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LIGHTWEIGHT PATHS IN GRAPHS

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Abstract. Let k be a positive integer, G be a graph on $V(G)$ containing a path on k vertices, and w be a weight function assigning each vertex $v \in V(G)$ a real weight $w(v)$. Upper bounds on the weight $w(P) = \sum_{v \in V(P)} w(v)$ of P are presented, where P is chosen among all paths of G on k vertices with smallest weight.

Keywords: weighted graph, lightweight path.

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1. INTRODUCTION

We use standard terminology of graph theory and consider finite and simple graphs, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. It is well known that every planar graph G contains a vertex v such that the degree $d_G(v)$ of v (in G) is at most 5. In 1955, Kotzig [7, 8] proved that every 3-connected planar graph G contains an edge uv such that $d_G(u) + d_G(v)$ is at most 13 in general and at most 11 in absence of 3-valent vertices. Moreover, these bounds are best possible. Given a positive integer k and a graph G, a *k-path of G* is a path of G on k vertices. Motivated by the previous results, for some positive integer k, upper bounds on a lightweight k-path of a planar graph were established, where the *weight* of a path P of a graph G is the sum of the degrees (in G) of the vertices of P. For example, Fabrici and Jendrol' [4] proved that any 3-connected planar graph containing a k-path has a k-path of weight at most $5k^2$. This result has been strengthened by Fabrici, Harant, and Jendrol' in [3] showing that the upper bound $5k^2$ can be replaced with $\frac{3}{2}k^2 + O(k)$ in general and with $k^2 + O(k)$ in the case of plane triangulations. Mohar [9] proved that any 4-connected planar graph of order at least k contains a k-path of weight at most $6k - 1$, which is tight.

Here the task is generalized by considering arbitrary graphs vertex-weighted by arbitrary real numbers. Let $w : V(G) \to R$ be a fixed weight function assigning each vertex $v \in V(G)$ of a graph G a real weight $w(v)$,

$$d_w = \frac{\sum_{v \in V(G)} w(v)}{|V(G)|}$$

be the *average weight* of G, and

$$w(P) = \sum_{v \in V(P)} w(v)$$

be the *weight of a path* P of G.

In the sequel, we are interested (for some k) in a k-path P of G of smallest weight. Obviously, we may assume that G is connected. If G is a tree, then P is a subpath of the (unique) path connecting two suitable leaves of G, thus, in this case it is easy to find P. Hence, throughout the paper, we assume that G is a connected graph with *size* $m = |E(G)|$ at least $n = |V(G)|$. Let $\mathcal{H}(G)$ be the set of subgraphs H of G of positive size such that every component of H is bridgeless. Since a cycle of G is a bridgeless subgraph of G of positive size, it follows that $\mathcal{H}(G)$ is not empty. By $girth(G)$ we denote the length of a shortest cycle of G.

The basic tool we use is the rotation of a k-path of G around a cycle of G on at least k vertices. This idea was introduced by Mohar in [9]. Since a 4-connected planar graph G contains a hamiltonian cycle C ([10]), i.e. $C \in \mathcal{H}(G)$, the above mentioned result of Mohar follows from the forthcoming Theorem 2.1, which is our main result.

2. RESULTS AND PROOFS

Theorem 2.1. *Let t be a real number, $H \in \mathcal{H}(G)$, and $1 \le k \le girth(H)$. Then H contains a k-path P such that*

$$w(P) \le \frac{\sum_{v \in V(H)} d_H(v) w(v)}{2|E(H)|} k = \left(d_w + \frac{\sum_{v \in V(H)} d_H(v)(w(v) - d_w)}{2|E(H)|} \right) k.$$

Moreover, if H is spanning, then

$$w(P) \le \left(d_w + \frac{\sum_{v \in V(G)} (d_H(v) - t)(w(v) - d_w)}{2|E(H)|} \right) k.$$

Proof. Given a positive integer s, a *cycle s-cover* of a graph G is a multiset of cycles of G that each edge of G is contained in exactly s of these cycles.

For instance, for any 2-connected planar graph, the faces provide a cycle 2-cover of the graph: each edge belongs to exactly two faces.

It is an unsolved problem (posed by G. Szekeres and P.D. Seymour and known as the *Cycle Double Cover Conjecture*), whether every bridgeless graph has a cycle

2-cover, however, Bermond, Jackson, and Jaeger [1] proved that every bridgeless graph has a cycle 4-cover.

For the proof of Theorem 2.1, we first construct a non-empty multiset Π of k-paths of H and show that the arithmetical mean of all values $w(P)$ taken over all $P \in \Pi$ equals $\frac{\sum_{v\in V(H)} d_H(v)w(v)}{2|E(H)|}k$.

Consider an arbitrary component F of H. If F consists of a single vertex only, then F does not contribute to the expression $\frac{\sum_{v\in V(H)} d_H(v)w(v)}{2|E(H)|}k$. Since $H \in \mathcal{H}(G)$, we may assume that $|V(F)| \geq 3$ and that F is bridgeless.

For a cycle C of a fixed cycle 4-cover of F, let R_C be the set of k-paths rotating around C (note that $|V(C)| \geq \mathit{girth}(H)$). If C is an i-cycle, then $|R_C| = i$. For the multiset $\Pi_F = \bigcup_C R_C$ of k-paths we have $|\Pi_F| = \sum_C |R_C| = 4|E(F)|$. Let $\Pi = \bigcup_{F,|V(F)|\geq 3} \Pi_F$ and it follows $|\Pi| = 4|E(H)|$.

Every vertex $v \in V(F)$ belongs to exactly $\frac{4d_H(v)}{2} = 2d_H(v)$ cycles of the cycle 4-cover of F, thus, $v \in V(F)$ belongs to exactly $2 \cdot d_H(v)k$ paths of Π, hence,

$$\sum_{P\in\Pi} w(P) = \Big(2 \sum_{v\in V(H)} d_H(v)w(v)\Big)k.$$

Eventually, the equality

$$\frac{\sum_{v\in V(H)} d_H(v)w(v)}{2|E(H)|} = d_w + \frac{\sum_{v\in V(H)} d_H(v)(w(v)-d_w)}{2|E(H)|}$$

is clear and, if H is spanning, then

$$d_w + \frac{\sum_{v\in V(G)} d_H(v)(w(v)-d_w)}{2|E(H)|} = d_w + \frac{\sum_{v\in V(G)} (d_H(v)-t)(w(v)-d_w)}{2|E(H)|}$$

because

$$\sum_{v\in V(G)} (w(v)-d_w) = 0. \qquad \square$$

We remark, that in the second part of Theorem 2.1 the assumption that H is spanning is not really a restriction. To see this, let v be a vertex of G not belonging to H. Then, as already mentioned, adding v to H as an additional component of H consisting of v only preserves all assumptions on H and does not change the value of $\frac{\sum_{v\in V(H)} d_H(v)w(v)}{2|E(H)|}$.

If G has a hamiltonian cycle H, then it follows by Theorem 2.1 that, for all $1 \leq k \leq n$, G contains a k-path P such that $w(P) \leq d_w \cdot k$. Clearly, if $w(v) = d_w$ for all $v \in V(G)$ (in this case G is called *w-regular*) or $k = n$ (i.e. P is a hamiltonian path of G), then the last inequality is tight.

At first, we prove Corollary 2.2 and Corollary 2.3 and show how Theorem 2.1 can be used to present inequalities $w(P) \leq c \cdot k$ for a k-path P of G, where c is a constant (depending on G and on w only) less than d_w. We have seen that this is possible only if G is not w-regular and $k < n$.

An edge $e = uv$ of G is *w-good* if $f(e) = 2d_w - w(u) - w(v) > 0$. Note that w-regular graphs do not contain w-good edges. On the other hand, it is easy to choose G and w such that all edges of G are w-good: let G be a star and $w(v) = w(u) + 1$ if $v \in V(G) \setminus \{u\}$, where u is the central vertex of G.

Corollary 2.2. *Let C be a hamiltonian cycle of G, M be a non-empty set of w-good chords of C, H be the subgraph of G with $V(H) = V(C)$ and $E(H) = E(C) \cup M$. If $1 \le k \le girth(H)$, then there is a k-path P of H such that*

$$w(P) \le \left(d_w - \frac{\sum_{e\in M} f(e)}{2(n + |M|)}\right)k < d_w \cdot k.$$

Proof. By Theorem 2.1 with $t = 2$, it follows

$$w(P) \le \left(d_w + \frac{\sum_{v\in V(G)}(d_H(v) - 2)(w(v) - d_w)}{2|E(H)|}\right)k$$

for all $1 \le k \le girth(G)$. Note that $d_H(v)-2$ is the number of edges in M incident with $v \in V(H)$.

If each vertex $v \in V(H)$ sends the value $w(v) - d_w$ to each edge of M incident with v, then

$$\sum_{v\in V(H)} (d_H(v) - 2)(w(v) - d_w) = \sum_{uv\in M} (w(u) + w(v) - 2d_w)$$

and, therefore,

$$w(P) \le \left(d_w + \frac{\sum_{uv\in M}(w(u) + w(v) - 2d_w)}{2|E(H)|}\right)k = \left(d_w - \frac{\sum_{e\in M} f(e)}{2(n + |M|)}\right)k < d_w \cdot k.$$

□

Throughout the paper, let C_w be a cycle of G such that

$$\frac{\sum_{v\in V(C_w)}(w(v) - d_w)}{|V(C_w)|} \le \frac{\sum_{v\in V(C)}(w(v) - d_w)}{|V(C)|}$$

for all cycles C of G. It is easy to see that C_w even can be a hamiltonian cycle of G: let G be obtained from a cycle C and an additional chord of C and $w(v) = d_G(v)$ for $v \in V(G)$.

Corollary 2.3. *If G contains at least n w-good edges and $1 \le k \le |V(C_w)|$, then there is a k-path P of G such that*

$$w(P) \le \left(d_w + \frac{\sum_{v\in V(C_w)}(w(v) - d_w)}{|V(C_w)|}\right)k < d_w \cdot k.$$

Proof. Obviously, G contains a cycle C containing w-good edges only, thus,

$$\frac{\sum_{v\in V(C_w)}(w(v)-d_w)}{|V(C_w)|} \le \frac{\sum_{v\in V(C)}(w(v)-d_w)}{|V(C)|} = \frac{\sum_{e\in E(C)}(-f(e))}{2|V(C)|} < 0.$$

We are done by Theorem 2.1 with $H = C_w$. □

Next, we ask which subgraph $H_w \in \mathcal{H}(G)$ in Theorem 2.1 is the best one, i.e.

$$\frac{\sum_{v\in V(H_w)} d_{H_w}(v)w(v)}{2|E(H_w)|} \le \frac{\sum_{v\in V(H)} d_H(v)w(v)}{2|E(H)|}$$

for all subgraphs $H \in \mathcal{H}(G)$.

Theorem 2.4. *$H_w = C_w$ and if $H \in \mathcal{H}(G)$, then H contains a cycle C such that*

$$\frac{\sum_{v\in V(C)} w(v)}{|V(C)|} \le \frac{\sum_{v\in V(H)} d_H(v)w(v)}{2|E(H)|}.$$

Proof. Let $\mathcal{C} = \{C_1, \dots, C_t\}$ be a cycle 4-cover of H. In the proof of Theorem 2.1, we have seen that $|V(C_1)| + \ldots + |V(C_t)| = 4|E(H)|$ and that a vertex $v \in V(G)$ belongs to exactly $2d_H(v)$ cycles of $\mathcal{C}$, thus,

$$2\sum_{v\in V(H)} d_H(v)w(v) = \Big(\sum_{v\in V(C_1)} w(v)\Big) + \ldots + \Big(\sum_{v\in V(C_t)} w(v)\Big)$$

and

$$\frac{\sum_{v\in V(H)} d_H(v)w(v)}{2|E(H)|} = \frac{(\sum_{v\in V(C_1)} w(v)) + \ldots + (\sum_{v\in V(C_t)} w(v))}{|V(C_1)| + \ldots + |V(C_t)|}.$$

Let $\mathcal{C} = \{C_1, \dots, C_t\}$ be ordered such that

$$\frac{\sum_{v\in V(C_1)} w(v)}{|V(C_1)|} \le \ldots \le \frac{\sum_{v\in V(C_t)} w(v)}{|V(C_t)|}.$$

It follows

$$\frac{\sum_{v\in V(C_1)} w(v)}{|V(C_1)|} \le \frac{(\sum_{v\in V(C_1)} w(v)) + \ldots + (\sum_{v\in V(C_t)} w(v))}{|V(C_1)| + \ldots + |V(C_t)|}$$

(can be seen easily by induction on t) and $H_w = C_w$. □

By Theorem 2.4, the best upper bound on the weight of a lightweight k-path presented by Theorem 2.1 is obtained if $H \in \mathcal{H}(G)$ is a cycle $C_{w,k}$ from the set $\mathcal{C}(G,k)$ of cycles of G on at least k vertices such that

$$\frac{\sum_{v\in V(C_{w,k})} w(v)}{|V(C_{w,k})|} \le \frac{\sum_{v\in V(C)} w(v)}{|V(C)|}$$

for $C \in \mathcal{C}(G,k)$.

It is clear that C_w is such a cycle $C_{w,k}$ if $k \le |V(C_w)|$.

It is known that, if $0 < c \le 1$ is a fixed absolute constant, then the problem to decide whether a graph G contains a cycle on at least $c \cdot n$ vertices is NP-complete. Thus, the problem to find a cycle $C_{w,k}$ is hard if k is large because the problem whether G contains a cycle on at least k vertices is a subproblem.

Using the observation

$$\frac{\sum_{v \in V(C_w)} w(v)}{|V(C_w)|} = \frac{\sum_{uv \in E(C_w)} (\frac{w(u)+w(v)}{2})}{|E(C_w)|}$$

and the polynomiality of the forthcoming undirected minimum mean cycle problem, it follows that C_w can be found in polynomial time.

Undirected minimum mean cycle problem: *Given an undirected graph G, $\sigma : E(G) \to R$, find a cycle C in G whose mean weight $\frac{\sum_{e \in E(C)} \sigma(e)}{|E(C)|}$ is minimum.*

There is an $O(n^5)$-algorithm solving the undirected minimum mean cycle problem ([6]), moreover, the time complexity can be improved to $O(n^2 m + n^3 \log n)$ (see also [5]).

We remark that this problem becomes already hard if C has to contain a specified vertex v of G. To see this, let $\sigma(e) = 1$ if e is incident with v, $\sigma(e) = 0$ otherwise, and C contain v. Then

$$\frac{\sum_{e \in E(C)} \sigma(e)}{|E(C)|} = \frac{2}{|E(C)|},$$

thus, C is a hamiltonian cycle of G if and only if G is hamiltonian. It is known, that the decision problem, whether a graph is hamiltonian, is NP-complete.

Corollary 2.5 presents easily calculable upper bounds on $\frac{\sum_{v \in V(C_w)} (w(v) - d_w)}{|V(C_w)|}$ (see Corollary 2.3) and on $\frac{\sum_{v \in V(C_w)} w(v)}{|V(C_w)|}$ (see Theorem 2.4) if the girth of G is known.

Corollary 2.5. *If the edges $e_1, \ldots, e_m$ of G are ordered such that $f(e_1) \ge \ldots \ge f(e_m)$, then*

$$\begin{aligned}\frac{\sum_{v \in V(C_w)} w(v)}{|V(C_w)|} &= d_w + \frac{\sum_{v \in V(C_w)} (w(v) - d_w)}{|V(C_w)|} \\ &\le d_w - \frac{f(e_{n-girth(G)+1}) + \ldots + f(e_n)}{2girth(G)}.\end{aligned}$$

Proof. Recall that $m \ge n$. Obviously, the subgraph F of G with $V(F) = V(G)$ and $E(F) = \{e_1, \ldots, e_n\}$ contains a cycle C. It follows

$$\frac{\sum_{v \in V(C_w)} w(v)}{|V(C_w)|} \le \frac{\sum_{v \in V(C)} w(v)}{|V(C)|} = \frac{\sum_{e \in E(C)} (2d_w - f(e))}{2|E(C)|}.$$

Note that $|E(C)| \ge girth(G)$ and that $2d_w - f(e_1) \le \ldots \le 2d_w - f(e_n)$.

Thus,

$$\begin{aligned}\frac{\sum_{e\in E(C)}(2d_w - f(e))}{2|E(C)|} &\le \frac{(2d_w - f(e_{n-|E(C)|+1})) + \ldots + (2d_w - f(e_n))}{2|E(C)|}\\ &\le \frac{(2d_w - f(e_{n-girth(G)+1})) + \ldots + (2d_w - f(e_n))}{2girth(G)}\\ &= d_w - \frac{f(e_{n-girth(G)+1}) + \ldots + f(e_n)}{2girth(G)}. \qquad \Box\end{aligned}$$

If G itself is bridgeless, then $G \in \mathcal{H}(G)$ and, by Theorem 2.1, it follows that G contains a k-path P such that

$$w_G(P) \le \frac{\sum_{v\in V(G)} d_G(v)w(v)}{2m}k$$

for $1 \le k \le girth(G)$. Figure 1 presents a graph G_0 showing that this is not true, if G contains bridges (let $w(v) = d_G(v)$ for $v \in V(G)$).

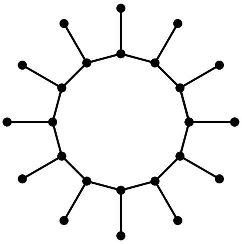

Fig. 1. The graph G_0

Obviously, $w_G(P) \le \Delta_w k$ for each k-path P of G, if $\Delta_w = \max_{v\in V(G)} w(v)$. Theorem 2.6 shows, how this trivial bound can be improved if G is bridgeless, $1 \le k \le girth(G)$, and G is not w-regular. Therefore, let δ be the minimum degree of G and $\Sigma_w = \sum_{v\in V(G)} w(v)$.

Theorem 2.6. *If G is a bridgeless graph of positive size m and $1 \le k \le girth(G)$, then G contains a k-path P such that*

$$w(P) \le \Big(\Delta_w - \frac{\delta}{2m}(\Delta_w n - \Sigma_w)\Big)k.$$

Proof. By Theorem 2.1, it follows with $H = G$ that

$$\begin{aligned} w(P) &\le \frac{\sum_{v\in V(G)} d_G(v)w(v)}{2m}k \\ &= \frac{1}{2m}\Big(\Delta_w \sum_{v\in V(G)} d_G(v) - \sum_{v\in V(G)} (\Delta_w - w(v)d_G(v))\Big)k \\ &\le \frac{1}{2m}\Big(\Delta_w \sum_{v\in V(G)} d_G(v) - \delta \sum_{v\in V(G)} (\Delta_w - w(v))\Big)k \\ &= \Big(\Delta_w - \frac{\delta}{2m}(\Delta_w n - \Sigma_w))\Big)k. \end{aligned}$$

□

Corollary 2.7 is a consequence of Theorem 2.6 if $w(v) = d_G(v)$ for $v \in V(G)$ or $w(v) = -d_G(v)$ for $v \in V(G)$.

Corollary 2.7. *If G is a bridgeless graph of positive size, $1 \le k \le girth(G)$, Δ and d are the maximum degree and the average degree of G, respectively, then G contains a k-path P and a k-path Q such that*

$$\sum_{v\in V(P)} d_G(v) \le \Big(\Delta - \delta\Big(\frac{\Delta}{d} - 1\Big)\Big)k \quad \textit{and} \quad \sum_{v\in V(Q)} d_G(v) \ge \delta\Big(2 - \frac{\delta}{d}\Big)k.$$

Obviously, $\Delta - \delta(\frac{\Delta}{d} - 1) \le \Delta$ and $\delta(2 - \frac{\delta}{d}) \ge \delta$ with equality if and only if G is regular. The same holds for the inequalities $d \le \Delta - \delta(\frac{\Delta}{d} - 1)$ and $d \ge \delta(2 - \frac{\delta}{d})$ because they are equivalent to $(\Delta - d)(d - \delta) \ge 0$ and $(d - \delta)^2 \ge 0$, respectively.

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