

Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday, with high feelings of esteem
for his contributions in Mathematics and great affection

CONCENTRATION–COMPACTNESS RESULTS FOR SYSTEMS IN THE HEISENBERG GROUP

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Communicated by P.A. Cojuhari

Abstract. In this paper we complete the study started in [P. Pucci, L. Temperini, *Existence for (p, q) critical systems in the Heisenberg group*, Adv. Nonlinear Anal. 9 (2020), 895–922] on some variants of the concentration-compactness principle in bounded PS domains Ω of the Heisenberg group \mathbb{H}^n . The concentration-compactness principle is a basic tool for treating nonlinear problems with lack of compactness. The results proved here can be exploited when dealing with some kind of elliptic systems involving critical nonlinearities and Hardy terms.

Keywords: Heisenberg group, concentration-compactness, critical exponents, Hardy terms.

Mathematics Subject Classification: 22E30, 35B33, 35J50, 58E30, 35H05, 35A23.

1. INTRODUCTION

In this paper we complete the study started in [19] on some important variants of the concentration-compactness principle in the Heisenberg group, which are crucial in the study of nonlinear critical elliptic systems, with Hardy terms.

In recent years, when dealing with nonlinear elliptic problems involving critical nonlinearities and Hardy terms, the concentration-compactness principle due to Lions, cf. [14, 15], has been a fundamental tool for proving existence of solutions. We just mention [2, 6, 13, 19] and the references therein.

Moreover, geometric Analysis in the Heisenberg group, and more in general in Carnot groups, represents one of the currently most active areas of mathematics. The main reason lies in the fact that the Heisenberg group \mathbb{H}^n plays an important role in several branches of mathematics, such as representation theory, partial differential equations, number theory, several complex variables and quantum mechanics. We refer, for example, to [1, 17–19].

In this paper, taking inspiration from [13] and following the basic ideas of the papers [14, 15] of Lions, we extend the vectorial concentration-compactness principle to the Heisenberg group setting. The methods used are based on the approach in [5, 13–15] on \mathbb{R}^n and in [2] on \mathbb{H}^n .

Throughout the paper, we assume that Ω is a bounded Poincaré–Sobolev domain of the Heisenberg group \mathbb{H}^n , briefly *PS* domain, that is Ω is an open set of \mathbb{H}^n with the property that there exist a covering $\{B\}_{B \in \mathcal{F}}$ of Ω by Carnot–Carathéodory balls and numbers $N > 0$, $a \geq 1$, and $b \geq 1$ such that

(i) for every $q \in \mathbb{H}^n$

$$\sum_{B \in \mathcal{F}} \mathbb{1}_{(a+1)B}(q) \leq N \mathbb{1}_\Omega(q),$$

(ii) there exists a ball $B_0 \in \mathcal{F}$ such that for all $B \in \mathcal{F}$ there is a finite chain $B_0, B_1, \dots, B_{s(B)}$, with $B_i \cap B_{i+1} \neq \emptyset$ and $|B_i \cap B_{i+1}| \geq \max\{|B_i|, |B_{i+1}|\}/N$,
 (iii) $B \subset bB_i$ for $i = 1, \dots, s(B)$.

The above definition can be found, together with a complete treatment of the topic, in [11]. In the context of Heisenberg groups one can produce a large class of *PS* domains as explained in details in [11].

Denote by $Q = 2n + 2$ the homogeneous dimension of \mathbb{H}^n , let \wp be an exponent with $1 < \wp < Q$, and let $\alpha > 1$ and $\beta > 1$ be such that $\alpha + \beta = \wp^*$, where

$$\wp^* = \frac{\wp Q}{Q - \wp}$$

is a critical exponent related to \wp , in a sense that will be explained later.

We denote by r the Heisenberg norm, or the *Korányi norm*,

$$r(q) = r(z, t) = (|z|^4 + t^2)^{1/4},$$

with $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}$, $|z|$ the Euclidean norm in \mathbb{R}^{2n} of z ,

$$D_H u = (X_1 u, \dots, X_n u, Y_1 u, \dots, Y_n u)$$

the horizontal gradient, $\{X_j, Y_j\}_{j=1}^n$ the basis of horizontal left invariant vector fields on \mathbb{H}^n , that is

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

for $j = 1, \dots, n$.

We prefer to use the Korányi norm and distance since they are much easier to compute than the Carnot–Carathéodory distance. Moreover, it is clear that the t -axis is a snowflake of \mathbb{R} . However, the Korányi distance does not reflect the sub-Riemannian structure of the Heisenberg group. Despite this, the two metrics are closely related. Interestingly, in the setting of the Heisenberg group it was shown by Yang in [24] that the L -gauge $d(x)$ – sometimes also called the Korányi–Folland or Kaplan gauge in this case – can be replaced by the Carnot–Carathéodory distance, and the Hardy

inequality in the Heisenberg group remains valid with the same best constant $p/(Q-p)$. Moreover the L -gauge $d(x)$ can be clearly replaced by another quasi-norm due to the equivalence of all homogeneous quasi-norms on stratified Lie groups, but this may change the best constant in a way which is not easy to trace. Excellent additions and related questions can be found in the recent paper [20].

In the paper, statements involving measure theory are always understood to be with respect to Haar measure on \mathbb{H}^n , which coincides with the $(2n+1)$ -dimensional Lebesgue measure, as explained in Section 2.

Assume that $1 < \varphi < Q$. Let us first introduce the Folland–Stein space $S_0^{1,\varphi}(\mathbb{H}^n)$, which is defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|D_H u\|_\varphi = \left(\int_\Omega |D_H u|_H^\varphi dq \right)^{1/\varphi}.$$

In order to state the main result of the paper, it is crucial to introduce also the Hardy inequality in the Heisenberg setting. To this purpose, consider the weight function ψ , defined as $\psi = |D_H r|_H$. The best Hardy–Sobolev constant $H_\varphi = H(\varphi, Q)$ is given by

$$H_\varphi = \inf_{\substack{u \in S_0^{1,\varphi}(\Omega) \\ u \neq 0}} \frac{\|D_H u\|_\varphi^\varphi}{\|u\|_{H_\varphi}^\varphi}, \quad \|u\|_{H_\varphi}^\varphi = \int_\Omega \psi^\varphi |u|^\varphi \frac{dq}{r^\varphi}. \tag{1.1}$$

For further details we refer to Section 2. One of the main difficulties of working with Hardy terms is that the Hardy embedding $S_0^{1,\varphi}(\Omega) \hookrightarrow L^\varphi(\Omega, \psi^\varphi r^{-\varphi} dq)$ is continuous but not compact, even locally in any neighborhood of O .

Let finally σ be a constant with $0 \leq \sigma < H_\varphi$. Now we can state the main result of the paper, which is a concentration–compactness principle for systems in $S = S_0^{1,\varphi}(\Omega) \times S_0^{1,\varphi}(\Omega)$.

Theorem 1.1. *Let Ω is a bounded PS domain of the Heisenberg group \mathbb{H}^n , with $O \in \Omega$. Let $\{(u_k, v_k)\}_k$ be a weakly convergent sequence in S , with weak limit (u, v) . Then there exist three nonnegative finite Radon measures μ, ν and ω in \mathbb{H}^n , such that*

$$\begin{aligned} (|D_H u_k|_H^\varphi + |D_H v_k|_H^\varphi) dq \xrightarrow{*} \mu, \quad (|u_k|^\varphi + |v_k|^\varphi) \psi^\varphi \frac{dq}{r^\varphi} \xrightarrow{*} \omega, \\ |u_k|^\alpha |v_k|^\beta dq \xrightarrow{*} \nu \quad \text{in } \mathcal{M}(\mathbb{H}^n), \end{aligned} \tag{1.2}$$

where $\mathcal{M}(\mathbb{H}^n)$ is the space of all finite Radon measures on \mathbb{H}^n . Furthermore, there exist an at most countable set J , a family of points $\{q_j\}_{j \in J} \subset \mathbb{H}^n$ and two families of nonnegative numbers $\{\mu_j\}_{j \in J \cup \{0\}}$ and $\{\nu_j\}_{j \in J \cup \{0\}}$ such that

$$d\nu = |u|^\alpha |v|^\beta dq + \nu_0 \delta_O + \sum_{j \in J} \nu_j \delta_{q_j}, \quad \nu_j = \nu(\{q_j\}), \tag{1.3}$$

$$d\mu \geq (|D_H u|_H^\varphi + |D_H v|_H^\varphi) dq + \mu_0 \delta_O + \sum_{j \in J} \mu_j \delta_{q_j}, \quad \mu_j = \mu(\{q_j\}), \tag{1.4}$$

$$d\omega = (|u|^\varphi + |v|^\varphi)\psi^\varphi \frac{dq}{r^\varphi} + \omega_0\delta_O, \tag{1.5}$$

$$\nu_j^{\varphi/\varphi^*} \leq \frac{\mu_j}{\mathcal{I}} \text{ for all } j \in J, \quad \nu_0^{\varphi/\varphi^*} \leq \frac{\mu_0 - \sigma\omega_0}{\mathcal{I}_\sigma}, \tag{1.6}$$

where δ_O and δ_{q_j} are the Dirac functions of mass 1 at the points O and q_j of \mathbb{H}^n , and

$$\mathcal{I}_\sigma = \inf_{\substack{(u,v) \in S \\ (u \neq 0) \wedge (v \neq 0)}} \frac{\|D_H u\|_\varphi^\varphi + \|D_H v\|_\varphi^\varphi - \sigma\|u\|_{H_\varphi}^\varphi - \sigma\|v\|_{H_\varphi}^\varphi}{\left(\int_\Omega |u|^\alpha |v|^\beta dq \right)^{\varphi/\varphi^*}}, \tag{1.7}$$

while $\mathcal{I} = \mathcal{I}_0$.

The assumption that Ω is bounded in Theorem 1.1 plays an essential role to get the compact embedding (2.3), which is widely used in the proof. Theorem 1.1 extends to the Heisenberg setting Theorem 2 of [13] and also Theorem 1.2 of [19]. The proof of Theorem 1.1 follows the arguments given in of Theorem 1.1 of [5] for the Euclidean setting, see also [14, 15], and extends Theorem 1.2 of [2]. In particular, we generalize the results of [2, 5] in two directions: first we consider the term depending on α and β and second we work in the vectorial case.

The paper is structured as follows. In Section 2, we recall the main features of the functional setting on the Heisenberg group \mathbb{H}^n and notations of the paper, while Section 3 is devoted the proof of Theorem 1.1.

2. PRELIMINARIES

In this section we present the basic properties of \mathbb{H}^n as a Lie group. Analysis on the Heisenberg group is very interesting because this space is topologically Euclidean, but analytically non-Euclidean, and so some basic ideas of analysis, such as dilatations, must be developed again. One of the main differences with the Euclidean case is that the homogeneous dimension $Q = 2n + 2$ of the Heisenberg group plays a role analogous to the topological dimension in the Euclidean context. For a complete treatment, we refer to [10–12, 23].

Let \mathbb{H}^n be the Heisenberg Lie group of topological dimension $2n + 1$, that is the Lie group which has \mathbb{R}^{2n+1} as a background manifold, endowed with the non-Abelian group law

$$q \circ q' = \left(z + z', t + t' + 2 \sum_{i=1}^n (y_i x'_i - x_i y'_i) \right)$$

for all points $q, q' \in \mathbb{H}^n$, with

$$q = (z, t) = (x_1, \dots, x_n, y_1, \dots, y_n, t) \quad \text{and} \quad q' = (z', t') = (x'_1, \dots, x'_n, y'_1, \dots, y'_n, t').$$

The inverse is given by $q^{-1} = -q$, so that $(q \circ q')^{-1} = (q')^{-1} \circ q^{-1}$.

The vector fields for $j = 1, \dots, n$

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t},$$

constitute a basis for the real Lie algebra of left-invariant vector fields on \mathbb{H}^n . This basis satisfies the Heisenberg canonical commutation relations

$$[X_j, Y_k] = -4\delta_{jk}T, \quad [Y_j, Y_k] = [X_j, X_k] = [Y_j, T] = [X_j, T] = 0.$$

A left invariant vector field X that is in the span of $\{X_j, Y_j\}_{j=1}^n$ is called *horizontal*.

We define the *horizontal gradient* of a C^1 function $u : \mathbb{H}^n \rightarrow \mathbb{R}$ by

$$D_H u = \sum_{j=1}^n [(X_j u)X_j + (Y_j u)Y_j].$$

Clearly, $D_H u$ is an element of the span of $\{X_j, Y_j\}_{j=1}^n$. Furthermore, if f is of class $C^1(\mathbb{R})$, then

$$D_H f(u) = f'(u)D_H u.$$

In the span of $\{X_j, Y_j\}_{j=1}^n \simeq \mathbb{R}^{2n}$ we consider the natural inner product given by

$$(X, Y)_H = \sum_{j=1}^n (x^j y^j + \tilde{x}^j \tilde{y}^j)$$

for $X = \{x^j X_j + \tilde{x}^j Y_j\}_{j=1}^n$ and $Y = \{y^j X_j + \tilde{y}^j Y_j\}_{j=1}^n$. The inner product $(\cdot, \cdot)_H$ produces the Hilbertian norm

$$|X|_H = \sqrt{(X, X)_H}$$

for the horizontal vector field X . Moreover, the Cauchy–Schwarz inequality

$$|(X, Y)_H| \leq |X|_H |Y|_H$$

holds for any horizontal vector fields X and Y .

For any horizontal vector field function $X = X(q)$, $X = \{x^j X_j + \tilde{x}^j Y_j\}_{j=1}^n$, of class $C^1(\mathbb{H}^n, \mathbb{R}^{2n})$, we define the *horizontal divergence* of X by

$$\operatorname{div}_H X = \sum_{j=1}^n [X_j(x^j) + Y_j(\tilde{x}^j)].$$

If furthermore $u \in C^1(\mathbb{H}^n)$, then the *Leibnitz formula* continues to be valid, that is

$$\operatorname{div}_H (uX) = u \operatorname{div}_H X + (D_H u, X)_H.$$

The *Korányi norm* is given by

$$r(q) = r(z, t) = (|z|^4 + t^2)^{1/4} \quad \text{for all } q = (z, t) \in \mathbb{H}^n.$$

The corresponding distance, the so called *Korányi distance*, is

$$d_K(q, q') = r(q^{-1} \circ q') \quad \text{for all } (q, q') \in \mathbb{H}^n \times \mathbb{H}^n.$$

This distance acts like the Euclidean distance in horizontal directions and behaves like the square root of the Euclidean distance in the missing direction. Consequently, the Korányi norm is homogeneous of degree 1, with respect to the family of dilations

$$\delta_R : (z, t) \mapsto (Rz, R^2t), \quad R > 0, \tag{2.1}$$

since

$$r(\delta_R(q)) = r(Rz, R^2t) = (|Rz|^4 + R^4t^2)^{1/4} = Rr(q)$$

for all $q = (z, t) \in \mathbb{H}^n$. It is easy to verify that the Jacobian determinant of dilatations $\delta_R : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is constant and equal to R^{2n+2} . This is why the natural number $Q = 2n + 2$ is called *homogeneous dimension* of \mathbb{H}^n .

Let $B_R(q_0) = \{q \in \mathbb{H}^n : d_K(q, q_0) < R\}$ be the *Korányi open ball* of radius R centered at q_0 . For simplicity B_R denotes the ball of radius R centered at $q_0 = O$, where $O = (0, 0)$ is the natural origin of \mathbb{H}^n .

The main geometrical function ψ , which appears in (1.1), is defined by

$$\psi(q) = |D_H r|_H = \frac{|z|}{r(q)} \quad \text{for all } q = (z, t) \in \mathbb{H}^n \setminus \{O\},$$

with $0 \leq \psi \leq 1$, $\psi(0, t) \equiv 0$, $\psi(z, 0) \equiv 1$. Furthermore, ψ^2 is the *density function*, which is homogeneous of degree 0, with respect to the dilatation δ_R introduced in (2.1). In Euclidean space the presence of the density ψ is outshone by the flat geometry of \mathbb{R}^n , which yields $\psi \equiv 1$.

The Lebesgue measure on \mathbb{R}^{2n+1} is invariant under left translations. Thus, from here on, we denote by dq the Haar measure on \mathbb{H}^n that coincides with the $(2n + 1)$ -Lebesgue measure, since the Haar measures on Lie groups are unique up to constant multipliers. We also denote by $|U|$ the $(2n + 1)$ -dimensional Lebesgue measure of any measurable set $U \subset \mathbb{H}^n$. Furthermore, the Haar measure on \mathbb{H}^n is Q -homogeneous with respect to dilations δ_R . Consequently,

$$|\delta_R(U)| = R^Q|U|, \quad d(\delta_R q) = R^Q dq.$$

In particular $|B_R| = |B_1|R^Q$.

Let us now introduce the Folland–Stein inequality in Ω , which is the analogous of the Sobolev inequality in the homogeneous setting. Let $1 < \wp < Q$. By [7, 8, 21, 22], we know that for all $\varphi \in C_0^\infty(\Omega)$

$$\|\varphi\|_{\wp^*} \leq C_{\wp^*} \|D_H \varphi\|_{\wp}, \quad \wp^* = \frac{\wp Q}{Q - \wp}, \tag{2.2}$$

where C_{\wp^*} is a positive constant depending only on Q and \wp .

Denote by $HW^{1,\varphi}(\Omega)$ the *horizontal Sobolev space* consisting of all functions $u \in L^\varphi(\Omega)$ such that $D_H u$ exists in the sense of distributions and $|D_H u|_H \in L^\varphi(\Omega)$, endowed with the natural norm

$$\|u\|_{HW^{1,\varphi}(\Omega)} = (\|u\|_\varphi^\varphi + \|D_H u\|_\varphi^\varphi)^{1/\varphi},$$

where

$$\|D_H u\|_\varphi = \left(\int_\Omega |D_H u|_H^\varphi dq \right)^{1/\varphi}.$$

Clearly, by (2.2), the embedding $S_0^{1,\varphi}(\Omega) \hookrightarrow HW^{1,\varphi}(\Omega)$ is continuous. Moreover, if Ω is a bounded PS domain in \mathbb{H}^n , the embedding

$$HW^{1,\varphi}(\Omega) \hookrightarrow L^s(\Omega) \tag{2.3}$$

is compact, when $1 \leq s < \varphi^*$ by [11, 12, 23]. In particular, since any Carnot–Carathéodory ball is a bounded PS domains of \mathbb{H}^n by [9, 12, 23], the property (2.3) holds for the Carnot–Carathéodory balls. Moreover, since the Carnot–Carathéodory distance and the Korányi distance are equivalent on \mathbb{H}^n , see [12], then (2.3) holds in particular when Ω is any Korányi ball. In conclusion, the embedding

$$HW^{1,\varphi}(B_R(q_0)) \hookrightarrow L^s(B_R(q_0)), \quad 1 \leq s < \varphi^*,$$

is compact for any $q_0 \in \mathbb{H}^n$ and $R > 0$.

Finally, we recall the Hardy–Sobolev inequality in Ω . From the Hardy–Sobolev Theorem 1 of [16], we know that

$$\int_\Omega \psi^\varphi |\varphi|^\varphi \frac{dq}{r^\varphi} \leq \left(\frac{\varphi}{Q - \varphi} \right)^\varphi \int_\Omega |D_H \varphi|_H^\varphi dq \tag{2.4}$$

for all $\varphi \in C_0^\infty(\Omega \setminus \{O\})$, with $O = (0, 0)$ the natural origin in \mathbb{H}^n . The above Hardy inequality was obtained in [10] when $\varphi = 2$ and in another version in [4] for all $\varphi > 1$. Clearly, (2.4) implies that

$$S_0^{1,\varphi}(\Omega) \hookrightarrow L^\varphi(\Omega, \psi^\varphi r^{-\varphi} dq). \tag{2.5}$$

However, as already noted, the above embedding is not compact, even locally in any neighborhood of O .

3. PROOF OF THEOREM 1.1

From here on we assume that Ω is a *bounded PS domain* of \mathbb{H}^n , with $O \in \Omega$. This section is devoted to the proof of Theorem 1.1, which concerns the study of the exact behavior of weakly convergent sequences in S in the space of measures. The proof follows the arguments given in the Euclidean, scalar setting in Theorem 1.1 of [5] and

in the Heisenberg, scalar setting in Theorem 1.2 of [2]. We extend these results in two directions: first we consider the term depending on α and β and second we work in the vectorial case.

Before proving Theorem 1.1, let us introduce some useful notations. Under the assumptions $1 < \wp < Q$, $\alpha > 1$, $\beta > 1$, $\alpha + \beta = \wp^*$, and $\sigma \in [0, H_\wp)$, the optimal constant \mathcal{I}_σ introduced in (1.7) is well defined. Indeed, since $\sigma \in [0, H_\wp)$, the quantity in (1.7) is nonnegative by (2.4). Moreover, the Hölder inequality and the Folland–Stein inequality (2.2) give

$$\int_\Omega |u|^\alpha |v|^\beta dq \leq \left(\int_\Omega |u|^{\wp^*} dq \right)^{\alpha/\wp^*} \left(\int_\Omega |v|^{\wp^*} dq \right)^{\beta/\wp^*} \tag{3.1}$$

$$\leq C_{\wp^*}^{\wp} \|D_H u\|_\wp^\alpha \|D_H v\|_\wp^\beta$$

for all $(u, v) \in S$, since $\alpha, \beta > 1$ and $\alpha + \beta = \wp^*$. Therefore, (3.1) and the Young inequality yield

$$\left(\int_\Omega |u|^\alpha |v|^\beta dq \right)^{\wp/\wp^*} \leq C_{\wp^*}^{\wp} \|D_H u\|_\wp^{\alpha\wp/\wp^*} \|D_H v\|_\wp^{\beta\wp/\wp^*} \tag{3.2}$$

$$\leq C_{\wp^*}^{\wp} \left(\frac{\alpha}{\wp^*} \|D_H u\|_\wp^\wp + \frac{\beta}{\wp^*} \|D_H v\|_\wp^\wp \right)$$

$$\leq C_{\wp^*}^{\wp} (\|D_H u\|_\wp^\wp + \|D_H v\|_\wp^\wp)$$

for all $(u, v) \in S$. Hence, by (3.2) and by the Hardy inequality, we get

$$\frac{\|D_H u\|_\wp^\wp + \|D_H v\|_\wp^\wp - \sigma \|u\|_{H_\wp}^\wp - \sigma \|v\|_{H_\wp}^\wp}{\left(\int_\Omega |u|^\alpha |v|^\beta dq \right)^{\wp/\wp^*}} \geq \frac{1 - \sigma/H_\wp}{C_{\wp^*}^{\wp}} > 0$$

since $\sigma < H_\wp$ and so $\mathcal{I}_\sigma > 0$. Taking inspiration from [2, 5, 13], we turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Fix a sequence $\{(u_k, v_k)\}_k$ in S , with $(u_k, v_k) \rightharpoonup (u, v)$ in S . Then $u_k \rightharpoonup u$, $v_k \rightharpoonup v$ in $L^\wp(\Omega, \psi^\wp r^{-\wp} dq)$ by (2.5). Moreover, by (3.1) the sequence $k \mapsto |u_k|^\alpha |v_k|^\beta$ is bounded in $L^1(\Omega)$. Since Ω is bounded, the measures

$$k \mapsto (|D_H u_k|_H^\wp + |D_H v_k|_H^\wp) dq, \quad k \mapsto (|u_k|^\wp + |v_k|^\wp) \psi^\wp \frac{dq}{r^\wp}, \quad k \mapsto |u_k|^\alpha |v_k|^\beta dq$$

in \mathbb{H}^n are uniformly tight in k . Therefore, there exist three nonnegative finite Radon measures μ, ν and ω in \mathbb{H}^n such that (1.2) holds.

Arguing as in Theorem 1.2 of [19], we obtain the validity of (1.3) and (1.4) as well as of the estimate $\nu_j^{\wp/\wp^*} \leq \mu_j/\mathcal{I}$ for all $j \in J$.

Put $\overline{u}_k = u_k - u$ and $\overline{v}_k = v_k - v$. Clearly, $\overline{u}_k \rightharpoonup 0$ and $\overline{v}_k \rightharpoonup 0$ in $S_0^{1,\varphi}(\Omega)$ as $k \rightarrow \infty$. Then, as observed above, we acquire the existence of two nonnegative finite Radon measures $\hat{\mu}$ and $\hat{\omega}$ on \mathbb{H}^n such that

$$\left(|D_H \overline{u}_k|_H^\varphi + |D_H \overline{v}_k|_H^\varphi\right) dq \xrightarrow{*} \hat{\mu} \quad \text{and} \quad \left(|\overline{u}_k|^\varphi + |\overline{v}_k|^\varphi\right) \psi^\varphi \frac{dq}{r^\varphi} \xrightarrow{*} \hat{\omega} \tag{3.3}$$

in $\mathcal{M}(\Omega)$. By (2.3), the sequences $(u_k)_k$ and $(v_k)_k$ strongly converges to u and v , respectively, in $L^\varphi(\Omega)$, being Ω a bounded PS domain. Thus, from Theorem 4.9 of [3], we get the existence of $g \in L^\varphi(\Omega)$ such that, up to subsequences,

$$u_k \rightarrow u \text{ a.e. in } \Omega, \quad |u_k| \leq g \text{ a.e. in } \Omega \text{ and for all } k. \tag{3.4}$$

Similarly, there exists $h \in L^\varphi(\Omega)$ such that, up to subsequences,

$$v_k \rightarrow v \text{ a.e. in } \Omega, \quad |v_k| \leq h \text{ a.e. in } \Omega \text{ and for all } k. \tag{3.5}$$

Therefore, the Brézis–Lieb lemma implies that for any $\varphi \in C_0^\infty(\Omega)$ it results

$$\begin{aligned} \lim_{k \rightarrow \infty} \|\varphi u_k\|_{H_\varphi}^\varphi - \|\varphi u\|_{H_\varphi}^\varphi &= \lim_{k \rightarrow \infty} \|\varphi \overline{u}_k\|_{H_\varphi}^\varphi, \\ \lim_{k \rightarrow \infty} \|\varphi v_k\|_{H_\varphi}^\varphi - \|\varphi v\|_{H_\varphi}^\varphi &= \lim_{k \rightarrow \infty} \|\varphi \overline{v}_k\|_{H_\varphi}^\varphi. \end{aligned}$$

A combination of the above formulas yields that

$$\int_\Omega |\varphi|^\varphi d\omega - \int_\Omega |\varphi|^\varphi (|u|^\varphi + |v|^\varphi) \psi^\varphi \frac{dq}{r^\varphi} = \int_\Omega |\varphi|^\varphi d\hat{\omega}.$$

Consequently,

$$\omega = \hat{\omega} + (|u|^\varphi + |v|^\varphi) \psi^\varphi \frac{dq}{r^\varphi},$$

since $\varphi \in C_0^\infty(\Omega)$ is arbitrary.

Let us now prove (1.5). To this purpose, fix $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$. Then, there exists $C_\varepsilon > 0$ such that $|\xi + \eta|^\varphi \leq (1 + \varepsilon)|\xi|^\varphi + C_\varepsilon|\eta|^\varphi$ for all numbers $\xi, \eta \in \mathbb{R}$. Hence, the Leibnitz formula gives for all k

$$\begin{aligned} \int_\Omega |D_H(\overline{u}_k \varphi)|_H^\varphi dq &\leq (1 + \varepsilon) \int_\Omega |D_H \overline{u}_k|_{\mathbb{H}^n}^\varphi |\varphi|^\varphi dq \\ &\quad + C_\varepsilon \int_\Omega |D_H \varphi|_H^\varphi |\overline{u}_k|^\varphi dq. \end{aligned} \tag{3.6}$$

Thus, (3.6) and the Hardy inequality (2.4), along the sequence $(\varphi \overline{u}_k)_k$ of $S_0^{1,p}(\Omega)$, imply that

$$H_\varphi \|\varphi \overline{u}_k\|_{H_\varphi}^\varphi \leq \|D_H(\overline{u}_k \varphi)\|_{H_\varphi}^\varphi \leq (1 + \varepsilon) \int_\Omega |D_H \overline{u}_k|_H^\varphi |\varphi|^\varphi dq + C_{\varepsilon,\varphi} \|\overline{u}_k\|_{H_\varphi}^\varphi, \tag{3.7}$$

for an appropriate constant $C_{\varepsilon, \varphi} > 0$. Replacing $\overline{u_k}$ by $\overline{v_k}$ and arguing in the same way we get (3.7) also in the v component. Thus,

$$\begin{aligned} & H_\varphi \int_\Omega |\varphi|^\varphi (|\overline{u_k}|_\varphi^\varphi + |\overline{v_k}|_\varphi^\varphi) \psi^\varphi \frac{dq}{r^\varphi} \\ & \leq (1 + \varepsilon) \int_\Omega (|D_H \overline{u_k}|_\varphi^\varphi + |D_H \overline{v_k}|_\varphi^\varphi) |\varphi|^\varphi dq \\ & \quad + C_{\varepsilon, \varphi} (\|\overline{u_k}\|_\varphi^\varphi + \|\overline{v_k}\|_\varphi^\varphi). \end{aligned} \tag{3.8}$$

Hence, passing to the limit in (3.8), using (3.3), (3.7) and the fact that $\overline{u_k} = u_k - u \rightarrow 0$ and $\overline{v_k} = v_k - v \rightarrow 0$ in $L^\varphi(\Omega)$ as $k \rightarrow \infty$, we have

$$\int_\Omega |\varphi|^\varphi d\hat{\omega} \leq \frac{1 + \varepsilon}{H_\varphi} \int_\Omega |\varphi|^\varphi d\hat{\mu}.$$

Therefore, by Lemma 1.4.6 of [12] the measure $\hat{\omega}$ is decomposed as sum of Dirac masses. Let us now prove that $\hat{\omega}$ is concentrated at O .

Fix $\varphi \in C_0^\infty(\Omega)$, with $O \notin \text{supp } \varphi$, so that $\psi^\varphi |\varphi|^\varphi / r^\varphi$ is in $L^\infty(\text{supp } \varphi)$. Then, (2.3) yields

$$\begin{aligned} \int_\Omega \psi^\varphi |\varphi|^\varphi (|\overline{u_k}|^\varphi + |\overline{v_k}|^\varphi) \frac{dq}{r^\varphi} &= \int_{\text{supp } \varphi} \psi^\varphi |\varphi|^\varphi (|\overline{u_k}|^\varphi + |\overline{v_k}|^\varphi) \frac{dq}{r^\varphi} \\ &\leq C \int_{\text{supp } \varphi} (|\overline{u_k}|^\varphi + |\overline{v_k}|^\varphi) dq \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. This, combined with (3.3), gives $\int_\Omega |\varphi|^\varphi d\hat{\omega} = 0$, that is $\hat{\omega}$ is a measure concentrated in O . Hence $\hat{\omega} = \omega_0 \delta_O$, and so (1.5) is proved.

It remains to show the second part of (1.6). From (1.7), for all $\varphi \in C_0^\infty(\Omega)$ it results

$$\begin{aligned} \mathcal{I}_\sigma \left(\int_\Omega |\varphi|^\varphi |u_k|^\alpha |v_k|^\beta dq \right)^{\varphi/\varphi^*} &\leq \int_\Omega (|D_H(\varphi u_k)|_\varphi^\varphi + |D_H(\varphi v_k)|_\varphi^\varphi) dq \\ &\quad - \sigma \int_\Omega \psi^\varphi |\varphi|^\varphi (|u_k|^\varphi + |v_k|^\varphi) \frac{dq}{r^\varphi}, \end{aligned} \tag{3.9}$$

since $\alpha + \beta = \varphi^*$. Then, using the Leibnitz formula (3.6) in (3.9), we get

$$\begin{aligned} \mathcal{I}_\sigma \left(\int_\Omega |\varphi|^\varphi |u_k|^\alpha |v_k|^\beta d\xi \right)^{\varphi/\varphi^*} &\leq (1 + \varepsilon) \int_\Omega (|D_H u_k|_\varphi^\varphi + |D_H v_k|_\varphi^\varphi) |\varphi|^\varphi dq \\ &\quad + C_{\varepsilon, \varphi} (\|u_k\|_\varphi^\varphi + \|v_k\|_\varphi^\varphi) \\ &\quad - \sigma \int_\Omega \psi^\varphi |\varphi|^\varphi (|u_k|^\varphi + |v_k|^\varphi) \frac{dq}{r^\varphi}. \end{aligned}$$

Passing to the limit as $k \rightarrow \infty$ in the above inequality and using (1.2), (3.4) and (3.5), we obtain

$$\begin{aligned} \mathcal{I}_\sigma \left(\int_\Omega |\varphi|^{\varphi^*} d\nu \right)^{\varphi/\varphi^*} &\leq (1 + \varepsilon) \int_\Omega |\varphi|^\varphi d\mu \\ &+ C_\varepsilon \int_\Omega |D_H \varphi|_H^\varphi (|u|^\varphi + |v|^\varphi) dq - \sigma \int_\Omega |\varphi|^\varphi d\omega. \end{aligned} \tag{3.10}$$

Fix now a test function $\varphi \in C_0^\infty(\Omega)$, with $0 \leq \varphi \leq 1$, $\varphi(0) = 1$, $\text{supp } \varphi = B_1$ and put $\varphi_{\tilde{\varepsilon}}(q) = \varphi(\delta_{1/\tilde{\varepsilon}}(q))$ for $\tilde{\varepsilon} > 0$ sufficiently small. Since $\nu \geq \nu_0 \delta_O$, choosing $\varphi_{\tilde{\varepsilon}}$ as test function in (3.10), we have

$$0 \leq \mathcal{I}_\sigma \nu_0^{\varphi/\varphi^*} \leq (1 + \varepsilon) \mu(B_{\tilde{\varepsilon}}) - \sigma \omega_0 + C_\varepsilon \int_\Omega |D_H \varphi_{\tilde{\varepsilon}}|^\varphi (|u|^\varphi + |v|^\varphi) dq. \tag{3.11}$$

The last term of the right-hand side of (3.11) goes to 0 as $\tilde{\varepsilon} \rightarrow 0^+$, thanks to the Hölder inequality. Hence, $0 \leq \mathcal{I}_\sigma \nu_0^{\varphi/\varphi^*} \leq \mu_0 - \sigma \omega_0$, letting $\tilde{\varepsilon} \rightarrow 0^+$ and also $\varepsilon \rightarrow 0^+$ in (3.11). By the Fatou lemma $\mu \geq |D_H u|_H^\varphi dq$ and this concludes the proof of (1.6), since $|D_H u|_H^\varphi dq$ and $\mu_0 \delta_O$ are orthogonal. \square

Acknowledgements

P. Pucci was partly supported by the Italian MIUR project titled Variational methods, with applications to problems in mathematical physics and geometry (2015KB9WPT_009) and by the Fondo Ricerca di Base di Ateneo – Esercizio 2017–2019 of the University of Perugia, named PDEs and Nonlinear Analysis.

Both authors are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

REFERENCES

- [1] S. Bordonì, P. Pucci, *Schrödinger–Hardy systems involving two Laplacian operators in the Heisenberg group*, Bull. Sci. Math. **146** (2018), 50–88.
- [2] S. Bordonì, R. Filippucci, P. Pucci, *Existence of solutions in problems on Heisenberg groups involving Hardy and critical terms*, J. Geom. Anal., Special Issue Perspectives of Geometric Analysis in PDEs (2019) 29 pp., <https://doi.org/10.1007/s12220-019-00295-z>.
- [3] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.
- [4] L. D’Ambrosio, *Hardy-type inequalities related to degenerate elliptic differential operators*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 451–486.
- [5] A. Fiscella, P. Pucci, *Kirchhoff Hardy fractional problems with lack of compactness*, Adv. Nonlinear Stud. **17** (2017), 429–456.
- [6] A. Fiscella, P. Pucci, *(p, q) systems with critical terms in \mathbb{R}^N* , Special Issue on Nonlinear PDEs and Geometric Function Theory, in honor of Carlo Sbordone on his 70th birthday, Nonlinear Anal. **177** (2018), Part B, 454–479.

- [7] G.B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1975), 161–207.
- [8] G.B. Folland, E.M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.
- [9] B. Franchi, C. Gutierrez, R.L. Wheeden, *Weighted Sobolev–Poincaré inequalities for Grushin type operators*, Comm. Partial Differential Equations **19** (1994), 523–604.
- [10] N. Garofalo, E. Lanconelli, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*, Ann. Inst. Fourier **40** (1990), 313–356.
- [11] N. Garofalo, D.–M. Nhieu, *Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces*, Comm. Pure Appl. Math. **49** (1996), 1081–1144.
- [12] S.P. Ivanov, D.N. Vassilev, *Extremals for the Sobolev Inequality and the Quaternionic Contact Yamabe Problem*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2011.
- [13] D. Kang, *Concentration compactness principles for the systems of critical elliptic equations*, Differ. Equ. Appl. **4** (2012), 435–444.
- [14] P.–L. Lions, *The concentration compactness principle in the calculus of variations. The limit case. Part 1*, Rev. Mat. Iberoamericana **1.1** (1985), 145–201.
- [15] P.–L. Lions, *The concentration compactness principle in the calculus of variations. The limit case. Part 2*, Rev. Mat. Iberoamericana **1.2** (1985), 45–121.
- [16] P. Niu, H. Zhang, Y. Wang, *Hardy-type and Rellich type inequalities on the Heisenberg group*, Proc. Amer. Math. Soc. **129** (2001), 3623–3630.
- [17] P. Pucci, *Critical Schrödinger–Hardy systems in the Heisenberg group*, Discrete Contin. Dyn. Syst. Ser. S, Special Issue on the occasion of the 60th birthday of Professor Vicentiu D. Radulescu, **12** (2019), 375–400.
- [18] P. Pucci, *Existence of entire solutions for quasilinear equations in the Heisenberg group*, Minimax Theory Appl., Special Issue on Nonlinear Phenomena: Theory and Applications **4** (2019), 32 pp.
- [19] P. Pucci, L. Temperini, *Existence for (p, q) critical systems in the Heisenberg group*, Adv. Nonlinear Anal. **9** (2020), 895–922.
- [20] M. Ruzhansky, D. Suragan, *Hardy and Rellich inequalities, identities, and sharp remainders on homogeneous groups*, Adv. Math. **317** (2017), 799–822.
- [21] N. Varopoulos, *Analysis on nilpotent Lie groups*, J. Funct. Anal. **66** (1986), 406–431.
- [22] N. Varopoulos, *Sobolev inequalities on Lie groups and symmetric spaces*, J. Funct. Anal. **86** (1989), 19–40.
- [23] D. Vassilev, *Existence of solutions and regularity near the characteristic boundary for sub-Laplacian equations on Carnot groups*, Pacific J. Math. **227** (2006), 361–397.
- [24] Q.–H. Yang, *Hardy type inequalities related to Carnot–Carathéodory distance on the Heisenberg group*, Proc. Amer. Math. Soc. **141** (2013), 351–362.

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Received: January 7, 2020.
Revised: January 27, 2020.
Accepted: January 27, 2020.