

MAXIMUM PACKINGS OF THE λ -FOLD COMPLETE 3-UNIFORM HYPERGRAPH WITH LOOSE 3-CYCLES

Ryan C. Bunge, Dontez Collins, Daryl Conko-Camel,
Saad I. El-Zanati, Rachel Liebrecht, and Alexander Vasquez

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Abstract. It is known that the 3-uniform loose 3-cycle decomposes the complete 3-uniform hypergraph of order v if and only if $v \equiv 0, 1, \text{ or } 2 \pmod{9}$. For all positive integers λ and v , we find a maximum packing with loose 3-cycles of the λ -fold complete 3-uniform hypergraph of order v . We show that, if $v \geq 6$, such a packing has a leave of two or fewer edges.

Keywords: maximum packing, λ -fold complete 3-uniform hypergraph, loose 3-cycle.

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1. INTRODUCTION

A *hypergraph* H consists of a finite nonempty set V of *vertices* and a finite collection $E = \{e_1, e_2, \dots, e_m\}$ of nonempty subsets of V called *hyperedges* or simply *edges*. For a given hypergraph H , we use $V(H)$ and $E(H)$ to denote the vertex set and the edge set (or multiset) of H , respectively. We call $|V(H)|$ and $|E(H)|$ the *order* and *size* of H , respectively. The *degree* of a vertex $v \in V(H)$ is the number of edges in $E(H)$ that contain v . A hypergraph H is *simple* if no edge appears more than once in $E(H)$. If for each $e \in E(H)$ we have $|e| = t$, then H is said to be *t -uniform*. Thus t -uniform hypergraphs are generalizations of the concept of a graph (where $t = 2$). Graphs with repeated edges are often called *multigraphs*. If H is a simple hypergraph and λ is a positive integer, then λ -fold H , denoted ${}^\lambda H$, is the multi-hypergraph obtained from H by repeating each edge exactly λ times. The hypergraph with vertex set V and edge set the set of all t -element subsets of V is called the *complete t -uniform hypergraph on V* and is denoted by $K_V^{(t)}$. If $v = |V|$, then ${}^\lambda K_v^{(t)}$ is called the *λ -fold complete t -uniform hypergraph of order v* and is used to denote any hypergraph isomorphic

to ${}^\lambda K_V^{(t)}$. When $t = 2$, we will use ${}^\lambda K_v$ in place of ${}^\lambda K_v^{(2)}$. Similarly, if $\lambda = 1$, then we will use $K_v^{(t)}$ in place of ${}^1 K_v^{(t)}$. If H' is a subhypergraph of H , then $H \setminus H'$ denotes the hypergraph obtained from H by deleting the edges of H' . We may refer to $H \setminus H'$ as the hypergraph H with a *hole* H' . The vertices in H' may be referred to as the vertices in the hole.

A commonly studied problem in combinatorics concerns decompositions of graphs or multigraphs into edge-disjoint subgraphs. A *decomposition* of a multigraph M is a set $\Delta = \{G_1, G_2, \dots, G_s\}$ of subgraphs of M such that $\{E(G_1), E(G_2), \dots, E(G_s)\}$ is a partition of $E(M)$. If each element of Δ is isomorphic to a fixed graph G , then Δ is called a *G-decomposition* of M . If L is a subgraph of M and Δ is a *G-decomposition* of $M \setminus L$, then Δ is called a *G-packing* of M with *leave* L . Such a *G-packing* is *maximum* if no other possible *G-packing* of M has a leave of a smaller size than that of L . Clearly, if $|E(L)| < |E(G)|$, then the *G-packing* is maximum. Moreover, a *G-decomposition* of M can be viewed as a maximum *G-packing* with an empty leave.

A *G-decomposition* of ${}^\lambda K_v$ is also known as a *G-design of order v and index λ*. A K_k -design of order v and index λ is usually known as a $2-(v, k, \lambda)$ *design* or as a *balanced incomplete block design of index λ* or a (v, k, λ) -*BIBD*. The problem of determining all v for which there exists a *G-design* of order v is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to hypergraphs. A *decomposition* of a hypergraph M is a set $\Delta = \{H_1, H_2, \dots, H_s\}$ of subhypergraphs of M such that $\{E(H_1), E(H_2), \dots, E(H_s)\}$ is a partition of $E(M)$. Any element of Δ isomorphic to a fixed hypergraph H is called an *H-block*. If all elements of Δ are *H-blocks*, then Δ is called an *H-decomposition* of M . If L is a subgraph of M and Δ is an *H-decomposition* of $M \setminus L$, then Δ is called an *H-packing* of M with *leave* L , where we again define such a packing to be *maximum* if L has the fewest edges possible. An *H-decomposition* of ${}^\lambda K_v^{(t)}$ is called an *H-design of order v and index λ*. The problem of determining all v for which there exists an *H-design* of order v and index λ is called the *λ-fold spectrum problem for H-designs*.

A $K_k^{(t)}$ -design of order v and index λ is a generalization of $2-(v, k, \lambda)$ designs and is known as a $t-(v, k, \lambda)$ *design* or simply as a *t-design*. A summary of results on *t-designs* appears in [15]. A $t-(v, k, 1)$ design is also known as a *Steiner system* and is denoted by $S(t, v, k)$ (see [8] for a summary of results on Steiner systems). Keevash [14] has recently shown that for all t and k the obvious necessary conditions for the existence of an $S(t, k, v)$ -design are sufficient for sufficiently large values of v . Similar results were obtained by Glock, Kühn, Lo, and Osthus [9, 10] and extended to include the corresponding asymptotic results for *H-designs* of order v for all uniform hypergraphs H . These results for *t-uniform hypergraphs* mirror the celebrated results of Wilson [23] for graphs. Although these asymptotic results assure the existence of *H-designs* for sufficiently large values of v for any uniform hypergraph H , the spectrum problem has been settled for very few hypergraphs of uniformity larger than 2.

In the study of graph decompositions, a fair amount of the focus has been on *G-decompositions* of K_v where G is a graph with a relatively small number of edges (see [1] and [5] for known results). Some authors have investigated the corresponding

problem for 3-uniform hypergraphs. For example, in [4], the 1-fold spectrum problem is settled for all 3-uniform hypergraphs on 4 or fewer vertices. More recently, the 1-fold spectrum problem was settled in [6] for all 3-uniform hypergraphs with at most 6 vertices and at most 3 edges. In [6], they also settle the 1-fold spectrum problem for the 3-uniform hypergraph of order 6 whose edges form the lines of the Pasch configuration. Authors have also considered H -designs where H is a 3-uniform hypergraph whose edge set is defined by the faces of a regular polyhedron. Let T , O , and I denote the tetrahedron, the octahedron, and the icosahedron hypergraphs, respectively. The hypergraph T is the same as $K_4^{(3)}$, and its spectrum was settled in 1960 by Hanani [11]. In another paper [12], Hanani settled the spectrum problem for O -designs and gave necessary conditions for the existence of I -designs. The 1-fold spectrum problem is also settled for a type of 3-uniform hyperstars which is part of a larger class of hypergraphs known as delta-systems. For a positive integer m , let $S_m^{(3)}$ denote the 3-uniform hypergraph of size m which consists of one vertex of degree m and $2m$ vertices of degree one. Necessary and sufficient conditions for the existence of $S_m^{(3)}$ -decompositions of $K_v^{(3)}$ are given in [21] for $m \in [4, 6]$ and for all m in [18]. Some results on maximum $S_m^{(3)}$ -packings of $K_v^{(3)}$ are given in [19]. Perhaps the best known general result on decompositions of complete t -uniform hypergraphs is Baranyai's result [3] on the existence of 1-factorizations of $K_{mt}^{(t)}$ for all positive integers m . There are, however, several articles on decompositions of complete t -uniform hypergraphs (see [2] and [20]) and of t -uniform t -partite hypergraphs (see [16] and [22]) into variations on the concept of a Hamilton cycle. There are also several results on decompositions of 3-uniform hypergraphs into structures known as Berge cycles with a given number of edges (see for example [13] and [17]). We note however that the Berge cycles in these decompositions are not required to be isomorphic.

In this paper we are interested in maximum H -packings of $\lambda K_v^{(3)}$, where H is a 3-uniform loose 3-cycle. For integer $m \geq 3$, a 3-uniform loose m -cycle, denoted $LC_m^{(3)}$, is a 3-uniform hypergraph with vertex set $\{v_1, v_2, \dots, v_{2m}\}$ and edge set $\{\{v_{2i-1}, v_{2i}, v_{2i+1}\} : 1 \leq i \leq m-1\} \cup \{v_{2m-1}, v_{2m}, v_1\}$. Thus $LC_3^{(3)}$ has vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $\{\{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_5, v_6, v_1\}\}$ for which we use $H[v_1, v_2, v_3, v_4, v_5, v_6]$ to denote (see Figure 1).

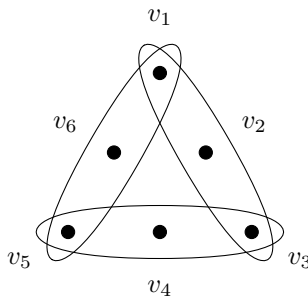


Fig. 1. The 3-uniform loose 3-cycle, $LC_3^{(3)}$, denoted by $H[v_1, v_2, v_3, v_4, v_5, v_6]$.

Since $LC_3^{(3)}$ has 3 edges and 6 vertices, it is one of the hypergraphs covered in the decomposition results by Bryant, Herke, Maenhaut, and Wannasit in [6]. It is shown in [6] that there exists an $LC_3^{(3)}$ -decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, \text{ or } 2 \pmod{9}$. Similarly, it is shown in [7] that there exists an $LC_4^{(3)}$ -decomposition of $K_v^{(3)}$ if and only if $v \equiv 0, 1, 2, 4, \text{ or } 6 \pmod{8}$ and $v \notin \{4, 6\}$. Here we focus on maximum $LC_3^{(3)}$ -packings of ${}^\lambda K_v^{(3)}$ and show that if λ and $v \geq 6$ are positive integers, then there exists a maximum $LC_3^{(3)}$ -packing of ${}^\lambda K_v^{(3)}$ where the leave has two or fewer edges.

1.1. ADDITIONAL NOTATION AND TERMINOLOGY

If a and b are integers with $a \leq b$, we define $[a, b]$ to be $\{a, a + 1, \dots, b\}$. We next define some notation for certain types of 3-uniform hypergraphs.

Let U_1, U_2, U_3 be pairwise disjoint sets. The hypergraph with vertex set $U_1 \cup U_2 \cup U_3$ and edge set consisting of all 3-element sets having exactly one vertex in each of U_1, U_2, U_3 is denoted by $K_{U_1, U_2, U_3}^{(3)}$. The hypergraph with vertex set $U_1 \cup U_2$ and edge set consisting of all 3-element sets having at most 2 vertices in each of U_1, U_2 is denoted by $L_{U_1, U_2}^{(3)}$. If $|U_i| = u_i$ for $i \in \{1, 2, 3\}$, we may use $K_{u_1, u_2, u_3}^{(3)}$ or $L_{u_1, u_2}^{(3)}$ to denote any hypergraph that is isomorphic to $K_{U_1, U_2, U_3}^{(3)}$ or $L_{U_1, U_2}^{(3)}$, respectively. From a hypergraph decomposition perspective, we note that if U_1, U'_1, U_2, U_3 are pairwise vertex disjoint, then

$$E(K_{U_1 \cup U'_1, U_2, U_3}^{(3)}) = E(K_{U_1, U_2, U_3}^{(3)}) \cup E(K_{U'_1, U_2, U_3}^{(3)}).$$

Thus, for any positive integer x , it is simple to see that $K_{u_1, u_2, u_3}^{(3)}$ decomposes $K_{u_1 \cdot x, u_2, u_3}^{(3)}$ and, in general, $K_{u_1+x, u_2, u_3}^{(3)}$ decomposes into one copy of $K_{u_1, u_2, u_3}^{(3)}$ and one copy of $K_{x, u_2, u_3}^{(3)}$.

2. MAIN CONSTRUCTIONS

The constructions in this section are dependent on many small examples. These examples are given in the last section. Throughout, we will often identify a hypergraph (e.g., a leave in a packing) with its edge set only. Since the hypergraphs presented here do not contain isolated vertices, this will uniquely define them.

We begin by proving a lemma that is fundamental to our constructions.

Lemma 2.1. *Let $n \geq 1$, $x \geq 0$, and $r \geq 0$ be integers and let $v = nx + r$. There exists a decomposition of $K_v^{(3)}$ into:*

- (i) 1 copy of $K_{n+r}^{(3)}$,
- (ii) $x - 1$ copies of $K_{n+r}^{(3)} \setminus K_r^{(3)}$ (these are isomorphic to $K_{n+r}^{(3)}$ if $r \in [0, 2]$),
- (iii) $\binom{x}{2}$ copies of $K_{r, n, n}^{(3)} \cup L_{n, n}^{(3)}$ (here $K_{r, n, n}^{(3)}$ is empty if $r = 0$), and
- (iv) $\binom{x}{3}$ copies of $K_{n, n, n}^{(3)}$.

Proof. If $x \in \{0, 1\}$, the decomposition is trivial. Thus we may assume that $x \geq 2$.

Let V_0, V_1, \dots, V_x be pairwise disjoint sets of vertices with $|V_0| = r, |V_1| = |V_2| = \dots = |V_x| = n$ and let $V = V_0 \cup V_1 \cup \dots \cup V_x$. Then, $K_V^{(3)}$ can be viewed as the (edge-disjoint) union

$$K_{V_1 \cup V_0}^{(3)} \cup \bigcup_{2 \leq i \leq x} \left(K_{V_i \cup V_0}^{(3)} \setminus K_{V_0}^{(3)} \right) \cup \bigcup_{1 \leq i < j \leq x} \left(K_{V_0, V_i, V_j}^{(3)} \cup L_{V_i, V_j}^{(3)} \right) \cup \bigcup_{1 \leq i < j < k \leq x} \left(K_{V_i, V_j, V_k}^{(3)} \right)$$

Thus the result follows. □

If $LC_3^{(3)}$ decomposes $K_v^{(3)}$, then we must have $3 \mid \binom{v}{3}$ and hence $18 \mid v(v-1)(v-2)$. Therefore we have $v \equiv 0, 1, \text{ or } 2 \pmod{9}$. In [6], it is shown that these necessary conditions are sufficient. Although a proof of Theorem 2.2 is given in [6], we include a proof here for the sake of completeness.

Theorem 2.2. *There exists an $LC_3^{(3)}$ -decomposition of K_v if and only if $v \equiv 0, 1, \text{ or } 2 \pmod{9}$.*

Proof. Let $v = 9x + r$, where $r \in [0, 2]$. If $x = 0$, the result is vacuously true. If $x = 1$, we give $LC_3^{(3)}$ -decompositions of $K_9^{(3)}$ in Example 3.1, of $K_{10}^{(3)}$ in Example 3.2, and of $K_{11}^{(3)}$ in Example 3.3. Thus we may assume that $x \geq 2$. By Lemma 2.1, it suffices to give $LC_3^{(3)}$ -decompositions of $K_{9+r}^{(3)}$, of $K_{9+r}^{(3)} \setminus K_r^{(3)}$ which is isomorphic to $K_{9+r}^{(3)}$ since $r \in [0, 2]$, of $K_{r,9,9}^{(3)} \cup L_{9,9}^{(3)}$, and of $K_{9,9,9}^{(3)}$. A decomposition of $K_{1,9,9}^{(3)} \cup L_{9,9}^{(3)}$ is given in Example 3.7, and a decomposition of $L_{9,9}^{(3)}$ is given in Example 3.6. Since $K_{2,3,3}^{(3)}$ decomposes $K_{2,9,9}^{(3)}$ and $K_{3,3,3}^{(3)}$ decomposes $K_{9,9,9}^{(3)}$ and since $LC_3^{(3)}$ -decompositions of $K_{2,3,3}^{(3)}$ of $K_{3,3,3}^{(3)}$ are given in Examples 3.4 and 3.5, we have that $LC_3^{(3)}$ decomposes both $K_{2,9,9}^{(3)}$ and $K_{9,9,9}^{(3)}$. Thus the result follows. □

Next, we give our main result on maximum $LC_3^{(3)}$ -packings of $K_v^{(3)}$.

Theorem 2.3. *If $v \geq 6$ is an integer, then there exists a maximum $LC_3^{(3)}$ -packing of $K_v^{(3)}$ where the leave has two or fewer edges.*

Proof. If $v \equiv 0, 1, \text{ or } 2 \pmod{9}$, then the result follows from the $LC_3^{(3)}$ -decomposition result in Theorem 2.2, which translates to a maximum $LC_3^{(3)}$ -packing with an empty leave. If $v \in [6, 8]$, a maximum $LC_3^{(3)}$ -packing of $K_v^{(3)}$ with a two edge leave is given in Examples 3.13–3.15. Hence, we need only consider when $v = 9x + r$ where $x \geq 1$ and $r \in [3, 8]$. By Lemma 2.1 it suffices to find

- (i) a maximum $LC_3^{(3)}$ -packing of $K_{9+r}^{(3)}$ with a leave consisting of two or fewer edges and
- (ii) $LC_3^{(3)}$ -decompositions of $K_{9+r}^{(3)} \setminus K_r^{(3)}$, $K_{r,9,9}^{(3)} \cup L_{9,9}^{(3)}$, and $K_{9,9,9}^{(3)}$.

We note that an $LC_3^{(3)}$ -decomposition of $K_{12}^{(3)} \setminus K_3^{(3)}$ is equivalent an $LC_3^{(3)}$ -packing of $K_{12}^{(3)}$ with a leave consisting of the single edge in the hole, which is given

in Example 3.16. Also, for $r \geq 3$, it is simple to see that $K_{r,9,9}^{(3)}$ is decomposable into copies of $K_{2,3,3}^{(3)}$ and $K_{3,3,3}^{(3)}$. Maximum $LC_3^{(3)}$ -packings (with leaves of two or fewer edges) of $K_{9+r}^{(3)}$, for $r \in [3, 8]$, are given in Examples 3.16–3.21. Similarly, $LC_3^{(3)}$ -decompositions of $K_{9+r}^{(3)} \setminus K_r^{(3)}$, for $r \in [4, 8]$, are given in Examples 3.8–3.12. Finally, an $LC_3^{(3)}$ -decomposition of $L_{9,9}^{(3)}$ is given in Example 3.6, and $LC_3^{(3)}$ -decompositions of $K_{2,3,3}^{(3)}$ and of $K_{3,3,3}^{(3)}$ are given in Examples 3.4 and 3.5, respectively. \square

Next, we give a lemma on maximum $LC_3^{(3)}$ -packings of ${}^2K_v^{(3)}$ for $v \in [6, 17]$.

Lemma 2.4. *If $v \in [6, 17]$, then there exists a maximum $LC_3^{(3)}$ -packing of ${}^2K_v^{(3)}$ where the leaf has two or fewer edges.*

Proof. Let $V({}^2K_v^{(3)}) = \mathbb{Z}_v$. If $v \in [9, 11]$, there exists an $LC_3^{(3)}$ -decomposition of $K_v^{(3)}$ and hence of ${}^2K_v^{(3)}$. Next, if $v \in [6, 8] \cup [15, 17]$, let Δ_1 be a maximum $LC_3^{(3)}$ -packing of $K_v^{(3)}$ where the leaf has edge set $\{\{0, 1, 2\}, \{2, 3, 4\}\}$ (which exists by Examples 3.13–3.15 and Examples 3.19–3.21) and let Δ_2 be another maximum $LC_3^{(3)}$ -packing of $K_v^{(3)}$ where the leaf has edge set $\{\{4, 5, 0\}, \{0, 1, 2\}\}$. Then $\Delta_1 \cup \Delta_2 \cup \{H[0, 1, 2, 3, 4, 5]\}$ is a maximum $LC_3^{(3)}$ -packing of ${}^2K_v^{(3)}$ where $\{0, 1, 2\}$ is the only edge in the leaf. Finally, if $v \in [12, 14]$, let Δ_1 be a maximum packing of $K_v^{(3)}$ where $\{0, 1, 2\}$ is the only edge in the leaf (which exists by Examples 3.16–3.18) and let Δ_2 be a maximum $LC_3^{(3)}$ -packing of $K_v^{(3)}$ where $\{2, 3, 4\}$ is the only edge in the leaf. Then $\Delta_1 \cup \Delta_2$ is a maximum $LC_3^{(3)}$ -packing of ${}^2K_v^{(3)}$ where the leaf has edge set $\{\{0, 1, 2\}, \{2, 3, 4\}\}$. \square

Now we extend our results to maximum $LC_3^{(3)}$ -packings of ${}^2K_v^{(3)}$ in general.

Theorem 2.5. *If $v \geq 6$, then there exists a maximum $LC_3^{(3)}$ -packing of ${}^2K_v^{(3)}$ where the leaf has two or fewer edges.*

Proof. If $v \equiv 0, 1, \text{ or } 2 \pmod{9}$, then the result follows from Theorem 2.2, which translates to a maximum $LC_3^{(3)}$ -packing with an empty leaf. If $v \in [6, 8]$, a maximum $LC_3^{(3)}$ -packing of ${}^2K_v^{(3)}$ with a one edge leaf is given in Lemma 2.4. Hence, we need only consider when $v \equiv r \pmod{9}, r \geq 3, v \geq 12$. Let $v = 9x + r$ where $x \geq 1$ and $r \in [3, 8]$. By Lemma 2.1 it suffices to find

- (i) a maximum $LC_3^{(3)}$ -packing of ${}^2K_{9+r}^{(3)}$ with a leaf consisting of two or fewer edges and
- (ii) $LC_3^{(3)}$ -decompositions of ${}^2K_{9+r}^{(3)} \setminus {}^2K_r^{(3)}, {}^2K_{r,9,9}^{(3)} \cup {}^2L_{9,9}^{(3)}$, and ${}^2K_{9,9,9}^{(3)}$.

But since $LC_3^{(3)}$ decomposes $K_{9+r}^{(3)} \setminus K_r^{(3)}, K_{r,9,9}^{(3)} \cup L_{9,9}^{(3)}$, and $K_{9,9,9}^{(3)}$ (see argument in proof of Theorem 2.3), $LC_3^{(3)}$ decomposes the 2-fold versions of these hypergraphs. Maximum $LC_3^{(3)}$ -packings (with leaves of two or fewer edges) of ${}^2K_{9+r}^{(3)}$, for $r \in [3, 8]$, are given in Lemma 2.4. The result now follows. \square

Next, we give a lemma on $LC_3^{(3)}$ -decompositions of ${}^3K_v^{(3)}$ for $v \in [6, 17]$.

Lemma 2.6. *If $v \in [6, 17]$, then there exists an $LC_3^{(3)}$ -decomposition of ${}^3K_v^{(3)}$.*

Proof. Let $V\left({}^3K_v^{(3)}\right) = \mathbb{Z}_v$. If $v \in [9, 11]$, there exists an $LC_3^{(3)}$ -decomposition of $K_v^{(3)}$ and hence of ${}^3K_v^{(3)}$. Next, if $v \in [6, 8] \cup [15, 17]$, let Δ_1 be a maximum packing of $K_v^{(3)}$ where the leave has edge set $\{\{2, 3, 4\}, \{4, 5, 0\}\}$ (which exists by Examples 3.13–3.15 and Examples 3.19–3.21) and let Δ_2 be a maximum $LC_3^{(3)}$ -packing of ${}^2K_v^{(3)}$ where $\{0, 1, 2\}$ is the only edge in the leave (which exists by Lemma 2.4). Then $\Delta_1 \cup \Delta_2 \cup \{H[0, 1, 2, 3, 4, 5]\}$ is an $LC_3^{(3)}$ -decomposition of ${}^3K_v^{(3)}$. Finally, if $v \in [12, 14]$, let Δ_1 be a maximum packing of $K_v^{(3)}$ where $\{4, 5, 0\}$ is the only edge in the leave (which exists by Examples 3.16–3.18) and let Δ_2 be a maximum $LC_3^{(3)}$ -packing of ${}^2K_v^{(3)}$ where the leave has edge set $\{\{0, 1, 2\}, \{2, 3, 4\}\}$ (which exists by Lemma 2.4). Then $\Delta_1 \cup \Delta_2 \cup \{H[0, 1, 2, 3, 4, 5]\}$ is an $LC_3^{(3)}$ -decomposition of ${}^3K_v^{(3)}$. \square

It is simple to see that if there is an $LC_3^{(3)}$ -decomposition of $\lambda K_v^{(3)}$, then we must have $v \notin [3, 5]$ and either $\lambda \equiv 0 \pmod{3}$ or $v \equiv 0, 1, 2 \pmod{9}$. Thus, in light of Theorem 2.2 and Lemmas 2.1 and 2.6 and because ${}^3K_v^{(3)}$ decomposes ${}^{3k}K_v^{(3)}$ for all positive integers k , we have the following obvious corollary.

Corollary 2.7. *Let λ and $v \notin [3, 5]$ be positive integers. There exists an $LC_3^{(3)}$ -decomposition of $\lambda K_v^{(3)}$ if and only if $\lambda \equiv 0 \pmod{3}$ or $v \equiv 0, 1, \text{ or } 2 \pmod{9}$.*

Finally we give our general main result.

Theorem 2.8. *If λ and $v \notin [3, 5]$ are positive integers, then there exists a maximum $LC_3^{(3)}$ -packing of $\lambda K_v^{(3)}$ where the leave has two or fewer edges.*

Proof. If $\lambda \in \{1, 2\}$, the result follows from Theorems 2.3 and 2.5. If $\lambda \equiv 0 \pmod{3}$, the result follows from Corollary 2.7. Suppose $\lambda \geq 4$ and let $\lambda = 3b + r$ for integers $b \geq 1$ and $r \in \{1, 2\}$. We can view $\lambda K_v^{(3)}$ as the edge disjoint union of ${}^{3b}K_v^{(3)}$ and ${}^rK_v^{(3)}$. An $LC_3^{(3)}$ -decomposition of ${}^{3b}K_v^{(3)}$ exists by Corollary 2.7 and a maximum $LC_3^{(3)}$ -packing of ${}^rK_v^{(3)}$ where the leave has two or fewer edges follows from Theorems 2.3 and 2.5. Thus the result follows. \square

3. SMALL EXAMPLES

Example 3.1. Let

$$V\left(K_9^{(3)}\right) = \mathbb{Z}_7 \cup \{\infty_1, \infty_2\}$$

and let

$$B = \{H[0, 1, 2, 4, 6, 3], H[\infty_1, 0, 1, \infty_2, 2, 4], H[0, 1, 3, \infty_2, 6, 4], H[\infty_1, 0, 3, 1, \infty_2, 2]\}.$$

Then an $LC_3^{(3)}$ -decomposition of $K_9^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{7}$.

Example 3.2. Let

$$V\left(K_{10}^{(3)}\right) = \mathbb{Z}_{10}$$

and let

$$B = \{H[0, 2, 1, 3, 4, 9], H[0, 7, 1, 2, 4, 5], H[0, 4, 2, 5, 7, 6], H[0, 6, 2, 4, 7, 3]\}.$$

Then an $LC_3^{(3)}$ -decomposition of $K_{10}^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j + 1 \pmod{10}$.

Example 3.3. Let

$$V\left(K_{11}^{(3)}\right) = \mathbb{Z}_{11}$$

and let

$$B = \{H[0, 8, 2, 6, 9, 1], H[0, 8, 1, 4, 7, 2], H[1, 0, 5, 3, 8, 9], H[0, 7, 1, 10, 5, 6], \\ H[0, 3, 1, 8, 10, 9]\}.$$

Then an $LC_3^{(3)}$ -decomposition of $K_{11}^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j + 1 \pmod{11}$.

Example 3.4. Let

$$V\left(K_{2,3,3}^{(3)}\right) = \mathbb{Z}_6 \cup \{\infty_1, \infty_2\}$$

with the vertex partition $\{\{\infty_1, \infty_2\}, \{0, 2, 4\}, \{1, 3, 5\}\}$ and let

$$B = \{H[\infty_1, 0, 1, \infty_2, 2, 5], H[5, \infty_2, 0, 1, \infty_1, 2, 5]\}.$$

Then an $LC_3^{(3)}$ -decomposition of $K_{2,3,3}^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 2 \pmod{6}$.

Example 3.5. Let

$$V\left(K_{3,3,3}^{(3)}\right) = \mathbb{Z}_9$$

with vertex partition $\{\{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\}\}$ and let

$$B = \{H[7, 3, 2, 1, 0, 5]\}.$$

Then an $LC_3^{(3)}$ -decomposition of $K_{3,3,3}^{(3)}$ consists of the $LC_3^{(3)}$ -block in B under the action of the map $j \mapsto j + 1 \pmod{9}$.

Example 3.6. Let

$$V\left(L_{9,9}^{(3)}\right) = \mathbb{Z}_{18}$$

with vertex partition $\{\{0, 2, \dots, 16\}, \{1, 3, \dots, 17\}\}$, and let

$$B = \{H[0, 16, 1, 4, 15, 2], H[14, 0, 1, 11, 16, 9], H[4, 0, 9, 1, 14, 3], H[9, 0, 1, 6, 17, 2], \\ H[0, 5, 1, 12, 15, 14], H[0, 1, 2, 5, 3, 15], H[0, 1, 15, 10, 4, 13], H[1, 5, 12, 7, 0, 13], \\ H[0, 1, 12, 3, 9, 16], H[10, 2, 17, 8, 1, 0], H[1, 0, 8, 2, 17, 4], H[1, 0, 7, 12, 6, 16]\}.$$

Then an $LC_3^{(3)}$ -decomposition of $L_{9,9}^{(3)} \cup K_{1,9,9}^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j + 1 \pmod{18}$.

Example 3.7. Let

$$V\left(L_{9,9}^{(3)} \cup K_{1,9,9}^{(3)}\right) = \mathbb{Z}_{18} \cup \{\infty\}$$

with vertex partition $\{\{\infty\}, \{0, 2, \dots, 16\}, \{1, 3, \dots, 17\}\}$, and let

$$\begin{aligned} B = & \{H[0, 16, 1, 4, 15, 2], H[14, 0, 1, 11, 16, 9], H[4, 0, 9, 1, 14, 3], H[9, 0, 1, 6, 17, 2], \\ & H[0, 5, 1, 12, 15, 14], H[0, 1, 2, \infty, 3, 15], H[0, 1, 15, 10, 4, 13], H[1, 5, 12, 7, 0, 13], \\ & H[0, \infty, 3, 12, 9, 16], H[14, 7, 17, 8, 1, \infty], H[1, \infty, 8, 2, 17, 4], H[1, 0, 7, 12, 6, 16]\}, \\ B' = & \{H[0, \infty, 9, 10, 1, 3], H[1, \infty, 10, 11, 2, 4], H[2, \infty, 11, 12, 3, 5], \\ & H[3, \infty, 12, 13, 4, 6], H[4, \infty, 13, 14, 5, 7], H[5, \infty, 14, 15, 6, 8], \\ & H[6, \infty, 15, 16, 7, 9], H[7, \infty, 16, 17, 8, 10], H[8, \infty, 17, 0, 9, 11], \\ & H[10, 0, 1, 2, 9, 3], H[11, 1, 2, 3, 10, 4], H[12, 2, 3, 4, 11, 5], \\ & H[13, 3, 4, 5, 12, 6], H[14, 4, 5, 6, 13, 7], H[15, 5, 6, 7, 14, 8], \\ & H[16, 6, 7, 8, 15, 9], H[17, 7, 8, 9, 16, 10], H[0, 8, 9, 10, 17, 11], \\ & H[12, 9, 10, 11, 0, 1], H[13, 10, 11, 12, 1, 2], H[14, 11, 12, 13, 2, 3], \\ & H[15, 12, 13, 14, 3, 4], H[16, 13, 14, 15, 4, 5], H[17, 14, 15, 16, 5, 6], \\ & H[0, 15, 16, 17, 6, 7], H[1, 16, 17, 0, 7, 8], H[2, 17, 0, 1, 8, 9]\}. \end{aligned}$$

Then an $LC_3^{(3)}$ -decomposition of $L_{9,9}^{(3)} \cup K_{1,9,9}^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{18}$, along with the $LC_3^{(3)}$ -blocks in B' .

Example 3.8. Let

$$V\left(K_{13}^{(3)} \setminus K_4^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$$

with $\infty_1, \dots, \infty_4$ being the vertices in the hole and let

$$\begin{aligned} B = & \{H[\infty_1, 0, \infty_2, 1, 4, 8], H[\infty_2, 0, \infty_3, 1, 4, 8], H[\infty_3, 0, \infty_4, 1, 4, 8], \\ & H[\infty_4, 0, \infty_1, 1, 4, 8], H[\infty_1, 0, \infty_3, 1, 2, 3], H[\infty_2, 0, \infty_4, 1, 2, 3], \\ & H[0, \infty_1, 3, \infty_2, 6, \infty_3], H[0, 1, 4, 8, 3, \infty_4], H[0, 2, 4, 6, 1, 8]\}, \\ B' = & \{H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \\ & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[6, 1, 0, 4, 3, 7], \\ & H[7, 2, 1, 5, 4, 8], H[8, 3, 2, 6, 5, 0], H[0, 7, 1, 8, 2, 3], H[3, 1, 4, 2, 5, 6], \\ & H[6, 4, 7, 5, 8, 0]\}. \end{aligned}$$

Then an $LC_3^{(3)}$ -decomposition of $K_{13}^{(3)} \setminus K_4^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{9}$ along with the $LC_3^{(3)}$ -blocks in B' .

Example 3.9. Let

$$V\left(K_{14}^{(3)} \setminus K_5^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$$

with $\infty_1, \dots, \infty_5$ being the vertices in the hole and let

$$\begin{aligned} B = & \{H[\infty_1, 0, \infty_2, 1, 2, 6], H[\infty_2, 0, \infty_3, 1, 2, 6], H[\infty_3, 0, \infty_4, 1, 2, 6], \\ & H[\infty_4, 0, \infty_5, 1, 2, 6], H[\infty_5, 0, \infty_1, 1, 2, 6], H[\infty_1, 0, \infty_3, 1, 3, 6], \\ & H[\infty_2, 0, \infty_4, 1, 3, 6], H[\infty_3, 0, \infty_5, 1, 3, 6], H[\infty_4, 0, \infty_1, 1, 3, 6], \\ & H[\infty_5, 0, \infty_2, 1, 3, 6], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6]\}, \\ B' = & \{H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \\ & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3], \\ & H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\}. \end{aligned}$$

Then an $LC_3^{(3)}$ -decomposition of $K_{14}^{(3)} \setminus K_5^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{9}$ along with the $LC_3^{(3)}$ -blocks in B' .

Example 3.10. Let

$$V\left(K_{15}^{(3)} \setminus K_6^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$$

with $\infty_1, \dots, \infty_6$ being the vertices in the hole and let

$$\begin{aligned} B = & \{H[\infty_1, 0, \infty_2, 1, \infty_3, 2], H[\infty_3, 0, \infty_4, 1, \infty_5, 2], H[\infty_5, 0, \infty_6, 1, \infty_1, 2], \\ & H[\infty_2, 0, \infty_4, 1, \infty_6, 2], H[\infty_1, 0, 4, \infty_2, 3, 6], H[\infty_2, 0, 4, \infty_3, 3, 6], \\ & H[\infty_3, 0, 4, \infty_4, 3, 6], H[\infty_4, 0, 4, \infty_5, 3, 6], H[\infty_5, 0, 4, \infty_6, 3, 6], \\ & H[\infty_6, 0, 4, \infty_1, 3, 6], H[0, 2, \infty_1, 1, \infty_4, 7], H[0, 2, \infty_2, 1, \infty_5, 7], \\ & H[0, 2, \infty_3, 1, \infty_6, 7], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6]\}, \\ B' = & \{H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \\ & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3], \\ & H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0]\}. \end{aligned}$$

Then an $LC_3^{(3)}$ -decomposition of $K_{15}^{(3)} \setminus K_6^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{9}$ along with the $LC_3^{(3)}$ -blocks in B' .

Example 3.11. Let

$$V\left(K_{16}^{(3)} \setminus K_7^{(3)}\right) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$$

with $\infty_1, \dots, \infty_7$ being the vertices in the hole and let

$$\begin{aligned}
 B = \{ & H[\infty_1, 0, \infty_2, 1, \infty_4, 2], H[\infty_2, 0, \infty_3, 1, \infty_5, 2], H[\infty_3, 0, \infty_4, 1, \infty_6, 2], \\
 & H[\infty_4, 0, \infty_5, 1, \infty_7, 2], H[\infty_5, 0, \infty_6, 1, \infty_1, 2], H[\infty_6, 0, \infty_7, 1, \infty_2, 2], \\
 & H[\infty_7, 0, \infty_1, 1, \infty_3, 2], H[\infty_1, 0, 1, \infty_2, 3, 7], H[\infty_2, 0, 1, \infty_3, 3, 7], \\
 & H[\infty_3, 0, 1, \infty_4, 3, 7], H[\infty_4, 0, 1, \infty_5, 3, 7], H[\infty_5, 0, 1, \infty_6, 3, 7], \\
 & H[\infty_6, 0, 1, \infty_7, 3, 7], H[\infty_7, 0, 1, \infty_1, 3, 7], H[0, \infty_1, 3, \infty_2, 6, \infty_3], \\
 & H[0, \infty_4, 3, \infty_5, 6, \infty_6], H[0, 1, 4, 8, 3, \infty_7], H[0, 2, 4, 6, 1, 8] \}, \\
 B' = \{ & H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \\
 & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[6, 1, 0, 4, 3, 7], \\
 & H[7, 2, 1, 5, 4, 8], H[8, 3, 2, 6, 5, 0], H[0, 7, 1, 8, 2, 3], H[3, 1, 4, 2, 5, 6], \\
 & H[6, 4, 7, 5, 8, 0] \}.
 \end{aligned}$$

Then an $LC_3^{(3)}$ -decomposition of $K_{16}^{(3)} \setminus K_7^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{9}$ along with the $LC_3^{(3)}$ -blocks in B' .

Example 3.12. Let

$$V(K_{17}^{(3)} \setminus K_8^{(3)}) = \mathbb{Z}_9 \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$$

with $\infty_1, \dots, \infty_8$ being the vertices in the hole and let

$$\begin{aligned}
 B = \{ & H[\infty_1, 0, \infty_2, 1, \infty_4, 2], H[\infty_2, 0, \infty_3, 1, \infty_5, 2], H[\infty_3, 0, \infty_4, 1, \infty_6, 2], \\
 & H[\infty_4, 0, \infty_5, 1, \infty_7, 2], H[\infty_5, 0, \infty_6, 1, \infty_8, 2], H[\infty_6, 0, \infty_7, 1, \infty_1, 2], \\
 & H[\infty_7, 0, \infty_8, 1, \infty_2, 2], H[\infty_8, 0, \infty_1, 1, \infty_3, 2], H[\infty_1, 0, 4, \infty_2, 3, 6], \\
 & H[\infty_2, 0, 4, \infty_3, 3, 6], H[\infty_3, 0, 4, \infty_4, 3, 6], H[\infty_4, 0, 4, \infty_5, 3, 6], \\
 & H[\infty_5, 0, 4, \infty_6, 3, 6], H[\infty_6, 0, 4, \infty_7, 3, 6], H[\infty_7, 0, 4, \infty_8, 3, 6], \\
 & H[\infty_8, 0, 4, \infty_1, 3, 6], H[\infty_1, 0, 2, 4, \infty_5, 1], H[\infty_2, 0, 2, 4, \infty_6, 1], \\
 & H[\infty_3, 0, 2, 4, \infty_7, 1], H[\infty_4, 0, 2, 4, \infty_8, 1], H[0, 2, 4, 6, 1, 8], H[1, 0, 4, 5, 2, 6] \}, \\
 B' = \{ & H[0, 2, 8, 5, 3, 6], H[1, 3, 0, 6, 4, 7], H[2, 4, 1, 7, 5, 8], H[5, 2, 3, 6, 4, 7], \\
 & H[8, 5, 6, 0, 7, 1], H[6, 2, 8, 4, 1, 3], H[7, 3, 0, 5, 2, 4], H[0, 7, 1, 8, 2, 3], \\
 & H[3, 1, 4, 2, 5, 6], H[6, 4, 7, 5, 8, 0] \}.
 \end{aligned}$$

Then an $LC_3^{(3)}$ -decomposition of $K_{17}^{(3)} \setminus K_8^{(3)}$ consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{9}$ along with the $LC_3^{(3)}$ -blocks in B' .

Example 3.13. Let

$$V(K_6^{(3)}) = \mathbb{Z}_6$$

and let

$$\begin{aligned}
 B = \{ & H[4, 0, 1, 5, 3, 2], H[5, 1, 2, 0, 4, 3], H[4, 5, 0, 3, 2, 1], H[5, 0, 1, 4, 3, 2], \\
 & H[0, 1, 2, 5, 4, 3], H[1, 2, 3, 0, 5, 4] \}.
 \end{aligned}$$

Then B is a maximum $LC_3^{(3)}$ -packing of $K_6^{(3)}$, where the leave has edge set $\{\{0, 1, 3\}, \{1, 2, 5\}\}$. Note that by renaming the vertices in this packing, any two hyperedges in $K_6^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_3^{(3)}$ -packing of $K_6^{(3)}$.

Example 3.14. Let

$$V(K_7^{(3)}) = \mathbb{Z}_7$$

and let

$$B = \{H[0, 1, 2, 4, 6, 3]\},$$

$$B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 0, 5, 1], H[1, 6, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 0]\}.$$

Then a maximum $LC_3^{(3)}$ -packing of $K_7^{(3)}$, where the leave has edge set $\{\{0, 1, 5\}, \{0, 2, 6\}\}$, consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j+1 \pmod{7}$ along with the $LC_3^{(3)}$ -blocks in B' . Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_7^{(3)}$ that intersect in a single vertex can be made into the leave edge set of a maximum $LC_3^{(3)}$ -packing of $K_7^{(3)}$.

Example 3.15. Let

$$V(K_8^{(3)}) = \mathbb{Z}_8$$

and let

$$B = \{H[6, 0, 7, 2, 3, 1], H[0, 2, 6, 7, 4, 1]\},$$

$$B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 0]\}.$$

Then a maximum $LC_3^{(3)}$ -packing of $K_8^{(3)}$, where the leave has edge set $\{\{1, 6, 7\}, \{0, 2, 7\}\}$, consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j+1 \pmod{8}$ along with the $LC_3^{(3)}$ -blocks in B' . Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_8^{(3)}$ that intersect in a single vertex can be made into the leave edge set of a maximum $LC_3^{(3)}$ -packing of $K_8^{(3)}$.

Example 3.16. Let

$$V(K_{12}^{(3)}) = \mathbb{Z}_{11} \cup \{\infty\}$$

and let

$$B = \{H[0, \infty, 1, 3, 2, 4], H[0, \infty, 2, 8, 5, 10], H[0, \infty, 3, 7, 4, 8], H[0, \infty, 4, 9, 5, 2], \\ H[0, \infty, 5, 10, 9, 6], H[0, 6, 2, 9, 4, 3]\},$$

$$B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 0], H[1, 10, 2, 0, 3, 4], \\ H[4, 2, 5, 3, 6, 7], H[8, 6, 9, 1, 10, 0], H[9, 7, 10, 2, 0, 1]\}.$$

Then a maximum $LC_3^{(3)}$ -packing of $K_{12}^{(3)}$, where the leave is the single edge $\{5, 7, 8\}$, consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty \mapsto \infty$ and $j \mapsto j + 1 \pmod{11}$ along with the $LC_3^{(3)}$ -blocks in B' . Note that by renaming the vertices in this packing, any edge in $K_{12}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$ -packing of $K_{12}^{(3)}$.

Example 3.17. Let

$$V(K_{13}^{(3)}) = \mathbb{Z}_{13}$$

and let

$$B = \{H[0, 3, 9, 12, 1, 11], H[0, 4, 8, 12, 1, 10], H[12, 4, 9, 0, 1, 7], H[12, 5, 8, 0, 1, 6], \\ H[7, 10, 4, 11, 0, 1], H[6, 10, 5, 2, 0, 1], H[4, 2, 3, 5, 1, 0]\}, \\ B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 0, 11, 1]\}.$$

Then a maximum $LC_3^{(3)}$ -packing of $K_{13}^{(3)}$, where the leave is the single edge $\{0, 2, 12\}$, consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j + 1 \pmod{13}$ along with the $LC_3^{(3)}$ -blocks in B' . Again, we note that by renaming the vertices in this packing, any edge in $K_{13}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$ -packing of $K_{13}^{(3)}$.

Example 3.18. Let

$$V(K_{14}^{(3)}) = \mathbb{Z}_{14}$$

and let

$$B = \{H[13, 0, 8, 3, 5, 12], H[0, 10, 1, 3, 5, 12], H[0, 8, 2, 3, 4, 13], H[0, 1, 11, 5, 6, 3], \\ H[0, 3, 8, 2, 9, 4], H[0, 2, 9, 13, 7, 3], H[0, 3, 10, 5, 8, 4], H[0, 4, 1, 7, 12, 8]\}, \\ B' = \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 13, 11, 0], \\ H[1, 13, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[11, 9, 12, 1, 13, 0], \\ H[12, 10, 13, 2, 0, 1]\}.$$

Then a maximum $LC_3^{(3)}$ -packing of $K_{14}^{(3)}$, where the leave is the single edge $\{8, 10, 11\}$, consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j + 1 \pmod{14}$ along with the $LC_3^{(3)}$ -blocks in B' . Again, we note that by renaming the vertices in this packing, any edge in $K_{14}^{(3)}$ can be made into the leave of a maximum $LC_3^{(3)}$ -packing of $K_{14}^{(3)}$.

Example 3.19. Let

$$V(K_{15}^{(3)}) = \mathbb{Z}_{13} \cup \{\infty_1, \infty_2\}$$

and let

$$\begin{aligned}
 B = & \{H[\infty_1, 0\infty_2, 1, 2, 3], H[0, \infty_1, 2\infty_2, 4, 8], H[0, \infty_1, 3\infty_2, 6, 12], \\
 & H[0, \infty_1, 4\infty_2, 8, 3], H[0, \infty_1, 5\infty_2, 10, 7], H[0, \infty_1, 6\infty_2, 12, 11], \\
 & H[0, 9, 11, 8, 4, 10], H[0, 5, 2, 4, 8, 6], H[0, 10, 2, 4, 11, 6], \\
 & H[0, 1, 4, 3, 8, 7], H[0, 1, 10, 9, 5, 6]\}, \\
 B' = & \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], H[12, 9, 10, 0, 11, 1], \\
 & H[1, 12, 2, 0, 3, 4], H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[10, 8, 11, 9, 12, 0]\}.
 \end{aligned}$$

Then a maximum $LC_3^{(3)}$ -packing of $K_{15}^{(3)}$, where the leave has edge set $\{\{0, 2, 14\}, \{2, 4, 5\}\}$, consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $\infty_i \mapsto \infty_i$ and $j \mapsto j + 1 \pmod{13}$ along with the $LC_3^{(3)}$ -blocks in B' . Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{15}^{(3)}$ that intersect in a single vertex can be made into the edge set the leave of a maximum $LC_3^{(3)}$ -packing of $K_{15}^{(3)}$.

Example 3.20. Let

$$V(K_{16}^{(3)}) = \mathbb{Z}_{16}$$

and let

$$\begin{aligned}
 B = & \{H[1, 0, 13, 10, 6, 4], H[0, 5, 10, 6, 11, 4], H[1, 14, 9, 3, 7, 13], \\
 & H[0, 11, 1, 12, 3, 10], H[0, 10, 2, 14, 7, 6], H[13, 5, 12, 0, 4, 7], \\
 & H[0, 3, 8, 15, 7, 14], H[15, 5, 14, 8, 6, 2], H[0, 13, 3, 8, 1, 6], \\
 & H[1, 0, 4, 8, 3, 7], H[0, 14, 12, 10, 11, 13]\}, \\
 B' = & \{H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], \\
 & H[12, 9, 10, 13, 11, 14], H[15, 12, 13, 0, 14, 1], H[1, 15, 2, 0, 3, 4], \\
 & H[4, 2, 5, 3, 6, 7], H[7, 5, 8, 6, 9, 10], H[10, 8, 11, 9, 12, 13], \\
 & H[13, 11, 14, 12, 15, 0]\}.
 \end{aligned}$$

Then a maximum $LC_3^{(3)}$ -packing of $K_{16}^{(3)}$, where the leave has edge set $\{\{0, 1, 14\}, \{0, 2, 15\}\}$, consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j + 1 \pmod{16}$ along with the $LC_3^{(3)}$ -blocks in B' . Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{16}^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_3^{(3)}$ -packing of $K_{16}^{(3)}$.

Example 3.21. Let

$$V\left(K_{17}^{(3)}\right) = \mathbb{Z}_{17}$$

and let

$$\begin{aligned} B = \{ & H[0, 15, 1, 16, 13, 3], H[0, 14, 1, 13, 5, 11], H[2, 13, 0, 4, 11, 16], \\ & H[11, 16, 1, 13, 0, 3], H[0, 12, 1, 11, 5, 10], H[0, 11, 1, 16, 10, 3], \\ & H[0, 10, 1, 16, 9, 4], H[14, 6, 0, 9, 1, 10], H[0, 8, 1, 6, 15, 7], \\ & H[15, 5, 1, 7, 0, 6], H[0, 6, 1, 4, 15, 5], H[0, 15, 4, 8, 5, 1], \\ & H[0, 15, 3, 5, 4, 2]\}, \\ B' = \{ & H[3, 0, 1, 4, 2, 5], H[6, 3, 4, 7, 5, 8], H[9, 6, 7, 10, 8, 11], \\ & H[12, 9, 10, 13, 11, 14], H[15, 12, 13, 16, 14, 0]\}. \end{aligned}$$

Then a maximum $LC_3^{(3)}$ -packing of $K_{17}^{(3)}$, where the leave has edge set $\{\{1, 15, 16\}, \{0, 2, 16\}\}$, consists of the $LC_3^{(3)}$ -blocks in B under the action of the map $j \mapsto j + 1 \pmod{17}$ along with the $LC_3^{(3)}$ -blocks in B' . Again, we note that by renaming the vertices in this packing, any two hyperedges in $K_{17}^{(3)}$ that intersect in a single vertex can be made into the edge set of the leave of a maximum $LC_3^{(3)}$ -packing of $K_{17}^{(3)}$.

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Ryan C. Bunge
rcbunge@ilstu.edu

Illinois State University
Normal, IL 61790, USA

Dontez Collins

Sussex Technical High School
Georgetown, DE 19947, USA

Daryl Conko-Camel

Salish Kootenai College
Pablo, MT 59855, USA

Saad I. El-Zanati (corresponding author)
saad@ilstu.edu

Illinois State University
Normal, IL 61790, USA

Rachel Liebrecht

Ohio Northern University
Ada, OH 45810, USA

Alexander Vasquez

Manhattan College
Bronx, NY 10471, USA

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