functions*. Independently, in [8], we studied free-probabilistic structures on *Hecke algebras* $\mathcal{H}(GL_2(\mathbb{Q}_p))$, for *primes* p (e.g., [2] and [26]). Number-theoretic results motivated such earlier works (see e.g., [11, 12], [13–20, 23], and [28]).

In [10], the authors constructed (weighted-)semicircular elements in a certain Banach $*$-algebra $\mathfrak{LS}_p$ induced by the $*$-*algebra* $\mathcal{M}_p$ of *measurable functions* on a p-adic number fields $\mathbb{Q}_p$, for a prime $p \in \mathcal{P}$. In [6], the first-named author constructed the *free product* Banach $*$-probability space $\left(\mathfrak{LS}, \tau^0\right)$ of the Banach $*$-algebras $\{\mathfrak{LS}_p\}_{p\in\mathcal{P}}$ of [10], and studied (weighted-)semicircular elements of $\mathfrak{LS}$ as *free generators*. As application, the asymptotic semicircular laws "over $\mathcal{P}$" are considered in [7].

To make this paper be as self-contained as possible, some main results from [6] will be re-considered below, in short Sections 1 through 7. In this paper, we are interested in the cases where the free product linear functional τ^0 on $\mathfrak{LS}$ of [6] is truncated over $\mathcal{P}$. How such truncations affect, or distort, the original free-distributional data? Especially, how such truncations distort the semicircular law on $\mathfrak{LS}$? The answers to these questions constitute major parts of our main results. As application, we characterize how our truncations distort *the circular law* on $\mathfrak{LS}$.

1.2. OVERVIEW

In Sections 2, we briefly introduce backgrounds of our works. In the short Sections 3 through 7, we construct our Banach $*$-probability space $(\mathfrak{LS}, \tau^0)$, and study (*weighted-*)*semicircular elements* induced from p-adic analysis on $\mathbb{Q}_p$, for primes p.

In Section 8, we define a free-probabilistic sub-structure $\mathbb{LS}_0 = \left(\mathbb{LS}, \tau^0\right)$ of $\left(\mathfrak{LS}, \tau^0\right)$, generated by the free reduced words of $\mathfrak{LS}$, having "non-zero" free distributions, and study free-probabilistic properties on $\mathbb{LS}_0$; and then, construct *truncated linear functionals of* τ^0 on $\mathbb{LS}$ to study how free-probabilistic data of such *free reduced words* are distorted from our truncations on primes, in Section 9.

In Section 10, we provide a different type of truncated linear functionals on $\mathbb{LS}$ over $\mathcal{P}$ under direct product, and investigate new free-probabilistic structures on $\mathbb{LS}$. Remark that the truncated free probabilistic structures of $\mathbb{LS}_0$ in Sections 9 and 10 are totally different from each other.

In Section 11, to distinguish-and-emphasize the differences between them, we provide some applications of our main results of Sections 8, 9 and 10; by taking truncated linear functionals of Sections 9 and 10 on $\mathbb{LS}$. In particular, we show how the circular law is distorted (or affected) by the truncations on $\mathcal{P}$.

Independently, in Section 11, a new type of free random variables is introduced. A free random variable x is said to be *followed by the semicircular law* in a topological $*$-probability space (A, ψ), if (i) x is not self-adjoint, as an operator, and (ii) the free distribution of x is characterized by the joint free moments of x and its adjoint x^*, satisfying

$$\psi\left(x^{r_1}x^{r_2}\dots x^{r_n}\right) = \omega_n c_{\frac{n}{2}},$$

for all $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where

$$\omega_k = \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd,} \end{cases}$$

and

$$c_k = \text{the } k\text{-th } \textit{Catalan number} = \frac{(2k)!}{k!(k+1)!}$$

for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. PRELIMINARIES

In this section, we offer about background for our work.

2.1. FREE PROBABILITY

For basic *free probability*, see [27] and [29] (and the cited papers therein). *Free probability* is the noncommutative operator-algebraic version of classical measure theory (including probability theory) and statistical analysis. As an independent branch of operator algebra theory, it has various applications not only in functional analysis (e.g., [21], [22, 24] and [25]), but also in related fields (e.g., [1] through [10]).

We here use combinatorial free probability of Speicher (e.g., [27]). In the text, without introducing detailed definitions and combinatorial backgrounds, *free moments* and *free cumulants* of operators will be computed to verify the *free distributions* of them. Also, we use *free product of $*$-probability spaces*, without precise introduction.

2.2. ANALYSIS ON $\mathbb{Q}_p$

For more about *p-adic analysis* and *Adelic analysis*, see e.g., [14, 17, 23, 29] and [28]. In this paper, we use same definitions, and notations of [28]. Let $p \in \mathcal{P}$ be a prime, and let $\mathbb{Q}$ be the set of all *rational numbers*. Define a *non-Archimedean norm* $|\cdot|_p$ on $\mathbb{Q}$ by

$$|x|_p = \left|p^k \frac{a}{b}\right|_p = \frac{1}{p^k},$$

whenever $x = p^k \frac{a}{b}$, where $k, a \in \mathbb{Z}$, and $b \in \mathbb{Z} \setminus \{0\}$. We call $|\cdot|_p$, the *p-norm on* $\mathbb{Q}$ (as in [28]), for all $p \in \mathcal{P}$.

The *p-adic number field* $\mathbb{Q}_p$ is the maximal p-norm closures in $\mathbb{Q}$, i.e., under norm topology, the set $\mathbb{Q}_p$ forms a *Banach space*, for $p \in \mathcal{P}$.

All elements x of $\mathbb{Q}_p$ are expressed by

$$x = \sum_{k=-N}^{\infty} x_k p^k, \text{ with } x_k \in \{0, 1, \ldots, p-1\},$$

for $N \in \mathbb{N}$, decomposed by

$$x = \sum_{l=-N}^{-1} x_l p^l + \sum_{k=0}^{\infty} x_k p^k.$$

If $x = \sum_{k=0}^{\infty} x_k p^k$ in $\mathbb{Q}_p$, then we call x, a *p-adic integer*. Remark that, $x \in \mathbb{Q}_p$ is a p-adic integer, if and only if $|x|_p \leq 1$. So, by collecting all p-adic integers in $\mathbb{Q}_p$, one can define the *unit disk* $\mathbb{Z}_p$ *of* $\mathbb{Q}_p$,

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Under the *p-adic addition* and the *p-adic multiplication* of [28], this Banach space $\mathbb{Q}_p$ forms a *field* algebraically, i.e., $\mathbb{Q}_p$ is a *Banach field*.

One can view this *Banach field* $\mathbb{Q}_p$ as a *measure space*,

$$\mathbb{Q}_p = (\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p),$$

where $\sigma(\mathbb{Q}_p)$ is the *σ-algebra of* $\mathbb{Q}_p$, consisting of all *μ_p-measurable subsets*, where μ_p is a left-and-right additive invariant *Haar measure* on $\mathbb{Q}_p$, satisfying

$$\mu_p(\mathbb{Z}_p) = 1.$$

If we define

$$U_k = p^k \mathbb{Z}_p = \{p^k x \in \mathbb{Q}_p : x \in \mathbb{Z}_p\}, \tag{2.1}$$

for all $k \in \mathbb{Z}$, then these μ_p-measurable subsets U_k's of (2.1) satisfy

$$\mathbb{Q}_p = \bigcup_{k \in \mathbb{Z}} U_k,$$

and

$$\mu_p(U_k) = \frac{1}{p^k} = \mu_p(x + U_k), \text{ for all } x \in \mathbb{Q}_p,$$

and

$$\ldots \subset U_2 \subset U_1 \subset U_0 = \mathbb{Z}_p \subset U_{-1} \subset U_{-2} \subset \ldots, \tag{2.2}$$

i.e., the family $\{U_k\}_{k \in \mathbb{Z}}$ of (2.1) forms a *basis* of the topology for $\mathbb{Q}_p$ (e.g., [28]).

Define now subsets $\partial_k \in \sigma(\mathbb{Q}_p)$ by

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}. \tag{2.3}$$

We call such μ_p-measurable subsets ∂_k of (2.3), the *k-th boundaries* (*of* U_k) in $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. By (2.2) and (2.3), one obtains that

$$\mathbb{Q}_p = \bigsqcup_{k \in \mathbb{Z}} \partial_k,$$

and

$$\mu_p(\partial_k) = \mu_p(U_k) - \mu_p(U_{k+1}) = \frac{1}{p^k} - \frac{1}{p^{k+1}}, \tag{2.4}$$

where $\sqcup$ means the *disjoint union*, for all $k \in \mathbb{Z}$.

Now, let $\mathcal{M}_p$ be the (pure-algebraic) *algebra*,

$$\mathcal{M}_p = \mathbb{C}\left[\{\chi_S : S \in \sigma(\mathbb{Q}_p)\}\right], \tag{2.5}$$

where χ_S are the usual *characteristic functions* of μ_p-measurable subsets S of $\mathbb{Q}_p$. So, $f \in \mathcal{M}_p$ if and only if

$$f = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S, \text{ with } t_S \in \mathbb{C}, \tag{2.6}$$

where $\sum$ is the *finite sum*. Remark that the algebra $\mathcal{M}_p$ of (2.5) forms a $*$-*algebra over* $\mathbb{C}$, with its well-defined *adjoint*,

$$\left(\sum_{S \in \sigma(G_p)} t_S \chi_S\right)^* \overset{def}{=} \sum_{S \in \sigma(G_p)} \overline{t_S} \chi_S,$$

where $t_S \in \mathbb{C}$ with their *conjugates* $\overline{t_S}$ in $\mathbb{C}$.

Let $f \in \mathcal{M}_p$ be in the sense of (2.6) Then one can define the *integral of* f by

$$\int_{\mathbb{Q}_p} f d\mu_p = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \mu_p(S). \tag{2.7}$$

Remark that, by (2.5), the integral (2.7) is unbounded on $\mathcal{M}_p$, i.e.,

$$\int_{\mathbb{Q}_p} \chi_{\mathbb{Q}_p} d\mu_p = \mu_p(\mathbb{Q}_p) = \infty, \tag{2.8}$$

by (2.2).

Note that, by (2.4), if $S \in \sigma(\mathbb{Q}_p)$, then there exists a unique subset Λ_S of $\mathbb{Z}$, such that

$$\Lambda_S = \{j \in \mathbb{Z} : S \cap \partial_j \neq \varnothing\}, \tag{2.9}$$

satisfying

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \int_{\mathbb{Q}_p} \sum_{j \in \Lambda_S} \chi_{S \cap \partial_j} d\mu_p = \sum_{j \in \Lambda_S} \mu_p(S \cap \partial_j)$$

by (2.7)

$$\leq \sum_{j\in\Lambda_S} \mu_p\left(\partial_j\right) = \sum_{j\in\Lambda_S} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{2.10}$$

by (2.4), for the subset Λ_S of $\mathbb{Z}$ of (2.9).

Remark again that the right-hand side of (2.10) can be ∞, for instance, $\Lambda_{\mathbb{Q}_p} = \mathbb{Z}$, e.g., see (2.4), (2.7) and (2.8). By (2.10), one obtains the following proposition.

Proposition 2.1. *Let $S \in \sigma(\mathbb{Q}_p)$, and let $\chi_S \in \mathcal{M}_p$. Then there exist $r_j \in \mathbb{R}$, such that*

$$0 \leq r_j = \frac{\mu_p(S \cap \partial_j)}{\mu_p(\partial_j)} \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_S,$$

and

$$\int_{\mathbb{Q}_p} \chi_S d\mu_p = \sum_{j\in\Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right). \tag{2.11}$$

3. FREE-PROBABILISTIC MODELS ON $\mathcal{M}_p$

Throughout this section, fix a prime $p \in \mathcal{P}$, and let $\mathbb{Q}_p$ be the corresponding p-adic number field, and let $\mathcal{M}_p$ be the $*$-algebra (2.5). In this section, we establish a suitable free-probabilistic model on $\mathcal{M}_p$. Remark that, since $\mathcal{M}_p$ is a "commutative" $*$-algebra, free probability theory is not needed to be used-or-applied, but, for our purposes, we here construct a free-probability-theoretic model on $\mathcal{M}_p$ under free-probabilistic language and terminology.

Let U_k be the basis elements (2.1), and ∂_k, their boundaries (2.3) of $\mathbb{Q}_p$, i.e.,

$$U_k = p^k \mathbb{Z}_p, \text{ for all } k \in \mathbb{Z}, \tag{3.1}$$

and

$$\partial_k = U_k \setminus U_{k+1}, \text{ for all } k \in \mathbb{Z}.$$

Define a linear functional $\varphi_p : \mathcal{M}_p \to \mathbb{C}$ by the *integration* (2.7), i.e.,

$$\varphi_p\left(f\right) = \int_{\mathbb{Q}_p} f d\mu_p, \text{ for all } f \in \mathcal{M}_p. \tag{3.2}$$

Then, by (2.11), one obtains that

$$\varphi_p\left(\chi_{U_j}\right) = \frac{1}{p^j}, \quad \text{and} \quad \varphi_p\left(\chi_{\partial_j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}},$$

since

$$\Lambda_{U_j} = \{k \in \mathbb{Z} : k \geq j\}, \text{ and } \Lambda_{\partial_j} = \{j\},$$

for all $j \in \mathbb{Z}$, where Λ_S are in the sense of (2.9) for all $S \in \sigma(\mathbb{Q}_p)$. Note that, by (2.8), this linear functional φ_p of (3.2) is unbounded on $\mathcal{M}_p$.

Definition 3.1. The pair $(\mathcal{M}_p, \varphi_p)$ is called the p-adic (unbounded-)measure space for $p \in \mathcal{P}$, where φ_p is the linear functional (3.2) on $\mathcal{M}_p$.

Let ∂_k be the k-th boundaries (3.1) of $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then, for $k_1, k_2 \in \mathbb{Z}$, one obtains that

$$\chi_{\partial_{k_1}} \chi_{\partial_{k_2}} = \chi_{\partial_{k_1} \cap \partial_{k_2}} = \delta_{k_1,k_2} \chi_{\partial_{k_1}},$$

and hence,

$$\varphi_p \left(\chi_{\partial_{k_1}} \chi_{\partial_{k_2}}\right) = \delta_{k_1,k_2} \varphi_p \left(\chi_{\partial_{k_1}}\right) = \delta_{k_1,k_2} \left(\frac{1}{p^{k_1}} - \frac{1}{p^{k_1+1}}\right). \tag{3.3}$$

Proposition 3.2. *Let $(j_1, \ldots, j_N) \in \mathbb{Z}^N$, for $N \in \mathbb{N}$. Then*

$$\prod_{l=1}^{N} \chi_{\partial_{j_l}} = \delta_{(j_1,\ldots,j_N)} \chi_{\partial_{j_1}} \text{ in } \mathcal{M}_p,$$

and hence,

$$\varphi_p \left(\prod_{l=1}^{N} \chi_{\partial_{j_l}}\right) = \delta_{(j_1,\ldots,j_N)} \left(\frac{1}{p^{j_1}} - \frac{1}{p^{j_1+1}}\right), \tag{3.4}$$

where

$$\delta_{(j_1,\ldots,j_N)} = \left(\prod_{l=1}^{N-1} \delta_{j_l, j_{l+1}}\right) \left(\delta_{j_N, j_1}\right).$$

Proof. The proof of (3.4) is done by induction on (3.3). □

Recall that, for any $S \in \sigma(\mathbb{Q}_p)$,

$$\varphi_p(\chi_S) = \sum_{j \in \Lambda_S} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{3.5}$$

for some $0 \le r_j \le 1$, for $j \in \Lambda_S$, by (2.11). So, by (3.5), if $S_1, S_2 \in \sigma(\mathbb{Q}_p)$, then

$$\begin{aligned} \chi_{S_1} \chi_{S_2} &= \left(\sum_{k \in \Lambda_{S_1}} \chi_{S_1 \cap \partial_k}\right) \left(\sum_{j \in \Lambda_{S_2}} \chi_{S_2 \cap \partial_j}\right) \\ &= \sum_{(k,j) \in \Lambda_{S_1} \times \Lambda_{S_2}} \delta_{k,j} \chi_{(S_1 \cap S_2) \cap \partial_j} = \sum_{j \in \Lambda_{S_1,S_2}} \chi_{(S_1 \cap S_2) \cap \partial_j}, \end{aligned} \tag{3.6}$$

where

$$\Lambda_{S_1,S_2} = \Lambda_{S_1} \cap \Lambda_{S_2},$$

by (2.4).

Proposition 3.3. *Let $S_l \in \sigma(\mathbb{Q}_p)$, and let $\chi_{S_l} \in (\mathcal{M}_p, \varphi_p)$, for $l = 1, \ldots, N$, for $N \in \mathbb{N}$. Let*

$$\Lambda_{S_1,\ldots,S_N} = \bigcap_{l=1}^{N} \Lambda_{S_l} \text{ in } \mathbb{Z},$$

where Λ_{S_l} are in the sense of (2.9), for $l = 1, \ldots, N$. Then there exist $r_j \in \mathbb{R}$, such that

$$0 \leq r_j \leq 1 \text{ in } \mathbb{R}, \text{ for all } j \in \Lambda_{S_1,\ldots,S_N},$$

and

$$\varphi_p \left(\prod_{l=1}^{N} \chi_{S_l} \right) = \sum_{j \in \Lambda_{S_1,\ldots,S_N}} r_j \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right). \tag{3.7}$$

Proof. The proof of (3.7) is done by the induction on (3.6), and by (3.4). □

4. REPRESENTATIONS OF $(\mathcal{M}_p, \varphi_p)$

Fix a prime $p \in \mathcal{P}$. Let $(\mathcal{M}_p, \varphi_p)$ be the p-adic measure space. By understanding $\mathbb{Q}_p$ as a measure space, construct the L^2-*space*,

$$H_p \overset{def}{=} L^2(\mathbb{Q}_p, \sigma(\mathbb{Q}_p), \mu_p) = L^2(\mathbb{Q}_p), \tag{4.1}$$

over $\mathbb{C}$. Then this L^2-space H_p of (4.1) is a well-defined *Hilbert space*, consisting of all square-integrable elements of $\mathcal{M}_p$, equipped with its *inner product* $\langle \cdot, \cdot \rangle_2$,

$$\langle f_1, f_2 \rangle_2 \overset{def}{=} \int_{\mathbb{Q}_p} f_1 f_2^* d\mu_p, \tag{4.2}$$

for all $f_1, f_2 \in H_p$, inducing the L^2-norm,

$$\|f\|_2 \overset{def}{=} \sqrt{\langle f, f \rangle_2}, \text{ for all } f \in H_p,$$

where $\langle \cdot, \cdot \rangle_2$ is the inner product (4.2) on H_p.

Definition 4.1. We call the Hilbert space H_p of (4.1), the p-adic Hilbert space.

By the definition (4.1) of the p-adic Hilbert space H_p, our $*$-algebra $\mathcal{M}_p$ acts on H_p, via an algebra-action α^p,

$$\alpha^p(f)(h) = fh, \text{ for all } h \in H_p, \tag{4.3}$$

for all $f \in \mathcal{M}_p$. i.e., the morphism α^p of (4.3) is a $*$-homomorphism from $\mathcal{M}_p$ to the *operator algebra* $B(H_p)$ consisting of all bounded linear operators on H_p. For instance,

$$\alpha^p(\chi_{\mathbb{Q}_p}) \left(\sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \right) = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_{\mathbb{Q}_p \cap S} = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S, \tag{4.4}$$

for all $h = \sum_{S \in \sigma(\mathbb{Q}_p)} t_S \chi_S \in H_p$, with $\|h\|_2 < \infty$, for $\chi_{\mathbb{Q}_p} \in \mathcal{M}_p$, even though $\chi_{\mathbb{Q}_p} \notin H_p$.

Indeed, it is not difficult to check that

$$\begin{aligned} \alpha^p(f_1 f_2) &= \alpha^p(f_1)\alpha^p(f_2) \text{ on } H_p, \text{ for all } f_1, f_2 \in \mathcal{M}_p, \\ (\alpha^p(f))^* &= \alpha(f^*) \text{ on } H_p, \text{ for all } f \in \mathcal{M}_p \end{aligned} \tag{4.5}$$

(e.g., see [6] and [10]).

Denote $\alpha^p(f)$ by α_f^p, for all $f \in \mathcal{M}_p$. Also, for convenience, denote $\alpha_{\chi_S}^p$ simply by α_S^p, for all $S \in \sigma(\mathbb{Q}_p)$.

Note that, by (4.4), one has a well-defined operator $\alpha_{\mathbb{Q}_p}^p = \alpha_{\chi_{\mathbb{Q}_p}}^p$ in $B(H_p)$, and it satisfies that

$$\alpha_{\mathbb{Q}_p}^p(h) = h = 1_{H_p}(h), \text{ for all } h \in H_p, \tag{4.6}$$

where $1_{H_p} \in B(H_p)$ is the identity operator on H_p.

Proposition 4.2. *The pair (H_p, α^p) is a well-determined Hilbert space representation of $\mathcal{M}_p$.*

Proof. It is sufficient to show that α^p is an algebra-action of $\mathcal{M}_p$ acting on H_p. But, by (4.5), this linear morphism α^p of (4.3) is indeed a $*$-homomorphism from $\mathcal{M}_p$ into $B(H_p)$. □

For a p-adic number fields, readers can check other types of representations in e.g., [18] and [20], different from our Hilbert-space representation (H_p, α^p).

Definition 4.3. The Hilbert-space representation (H_p, α^p) is said to be the p-adic (Hilbert-space) representation of $\mathcal{M}_p$.

Depending on the p-adic representation (H_p, α^p) of $\mathcal{M}_p$, one can construct the C^*-subalgebra M_p of $B(H_p)$ as follows.

Definition 4.4. Let M_p be the operator-norm closure of $\mathcal{M}_p$ in the operator algebra $B(H_p)$, i.e.,

$$M_p \overset{def}{=} \overline{\alpha^p(\mathcal{M}_p)} = \overline{\mathbb{C}\left[\alpha_f^p : f \in \mathcal{M}_p\right]} \tag{4.7}$$

in $B(H_p)$, where $\overline{X}$ mean the operator-norm closures of subsets X of $B(H_p)$. This C^*-algebra M_p of (4.7) is called the p-adic C^*-algebra of $(\mathcal{M}_p, \varphi_p)$.

By the definition (4.7) of the p-adic C^*-algebra M_p, it is a unital C^*-algebra, containing its unity (or the unit, or the multiplication-identity) $1_{H_p} = \alpha_{\mathbb{Q}_p}^p$, by (4.6).

5. FREE-PROBABILISTIC MODELS ON M_p

Throughout this section, let us fix a prime $p \in \mathcal{P}$, and let $(\mathcal{M}_p, \varphi_p)$ be the corresponding p-adic measure space, and let (H_p, α^p) be the p-adic representation of $\mathcal{M}_p$, inducing the corresponding p-adic C^*-algebra M_p of (4.7). We here consider suitable (non-traditional) free-probabilistic models on M_p.

Define a linear functional $\varphi_j^p : M_p \to \mathbb{C}$ by a linear morphism,

$$\varphi_j^p(a) \overset{def}{=} \left\langle a(\chi_{\partial_j}), \chi_{\partial_j} \right\rangle_2, \text{ for all } a \in M_p, \tag{5.1}$$

for $\chi_{\partial_j} \in H_p$, where $\langle \cdot, \cdot \rangle_2$ is the inner product (4.2) on the p-adic Hilbert space H_p of (4.1), and ∂_j are the j-th boundaries (3.1) of $\mathbb{Q}_p$, for all $j \in \mathbb{Z}$. It is not hard to check such a linear functional φ_j^p on M_p is bounded, since

$$\begin{aligned}\varphi_j^p(\alpha_S^p) &= \left\langle \alpha_S^p\left(\chi_{\partial_j}\right), \chi_{\partial_j}\right\rangle_2 = \left\langle \chi_S\chi_{\partial_j}, \chi_{\partial_j}\right\rangle_2 = \left\langle \chi_{S\cap\partial_j}, \chi_{\partial_j}\right\rangle_2 \\ &= \int_{\mathbb{Q}_p} \chi_{S\cap\partial_j}\, d\mu_p = \mu_p\left(S\cap\partial_j\right) \leq \mu_p\left(\partial_j\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}}\end{aligned}$$

for all $S \in \sigma(\mathbb{Q}_p)$, for any fixed $j \in \mathbb{Z}$.

Remark that, if $a \in M_p$, then

$$a = \sum_{S\in\sigma(\mathbb{Q}_p)} t_S\alpha_S^p \text{ in } M_p \quad (t_S \in \mathbb{C}),$$

where $\sum$ is finite or infinite (limit of finite) sum(s) under C^*-topology of M_p, and hence, the morphisms φ_j^p of (5.1) are indeed well-defined bounded linear functionals on M_p, for all $j \in \mathbb{Z}$.

Definition 5.1. Let φ_j^p be bounded linear functionals (5.1) on the p-adic C^*-algebra M_p, for all $j \in \mathbb{Z}$. Then the pairs $\left(M_p, \varphi_j^p\right)$ are said to be the j-th p-adic C^*-measure spaces, for all $j \in \mathbb{Z}$.

So, one can get the system

$$\{(M_p, \varphi_j^p) : j \in \mathbb{Z}\}$$

of the j-th p-adic C^*-measure spaces (M_p, φ_j^p)'s.

Note that, for any fixed $j \in \mathbb{Z}$, and (M_p, φ_j^p), the unity

$$1_{M_p} \overset{\text{denote}}{=} 1_{H_p} = \alpha_{\mathbb{Q}_p}^p \text{ of } M_p$$

satisfies that

$$\varphi_j^p\left(1_{M_p}\right) = \left\langle \chi_{\mathbb{Q}_p\cap\partial_j}, \chi_{\partial_j}\right\rangle_2 = \left\|\chi_{\partial_j}\right\|^2 = \frac{1}{p^j} - \frac{1}{p^{j+1}}.$$

So, the j-th p-adic C^*-measure space (M_p, φ_j^p) is a "bounded" measure space, but not a (classical) probability space, in general.

Now, fix $j \in \mathbb{Z}$, and take the corresponding j-th p-adic C^*-measure space $\left(M_p, \varphi_j^p\right)$. For $S \in \sigma\left(\mathbb{Q}_p\right)$, and an element $\alpha_S^p \in M_p$, one has that

$$\begin{aligned}\varphi_j^p(\alpha_S^p) &= \left\langle \alpha_S^p(\chi_{\partial_j}), \chi_{\partial_j}\right\rangle_2 = \left\langle \chi_{S\cap\partial_j}, \chi_{\partial_j}\right\rangle_2 \\ &= \int_{\mathbb{Q}_p} \chi_{S\cap\partial_j}\, d\mu_p = \mu_p\left(S\cap\partial_j\right) = r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right),\end{aligned} \tag{5.2}$$

by (3.7), for some $0 \leq r_S \leq 1$ in $\mathbb{R}$.

Proposition 5.2. *Let $S \in \sigma(\mathbb{Q}_p)$, and $\alpha_S^p \in (M_p, \varphi_j^p)$, for a fixed $j \in \mathbb{Z}$. Then there exists $r_S \in \mathbb{R}$, such that*

$$0 \leq r_S \leq 1 \text{ in } \mathbb{R},$$

and

$$\varphi_j^p\left((\alpha_S^p)^n\right) = r_S\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \text{ for all } n \in \mathbb{N}. \tag{5.3}$$

Proof. Remark that the element α_S^p is a projection in M_p, in the sense that

$$(\alpha_S^p)^* = \alpha_S^p = (\alpha_S^p)^2, \text{ in } M_p,$$

and hence,

$$(\alpha_S^p)^n = \alpha_S^p, \text{ for all } n \in \mathbb{N}.$$

Thus, we obtain the formula (5.3) by (5.2). □

As a corollary of (5.3), we obtain the following results.

Corollary 5.3. *Let ∂_k be the k-th boundaries* (3.1) *of $\mathbb{Q}_p$, for all $k \in \mathbb{Z}$. Then*

$$\varphi_j^p\left((\alpha_{\partial_k}^p)^n\right) = \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right), \tag{5.4}$$

for all $n \in \mathbb{N}$, for $k \in \mathbb{Z}$.

6. SEMIGROUP C^*-SUBALGEBRAS $\mathfrak{S}_p$ of M_p

Let M_p be the p-adic C^*-algebra (4.7) for $p \in \mathcal{P}$. Take operators

$$P_{p,j} = \alpha_{\partial_j}^p \in M_p, \tag{6.1}$$

for all $j \in \mathbb{Z}$.

As we have seen in (5.3) and (5.4), these operators $P_{p,j}$ are *projections* on the p-adic Hilbert space H_p in M_p, for all $\in \mathcal{P}, j \in \mathbb{Z}$. We now restrict our interests to these projections $P_{p,j}$ of (6.1).

Definition 6.1. Fix $p \in \mathcal{P}$. Let $\mathfrak{S}_p$ be the C^*-subalgebra

$$\mathfrak{S}_p = C^*\left(\{P_{p,j}\}_{j\in\mathbb{Z}}\right) = \overline{\mathbb{C}\left[\{P_{p,j}\}_{j\in\mathbb{Z}}\right]} \text{ of } M_p, \tag{6.2}$$

where $P_{p,j}$ are projections (6.1), for all $j \in \mathbb{Z}$. We call this C^*-subalgebra $\mathfrak{S}_p$, the p-adic boundary (C^*-)subalgebra of M_p.

The p-adic boundary subalgebra $\mathfrak{S}_p$ acts like a diagonal subalgebra of the p-adic C^*-algebra M_p.

Proposition 6.2. *Let $\mathfrak{S}_p$ be the p-adic boundary subalgebra* (6.2) *of the p-adic C^*-algebra M_p. Then*

$$\mathfrak{S}_p \overset{*\text{-iso}}{=} \underset{j\in\mathbb{Z}}{\oplus}\left(\mathbb{C}\cdot P_{p,j}\right) \overset{*\text{-iso}}{=} \mathbb{C}^{\oplus\mathbb{Z}}, \tag{6.3}$$

in M_p.

Proof. It suffices to show that the generating projections $\{P_{p,j}\}_{j\in\mathbb{Z}}$ of the p-adic boundary subalgebra $\mathfrak{S}_p$ are mutually orthogonal from each other. But, one can get that

$$P_{p,j_1}P_{p,j_2} = \alpha^p\left(\chi_{\partial_{j_1}^p \cap \partial_{j_2}^p}\right) = \delta_{j_1,j_2}\alpha^p_{\partial^p_{j_1}} = \delta_{j_1,j_2}P_{p,j_1},$$

in $\mathfrak{S}_p$, for all $j_1, j_2 \in \mathbb{Z}$. Therefore, the structure theorem (6.3) holds. □

Since the p-adic boundary subalgebra $\mathfrak{S}_p$ of (6.2) is a C^*-subalgebra of M_p, one can naturally obtain the measure spaces,

$$\mathfrak{S}_{p,j} \overset{denote}{=} \left(\mathfrak{S}_p, \varphi_j^p\right), \text{ for all } j \in \mathbb{Z}, \text{ for } p \in \mathcal{P}, \tag{6.4}$$

where the linear functionals φ_j^p of (6.4) are the restrictions $\varphi_j^p \mid_{\mathfrak{S}_p}$ of (5.1), for all $p \in \mathcal{P}, j \in \mathbb{Z}$.

7. WEIGHTED-SEMICIRCULAR ELEMENTS

Fix $p \in \mathcal{P}$, and let $\mathfrak{S}_p$ be the p-adic boundary subalgebra of the p-adic C^*-algebra M_p, satisfying the structure theorem (6.3). Recall that the generating projections $P_{p,j}$ of $\mathfrak{S}_p$ satisfy

$$\varphi_j^p\left(P_{p,j}\right) = \frac{1}{p^j} - \frac{1}{p^{j+1}}, \text{ for all } j \in \mathbb{Z}, \tag{7.1}$$

by (5.3) and (5.4).

Now, let ϕ be the *Euler totient function*, the *arithmetic function*,

$$\phi : \mathbb{N} \to \mathbb{C}, \tag{7.2}$$

defined by

$$\phi(n) = \left|\{k \in \mathbb{N} : k \leq n, \gcd(n,k) = 1\}\right|,$$

for all $n \in \mathbb{N}$, where gcd means the *greatest common divisor*.

By (7.2), one has

$$\phi(p) = p - 1 = p\left(1 - \frac{1}{p}\right), \text{ for all } p \in \mathcal{P}. \tag{7.3}$$

So, we have

$$\begin{aligned}\varphi_j^p\left(P_{p,k}\right) &= \delta_{j,k}\left(\frac{1}{p^j} - \frac{1}{p^{j+1}}\right) = \frac{\delta_{j,k}}{p^j}\left(1 - \frac{1}{p}\right) \\ &= \delta_{j,k}\left(\frac{p}{p^{j+1}}\left(1 - \frac{1}{p}\right)\right) = \delta_{j,k}\left(\frac{\phi(p)}{p^{j+1}}\right),\end{aligned} \tag{7.4}$$

by (7.1) and (7.3), for $P_{p,k} \in \mathfrak{S}_p$, for all $k \in \mathbb{Z}$.

Now, for a fixed prime p, define new linear functionals τ_j^p on $\mathfrak{S}_p$ by

$$\tau_j^p = \frac{1}{\phi(p)} \varphi_j^p, \text{ on } \mathfrak{S}_p, \tag{7.5}$$

for all $j \in \mathbb{Z}$, where φ_j^p are in the sense of (6.4).

Then one obtains new free-probabilistic models of $\mathfrak{S}_p$,

$$\{\mathfrak{S}_p(j) = \left(\mathfrak{S}_p, \tau_j^p\right) : p \in \mathcal{P}, j \in \mathbb{Z}\}, \tag{7.6}$$

where τ_j^p are in the sense of (7.5).

Proposition 7.1. *Let* $\mathfrak{S}_p(j) = (\mathfrak{S}_p, \tau_j^p)$ *be in the sense of* (7.6), *and let* $P_{p,k}$ *be generating operators* (6.1) *of* $\mathfrak{S}_p(j)$, *for* $p \in \mathcal{P}, j \in \mathbb{Z}$. *Then*

$$\tau_j^p \left(P_{p,k}^n\right) = \frac{\delta_{j,k}}{p^{j+1}}, \text{ for all } n \in \mathbb{N}. \tag{7.7}$$

Proof. The formula (7.7) is proven by (7.4) and (7.5), since $P_{p,k}^n = P_{p,k}$ for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$. □

7.1. SEMICIRCULAR AND WEIGHTED-SEMICIRCULAR ELEMENTS

Let (A, φ) be an arbitrary *topological $*$-probability space* (C^*-probability space, or W^*-probability space, or Banach $*$-probability space, etc.), equipped with a topological $*$-algebra A (C^*-algebra, resp., W^*-algebra, resp., Banach $*$-algebra, etc.), and a (bounded or unbounded) linear functional φ on A. If an operator $a \in A$ is regarded as an element of (A, φ), we call a, a *free random variable* of (A, φ).

Definition 7.2. Let a be a self-adjoint free random variable in (A, φ). It is said to be semicircular in (A, φ), if

$$\varphi(a^n) = \omega_n c_n, \text{ for all } n \in \mathbb{N}, \tag{7.8}$$

with

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

for all $n \in \mathbb{N}$, and

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1}\frac{(2n)!}{(n!)^2} = \frac{(2n)!}{n!(n+1)!}$$

are the n-th Catalan numbers, for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It is well-known that, if $k_n(\cdot)$ is the *free cumulant on A in terms of a linear functional φ* (in the sense of [27]), then a self-adjoint free random variable a is *semicircular* in (A, φ), if and only if

$$k_n(\underbrace{a, a, \ldots, a}_{n\text{-times}}) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.9}$$

for all $n \in \mathbb{N}$. The above equivalent free-distributional data (7.9) of the semicircularity (7.8) is obtained by the *Möbius inversion of* [27].

Motivated by (7.9), one can define the *weighted-semicircularity.*

Definition 7.3. Let $a \in (A, \varphi)$ be a self-adjoint free random variable. It is said to be weighted-semicircular in (A, φ) with its weight t_0 (in short, t_0-semicircular), if there exists $t_0 \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, such that

$$k_n(\underbrace{a, a, \ldots, a}_{n\text{-times}}) = \begin{cases} t_0 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.10}$$

for all $n \in \mathbb{N}$, where $k_n(\cdot)$ is the free cumulant on A in terms of φ.

By (7.9) and (7.10), every 1-semicircular element is semicircular. By the definition (7.10), and by the Möbius inversion of [27], a self-adjoint free random variable a is t_0-*semicircular* in (A, φ), if and only if there exists $t_0 \in \mathbb{C}^\times$, such that

$$\varphi(a^n) = \omega_n t_0^{\frac{n}{2}} c_{\frac{n}{2}}, \tag{7.11}$$

where ω_n and $c_{\frac{n}{2}}$ are in the sense of (7.8), for all $n \in \mathbb{N}$.

7.2. TENSOR PRODUCT BANACH $*$-ALGEBRA $\mathfrak{L}\mathfrak{S}_p$

Let $\mathfrak{S}_p(k) = (\mathfrak{S}_p, \tau_k^p)$ be in the sense of (7.6), for $p \in \mathcal{P}, k \in \mathbb{Z}$. Define now a *bounded linear transformations* $\mathbf{c}_p$ and $\mathbf{a}_p$ "acting on $\mathfrak{S}_p$", by the linear morphisms satisfying

$$\mathbf{c}_p\left(P_{p,j}\right) = P_{p,j+1} \quad \text{and} \quad \mathbf{a}_p\left(P_{p,j}\right) = P_{p,j-1}, \tag{7.12}$$

on $\mathfrak{S}_p$, for all $j \in \mathbb{Z}$.

By the definition (7.12), these linear transformations $\mathbf{c}_p$ and $\mathbf{a}_p$ are bounded under the operator-norm induced by the C^*-norm on $\mathfrak{S}_p$. So, the linear transformations $\mathbf{c}_p$ and $\mathbf{a}_p$ are regarded as *Banach-space operators* acting "on $\mathfrak{S}_p$", by regarding the C^*-algebra $\mathfrak{S}_p$ as a *Banach space* equipped with its C^*-norm, i.e., $\mathbf{c}_p$ and $\mathbf{a}_p$ are elements of the *operator space* $B\left(\mathfrak{S}_p\right)$ consisting of all bounded linear transformations on the Banach space $\mathfrak{S}_p$.

Definition 7.4. The Banach-space operators $\mathbf{c}_p$ and $\mathbf{a}_p$ of (7.12) are called the p-creation, respectively, the p-annihilation on $\mathfrak{S}_p$, for $p \in \mathcal{P}$. Define a new Banach-space operator $\mathbf{l}_p \in B\left(\mathfrak{S}_p\right)$, by

$$\mathbf{l}_p = \mathbf{c}_p + \mathbf{a}_p \text{ on } \mathfrak{S}_p. \tag{7.13}$$

We call it the p-radial operator on $\mathfrak{S}_p$.

Let $\mathbf{l}_p$ be the p-radial operator $\mathbf{c}_p + \mathbf{a}_p$ of (7.13) on $\mathfrak{S}_p$. Construct a *closed subspace* $\mathfrak{L}_p$ of $B\left(\mathfrak{S}_p\right)$ by

$$\mathfrak{L}_p = \overline{\mathbb{C}[\{\mathbf{l}_p\}]} \text{ in } B(\mathfrak{S}_p), \tag{7.14}$$

where $\overline{Y}$ mean the operator-norm-topology closures of all subsets Y of $B(\mathfrak{S}_p)$.

By the definition (7.14), $\mathfrak{L}_p$ is not only a closed subspace of the topological vector space $B(\mathfrak{S}_p)$, but also an algebra embedded in $B(\mathfrak{S}_p)$. On this Banach algebra $\mathfrak{L}_p$, define the adjoint $*$ by

$$\sum_{k=0}^{\infty} s_k \mathbf{l}_p^k \in \mathfrak{L}_p \longmapsto \sum_{k=0}^{\infty} \overline{s_k} \mathbf{l}_p^k \in \mathfrak{L}_p, \tag{7.15}$$

where $s_k \in \mathbb{C}$ with their conjugates $\overline{s_k} \in \mathbb{C}$ (e.g., [6]).

Then, equipped with the adjoint (7.15), this Banach algebra $\mathfrak{L}_p$ of (7.14) forms a *Banach $*$-algebra*.

Definition 7.5. Let $\mathfrak{L}_p$ be a Banach $*$-algebra (7.14) in the operator space $B(\mathfrak{S}_p)$, for $p \in \mathcal{P}$. We call it the p-radial (Banach-$*$-)algebra on $\mathfrak{S}_p$.

Let $\mathfrak{L}_p$ be the p-radial algebra (7.14) on $\mathfrak{S}_p$. Construct now the tensor product Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p$ by

$$\mathfrak{L}\mathfrak{S}_p = \mathfrak{L}_p \otimes_{\mathbb{C}} \mathfrak{S}_p, \tag{7.16}$$

where $\otimes_{\mathbb{C}}$ means the *tensor product of Banach $*$-algebras*.

Note that the operators $\mathbf{l}_p^k \otimes P_{p,j}$ generate the Banach $*$-algebra $\mathfrak{L}\mathfrak{S}_p$ of (7.16), for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $j \in \mathbb{Z}$, where $P_{p,j}$ are the generating projections of (6.1) in $\mathfrak{S}_p$, with axiomatization:

$$l_p^0 = 1_{\mathfrak{S}_p}, \text{ the identity operator on } \mathfrak{S}_p,$$

in $B(\mathfrak{S}_p)$, satisfying

$$1_{\mathfrak{S}_p}(T) = T, \text{ for all } T \in \mathfrak{S}_p,$$

for all $j \in \mathbb{Z}$.

Define now a linear morphism

$$E_p : \mathfrak{L}\mathfrak{S}_p \to \mathfrak{S}_p$$

by a linear transformation satisfying that

$$E_p\left(\mathbf{l}_p^k \otimes P_{p,j}\right) = \frac{\left(p^{j+1}\right)^{k+1}}{\left[\frac{k}{2}\right]+1} \mathbf{l}_p^k(P_{p,j}), \tag{7.17}$$

for all $k \in \mathbb{N}_0$, $j \in \mathbb{Z}$, where $\left[\frac{k}{2}\right]$ is the *minimal integer greater than or equal to* $\frac{k}{2}$, for all $k \in \mathbb{N}_0$; for example,

$$\left[\frac{3}{2}\right] = 2 = \left[\frac{4}{2}\right].$$

By the cyclicity (7.14) of the tensor factor $\mathfrak{L}_p$ of $\mathfrak{L}\mathfrak{S}_p$, and by the structure theorem (6.3) of the other tensor factor $\mathfrak{S}_p$ of $\mathfrak{L}\mathfrak{S}_p$, the above morphism E_p of (7.17) is a well-defined bounded surjective linear transformation.

Now, consider how our p-radial operator $\mathbf{l}_p$ acts on $\mathfrak{S}_p$. If $\mathbf{c}_p$ and $\mathbf{a}_p$ are the p-creation, respectively, the p-annihilation on $\mathfrak{S}_p$, then

$$\mathbf{c}_p \mathbf{a}_p\left(P_{p,j}\right) = P_{p,j} = \mathbf{a}_p \mathbf{c}_p\left(P_{p,j}\right),$$

for all $j \in \mathbb{Z}, p \in \mathcal{P}$, and hence,

$$\mathbf{c}_p \mathbf{a}_p = 1_{\mathfrak{S}_p} = \mathbf{a}_p \mathbf{c}_p \text{ on } \mathfrak{S}_p. \tag{7.18}$$

Lemma 7.6. *Let $\mathbf{c}_p, \mathbf{a}_p$ be the p-creation, respectively, the p-annihilation on $\mathfrak{S}_p$. Then*

$$\mathbf{c}_p^n \mathbf{a}_p^n = (\mathbf{c}_p \mathbf{a}_p)^n = 1_{\mathfrak{S}_p} = (\mathbf{a}_p \mathbf{c}_p)^n = \mathbf{a}_p^n \mathbf{c}_p^n,$$

and

$$\mathbf{c}_p^{n_1} \mathbf{a}_p^{n_2} = \mathbf{a}_p^{n_2} \mathbf{c}_p^{n_1} \text{ on } \mathfrak{S}_p, \tag{7.19}$$

for all $n, n_1, n_2 \in \mathbb{N}_0$.

Proof. The formula (7.19) holds by (7.18). □

By (7.19), one can get that

$$\mathbf{l}_p^n = (\mathbf{c}_p + \mathbf{a}_p)^n = \sum_{k=0}^{n} \binom{n}{k} \mathbf{c}_p^k \mathbf{a}_p^{n-k}, \tag{7.20}$$

with

$$\mathbf{c}_p^0 = 1_{\mathfrak{S}_p} = \mathbf{a}_p^0,$$

for all $n \in \mathbb{N}$, where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for all } k \leq n \in \mathbb{N}_0.$$

Thus, one obtains the following proposition.

Proposition 7.7. *Let $\mathbf{l}_p \in \mathfrak{L}_p$ be the p-radial operator on $\mathfrak{S}_p$. Then, for all $m \in \mathbb{N}$,*

(i) $\mathbf{l}_p^{2m-1}$ *does not contain* $1_{\mathfrak{S}_p}$*-term,*
(ii) $\mathbf{l}_p^{2m}$ *contains its* $1_{\mathfrak{S}_p}$*-term,* $\binom{2m}{m} \cdot 1_{\mathfrak{S}_p}$.

Proof. The proofs of (i) and (ii) are done by straightforward computations under (7.19) and (7.20). See [6] for more details. □

7.3. WEIGHTED-SEMICIRCULAR ELEMENTS $Q_{p,j}$ in $\mathfrak{L}\mathfrak{S}_p$

Fix $p \in \mathcal{P}$, and let $\mathfrak{L}\mathfrak{S}_p = \mathfrak{L}_p \otimes_{\mathbb{C}} \mathfrak{S}_p$ be the tensor product Banach $*$-algebra (7.16), and let E_p be the linear transformation (7.17) from $\mathfrak{L}\mathfrak{S}_p$ onto $\mathfrak{S}_p$. Throughout this section, fix a generating operator

$$Q_{p,j} = \mathbf{l}_p \otimes P_{p,j} \text{ of } \mathfrak{L}\mathfrak{S}_p, \tag{7.21}$$

for $j \in \mathbb{Z}$, where $P_{p,j}$ are projections (6.1) generating $\mathfrak{S}_p$.

If $Q_{p,j} \in \mathfrak{L}\mathfrak{S}_p$ is in the sense of (7.21) for $j \in \mathbb{Z}$, then

$$E_p\left(Q_{p,j}^n\right) = E_p\left(\mathbf{l}_p^n \otimes P_{p,j}\right) = \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1} \mathbf{l}_p^n\left(P_{p,j}\right), \tag{7.22}$$

by (7.17), for all $n \in \mathbb{N}$.

Now, for a fixed $j \in \mathbb{Z}$, define a linear functional $\tau_{p,j}^{0}$ on $\mathfrak{LS}_p$ by

$$\tau_{p,j}^{0} = \tau_j^p \circ E_p \text{ on } \mathfrak{LS}_p, \tag{7.23}$$

where $\tau_j^p = \frac{1}{\phi(p)}\varphi_j^p$ is in the sense of (7.5).

By the bounded-linearity of both τ_j^p and E_p, the morphism $\tau_{p,j}^{0}$ of (7.23) is a bounded linear functional on $\mathfrak{LS}_p$. By (7.22) and (7.23), if $Q_{p,j}$ is in the sense of (7.21), then

$$\tau_{p,j}^{0}\left(Q_{p,j}^{n}\right) \quad = \frac{\left(p^{j+1}\right)^{n+1}}{\left[\frac{n}{2}\right]+1}\tau_j^p\left(\mathbf{l}_p^n(P_{p,j})\right), \tag{7.24}$$

for all $n \in \mathbb{N}$.

Theorem 7.8. *Let $Q_{p,j} = \mathbf{l}_p \otimes P_{p,j} \in \left(\mathfrak{LS}_p, \tau_{p,j}^{0}\right)$, for a fixed $j \in \mathbb{Z}$. Then*

$$\tau_{p,j}^{0}\left(Q_{p,j}^{n}\right) = \omega_n c_{\frac{n}{2}}\left(p^{2(j+1)}\right)^{\frac{n}{2}}, \tag{7.25}$$

for all $n \in \mathbb{N}$, where ω_n are in the sense of (7.11).

Proof. The formula (7.25) is obtained by Proposition 7.7 and (7.24). See [10] for details. □

8. SEMICIRCULARITY ON $\mathfrak{LS}$

For all $p \in \mathcal{P}$, $j \in \mathbb{Z}$, let

$$\mathfrak{LS}_p(j) = \left(\mathfrak{LS}_p, \tau_{p,j}^{0}\right) \tag{8.1}$$

be the measure-theoretic structures of the tensor product Banach $*$-algebra $\mathfrak{LS}_p$ of (7.16), and the linear functional $\tau_{p,j}^{0}$ of (7.24).

Definition 8.1. We call such pairs $\mathfrak{LS}_p(j)$ of (8.1), the *j-th p-adic filter*, for all $p \in \mathcal{P}, j \in \mathbb{Z}$.

Let $Q_{p,k} = \mathbf{l}_p \otimes P_{p,k}$ be the k-th generating elements of the j-th p-adic filter $\mathfrak{LS}_p(j)$ of (8.1), for all $k \in \mathbb{Z}$, for fixed $p \in \mathcal{P}, j \in \mathbb{Z}$. Then they satisfy

$$\tau_{p,j}^{0}\left(Q_{p,k}^{n}\right) = \delta_{j,k}\left(\omega_n\left(p^{2(j+1)}\right)^{\frac{n}{2}} c_{\frac{n}{2}}\right), \tag{8.2}$$

by (7.23) and (7.25), for all $n \in \mathbb{N}$.

For the family

$$\left\{\mathfrak{LS}_p(j) = \left(\mathfrak{LS}_p, \tau_{p,j}^{0}\right) : p \in \mathcal{P}, j \in \mathbb{Z}\right\}$$

of p-adic filters of (8.1), define the *free product Banach $*$-probability space*,

$$\mathfrak{LS} \overset{denote}{=} \left(\mathfrak{LS}, \tau^{0}\right) \overset{def}{=} \underset{p \in \mathcal{P}, j \in \mathbb{Z}}{\star} \mathfrak{LS}_p(j). \tag{8.3}$$

as in [27] and [29], with

$$\mathfrak{LS} = \underset{p\in\mathcal{P},j\in\mathbb{Z}}{\star} \mathfrak{LS}_p, \text{ and } \tau^0 = \underset{p\in\mathcal{P},j\in\mathbb{Z}}{\star} \tau^0_{p,j}.$$

Note that the Banach $*$-probability space $\mathfrak{LS}$ of (8.3) is a well-defined Banach $*$-probability space with its *free blocks* $\mathfrak{LS}_p(j)$, for all $p \in \mathcal{P}, j \in \mathbb{Z}$. For more about (free-probabilistic) *free product*, see [27] and [29].

Definition 8.2. The Banach $*$-probability space $\mathfrak{LS} = \left(\mathfrak{LS}, \tau^0\right)$ of (8.3) is called the free Adelic filterization.

Let $\mathfrak{LS}$ be the free Adelic filterization (8.3). Then, by (8.2), we obtain a subset

$$\mathcal{Q} = \{Q_{p,j} = \mathbf{1}_p \otimes P_{p,j} \in \mathfrak{LS}_p(j)\}_{p\in\mathcal{P},j\in\mathbb{Z}}$$

in $\mathfrak{LS}$.

Since all entries $Q_{p,j}$ of the above family $\mathcal{Q}$ are taken from the j-th p-adic filters $\mathfrak{LS}_p(j)$, which are the free blocks of $\mathfrak{LS}$, they are free from each other in $\mathfrak{LS}$, for all $p \in \mathcal{P}, j \in \mathbb{Z}$. Also, since $Q^n_{p,j} \in \mathfrak{LS}_p(j)$ in $\mathfrak{LS}$, for all $n \in \mathbb{N}$, they are free reduced words with their lengths-1, and hence,

$$\tau^0\left(Q^n_{p,j}\right) = \tau^0_{p,j}\left(Q^n_{p,j}\right) = \omega_n p^{n(j+1)} c_{\frac{n}{2}},$$

by (8.2) and (8.3), for all $n \in \mathbb{N}$.

Lemma 8.3. *Let $\mathcal{Q}$ be the subset of the free Adelic filterization $\mathfrak{LS}$ introduced in the above paragraph. Then all elements $Q_{p,j} \in \mathcal{Q}$ are $p^{2(j+1)}$-semicircular in $\mathfrak{LS}$.*

Proof. As we discussed in the very above paragraphs, it is shown by (7.11), (8.2) and (8.3). □

Recall that a subset S of an arbitrary (topological or pure-algebraic) $*$-probability space (A, φ) is said to be a *free family*, if all elements of S are mutually free from each other (e.g., [27] and [28]).

Definition 8.4. Let S be a free family in an arbitrary topological $*$-probability space (A, φ). This family S is called a free (weighted-)semicircular family, if every element of S is (weighted-)semicircular in (A, φ).

By the above lemma, we obtain the following fact.

Theorem 8.5. *Let $\mathfrak{LS}$ be the free Adelic filterization* (8.3)*, and let*

$$\mathcal{Q} = \{Q_{p,j} \in \mathfrak{LS}_p(j)\}_{p\in\mathcal{P},j\in\mathbb{Z}} \subset \mathfrak{LS}, \tag{8.4}$$

where $\mathfrak{LS}_p(j)$ are the j-th p-adic filters, the free blocks of $\mathfrak{LS}$. Then this family $\mathcal{Q}$ is a free weighted-semicircular family in $\mathfrak{LS}$.

Proof. Let $\mathcal{Q}$ be a subset (8.4) of $\mathfrak{LS}$. Then, by the above lemma, all elements $Q_{p,j}$ of $\mathcal{Q}$ are $p^{2(j+1)}$-semicircular in $\mathfrak{LS}$, for all $p \in \mathcal{P}, j \in \mathbb{Z}$. Also, they are mutually free from each other in $\mathfrak{LS}$, because all entries $Q_{p,j}$ are contained in the mutually distinct free blocks $\mathfrak{LS}_p(j)$ of $\mathfrak{LS}$, for all $p \in \mathcal{P}, j \in \mathbb{Z}$. Therefore, the family $\mathcal{Q}$ forms a free weighted-semicircular family in $\mathfrak{LS}$. □

Now, take elements

$$\Theta_{p,j} \overset{def}{=} \frac{1}{p^{j+1}} Q_{p,j}, \text{ for all } p \in \mathcal{P}, j \in \mathbb{Z}, \tag{8.5}$$

in $\mathfrak{LS}$, where $Q_{p,j} \in \mathcal{Q}$, where $\mathcal{Q}$ is the free weighted-semicircular family (8.4) in $\mathfrak{LS}$.

Then, by the self-adjointness of $Q_{p,j}$, these operators $\Theta_{p,j}$ of (8.5) are self-adjoint in $\mathfrak{LS}$, too, because

$$p^{j+1} \in \mathbb{R} \text{ in } \mathbb{C}^{\times},$$

satisfying $\overline{p^{j+1}} = p^{j+1}$, for all $p \in \mathcal{P}, j \in \mathbb{Z}$.

Theorem 8.6. *Let $\Theta_{p,j} \in \mathfrak{LS}_p(j)$ be free random variables* (8.5) *of the free Adelic filterization $\mathfrak{LS}$, for all $p \in \mathcal{P}, j \in \mathbb{Z}$. Then the family*

$$\Theta = \{\Theta_{p,j} \in \mathfrak{LS}_p(j) : p \in \mathcal{P}, j \in \mathbb{Z}\} \tag{8.6}$$

forms a free semicircular family in $\mathfrak{LS}$.

Proof. Let Θ be the family (8.6). Then it forms a free family in $\mathfrak{LS}$, because $\Theta_{p,j} \in \Theta$ are the scalar-product of $Q_{p,j} \in \mathcal{Q}$, and the family $\mathcal{Q}$ of (8.4) is a free family in $\mathfrak{LS}$. Observe now that

$$\begin{aligned} \tau^0\left(\Theta_{p,j}^n\right) &= \tau^0\left(\left(\frac{1}{p^{j+1}}\right)^n Q_{p,j}^n\right) \\ &= \left(\frac{1}{p^{j+1}}\right)^n \tau^0\left(Q_{p,j}^n\right) = \left(\frac{1}{p^{j+1}}\right)^n \left(\omega_n p^{n(j+1)} c_{\frac{n}{2}}\right) \end{aligned}$$

by the $p^{2(j+1)}$-semicircularity of $Q_{p,j} \in \mathcal{Q}$

$$= \omega_n c_{\frac{n}{2}}, \tag{8.7}$$

for all $n \in \mathbb{N}$, for all $p \in \mathcal{P}, j \in \mathbb{Z}$.

Thus, all entries $\Theta_{p,j}$ of the free family Θ are semicircular by (7.8) and (8.7). Therefore, this free family Θ of (8.6) forms a free semicircular family in $\mathfrak{LS}$. □

Define a Banach $*$-subalgebra $\mathbb{LS}$ of $\mathfrak{LS}$ by

$$\mathbb{LS} \overset{def}{=} \overline{\mathbb{C}[\mathcal{Q}]} \text{ in } \mathfrak{LS}, \tag{8.8}$$

where $\mathcal{Q}$ is our free weighted-semicircular family (8.4), and $\overline{Y}$ mean the Banach topology closures of subsets Y of $\mathfrak{LS}$.

Then one can obtain the following structure theorem for the Banach $*$-algebra $\mathbb{LS}$ of (8.8) in $\mathfrak{LS}$.

Theorem 8.7. *Let $\mathbb{LS}$ be the Banach $*$-subalgebra* (8.8) *of the free Adelic filterization $\mathfrak{LS}$ generated by the free weighted-semicircular family $\mathcal{Q}$ of* (8.4). *Then*

$$\mathbb{LS} = \overline{\mathbb{C}\left[\Theta\right]} \textit{ in } \mathfrak{LS}, \tag{8.9}$$

where Θ is the free semicircular family (8.6).

Moreover,

$$\mathbb{LS} \overset{*\text{-}iso}{=} \underset{p\in\mathcal{P},j\in\mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{p,j}\}\right]} \overset{*\text{-}iso}{=} \overline{\mathbb{C}\left[\underset{p\in\mathcal{P},j\in\mathbb{Z}}{\star}\{Q_{p,j}\}\right]}, \tag{8.10}$$

in $\mathfrak{LS}$, *where* "$\overset{*\text{-}iso}{=}$" *means "being Banach-∗-isomorphic", and*

$$\overline{\mathbb{C}\left[\{Q_{p,j}\}\right]} \textit{ are Banach } *\textit{-subalgebras of } \mathfrak{LS}_p(j),$$

for all $p \in \mathcal{P}, j \in \mathbb{Z}$, *in* $\mathfrak{LS}$. *Here,* $\star$ *in the first* $*$*-isomorphic relation of* (8.10) *is the free-probability-theoretic free product* (*of* [27] *and* [29]), *and* $\star$ *in the second* $*$*-isomorphic relation of* (8.10) *is the pure-algebraic free product* (*generating noncommutative algebraic free words in* $\mathcal{Q}$).

Proof. Let $\mathbb{LS}$ be the Banach $*$-subalgebra (8.8) of $\mathfrak{LS}$. Since the generator set $\mathcal{Q}$ of $\mathbb{LS}$ is a free family, as an embedded sub-structure of $\mathfrak{LS}$, we have that

$$\mathbb{LS} \overset{*\text{-iso}}{=} \underset{p\in\mathcal{P},j\in\mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{p,j}\}\right]} \text{ in } \mathfrak{LS}, \tag{8.11}$$

by (8.3).

Since every free block $\overline{\mathbb{C}\left[\{Q_{p,j}\}\right]}$ of (8.11) is generated by a single self-adjoint (weighted-semicircular) element $Q_{p,j}$, every operator T of $\mathbb{LS}$ is a limit of linear combinations of operator products spanned by the family $\mathcal{Q}$ of (8.4), which form noncommutative free reduced words (in the sense of [27] and [29]) in $\mathbb{LS}$. Note that every (pure-algebraic) free word in $\mathcal{Q}$ has a unique free reduced word in $\mathbb{LS}$, as an operator. So, the $*$-isomorphic relation (8.11) guarantees that

$$\mathbb{LS} \overset{*\text{-iso}}{=} \overline{\mathbb{C}\left[\underset{p\in\mathcal{P},j\in\mathbb{Z}}{\star}\{Q_{p,j}\}\right]}, \tag{8.12}$$

where the free product ($\star$) in (8.12) is pure-algebraic.

Therefore, by (8.11) and (8.12), the structure theorem (8.10) holds true.

Note now that

$$Q_{p,j} = p^{j+1}\Theta_{p,j} \in \mathcal{Q}, \text{ for all } p \in \mathcal{P}, j \in \mathbb{Z},$$

by (8.5), where $\Theta_{p,j} \in \Theta$ are the semicircular elements of (8.6). So,

$$\mathbb{LS} \overset{def}{=} \overline{\mathbb{C}[\mathcal{Q}]} = \overline{\mathbb{C}\left[\{p^{j+1}\Theta_{p,j} : \Theta_{p,j} \in \Theta\}\right]} = \overline{\mathbb{C}[\Theta]}, \tag{8.13}$$

in $\mathfrak{LS}$. Therefore, the equality (8.9) holds by (8.13). □

As a sub-structure of the free Adelic filterization $\mathfrak{LS}$, one gets the Banach $*$-probability space,

$$\left(\mathbb{LS}, \tau^0 \overset{denote}{=} \tau^0 \mid_{\mathbb{LC}}\right). \tag{8.14}$$

Definition 8.8. Let $\mathbb{LS}$ be the Banach $*$-subalgebra (8.8) of $\mathfrak{LG}$. Then we call

$$\mathbb{LS}_0 \overset{denote}{=} \left(\mathbb{LS}, \tau^0\right) \text{ of (8.14)},$$

the (free) semicircular (Adelic sub-)filterization of the free Adelic filterization $\mathfrak{LG}$.

Note that, by (8.3) and (8.10), all elements of the semicircular filterization $\mathbb{LS}$ provide possible non-zero free distributions in the free Adelic filterization $\mathfrak{LG}$. More precisely, a free reduced word of $\mathfrak{LG}$ has its nonzero free distribution, if and only if it is a free reduced words in $\mathcal{Q} \cup \Theta$, if and only if it is contained in $\mathbb{LS}_0$. Therefore, we now focus on free probability on the semicircular filterization $\mathbb{LS}_0$ of (8.14).

9. TRUNCATED LINEAR FUNCTIONALS $\tau_{t<s}$ ON $\mathbb{LS}$

In *number theory*, one of the most interesting topics is finding the number of primes, or the density of primes, contained in a closed interval $[t_1, t_2]$ of the real numbers $\mathbb{R}$ (e.g., [11–13] and [19]). Motivated by this theory, we consider "suitable" *truncated linear functionals* on our semicircular filterization $\mathbb{LS}_0$ of (8.10).

9.1. LINEAR FUNCTIONALS $\{\tau_{(t)}\}_{t\in\mathbb{R}}$ on $\mathbb{LS}$

Let $\mathbb{LS}_0$ be the semicircular filterization $\left(\mathbb{LS},\ \tau^0\right)$ of the free Adelic filterization $\mathfrak{LG}$, where $\mathbb{LS}$ is the Banach $*$-subalgebra (8.8) of $\mathfrak{LG}$, satisfying (8.10). We now truncate τ^0 on $\mathbb{LS}$, in terms of a fixed real number $t \in \mathbb{R}$.

First, recall and remark that

$$\tau^0 = \underset{p\in\mathcal{P},j\in\mathbb{Z}}{\star} \tau^0_{p,j} \text{ on } \mathbb{LS},$$

by (8.3) and (8.14). So, one can sectionize τ^0 in terms of $\mathcal{P}$ as follows:

$$\tau^0 = \underset{p\in\mathcal{P}}{\star} \tau^0_p \text{ on } \mathbb{LS}, \tag{9.1}$$

with

$$\tau^0_p = \underset{j\in\mathbb{Z}}{\star} \tau^0_{p,j} \text{ on } \mathbb{LS}_p, \text{ for } p \in \mathcal{P},$$

where

$$\mathbb{LS}_p \overset{def}{=} \underset{j\in\mathbb{Z}}{\star} \overline{\mathbb{C}[\{\Theta_{p,j}\}]} \text{ in } \mathbb{LS} \subset \mathfrak{LG}, \tag{9.2}$$

for each $p \in \mathcal{P}$.

Such a sectionization (9.1) and (9.2) can be done by the structure theorem (8.10) of $\mathbb{LS}$ in $\mathfrak{LG}$.

By the very constructions (8.14) and (9.2), one can get the following lemma.

Lemma 9.1. *Let $\mathbb{LS}_{p_l}$ be $*$-subalgebras (9.2) of the semicircular filterization $\mathbb{LS}_0$, for $l = 1, 2$. Then $\mathbb{LS}_{p_1}$ and $\mathbb{LS}_{p_2}$ are free in $\mathbb{LS}_0$, if and only if $p_1 \neq p_2$ in $\mathcal{P}$.*

Proof. The proof of this freeness condition in $\mathbb{LS}_0$ is clear by (8.3), (8.14) and (9.2). Indeed, $p_1 \neq p_2$ in $\mathcal{P}$, if and only if all free blocks $\left\{\overline{\mathbb{C}[\{\Theta_{p_1,j}\}]}\right\}_{j\in\mathbb{Z}}$ of $\mathbb{LS}_{p_1}$, and those $\left\{\overline{\mathbb{C}[\{\Theta_{p_2,j}\}]}\right\}_{j\in\mathbb{Z}}$ of $\mathbb{LS}_{p_2}$ are disjoint from each other in $\mathbb{LS}_0$, if and only if $\mathbb{LS}_{p_1}$ and $\mathbb{LS}_{p_2}$ are free in $\mathbb{LS}_0$ by (8.10). $\square$

Fix now $t \in \mathbb{R}$, and define a new linear functional $\tau_{(t)}$ on $\mathbb{LS}$ by

$$\tau_{(t)} \overset{def}{=} \begin{cases} \underset{p\leq t}{\star} \tau_p^0 & \text{on } \underset{p\leq t}{\star} \mathbb{LS}_p \text{ in } \mathbb{LS}, \\ O & \text{otherwise}, \end{cases} \tag{9.3}$$

where τ_p^0 are the linear functionals (9.1) on the Banach $*$-subalgebras $\mathbb{LS}_p$ of (9.2) in $\mathbb{LS}$, for all $p \in \mathcal{P}$, and O is the *zero linear functional*, satisfying $O(T) = 0$, for all $T \in \mathbb{LS}$.

By the definition (9.3), one can easily verify that, if $t < 2$ in $\mathbb{R}$, then the corresponding linear functional $\tau_{(t)}$ is defined to the zero linear functional O on $\mathbb{LS}$. From below, if there is no confusion, we simply write the above conditional definition (9.3) by

$$\tau_{(t)} \overset{denote}{=} \underset{p\leq t}{\star} \tau_p^0 \text{ on } \mathbb{LS}, \tag{9.4}$$

for all $t \in \mathbb{R}$. For example,

$$\tau_{\left(\frac{\sqrt{2}}{2}\right)} = O, \quad \tau_{(2.0001)} = \tau_2^0, \quad \text{and} \quad \tau_{(5)} = \tau_2^0 \star \tau_3^0 \star \tau_5^0,$$

etc., on $\mathbb{LS}$, in the sense of (9.4) representing (9.3).

Theorem 9.2. *Let $Q_{p,j} \in \mathcal{Q}$, and $\Theta_{p,j} \in \Theta$ in the semicircular filterization $\mathbb{LS}_0$, for $p \in \mathcal{P}$, $j \in \mathbb{Z}$, where $\mathcal{Q}$ is the free weighted-semicircular family* (8.4) *and Θ is the semicircular family* (8.6)*, generating $\mathbb{LS}_0$. Let $t \in \mathbb{R}$, and $\tau_{(t)}$, the corresponding linear functional* (9.4) *on $\mathbb{LS}$. Then*

$$\tau_{(t)}\left(Q_{p,j}^n\right) = \begin{cases} \omega_n p^{2(j+1)} c_{\frac{n}{2}} & \text{if } t \geq p, \\ 0 & \text{if } t < p, \end{cases} \quad \text{and} \quad \tau_{(t)}\left(\Theta_{p,j}^n\right) = \begin{cases} \omega_n c_{\frac{n}{2}} & \text{if } t \geq p, \\ 0 & \text{if } t < p, \end{cases} \tag{9.5}$$

for all $n \in \mathbb{N}$.

Proof. By the $p^{2(j+1)}$-semicircularity of $Q_{p,j} \in \mathcal{Q}$, and the semicircularity of $\Theta_{p,j} \in \Theta$ in $\mathbb{LS}_0$, and by the definition (9.3) or (9.4), one obtains that: if $t \geq p$ in $\mathbb{R}$, then

$$\begin{aligned} \tau_{(t)}\left(Q_{p,j}^n\right) &= \tau_p^0\left(Q_{p,j}^n\right) = \tau_{p,j}^0\left(Q_{p,j}^n\right) \\ &= \tau^0\left(Q_{p,j}^n\right) = \omega_n p^{2(j+1)} c_{\frac{n}{2}}, \end{aligned}$$

and

$$\begin{aligned} \tau_{(t)}\left(\Theta_{p,j}^n\right) &= \tau_p^0\left(\Theta_{p,j}^n\right) = \tau_{p,j}^0\left(\Theta_{p,j}^n\right) \\ &= \tau^0\left(\Theta_{p,j}^n\right) = \omega_n c_{\frac{n}{2}}, \end{aligned}$$

by (9.2) and (9.3), for all $n \in \mathbb{N}$.

If $t < p$, then

$$\tau_{(t)} = \underset{2 \leq q < p \text{ in } \mathcal{P}}{\star} \tau_q^0, \text{ or } O, \text{ on } \mathbb{LS}.$$

So, in such cases,

$$\tau_{(t)} \left(Q_{p,j}^n \right) = \tau_{(t)} \left(\Theta_{p,j}^n \right) = 0, \text{ for all } n \in \mathbb{N},$$

by (9.3). Therefore, the free-momental data (9.5) for the linear functional $\tau_{(t)}$ holds. □

Definition 9.3. Let $\mathbb{LS}_0 = (\mathbb{LS}, \tau^0)$ be the semicircular filterization, and let $\tau_{(t)}$ be a linear functionals (9.4) on $\mathbb{LS}$, for $t \in \mathbb{R}$. Then the new Banach $*$-probability spaces,

$$\mathbb{LS}_{(t)} \overset{denote}{=} \left(\mathbb{LS}, \tau_{(t)} \right), \tag{9.6}$$

are called the semicircular t-filterizations of $\mathbb{LS}_0$, for all $t \in \mathbb{R}$.

Note that if t is suitable in the sense that "$\tau_{(t)} \neq O$ on $\mathbb{LS}$", then the free-probabilistic structure $\mathbb{LS}_{(t)}$ of (9.6) is meaningful (or non-trivial).

Notation and Assumption 9.4. (in short, NA 9.4, from below) In the following, we will say "$t \in \mathbb{R}$ is suitable", if the semicircular t-filterization "$\mathbb{LS}_{(t)}$ of (9.6) is meaningful", in the sense that $\tau_{(t)} \neq O$ on $\mathbb{LS}$.

Now, let us consider the following concept.

Definition 9.5. Let (A_k, φ_k) be Banach $*$-probability spaces (or C^*-probability spaces, or W^*-probability spaces, etc.), for $k = 1, 2$. A Banach $*$-probability space (A_1, φ_1) is said to be free-homomorphic to a Banach $*$-probability space (A_2, φ_2), if there exists a bounded $*$-homomorphism

$$\Phi : A_1 \to A_2,$$

such that

$$\varphi_2 \left(\Phi(a) \right) = \varphi_1 \left(a \right),$$

for all $a \in A_1$. The $*$-homomorphism Φ is called a free-homomorphism.

If Φ is a $*$-isomorphism, then it is called a free-isomorphism; and (A_1, φ_1) and (A_2, φ_2) are said to be free-isomorphic.

By (9.5), we obtain the following free-probabilistic-structural theorem.

Theorem 9.6. *Let*

$$\mathbb{LS}_q = \underset{j \in \mathbb{Z}}{\star} \overline{\mathbb{C}\left[\{Q_{q,j}\}\right]}$$

be Banach $*$-subalgebras (9.2) *of* $\mathbb{LS}$, *for all* $q \in \mathcal{P}$, *and let* $t \in \mathbb{R}$ *be suitable in the sense of NA 9.4. Construct a Banach $*$-probability space* $\mathbb{LS}^t$ *by a Banach $*$-probabilistic sub-structure of the semicircular filterization* $\mathbb{LS}_0$,

$$\mathbb{LS}^t \overset{def}{=} \underset{p \leq t}{\star} \left(\mathbb{LS}_p, \tau_p^0\right) = \left(\underset{p \leq t}{\star} \mathbb{LS}_p, \underset{p \leq t}{\star} \tau_p^0\right) \tag{9.7}$$

where $\tau_p^0 = \underset{j \in \mathbb{Z}}{\star} \tau_{p,j}^0$ *are in the sense of* (9.1), *and* $\mathbb{LS}_p$ *are in the sense of* (9.2) *in* $\mathbb{LS}$. *Then, for suitable* $t \in \mathbb{R}$

$$\mathbb{LS}^t \text{ of } (9.7) \text{ is free-homomorphic to } \mathbb{LS}_{(t)}. \tag{9.8}$$

Proof. Let $\mathbb{LS}_{(t)}$ be the semicircular t-filterization (9.6) of the semicircular filterization $\mathbb{LS}_0$, and let $\mathbb{LS}^t$ be a Banach $*$-probability space (9.7), for a fixed suitable $t \in \mathbb{R}$. Define a bounded linear morphism

$$\Phi_t : \mathbb{LS}^t \to \mathbb{LS}_{(t)},$$

by the canonical embedding map,

$$\Phi_t(T) = T \text{ in } \mathbb{LS}_{(t)}, \text{ for all } T \in \mathbb{LS}^t. \tag{9.9}$$

Then it is a well-defined injective bounded $*$-homomorphism from $\mathbb{LS}^t$ into $\mathbb{LS}_{(t)}$, by (8.8), (8.11), (9.2) and (9.7).

Therefore, we obtain that

$$\tau_{(t)}\left(\Phi(T)\right) = \tau_{(t)}(T) = \tau^0(T) = \tau^t(T),$$

for all $T \in \mathbb{LS}^t$, where

$$\tau^t = \underset{p \leq t}{\star} \tau_p^0 \text{ on } \mathbb{LS}^t,$$

in the sense of (9.7), by (9.5).

It shows that the Banach $*$-probability space $\mathbb{LS}^t$ of (9.7) is free-homomorphic to the semicircular t-filterization $\mathbb{LS}_{(t)}$ of (9.6). Therefore, the statement (9.8) holds by a free-homomorphism Φ_t of (9.9). □

The above theorem shows that the Banach $*$-probability spaces $\mathbb{LS}^t$ of (9.7) are free-homomorphic to the semicircular t-filterizations $\mathbb{LS}_{(t)}$ of (9.6), for any suitable $t \in \mathbb{R}$.

Corollary 9.7. *All free reduced words T of the semicircular t-filterization* $\mathbb{LS}_{(t)}$, *having non-zero free distributions, are contained in the Banach $*$-probability space* $\mathbb{LS}^t$ *of* (9.7), *whenever t is suitable. The converse holds true, too.*

Proof. The proof of this characterization is done by (9.3), (9.5), (9.7), (9.8), and (9.9). □

So, if T are non-zero-free-distribution-having free reduced words of our semicircular t-filterization $\mathbb{LS}_{(t)}$, then such operators T are regarded as free random variables of the Banach $*$-probability space $\mathbb{LS}^t$ of (9.7).

Remark 9.8. Let F_n be the free groups with n-generators, for all

$$n \in \mathbb{N}^{\infty}_{>1} = (\mathbb{N} \setminus \{1\}) \cup \{\infty\},$$

and let $L(F_n)$ be the corresponding free group factors (the group von Neumann algebras generated by F_n, equipped with their canonical traces), for all $n \in \mathbb{N}^{\infty}_{>1}$.

In [25], Radulescu showed that either (9.10) or (9.11) holds, where

$$L(F_n) \overset{*\text{-iso}}{=} L(F_\infty), \text{ for all } n \in \mathbb{N}^{\infty}_{>1}, \tag{9.10}$$

$$L(F_{n_1}) \overset{*\text{-iso}}{\neq} L(F_{n_2}) \text{ if and only if } n_1 \neq n_2 \text{ in } \mathbb{N}^{\infty}_{>1}. \tag{9.11}$$

We do not know which one holds true at this moment.

In our case, we have similar difficulties to check $\mathbb{L}^t$ and $\mathbb{L}_{(t)}$ are $*$-isomorphic (and hence, free-isomorphic) or not. One thing clear now is that $\mathbb{LS}^t$ is free-homomorphic to $\mathbb{LS}_{(t)}$ by (9.8), for any suitable $t \in \mathbb{R}$.

Conjecture 9.9. *Let $t \in \mathbb{R}$ be suitable in the sense of NA 9.4, and assume that there are more than one primes less than or equal to t. Even though the Banach $*$-algebras $\mathbb{LS}^t = \underset{p \leq t}{\star} \mathbb{LS}_p$ and $\mathbb{LS}$ are $*$-isomorphic (which we are not sure either), the Banach $*$-probability spaces $\mathbb{LS}^t$ and $\mathbb{LS}_{(t)}$ are not free-isomorphic.*

9.2. TRUNCATED LINEAR FUNCTIONALS $\tau_{t_1<t_2}$ ON $\mathbb{LS}$

In this section, we generalize the semicircular t-filterizations $\mathbb{LS}_{(t)}$, for suitable $t \in \mathbb{R}$. Throughout this section, let $[t_1, t_2]$ be a *closed interval* in $\mathbb{R}$, for $t_1 < t_2 \in \mathbb{R}$. For a fixed closed interval $[t_1, t_2]$, define the corresponding linear functional $\tau_{t_1<t_2}$ on the Banach $*$-algebra $\mathbb{LS}$ by

$$\tau_{t_1<t_2} \overset{def}{=} \begin{cases} \underset{t_1 \leq p \leq t_2 \text{ in } \mathcal{P}}{\star} \tau_p^0 & \text{on } \underset{t_1 \leq p \leq t_2}{\star} \mathbb{LS}_p \text{ in } \mathbb{LS}, \\ \\ O & \text{otherwise}, \end{cases} \tag{9.12}$$

where τ_p^0 are the linear functionals (9.1) on the Banach $*$-subalgebras $\mathbb{LS}_p$ of (9.2) in $\mathbb{LS}$, for $p \in \mathcal{P}$.

As in Section 9.1, if there is no confusion, we write the conditional definition (9.12) of $\tau_{t_1<t_2}$ as

$$\tau_{t_1<t_2} = \underset{t_1 \leq p \leq t_2 \text{ in } \mathcal{P}}{\star} \tau_p^0 \text{ on } \mathbb{LS}. \tag{9.13}$$

To make a linear functional $\tau_{t_1<t_2}$ of (9.12) be a non-zero-linear functional on $\mathbb{LS}$, the interval $[t_1, t_2]$ must be taken "suitably" in $\mathbb{R}$. For example,

$$\tau_{t_1<t_2} = O, \text{ whenever } t_2 < 2,$$

and

$$\tau_{8<10} = O, \quad \tau_{14<16} = O, \quad \text{and} \quad \tau_{\frac{3}{7}<\frac{3}{2}} = O, \quad \text{etc.},$$

but

$$\tau_{\frac{3}{2}<8} = \tau_{(8)} = \tau_2^0 \star \tau_3^0 \star \tau_5^0 \star \tau_7^0$$

and

$$\tau_{7<14} = \tau_7^0 \star \tau_{11}^0 \star \tau_{13}^0,$$

on $\mathbb{LS}$ in the sense of (9.13), representing (9.12).

It is not difficult to check that the concept of truncated linear functionals $\tau_{t_1<t_2}$ of (9.12) covers the definition of the linear functionals $\tau_{(t)}$ of (9.3). In particular, if $\tau_{(t)}$ is "suitable" in the sense of NA 9.4, then one may understand

$$\tau_{(t)} = \tau_{s<t}, \text{ for } 2 \geq s < t \in \mathbb{R},$$

with axiomatization:

$$\tau_{p<p} = \tau_p^0 \text{ on } \mathbb{LS}, \text{ for all } p \in \mathcal{P} \subset \mathbb{R},$$

in the sense of (9.13). Remark that the above axiomatization is only for the case where $p \in \mathcal{P}$.

Definition 9.10. Let $[t_1, t_2]$ be a given interval in $\mathbb{R}$, and $\tau_{t_1<t_2}$, the corresponding linear functional (9.12) on $\mathbb{LS}$. Then we call it the $[t_1, t_2]$(-truncated)-linear functional on $\mathbb{LS}$. The Banach $*$-probability space

$$\mathbb{LS}_{t_1<t_2} \overset{denote}{=} (\mathbb{LS}, \tau_{t_1<t_2}) \tag{9.14}$$

is said to be the semicircular $[t_1, t_2]$(-truncated)-filterization of the semicircular filterization $\mathbb{LS}_0 = (\mathbb{LS}, \tau^0)$.

As we discussed in the above paragraphs, a semicircular $[t_1, t_2]$-filterization $\mathbb{LS}_{t_1<t_2}$ of (9.14) is "meaningful", if $t_1 < t_2$ are suitable in $\mathbb{R}$, like in NA 9.4.

Notation and Assumption 9.11. (in short, NA 9.11, from below) In the rest of this paper, if we write "$t_1 < t_2$ are suitable in $\mathbb{R}$," then it means "$\mathbb{LS}_{t_1<t_2}$ is meaningful", in the sense that: $\tau_{t_1<t_2} \neq O$ on $\mathbb{LS}$.

Remark 9.12. In fact, the study of such "suitability" of $t_1 < t_2$ in $\mathbb{R}$ is to study the density of primes in $[t_1, t_2]$ in number theory. e.g., see [11–13] and [19].

If $t_1 \leq 2$, and if $t_1 < t_2$ is suitable in $\mathbb{R}$, then the semicircular $[t_1, t_2]$-filterization $\mathbb{LS}_{t_1<t_2}$ of (9.14) is identified with the semicircular t_2-filterization $\mathbb{LS}_{(t_2)}$ of (9.6).

Theorem 9.13. *Let $t_1 \leq 2$, and t_2 is suitable in $\mathbb{R}$ in the sense of NA 9.4.*

(i) *$\mathbb{LS}_{t_1<t_2}$ is not only suitable in the sense of NA 9.11, but also it is free-isomorphic to $\mathbb{LS}_{(t_2)}$.*

(ii) *The Banach $*$-probability space $\mathbb{LS}^{t_2}$ of (9.7) is free-homomorphic to $\mathbb{LS}_{t_1<t_2}$.*

Proof. Suppose $t_1 \leq 2$, and t_2 is suitable in $\mathbb{R}$ in the sense of NA 9.4. Then $t_1 < t_2$ are suitable in $\mathbb{R}$ in the sense of NA 9.11. Since t_1 is assumed to be less than or equal to 2, the linear functional $\tau_{t_1<t_2} = \tau_{(t_2)}$, by (9.3) and (9.12). So,

$$\mathbb{LS}_{t_1<t_2} = (\mathbb{LS}, \tau_{t_1<t_2}) = \left(\mathbb{LS}, \tau_{(t_2)}\right) = \mathbb{LS}_{(t_2)}.$$

Therefore, the free-isomorphic relation (i) holds by taking the free-isomorphism as the identity map on $\mathbb{LS}$.

By (9.8), the Banach $*$-probability space $\mathbb{LS}^{t_2}$ of (9.7) is free-homomorphic to the semicircular t_2-filterization $\mathbb{LS}_{(t_2)}$. Therefore, $\mathbb{LS}^{t_2}$ is free-homomorphic to $\mathbb{LS}_{t_1<t_2}$, by (i), i.e., the statement (ii) holds. □

The above theorem characterizes the free-probabilistic structures for $\mathbb{LS}_{t_1<t_2}$, whenever $t_1 \leq 2$, and t_2 is suitable, by (i) and (ii). So, we restrict our interests to the cases where $t_1 \geq 2$ in $\mathbb{R}$.

Theorem 9.14. *Let $2 \leq t_1 < t_2$ be suitable in $\mathbb{R}$, and let $\mathbb{LS}_{t_1<t_2}$ be the semicircular $[t_1, t_2]$-filterization of (9.14). Then the Banach $*$-probability space*

$$\mathbb{LS}^{t_1<t_2} = \underset{t_1 \leq p \leq t_2 \text{ in } \mathcal{P}}{\star} \left(\mathbb{LS}_p, \tau_p^0\right) = \left(\underset{t_1 \leq p \leq t_2}{\star} \mathbb{LS}_p, \underset{t_1 \leq p \leq t_2}{\star} \tau_p^0\right) \tag{9.15}$$

is free-homomorphic to $\mathbb{LS}_{t_1<t_2}$ in the semicircular filterization $\mathbb{LS}_0$. i.e.,

$$\mathbb{LS}^{t_1<t_2} \text{ of (9.15) is free-homomorphic to } \mathbb{LS}_{t_1<t_2}. \tag{9.16}$$

Proof. Let $\mathbb{LS}^{t_1<t_2}$ be in the sense of (9.15) in $\mathbb{LS}_0$, i.e.,

$$\mathbb{LS}^{t_1<t_2} = \left(\underset{t_1 \leq p \leq t_2}{\star} \mathbb{LS}_p, \underset{t_1 \leq p \leq t_2}{\star} \tau_p^0\right),$$

is a free-probabilistic sub-structure of the semicircular filterization $\mathbb{LS}_0$.

By (9.14), one can define the canonical embedding map Φ from $\mathbb{LS}^{t_1<t_2}$ into $\mathbb{LS}$, satisfying

$$\Phi(T) = T, \text{ for all } T \in \mathbb{LS}^{t_1<t_2}.$$

For any $T \in \mathbb{LS}^{t_1<t_2}$, one can get that

$$\tau^{t_1<t_2}(T) = \tau^0(T) = \tau_{t_1<t_2}(T).$$

Therefore, the Banach $*$-probability space $\mathbb{LS}^{t_1<t_2}$ is free-homomorphic to $\mathbb{LS}_{t_1<t_2}$ in $\mathbb{LS}$, whenever $2 \leq t_1 < t_2$ are suitable in $\mathbb{R}$. Therefore, the relation (9.16) holds. □

Note again that we are not sure $\mathbb{LS}^{t_1<t_2}$ and $\mathbb{LS}_{t_1<t_2}$ are free-isomorphic or not at this moment. But if the conjecture of Section 9.1 is positive, then they may not be free-isomorphic.

Corollary 9.15. *Let T be a free reduced word of the semicircular $[t_1, t_2]$-filterization $\mathbb{LS}_{t_1<t_2}$, and assume that the free distribution of T is not the zero free distribution. Then T is a free random variable of the Banach $*$-probability space $\mathbb{LS}^{t_1<t_2}$ of (9.15).*

10. LINEAR FUNCTIONALS $\tau^{+}_{t<s}$ on $\mathbb{LS}$ UNDER TRUNCATION ON PRIMES

Throughout this section, let $\mathbb{LS}_0 = \left(\mathbb{LS}, \tau^0\right)$ be the semicircular filterization of the free Adelic filterization $\mathfrak{LS}$, and assume that $t < s$ be arbitrarily fixed suitable quantities of $\mathbb{R}$ in the sense of NA 9.11. Different from the truncated linear functionals (9.12),

$$\tau_{t<s} = \underset{t \leq p \leq s \text{ in } \mathcal{P}}{\star} \tau^0_p \text{ on } \mathbb{LS},$$

(in the sense of (9.13)), we here introduce and consider a new type of the linear functionals $\tau^{+}_{t<s}$ defined by

$$\tau^{+}_{t<s} \overset{def}{=} \begin{cases} \sum_{t \leq p \leq s \text{ in } \mathcal{P}} \tau^0_p & \text{on } \underset{t \leq p \leq s}{\oplus} \mathbb{LS}_p \text{ in } \mathbb{LS}, \\ O & \text{otherwise}, \end{cases} \tag{10.1}$$

where $\tau^0_q = \underset{k \in \mathbb{Z}}{\star} \tau^0_{q,k}$ are the linear functionals (9.1) on the Banach $*$-subalgebra $\mathbb{LS}_q$ of (9.2) in $\mathbb{LS}_0$, for all $q \in \mathcal{P}$, where "$\oplus$" is the *direct product of Banach $*$-algebras.*

If there is no confusion, we write the conditional definition (10.1) simply as

$$\tau^{+}_{t<s} = \sum_{t \leq p \leq s \text{ in } \mathcal{P}} \tau^0_p \text{ on } \mathbb{LS}. \tag{10.2}$$

Definition 10.1. Let $\tau^{+}_{t<s}$ be a linear functional (10.1) on $\mathbb{LS}$, for suitable $t < s \in \mathbb{R}$ in the sense of NA 9.11. Then it is called the $[t, s]$-truncated "additive" linear functional on $\mathbb{LS}$. And the corresponding Banach $*$-probability space,

$$\mathbb{LS}^{+}_{t<s} \overset{denote}{=} (\mathbb{LS}, \tau^{+}_{t<s}), \tag{10.3}$$

is said to be the $[t, s]$(-truncated)-$(+)$(-semicircular)-filterization of $\mathbb{LS}_0$.

By the definition (10.1), two Banach $*$-probability spaces, the $[t,\ s]$-filterization $\mathbb{LS}_{t<s}$ of (9.14), and the $[t, s]$-$(+)$-filterization $\mathbb{LS}^{+}_{t<s}$ of (10.3) are different free-probabilistic objects in the semicircular filterization $\mathbb{LS}_0$, in general. More precisely, one can get the following result.

Theorem 10.2. *Let $\mathbb{LS}_{t<s}$ be the $[t, s]$-filterization* (9.14), *and let $\mathbb{LS}^{+}_{t<s}$ be the $[t, s]$-$(+)$-filterization* (10.3), *for suitable $t < s$ in $\mathbb{R}$.*

(i) *If there are multi-primes in $[t, s]$, then $\mathbb{LS}_{t<s}$ and $\mathbb{LS}^{+}_{t<s}$ are not free-homomorphic.*
(ii) *If $[t, s]$ contains only one prime p, then $\mathbb{LS}_{t<s}$ and $\mathbb{LS}^{+}_{t<s}$ are free-isomorphic.*

Proof. First of all, let us prove the statement (ii). Suppose $t < s$ are suitable in $\mathbb{R}$, and assume that $p \in \mathcal{P}$ is the only prime satisfying

$$t \leq p \leq s.$$

Then, by the definitions (9.12) and (10.1), we have

$$\tau_{t<s} = \tau_{p<p} = \tau^0_p = \tau^{+}_{t<s} \text{ on } \mathbb{LS},$$

in the sense of (9.13) and (10.2), where $\tau_{p<p}$ is axiomatized to be τ_p^0 on $\mathbb{LS}$ in the sense of (9.4).

It shows that

$$\mathbb{LS}_{t<s} = (\mathbb{LS}, \tau_p^0) = \mathbb{LS}_{t<s}^{+}.$$

Therefore, if p is the only prime in $[t, s]$, then $\mathbb{LS}_{t<s}$ and $\mathbb{LS}_{t<s}^{+}$ are free-isomorphic in the semicircular filterization $\mathbb{LS}_0$, with a free-isomorphism, the identity map on $\mathbb{LS}$. Thus, the statement (ii) holds.

Now, assume that there are N-many primes $q_1, \ldots, q_N$ are contained in $[t, s]$, where $N > 1$ in $\mathbb{N}$. Thus,

$$\tau_{t<s} = \underset{k=1}{\overset{N}{\star}} \tau_{q_k}^0, \quad \text{and} \quad \tau_{t<s}^{+} = \sum_{k=1}^{N} \tau_{q_k}^0,$$

on the Banach $*$-algebra $\mathbb{LS}$ in the sense of (9.13), respectively, (10.2).

Take an arbitrary free reduced word T with its length-n,

$$T = Q_{p_1,j_1}^{n_1} Q_{p_2,j_2}^{n_2} \ldots Q_{p_n,j_n}^{n_n} \tag{10.4}$$

of $\mathbb{LS}_0$ in the free weighted-semicircular family $\mathcal{Q}$, for $1 < n \in \mathbb{N}$, where either

$$(p_1, \ldots, p_n), \text{ or } (j_1, \ldots, j_n)$$

consists of "mutually distinct" $p_1, \ldots, p_n$ in $\mathcal{P}$, respectively, consists of "mutually distinct" $j_1, \ldots, j_n$ in $\mathbb{Z}$, for $n_1, \ldots, n_n \in \mathbb{N}$. Also, for convenience, assume further that

$$p_1, \ldots, p_n \in \{q_1, \ldots, q_N\}, \tag{10.5}$$

and

$$n_1, \ldots, n_n \in 2\mathbb{N} = \{2n : n \in \mathbb{N}\},$$

for $1 < n \leq N$ in $\mathbb{N}$.

For any $*$-homomorphisms Ω from $\mathbb{LS}_{t<s}$ to $\mathbb{LS}_{t<s}^{+}$ (i.e., for any $*$-homomorphisms Ω on $\mathbb{LS}$), the corresponding images $\Omega(T)$ of the free reduced word T of (10.4) would be the free reduced word T' with its length-n', where

$$n' \leq n \leq N \text{ in } \mathbb{N}.$$

One may write this image T' of T as

$$T' = Q_{r_1,i_1}^{k_1} Q_{r_2,i_2}^{k_2} \ldots Q_{r_{n'},i_{n'}}^{k_{n'}}, \tag{10.6}$$

for $r_1, \ldots, r_{n'} \in \mathcal{P}$, $i_1, \ldots, i_{n'} \in \mathbb{Z}$, and $k_1, \ldots, k_{n'} \in \mathbb{N}$, as a free reduced word of $\mathbb{LS}$.

Observe now that if T is in the sense of (10.4), then

$$\tau_{t<s}(T) = \prod_{k=1}^{n} \left(p_k^{n_k(j_k+1)} c_{\frac{n_k}{2}} \right) \neq 0. \tag{10.7}$$

by (10.5), because all factors of T are mutually free from each other; meanwhile, if T' is in the sense of (10.6), then

$$\tau_{t<s}^{+}\left(T'\right) = \begin{cases} 0 & \text{if } n' > 1, \\ \sum_{l=1}^{N} \delta_{q_l, r_1} \omega_{k_1} q_l^{k_1(i_1+1)} c_{\frac{k_1}{2}} & \text{if } n' = 1, \end{cases} \tag{10.8}$$

by (10.1).

So, $\mathbb{LS}_{t<s}$ is not free-homomorphic to $\mathbb{LS}_{t<s}^{+}$ by (10.7) and (10.8).

Similarly, let us take a free reduced word T of (10.4), now in the $[t,s]$-(+)--filterization $\mathbb{LS}_{t<s}^{+}$, satisfying (10.5). Then, since $N > 1$ in $\mathbb{N}$,

$$\tau_{t<s}^{+}(T) = 0,$$

more precisely,

$$\tau_{t<s}^{+}(T^n) = \tau_{t<s}^{+}\left((T^*)^n\right) = \tau_{t<s}^{+}\left(T^{s_1} T^{s_2} \dots T^{s_n}\right) = 0, \tag{10.9}$$

for all $(s_1, \dots, s_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$. It shows that, as an element of $\mathbb{LS}_{t<s}^{+}$, the free reduced word T, whose length is $N > 1$, follows the zero free distribution. For any $*$-homomorphism from $\mathbb{LS}_{t<s}^{+}$ to $\mathbb{LS}_{t<s}$, the images T' of them (in the sense of (10.6), as elements of $\mathbb{LS}_{t<s}$) satisfy

$$\tau_{t<s}\left(T'\right) = \delta_{(q_1, \dots, q_N : r_1, \dots, r_{n'})} \prod_{l=1}^{n'} \left(\omega_{k_l} r_l^{k_l(i_l+1)} c_{\frac{k_l}{2}} \right), \tag{10.10}$$

by (10.7), where

$$\delta_{(q_1, \dots, q_N : r_1, \dots, r_{n'})} = \begin{cases} 1 & \text{if } r_1, \dots, r_{n'} \in \{q_1, \dots, q_N\}, \\ 0 & \text{otherwise.} \end{cases}$$

The formulas (10.9) and (10.10) demonstrate that $\mathbb{LS}_{t<s}^{+}$ is not free-homomorphic to $\mathbb{LS}_{t<s}$.

Therefore, the $[t,s]$-filterization $\mathbb{LS}_{t<s}$ and the $[t,s]$-(+)-filterization $\mathbb{LS}_{t<s}^{+}$ are not free-homomorphic from each other, whenever there are multi-primes in $[t,s]$. So, the statement (ii) of this theorem holds true. □

By (10.1), we obtain a following free-homomorphic relation.

Theorem 10.3. *Let $\mathbb{LS}_q$ be in the sense of (9.2) in the semicircular filterization $\mathbb{LS}_0$, for $q \in \mathcal{P}$, and let*

$$\mathcal{P}_{[t,s]} = \mathcal{P} \cap [t,s], \tag{10.11}$$

for suitable $t < s \in \mathbb{R}$. Define a Banach $*$-probabilistic sub-structure $\mathbb{LS}_{[t,s]}$ of $\mathbb{LS}_0$ by

$$\mathbb{LS}_{[t,s]} \overset{def}{=} \left(\underset{p \in \mathcal{P}_{[t,s]}}{\oplus} \mathbb{LS}_p, \tau_{[t,s]} = \sum_{p \in \mathcal{P}_{[t,s]}} \tau_p^0 \right). \tag{10.12}$$

Then $\mathbb{LS}_{[t,s]}$ of (10.12) is free-homomorphic to the $[t,s]$-(+)-filterization $\mathbb{LS}_{t<s}^{+}$, in $\mathbb{LS}$.

Proof. Let $\mathbb{LS}_{[t,s]}$ be in the sense of (10.12) embedded in the semicircular filterization $\mathbb{LS}_0$. Define now a bounded linear transformation

$$\Psi : \mathbb{LS}_{[t,s]} \to \mathbb{LS}^{+}_{t<s}$$

by the canonical embedding map,

$$\Psi(T) = T \text{ in } \mathbb{LS}^{+}_{t<s}, \text{ for all } T \in \mathbb{LS}_{[t,s]}. \tag{10.13}$$

For any $T \in \mathbb{LS}_{[t,s]}$, one has that

$$\tau^{+}_{t<s}\left(\Psi(T)\right) = \tau^{+}_{t<s}\left(T\right) = \tau^{+}_{t<s}\left(\underset{q\in\mathcal{P}_{[t,s]}}{\oplus} T_q\right)$$

since $T = \Psi(T) \in \mathbb{LS}_{[t,s]} \subset \mathbb{LS}^{+}_{t<s}$, and hence, there exist unique $T_q \in \mathbb{LS}_q$, for all $q \in \mathcal{P}_{[t,s]}$, such that $T = \underset{q\in\mathcal{P}_{[t,s]}}{\oplus} T_q$, and hence, the above formula goes to

$$\begin{aligned} &= \sum_{q\in\mathcal{P}_{[t,s]}} \tau^{0}_{q}\left(T_q\right) = \left(\sum_{q\in\mathcal{P}_{[t,w]}} \tau^{0}_{q}\right)\left(\underset{q\in\mathcal{P}_{[t,s]}}{\oplus} T_q\right) \\ &= \tau_{[t,s]}(T), \end{aligned} \tag{10.14}$$

by (10.11) and (10.12). Therefore, the $*$-homomorphism Ψ of (10.13) is free--distribution-preserving by (10.14). Equivalently, it is a free-homomorphism. □

11. APPLICATION: CIRCULARITY ON $\mathbb{LS}_0, \mathbb{LS}_{t<s}$, and $\mathbb{LS}^{+}_{t<s}$

Throughout this section, we use same definitions, and notations introduced in previous sections. Let $\mathbb{LS}_0 = (\mathbb{LS}, \tau^0)$ be the semicircular filterization in the free Adelic filterization $\mathfrak{LS}$, and let $t < s$ be suitable in $\mathbb{R}$ in the sense of NA 9.11, and

$$\mathbb{LS}_{t<s} = (\mathbb{LS}, \tau_{t<s}), \text{ and } \mathbb{LS}^{+}_{t<s} = (\mathbb{LS}, \tau^{+}_{t<s})$$

are the $[t,s]$-filterization (9.14), respectively, the $[t,s]$-(+)-filterization (10.3) of $\mathbb{LS}_0$.

In this section, we apply our main results of Sections 8, 9 and 10 to the case where we have the operators $X \in \mathbb{LS}$,

$$X = \frac{1}{\sqrt{2}}\left(\Theta_{p_1,j_1} + i\Theta_{p_2,j_2}\right), \tag{11.1}$$

where $i = \sqrt{-1}$ in $\mathbb{C}$,

$$\Theta_{p_l,j_l} = \frac{1}{p_l^{j_l+1}} Q_{p_l,j_l} \in \Theta, \text{ for all } l = 1, 2,$$

and where either

$$p_1 \neq p_2 \in \mathcal{P}, \text{ or } j_1 \neq j_2 \in \mathbb{Z}, \tag{11.2}$$

where Θ is the free semicircular family (8.6) generating $\mathbb{LS}_0$.

By the condition (11.2), the summands Θ_{p_1,j_1} and $i\Theta_{p_2,j_2}$ of the operators X of (11.1) are free in the semicircular filterization $\mathbb{LS}_0$.

Definition 11.1. Let (A, ψ) be an arbitrary topological $*$-probability space, and let s_1 and s_2 be semicircular elements in (A, ψ). Assume these two semicircular elements s_1 and s_2 are free in (A, ψ). Then the free random variable

$$x = \frac{1}{\sqrt{2}}(s_1 + is_2) \in (A, \psi), \tag{11.3}$$

is called the circular element induced by s_1 and s_2 in (A, ψ) (e.g., [21, 22, 24] and [29]). The free distributions of such circular elements x of (11.3) are called the circular law.

The circular law is characterized by the very semicircularity under free sum (e.g., [21, 22] and [24]). In particular, the circular law is characterized by the joint free-moments of a circular element x of (11.3), and its adjoint x^* under identically-free-distributedness, since x is not self-adjoint in A, i.e.,

$$x^* = \frac{1}{\sqrt{2}}\left(s_1 - is_2\right) \neq x \text{ in } (A, \psi).$$

Recall that two free random variables a_l of topological $*$-probability spaces (A_l, ψ_l), for $l = 1, 2$, are said to be *identically free-distributed*, if

$$\psi_1\left(a_1^{r_1} a_1^{r_2} \ldots a_1^{r_n}\right) = \psi_2\left(a_2^{r_1} a_2^{r_2} \ldots a_2^{r_n}\right), \tag{11.4}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$. For instance, if a_1 and a_2 are self-adjoint in A_1, respectively, in A_2, then they are identically free-distributed, if and only if

$$\psi_1(a_1^n) = \psi_2(a_2^n), \text{ for all } n \in \mathbb{N}$$

(e.g., [1] and [29]).

Note that the semicircular law, and the circular law are characterized under identically free-distributedness universally (different from weighted-semicircular laws). i.e., "all" circular elements (resp., "all" semicircular elements) have the same free distributions, the circular law (resp., the semicircular law).

11.1. CIRCULARITY ON $\mathbb{LS}_0$

Let X be an operator (11.1), satisfying the condition (11.2) in the semicircular filterization $\mathbb{LS}_0$. Then it is a circular element in $\mathbb{LS}_0$ by (11.3).

Proposition 11.2. *Let $\Theta_{p_1,j_1}, \Theta_{p_2,j_2} \in \Theta$ be semicircular elements of $\mathbb{LS}_0$, where either*

$$p_1 \neq p_2 \text{ in } \mathcal{P}, \text{ or } j_1 \neq j_2 \text{ in } \mathbb{Z}.$$

Then the operator X,

$$X = \frac{1}{\sqrt{2}}\left(\Theta_{p_1,j_1} + i\Theta_{p_2,j_2}\right) \in \mathbb{LS}_0 \tag{11.5}$$

is a circular element in $\mathbb{LS}_0$.

Proof. Suppose $\Theta_{p_1,j_1}, \Theta_{p_2,j_2} \in \Theta$ are semicircular elements of $\mathbb{LS}_0$, and the above condition is satisfied. Then, these two semicircular elements are free in $\mathbb{LS}_0$. So, by the circularity (11.3), the operator X of (11.5) is circular in $\mathbb{LS}_0$. □

11.2. CIRCULARITY ON $\mathbb{LS}_0$ IN $\mathbb{LS}_{t<s}$

Let $t < s$ be suitable real numbers in $\mathbb{R}$ under NA 9.11, and $\mathbb{LS}_{t<s} = (\mathbb{LS}, \tau_{t<s})$, the corresponding $[t,s]$-filterization of the semicircular filterization $\mathbb{LS}_0$. Let X be an operator (11.1) satisfying the condition (11.2) in the Banach $*$-algebra $\mathbb{LS}$, and let

$$\mathcal{P}_{[t,s]} = \mathcal{P} \cap [t,s].$$

Before considering the free-distributional data of X in $\mathbb{LS}_{t<s}$, let us introduce the following concept.

Definition 11.3. Let (A,ψ) be an arbitrary topological $*$-probability space, and suppose $x \in (A,\psi)$ is "not" self-adjoint. We will say that the free distribution of x is followed by the semicircular law, if

$$\psi\left(x^{r_1} x^{r_2} \dots x^{r_n}\right) = \omega_n c_{\frac{n}{2}},$$

for all $(r_1, \dots, r_n) \in \{1,*\}^n$, for all $n \in \mathbb{N}$.

Suppose a free random variable x is not self-adjoint in a topological $*$-probability space (A,ψ). Then it cannot be a semicircular element by (7.5), (7.8) and (7.9). But, does a free random variable x whose free distribution is followed by the semicircular law in the above sense exist? The following theorem not only characterizes the free distribution of an operator X of (11.1) in the $[t,s]$-filterization $\mathbb{LS}_{t<s}$, but also provides the positive answer of this question.

Theorem 11.4. *Let X be a circular element* (11.5) *in* $\mathbb{LS}_0$, *and let* $\mathcal{P}_{[t,s]} = \mathcal{P} \cap [t,s]$, *where $[t,s]$ is a closed interval of $\mathbb{R}$. Then the following assertions hold.*

(i) *If $p_1, p_2 \in \mathcal{P}_{[t,s]}$, then X is circular in the $[t,s]$-filterization $\mathbb{LS}_{t<s}$.*
(ii) *If $p_1 \in \mathcal{P}_{[t,s]}$, and $p_2 \notin \mathcal{P}_{[t,s]}$, then the free distribution of $\sqrt{2}X$ is followed by the semicircular law in $\mathbb{LS}_{t<s}$.*
(iii) *If $p_1 \notin \mathcal{P}_{[t,s]}$, and $p_2 \in \mathcal{P}_{[t,s]}$, then the free distribution of $-i\sqrt{2}X$ is followed by the semicircular law in $\mathbb{LS}_{t<s}$.*
(iv) *If $p_1 \notin \mathcal{P}_{[t,s]}$, and $p_2 \notin \mathcal{P}_{[t,s]}$, then X has the zero free distribution in $\mathbb{LS}_{t<s}$.*

Proof. Suppose first that

$$p_1, p_2 \in \mathcal{P}_{[t,s]}.$$

Then the summands Θ_{p_l,j_l} are free in $\mathbb{LS}_{t<s}$, by Lemma 9.1, for all $l = 1,2$. So, by (9.16) and (11.5), the operator X is circular in the $[t,s]$-filterization $\mathbb{LS}_{t<s}$, too. So, the statement (i) holds.

Assume that

$$p_1 \in \mathcal{P}_{[t,s]}, \text{ and } p_2 \notin \mathcal{P}_{[t,s]},$$

and regard X as a free random variable of $\mathbb{LS}_{t<s}$.

Now, let

$$T = \sqrt{2}X = \Theta_{p_1,j_1} + i\Theta_{p_2,j_2} \in \mathbb{LS}_{t<s}.$$

Observe that if there are free reduced words

$$W_{p_2,j_2} = \Theta_{q_1,j_1}^{n_1} \dots \Theta_{p_2,j_2}^{n} \dots \Theta_{q_2,j_2}^{n_N} \in \mathbb{LS}_{t<s},$$

containing at least one free-factor Θ_{p_2,j_2}^n for $n \in \mathbb{N}$, then

$$\tau_{t<s}\left(W_{p_2,j_2}\right) = 0, \text{ for all } N \in \mathbb{N},$$

by (9.12) and (9.13). Therefore, one can get that

$$\tau_{t<s}\left(T^n\right) = \tau_{p_1}^0\left(\Theta_{p_1,j_1}^n\right) = \tau_{t<s}\left((T^*)^n\right),$$

and

$$\tau_{t<s}\left(T^{r_1}T^{r_2}\dots T^{r_n}\right) = \tau_{p_1}^0\left(\Theta_{p_1,j_1}^n\right),$$

for all mixed $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Note that

$$\Theta_{p_1,j_1} \in \mathbb{LS}_{p_1} \subset \mathbb{LS}_{t<s} \overset{\text{free-homo}}{\subseteq} \mathbb{LS}_0,$$

where "$\overset{\text{free-homo}}{\subseteq}$" means "being free-homomorphic", and hence, it is semicircular. Therefore, the free distribution of $T = \sqrt{2}X$ is followed by the semicircular law in $\mathbb{LS}_{t<s}$, by (11.2). (Remark that this operator T is not semicircular in $\mathbb{LS}_{t<s}$, but, the free distribution of T is followed by the semicircular law.) It shows that the statement (ii) holds.

Let $p_1 \notin \mathcal{P}_{[t,s]}$ and $p_2 \in \mathcal{P}_{[t,s]}$, and let

$$S = -\sqrt{2}iX = -i\Theta_{p_1,j_1} + \Theta_{p_2,j_2} \in \mathbb{LS}_{t<s}.$$

Then, similar to (11.2), one obtains that

$$\tau_{t<s}\left(S^n\right) = \tau_{p_2}^0\left(\Theta_{p_2,j_2}^n\right) = \tau_{t<s}\left((S^*)^n\right), \tag{11.6}$$

and

$$\tau_{t<s}\left(S^{r_1}S^{r_2}\dots S^{r_n}\right) = \tau_{p_2}^0\left(\Theta_{p_2,j_2}^n\right),$$

for all mixed $(r_1, \dots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$. So, like in the proof of (ii), the free distribution of $S = -\sqrt{2}iX$ is followed by the semicircular law in $\mathbb{LS}_{t<s}$, by (11.6). Thus, the statement (iii) holds.

Finally, assume that

$$p_1 \notin \mathcal{P}_{[t,s]}, \text{ and } p_2 \notin \mathcal{P}_{[t,s]}.$$

Then $X \notin \mathbb{LS}^{t<s}$, where

$$\mathbb{LS}^{t<s} = \left(\underset{q \in \mathcal{P}_{[t,s]}}{\star} \mathbb{LS}_q, \underset{q \in \mathcal{P}_{[t,s]}}{\star} \tau_q^0\right)$$

is the Banach $*$-probability space (9.15) in $\mathbb{LS}$. Therefore, by the free-homomorphic relation (9.16), this operator X has the zero free distribution in the $[t, s]$-filterization $\mathbb{LS}_{t<s}$. Therefore, the statement (iv) holds true. □

The above theorem illustrates the difference between original free-distributional data on the semicircular filterization $\mathbb{LS}_0$, and those on the $[t, s]$-filterization $\mathbb{LS}_{t<s}$ under suitable truncations for $[t, s]$. In particular, the circularity (11.5) of $\mathbb{LS}_0$ is affected by the truncations for $[t, s]$ by (i)–(iv).

The following corollary is a direct consequence of the above theorem.

Corollary 11.5. *Let* $X = \frac{1}{\sqrt{2}}(\Theta_{p_1,j_1} + i\Theta_{p_2,j_2})$ *be a circular element* (11.5) *of* $\mathbb{LS}_0$. *Suppose* $t < s$ *are suitable in* $\mathbb{R}$, *and assume either*

$$p_1 \notin \mathcal{P}_{[t,s]}, \text{ or } p_2 \notin \mathcal{P}_{[t,s]} \text{ in } \mathcal{P}.$$

Then X *is not circular in* $\mathbb{LS}_{t<s}$. *i.e., the circular law is distorted by the truncation for* $[t, s]$.

Proof. Let $X \in \mathbb{LS}_0$ be a circular element (11.5). Assume that either

$$p_1 \notin \mathcal{P}_{[t,s]}, \text{ or } p_2 \notin \mathcal{P}_{[t,s]} \text{ in } \mathcal{P}.$$

Then X is not circular in $\mathbb{LS}_{t<s}$ by (ii)–(iv) of Theorem 11.4. □

11.3. CIRCULARITY OF $\mathbb{LS}_0$ in $\mathbb{LS}^+_{t<s}$

Let $\mathbb{LS}^+_{t<s}$ be the $[t, s]$-(+)-filterization of the semicircular filterization $\mathbb{LS}_0$, for suitable $t < s$ in $\mathbb{R}$ under NA 9.11, and let X be a circular element (11.5) of the semicircular filterization $\mathbb{LS}_0$ under (11.2).

Lemma 11.6. *Let* $X = \frac{1}{\sqrt{2}}(\Theta_{p_1,j_1} + i\Theta_{p_2,j_2})$ *be a circular element* (11.5) *in* $\mathbb{LS}_0$. *If we regard* X *as a free random variable of the* $[t, s]$-(+)-*filterization* $\mathbb{LS}^+_{t<s}$, *then one obtains the following free-distributional data.*

(i) *If* $p_1, p_2 \in \mathcal{P}_{[t,s]} = \mathcal{P} \cap [t, s]$, *then*

$$\tau^+_{t<s}(X^n) = \omega_n \left(\frac{1}{\sqrt{2}}\right)^n (1 + i^n)\, c_{\frac{n}{2}},$$

and

$$\tau^+_{t<s}((X^*)^n) = \omega_n \left(\frac{1}{\sqrt{2}}\right)^n (1 + (-i)^n)\, c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$.

(ii) *If* $p_1 \in \mathcal{P}_{[t,s]}$, *and* $p_2 \notin \mathcal{P}_{[t,s]}$, *then*

$$\tau^+_{t<s}(X^n) = \tau^+_{t<s}((X^*)^n) = \omega_n \left(\frac{1}{\sqrt{2}}\right)^n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$.

(iii) *If* $p_1 \notin \mathcal{P}_{[t,s]}$ *and* $p_2 \in \mathcal{P}_{[t,s]}$, *then*

$$\tau^{+}_{t<s}(X^n) = \omega_n \left(\frac{i}{\sqrt{2}}\right)^n c_{\frac{n}{2}},$$

and

$$\tau^{+}_{t<s}\left((X^*)^n\right) = \omega_n \left(\frac{-i}{\sqrt{2}}\right)^n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$.

(iv) *If* $p_1 \notin \mathcal{P}_{[t,s]}$, *and* $p_2 \notin \mathcal{P}_{[t,s]}$, *then* X *has the zero free distribution on* $\mathbb{LS}^{+}_{t<s}$.

Proof. Suppose $p_1, p_2 \in \mathcal{P}_{[t,s]}$. Then, by (10.12), (10.13) and (10.14),

$$\tau^{+}_{t<s}\left(X^n\right) = \tau_{[t,s]}\left(\left(\frac{1}{\sqrt{2}}(\Theta_{p_1,j_1} \oplus i\Theta_{p_2,j_2}\right)^n\right)$$

where $\tau_{[t,s]} = \sum_{q \in \mathcal{P}_{[t,s]}} \tau^0_q$ is in the sense of (10.12)

$$\begin{aligned}
&= \left(\frac{1}{\sqrt{2}}\right)^n \tau_{[t,s]}\left(\left(\Theta^n_{p_1,j_1} \oplus i^n \Theta^n_{p_2,j_2}\right)\right) \\
&= \left(\frac{1}{\sqrt{2}}\right)^n \left(\tau^0_{p_1}\left(\Theta^n_{p_1,j_1}\right) + i^n \tau^0_{p_2}\left(\Theta^n_{p_2,j_2}\right)\right) \\
&= \left(\frac{1}{\sqrt{2}}\right)^n \left(\omega_n c_{\frac{n}{2}} + i^n \omega_n c_{\frac{n}{2}}\right)
\end{aligned}$$

by the semicircularity of Θ_{p_l,j_l} in $\mathbb{LS}_0$ (and hence, in $\mathbb{LS}^{+}_{t<s}$)

$$= \omega_n \left(\frac{1}{\sqrt{2}}\right)^n (1 + i^n) c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$.

Similarly,

$$\begin{aligned}
\tau^{+}_{t<s}\left((X^*)^n\right) &= \left(\frac{1}{\sqrt{2}}\right)^n \tau_{[t,s]}\left(\left(\Theta_{p_1,j_1} \oplus (-i\Theta_{p_2,j_2})\right)^n\right) \\
&= \left(\frac{1}{\sqrt{2}}\right)^n \left(\tau^0_{p_1}\left(\Theta^n_{p_1,j_1}\right) + (-i)^n \tau^0_{p_2}\left(\Theta^n_{p_2,j_2}\right)\right) \\
&= \omega_n \left(\frac{1}{\sqrt{2}}\right)^n \left(1 + (-i)^n\right) c_{\frac{n}{2}},
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, the statement (i) holds.

Suppose $p_1 \in \mathcal{P}_{[t,s]}$, and $p_2 \notin \mathcal{P}_{[t,s]}$. Then

$$\begin{aligned}\tau^{+}_{t<s}\left(X^n\right) &= \tau_{[t,s]}\left(\left(\frac{1}{\sqrt{2}}\left(\Theta_{p_1,j_1} + i\Theta_{p_2,j_2}\right)\right)^n\right) \\ &= \left(\frac{1}{\sqrt{2}}\right)^n \tau_{[t,s]}\left(\Theta^n_{p_1,j_1}\right) \\ &= \left(\frac{1}{\sqrt{2}}\right)^n \tau^0_{p_1}\left(\Theta^n_{p_1,j_2}\right) = \omega_n \left(\frac{1}{\sqrt{2}}\right)^n c_{\frac{n}{2}} \\ &= \tau^{+}_{t<s}\left(\left(X^*\right)^n\right),\end{aligned}$$

for all $n \in \mathbb{N}$. Thus, the statement (ii) holds.

Assume now that $p_1 \notin \mathcal{P}_{[t,s]}$, and $p_2 \in \mathcal{P}_{[t,s]}$. Then, similar to the proof of (ii), one can get that

$$\tau^{+}_{t<s}(X^n) = \omega_n \left(\frac{i}{\sqrt{2}}\right)^n c_{\frac{n}{2}},$$

and

$$\tau^{+}_{t<s}\left(\left(X^*\right)^n\right) = \omega_n \left(\frac{-i}{\sqrt{2}}\right)^n c_{\frac{n}{2}},$$

for all $n \in \mathbb{N}$. It guarantees the statement (iii) holds true.

Finally, assume that $p_1 \notin \mathcal{P}_{[t,s]}$, and $p_2 \notin \mathcal{P}_{[t,s]}$. Then, by (10.13) and (10.14), the operator X has the zero free distribution on $\mathbb{LS}^{+}_{t<s}$. Equivalently, the statement (iv) holds. □

By the above lemma, one immediately obtains the following result.

Theorem 11.7. *Let X be a circular element* (11.5) *of the semicircular filterization $\mathbb{LS}_0$. If X is regarded as a free random variable of the* $[t,s]$-$(+)$-*filterization $\mathbb{LS}^{+}_{t<s}$, then X is not circular in $\mathbb{LS}^{+}_{t<s}$, i.e.,*

$$X \text{ cannot be a circular element in } \mathbb{LS}^{+}_{t<s}. \tag{11.7}$$

Proof. Let X be given as above in $\mathbb{LS}^{+}_{t<s}$. Then it cannot be circular in $\mathbb{LS}^{+}_{t<s}$, by (i)–(iv) of Lemma 11.6. So, the statement (11.7) is proven. □

It shows that a circular element X of the semicircular filterization $\mathbb{LS}_0$ cannot be circular in all $[t,s]$-$(+)$-filterizations $\mathbb{LS}^{+}_{t<s}$, whenever $-\infty < t < s < \infty$ in $\mathbb{R}$.

11.4. DISCUSSION

In Sections 11.1, 11.2 and 11.3, we applied the main results of Sections 8, 9 and 10 to circular elements of the semicircular filterization $\mathbb{LS}_0$. Especially, the distorted circularity is observed in $\mathbb{LS}_{t<s}$, and in $\mathbb{LS}^{+}_{t<s}$, where $t < s$ are suitable in the sense of NA 9.11, i.e., the circularity (11.5) of $\mathbb{LS}_0$ is affected by our truncations in $\mathbb{LS}_{t<s}$ by (i)–(iv) of Theorem 11.4, meanwhile, it is distorted by truncations in $\mathbb{LS}^{+}_{t<s}$, by (11.7).

In the middle of studying such distortions, the existence of a certain type of free random variables, whose free distributions are followed by the semicircular law, is shown (in Section 11.2).

Proposition 11.8. *There exist a topological $*$-probability space (A, ψ), and free random variables $x \in (A, \psi)$, such that:*

(i) *x is not self-adjoint (and hence, not semicircular),*
(ii) *the free distribution of x is followed by the semicircular law in the sense that:*

$$\psi\left(x^{r_1} x^{r_2} \ldots x^{r_n}\right) = \omega_n c_{\frac{n}{2}}, \tag{11.8}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

Proof. The proof is done by construction. Let

$$\mathbb{LS}_{t<s} = (\mathbb{LS}, \tau_{t<s})$$

be the $[t, s]$-filterization of the semicircular filterization $\mathbb{LS}_0$, where $t < s$ are suitable in $\mathbb{R}$. Let us take a free random variable

$$T = \Theta_{p_1, j_1} + t\Theta_{p_2, j_2}$$

in $\mathbb{LS}_{t<s}$, for $t \in \mathbb{C}$, where $\Theta_{p_l, j_l} \in \Theta$ are two distinct (and hence, free) semicircular elements in $\mathbb{LS}_0$, for $l = 1, 2$, and

$$p_1 \in \mathcal{P}_{[t,s]} = \mathcal{P} \cap [t, s], \text{ and } p_2 \notin \mathcal{P}_{[t,s]}.$$

Then, similar to the proofs of (ii) and (iii) of Theorem 11.4, the free distributions of T are characterized by the joint free moments of $\{T, T^*\}$ satisfying

$$\tau_{t<s}\left(T^{r_1} T^{r_2} \ldots T^{r_n}\right) = \omega_n c_{\frac{n}{2}},$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$.

It guarantees the existence of non-self-adjoint free random variables whose free distributions are followed by the semicircular law. □

The above proposition provides an interesting class of free random variables of topological $*$-probability spaces. By the Möbius inversion of [27], one can get the following equivalent result of the above proposition.

Corollary 11.9. *There exist topological $*$-probability spaces (A, ψ), and free random variables $x \in (A, \psi)$, such that*

(i) *x is not self-adjoint,*
(ii) *the free distribution of x is followed by the semicircular law in the sense that:*

$$k_n^{\psi}\left(x^{r_1}, \ldots, x^{r_n}\right) = \begin{cases} 1 & \text{if } n = 2, \\ 0 & \text{otherwise,} \end{cases} \tag{11.9}$$

for all $(r_1, \ldots, r_n) \in \{1, *\}^n$, for all $n \in \mathbb{N}$, where $k_n^{\psi}(\cdot)$ is the free cumulant on A in terms of the linear functionals ψ.

Proof. The proof of (11.9) is done by (11.8) under the Möbius inversion of [27]. □

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