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FACIAL RAINBOW EDGE-COLORING OF SIMPLE 3-CONNECTED PLANE GRAPHS

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Abstract. A facial rainbow edge-coloring of a plane graph G is an edge-coloring such that any two edges receive distinct colors if they lie on a common facial path of G. The minimum number of colors used in such a coloring is denoted by $\mathrm{erb}(G)$. Trivially, $\mathrm{erb}(G) \geq \mathrm{L}(G) + 1$ holds for every plane graph without cut-vertices, where $\mathrm{L}(G)$ denotes the length of a longest facial path in G. Jendroľ in 2018 proved that every simple 3-connected plane graph admits a facial rainbow edge-coloring with at most $\mathrm{L}(G) + 2$ colors, moreover, this bound is tight for $\mathrm{L}(G) = 3$. He also proved that $\mathrm{erb}(G) = \mathrm{L}(G) + 1$ for $\mathrm{L}(G) \notin \{3, 4, 5\}$. He posed the following conjecture: There is a simple 3-connected plane graph G with $\mathrm{L}(G) = 4$ and $\mathrm{erb}(G) = \mathrm{L}(G)+2$. In this note we answer the conjecture in the affirmative.

Keywords: plane graph, facial path, edge-coloring.

Mathematics Subject Classification: 05C10, 05C15.

1. INTRODUCTION

We use standard graph theory terminology according to [2]. However, the most frequent notions of the paper are defined through it. A plane graph is a particular drawing of a planar graph in the Euclidean plane such that no edges intersect. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. The boundary of a face f is the boundary in the usual topological sense. It is the collection of all edges and vertices contained in the closure of f that can be organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of initial vertex and direction, and is called the boundary walk of the face f (see [9, p. 101]). Let f be a face having the boundary walk $v_0v_1 \dots v_{k-1}v_0$ with $v_i \in V(G)$ and $v_iv_{i+1} \in E(G)$, $i = 0, \dots, k-1$, subscripts taken modulo k. A facial path of f is a subpath $v_mv_{m+1} \dots v_n$ of the boundary walk of f (i.e. a facial path is any path which is a consecutive part of the boundary walk of a face).

Two vertices (two edges) are adjacent if they are connected by an edge (have a common endvertex). A vertex and an edge are incident if the vertex is an endvertex of the edge. A vertex (or an edge) and a face are incident if the vertex (or the edge) lies on the boundary of the face.

The dual G^* of a plane graph G is obtained as follows: Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G (an edge separates the faces incident with it). It is easy to see that the dual of a plane graph is itself a planar graph; in fact, there is a natural embedding of G^* in the plane. We place each vertex f^* in the corresponding face f of G, and then draw each edge e^* in such a way that it crosses the corresponding edge e of G exactly once (and crosses no other edge of G).

An edge-coloring of a graph G is an assignment of colors to the edges, one color to each edge. An edge-coloring c of a graph G is proper if for any two adjacent edges e_1 and e_2 of G, $c(e_1) \neq c(e_2)$ holds. The chromatic index of G, denoted by $\chi'(G)$, is the minimum number of colors needed for a proper edge-coloring of G. Clearly, at least Δ colors are required for any proper edge-coloring of a graph with maximum degree Δ. By the well-known Vizing's theorem [23], $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ holds for every simple graph G. This leads to a natural classification of simple graphs into two classes. A simple graph G is said to be of class one if $\chi'(G) = \Delta(G)$ and of class two if $\chi'(G) = \Delta(G) + 1$. By a result of Holyer [12], the problem of determining the chromatic index of an arbitrary simple graph is NP-complete. In fact, the problem is NP-complete even for simple cubic graphs. Simple graphs of class two are relatively scarce. Erdős and Wilson [7] proved that almost all simple graphs are of class one. Vizing [24] showed that every simple planar graph with maximum degree $\Delta \geq 8$ is of class one and conjectured that the same holds for $6 \leq \Delta \leq 7$. The first part of this conjecture was proved independently by Sanders and Zhao [20] and Zhang [29]. For every $2 \leq \Delta \leq 5$, there are simple planar graphs of class two with maximum degree Δ. Such graphs can be obtained from any Δ-regular simple planar graph with an even number of vertices by subdividing one of its edges.

A facial rainbow edge-coloring of a plane graph G is an edge-coloring (not necessarily proper) such that any two edges receive distinct colors if they lie on a common facial path of G. The minimum number of colors used in such a coloring is denoted by $\mathrm{erb}(G)$. Trivially, $\mathrm{erb}(G) \geq \mathrm{L}(G)$, where $\mathrm{L}(G)$ denotes the length of a longest facial path in G. If G is without cut-vertices, then $\mathrm{erb}(G) \geq \mathrm{L}(G) + 1$ (since in a 2-connected plane graph every face is bounded by a cycle). This type of coloring was introduced by Jendroľ [14]. He proved that every connected loopless plane graph G admits a facial rainbow edge-coloring with at most $\left\lfloor \frac{3}{2} \cdot (\mathrm{L}(G) + 1) \right\rfloor$ colors. Moreover, the bound is tight. For simple 3-connected plane graphs he obtained the following result.

Theorem 1.1 ([14]). *If G is a simple 3-connected plane graph, then*

(i) $\mathrm{erb}(G) = \mathrm{L}(G) + 1$ *for* $\mathrm{L}(G) \notin \{3, 4, 5\}$, *and*

(ii) $\mathrm{L}(G) + 1 \leq \mathrm{erb}(G) \leq \mathrm{L}(G) + 2$ *for* $\mathrm{L}(G) \in \{3, 4, 5\}$.

Moreover, the lower bound is tight for all $\mathrm{L}(G)$, *the upper bound in* (ii) *is tight for* $\mathrm{L}(G) = 3$.

He posed the following conjecture.

Conjecture 1.2 ([14]).

(i) *There is a simple* 3*-connected plane graph* G *with* $\mathrm{L}(G) = 4$ *and* $\mathrm{erb}(G) = \mathrm{L}(G)+2$.

(ii) *There is no simple* 3*-connected plane graph* G *with* $\mathrm{L}(G) = 5$ *and* $\mathrm{erb}(G) = \mathrm{L}(G) + 2$.

If G is a simple 3-connected plane graph, then its dual G^* is also simple and 3-connected, see [15, p. 46]. In every facial rainbow edge-coloring of a 3-connected plane graph G the edges bounding every face are colored distinctly, hence any such coloring of G induces a proper edge-coloring of its dual graph G^* and vice versa, i.e. $\mathrm{erb}(G) = \chi'(G^*)$. Therefore, Conjecture 1.2 part (ii) is the "3-connected case" of Vizing's Planar Graph Conjecture: Every simple planar graph with maximum degree 6 is of class one. There are many papers, published in recent years, answering Vizing's conjecture in the affirmative, provided some additional conditions regarding the absence of cycles of given length is guaranteed. It is shown that every simple planar graph G with $\Delta = 6$ is of class one if it is without 3-cycles, 4-cycles, or 5-cycles [10], 6-cycles [3], 7-cycles [13], chordal 4-cycles [3], chordal 5-cycles [25], chordal 6-cycles [16], 5-cycles with two chords [27], 6-cycles with two chords [28], 6-cycles with three chords [32], 7-cycles with three chords [30]. Vizing's Planar Graph Conjecture also holds for simple planar graphs in which no vertex is incident with four faces of size 3 [26], no 4-cycle is adjacent to a 5-cycle [17], no 7-cycles are adjacent [31], any k-cycle is adjacent to at most one k-cycle for some k ($k = 3, 4, 5$) [18]. Vizing's conjecture is still open in general.

In this note we affirm the first part of Conjecture 1.2.

2. RESULTS

Theorem 2.1. *For every positive integer* n*, there is a simple* 3*-connected plane graph* G *on at least* n *vertices such that* $\mathrm{L}(G) = 4$ *and* $\mathrm{erb}(G) = \mathrm{L}(G) + 2$.

As it was mentioned above, $\mathrm{erb}(G) = \chi'(G^*)$ holds for every simple 3-connected plane graph G. Therefore it is sufficient to find simple 3-connected planar graphs with maximum degree five and chromatic index six (then their duals fulfill the conditions of Theorem 2.1).

In the following, we prove that there are infinitely many simple 3-connected planar graphs with maximum degree five, chromatic index six, and minimum degree δ, for every $\delta \in \{3, 4, 5\}$ (clearly, the minimum degree of every 3-connected graph is at least three).

Lemma 2.2. *Let* G *be a simple graph with* $2k+1$ *vertices. If* G *is of class one, then it has at most* $k \cdot \Delta(G)$ *edges.*

Proof. Consider a proper edge-coloring of G with $\Delta(G)$ colors. Since the edges of the same color are independent (no two of them have a common endvertex), there are at most k edges in each color class. Consequently, we can color at most $k \cdot \Delta(G)$ edges with $\Delta(G)$ colors. □

Corollary 2.3. *If* G *is a simple graph with* $2k + 1$ *vertices and at least* $k \cdot \Delta(G) + 1$ *edges, then it is of class two.*

2.1. SIMPLE 3-CONNECTED PLANAR GRAPHS WITH $\delta(G) = 3$, $\Delta(G) = 5$, AND $\chi'(G) = 6$

The operation of vertex splitting in a graph G replaces a vertex v of degree at least 4 by two vertices x and y, inserts the edge xy and replaces every former edge uv in G, with either the edge ux or uy such that the degree of x and y is at least three in the new graph. As proven by Tutte [22], vertex splitting preserves 3-connectivity.

Theorem 2.4 ([22]). *Applying a vertex splitting on a 3-connected graph generates a 3-connected graph.*

Lemma 2.5. *There are infinitely many simple 3-connected planar graphs with $\delta(G) = 3$, $\Delta(G) = 5$, and $\chi'(G) = 6$.*

Proof. Owens [19] showed that a simple 5-regular planar graph with n vertices exists if and only if n is even, $n \geq 12$, and $n \neq 14$. From [11] it follows that there is a simple 3-connected 5-regular planar graph with n vertices for any such n.

First we take a simple 3-connected 5-regular planar graph H on $2n$ vertices. It has $5n$ edges, since

$$2|E(H)| = \sum_{v \in V(H)} \deg(v) = 5 \cdot 2n.$$

Now we split any vertex of H and obtain a new graph G. The graph G is 3-connected by Theorem 2.4. The minimum and maximum degree of G is three and five, respectively. The chromatic index of G is five or six, by Vizing's theorem. The graph G has $2n + 1$ vertices and $5n + 1$ edges, therefore it is of class two by Corollary 2.3. □

2.2. SIMPLE 3-CONNECTED PLANAR GRAPHS WITH $\delta(G) = 4$, $\Delta(G) = 5$, AND $\chi'(G) = 6$

In a graph G, subdivision of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w.

The Barnette and Grünbaum operations (BG-operations) consist of the following operations on a graph:

(i) add an edge xy (possibly a parallel edge),
(ii) subdivide an edge ab by a vertex x and add an edge xy for $y \notin \{a, b\}$,
(iii) subdivide two distinct, non-parallel edges by vertices x and y, respectively, and add the edge xy.

The following result was proven by Barnette and Grünbaum [1].

Theorem 2.6 ([1]). *Applying a BG-operation on a 3-connected graph generates a 3-connected graph.*

Lemma 2.7. *There are infinitely many simple 3-connected planar graphs with $\delta(G) = 4$, $\Delta(G) = 5$, and $\chi'(G) = 6$.*

Proof. First we take a plane drawing of a simple 3-connected 5-regular planar graph H on $2n$ vertices. By Euler's formula, $|V(H)| - |E(H)| + |F(H)| = 2$. Using the Handshaking Lemma

$$\sum_{v \in V(H)} \deg(v) = 2|E(H)| = \sum_{f \in F(H)} \deg(f),$$

we have

$$\sum_{v \in V(H)} (\deg(v) - 4) + \sum_{f \in F(H)} (\deg(f) - 4) = -8.$$

Therefore, H has a face f of size three. Let e_1, e_2, e_3 be the edges incident with f. Now we subdivide the edges e_1 and e_2 with vertices v_1 and v_2, and add the edge v_1v_2. After that we subdivide the edge e_3 with vertex v_3 and add the edge v_2v_3. Finally, we add the edge v_1v_3. The obtained graph G is simple, planar, and 3-connected, since it was obtained from H using BG-operations. The graph G has $2n + 3$ vertices and $5n + 6$ edges, therefore it is of class two by Corollary 2.3 (with $k = n + 1$). □

2.3. SIMPLE 3-CONNECTED PLANAR GRAPHS WITH $\delta(G) = 5$, $\Delta(G) = 5$, AND $\chi'(G) = 6$

In the proof of Lemma 2.7 we constructed simple 3-connected plane graphs with $\delta(G) = 4$, $\Delta(G) = 5$, and $\chi'(G) = 6$ such that each of them contains only three vertices of degree four, moreover, these three vertices form a face f of size three. Since any two planar embeddings of a simple 3-connected planar graph are equivalent (see [2, p. 267]), we can assume that f is the outer face. We will use the notation $\triangle$-graph for any such graph.

Lemma 2.8. *Let G be a simple 3-connected plane graph with a face f of size three. Let H be a $\triangle$-graph. If we glue G and H by identifying the boundary of f with the boundary of the outer face of H, then the obtained graph is also 3-connected.*

Proof. This follows from the fact that such a gluing of two simple 3-connected plane graphs corresponds to a connected sum of two 3-dimensional polytopes (see [8, p. 29]). Steinitz [21] proved that a planar graph is simple and 3-connected if and only if it is the edge graph of a 3-dimensional polytope. □

Lemma 2.9. *There are infinitely many simple 3-connected planar graphs with $\delta(G) = 5$, $\Delta(G) = 5$, and $\chi'(G) = 6$.*

Proof. First we take a plane drawing of a simple 3-connected 3-regular planar graph. Then, for each vertex v we subdivide two incident edges by vertices v_1 and v_2 and add the edge v_1v_2. In such a way we obtain a new plane graph H which is also 3-connected (see Theorem 2.6). Now we glue a $\triangle$-graph to each face of H of size three in order to obtain a simple 5-regular plane graph G, see Figure 1 for illustration.

The chromatic index of G is five or six, by Vizing's theorem. Since $\triangle$-graphs admit no proper edge-coloring with five colors, we have $\chi'(G) = 6$. From Lemma 2.8 it follows that G is 3-connected. □

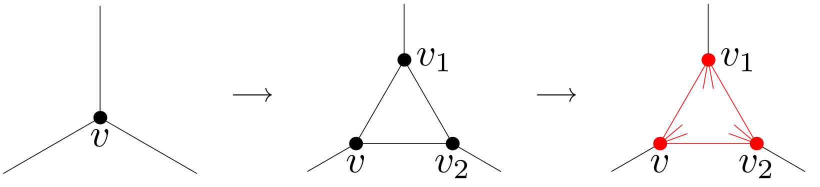

Fig. 1. An operation used to construct simple 3-connected 5-regular plane graphs from simple 3-connected 3-regular plane graphs.

Note that using the construction described in the proof of Lemma 2.9, we can obtain simple 3-connected planar graphs of class two with maximum degree five and minimum degree δ for every $\delta \in \{3,4,5\}$. First we take a simple 3-connected planar graph with vertices of degree 3 and δ (such graphs can be obtain from the complete graph on four vertices using BG-operations) and then replace some vertices of degree three by $\triangle$-graphs.

3. DISCUSSION

The present paper brings a contribution to the theory of proper edge-colorings and also to the theory of facial edge-colorings of plane graphs. We constructed simple 3-connected planar graphs with maximum degree five and chromatic index six. The existence of such graphs affirm a conjecture of Jendroľ about facial rainbow edge-coloring: There are simple 3-connected plane graphs G with $\mathrm{L}(G) = 4$ and $\mathrm{erb}(G) = \mathrm{L}(G) + 2$. We recommend to the reader recent survey papers about (facial) edge-colorings [4–6].

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