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PROPERTIES OF SOLUTIONS TO SOME WEIGHTED p-LAPLACIAN EQUATION

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Abstract. In this paper, we prove some qualitative properties for the positive solutions to some degenerate elliptic equation given by

$$-\mathrm{div}\big(w|\nabla u|^{p-2}\nabla u\big) = f(x,u), \quad w \in \mathcal{A}_p,$$

on smooth domain and for varying nonlinearity f.

Keywords: p-Laplacian, degenerate elliptic equations, weighted Sobolev space.

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1. INTRODUCTION

In this paper we are interested in studying the properties of positive solutions to the following class of degenerate elliptic equations given by

$$-\Delta_{p,w} u = f(x,u), \quad 1 < p < \infty, \tag{1.1}$$

where the weight function $w \in \mathcal{A}_p$ (defined in section 2) and $\Delta_{p,w}$ is the weighted p-Laplace operator defined by

$$\Delta_{p,w} u := \mathrm{div}\big(w|\nabla u|^{p-2}\nabla u\big)$$

in a bounded smooth domain $\Omega\,(\subset \mathbb{R}^N)$ for varying degree of nonlinearity f. When $w = 1$, $\Delta_{p,w}$ becomes the p-Laplace operator Δ_p defined by $\Delta_p u := \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)$ which further reduces to the classical Laplacian Δ for $p = 2$.

Before proceeding to state our main results, let us start with a brief background on the problem (1.1) already available in the literature which motivates the framework of our present study. In the pioneering work of Fabes *et al.* [10], the authors established a local Hölder regularity result and some maximum principle together with Poincaré type inequalities for Muckenhoupt weights. There has been a huge surge in interest to study problems on degenerate elliptic operators with Muckenhoupt weights. To note, results concerning weighted Poincaré and Sobolev inequalities was obtained by Chanillo and Wheeden [6], whereas a Liouville theorem was proved for the weight $w(x) = |x|^r$ with $r > -N$ and $N > 2$ in De Cicco–Vivaldi [7]. Related degenerate eigenvalue problem was studied in Kawohl *et al.* [19]. For more information on this field one can refer to [11–18].

The goal of our paper is to improve and complement the previously obtained results wherever possible. To be more precise, when $w = 1$, several qualitative properties of solutions to the equation (1.1) are well understood over the last three decades. In this context, we refer the reader to a nice survey by Allegretto-Huang [1]. We would also like to point out that similar results are investigated in the presence of a general class of nonlinearity and a general operator in Bal [2] and Tyagi [9] respectively.

In this paper, we provide sufficient conditions on the weight function w (which may vanish or blow up near the origin, see section 2 for definition) to establish nonexistence result, Hardy type inequality, comparison theorem, simplicity and monotonicity properties of the first eigenvalue (Theorem 4.1–Theorem 4.7) for the weighted p-Laplace equation (1.1) in the framework of weighted Sobolev space (see section 2 for definition). We start with a weighted version of Picone's identity (Lemma 3.1) although a direct consequence of [1,2] will play a very crucial role to obtain the main results. Furthermore, we prove some preliminary results (Lemma 3.3–3.4) which guarantee the existence of certain test function being an important ingredient in proving the main results.

Throughout the paper, we assume $N \geq 1$, $p > 1$ and Ω will denote a bounded smooth domain in $\mathbb{R}^N$ unless otherwise stated.

This paper is organized as follows. In Section 2, some preliminaries are mentioned. Some essential lemmas have been obtained in Section 3. In Section 4, our main results are stated and in Section 5, we prove our main results.

2. PRELIMINARIES

We begin this section by presenting some facts about the weighted Sobolev space.

Definition 2.1. A weight function w on any domain Ω of $\mathbb{R}^N$ is a real-valued function which is Lebesgue measurable and positive a.e. in Ω.

Definition 2.2. The class of weights $\mathcal{A}_p$ is defined by

$$\mathcal{A}_p := \Big\{ w \text{ is a weight on } \Omega : \, w \in L^1_{loc}(\Omega), \, w^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega) \Big\}.$$

For example,

$$w(x) = |x|^\alpha \in \mathcal{A}_p \quad \text{if} \quad -N < \alpha < N(p-1).$$

Definition 2.3 (Weighted Sobolev space)**.** For $w \in \mathcal{A}_p$, we define the weighted Sobolev space $W^{1,p}(\Omega, w)$ to be the set of all real-valued Lebesgue measurable functions u defined a.e. in Ω for which

$$\|u\|_{1,p,w} := \left(\int\limits_{\Omega} |u(x)|^p \, dx + \int\limits_{\Omega} w(x)|\nabla u(x)|^p \, dx \right)^{\frac{1}{p}} < +\infty. \tag{2.1}$$

Remark 2.4. The fact $w \in L^1_{loc}(\Omega)$ implies $C_c^\infty(\Omega) \subset W^{1,p}(\Omega, w)$. As a consequence one can introduce the space

$$W_0^{1,p}(\Omega, w) := \overline{\{C_c^\infty(\Omega), \| \cdot \|_{1,p,w}\}}.$$

Moreover, for $w \in \mathcal{A}_p$ both the spaces $W^{1,p}(\Omega, w)$ and $W_0^{1,p}(\Omega, w)$ are uniformly convex (and hence reflexive) Banach space, for details see Drábek *et al.* [8].

Definition 2.5. We define a new class of weights

$$\mathcal{A}_t := \left\{ w \in \mathcal{A}_p : \text{for some } s \in \left(\frac{N}{p}, \infty \right) \cap \left[\frac{1}{p-1}, \infty \right), \ w^{-s} \in L^1(\Omega) \right\}.$$

Assume that $p > 2N$. Then for $s = \frac{N}{p-2N}$, we have $p_s = \frac{ps}{s+1} > N$. Hence $|x|^\alpha \in \mathcal{A}_t$ with $p_s > N$ if $-N < \alpha < p - 2N$.

Remark 2.6. For $w \in \mathcal{A}_t$, we have the compact embedding $W^{1,p}(\Omega, w) \hookrightarrow L^{p+\eta}(\Omega)$ for $0 \le \eta < p_s^* - p$, where $p_s^* = \frac{Np_s}{N-p_s}$ for $1 \le p_s < N$. Moreover for $p_s > N$, the continuous embedding $W^{1,p}(\Omega, w) \hookrightarrow C(\bar{\Omega})$ holds for any $w \in \mathcal{A}_t$. The same result holds, if we replace the space $W^{1,p}(\Omega, w)$ by $W_0^{1,p}(\Omega, w)$.

For the proof of the above embedding results and more information regarding the weighted Sobolev space see [8, 11] and the references therein.

Let us introduce the following notation:

(i) For any set S, we denote by S^+ the set of all nonnegative elements in S.
(ii) $|S|$ will denote the Lebesgue measure of S.
(iii) We write C to denote a positive constant which may vary from line to line or even in the same line depending on the situation.

3. ESSENTIAL LEMMAS

We start this section by proving some lemmas.

For the rest of the paper, we will assume $h \in \mathbb{Q}$ and $f \in \mathbb{M}$ unless otherwise stated where $\mathbb{Q}$ and $\mathbb{M}$ are defined as follows:

$$\mathbb{Q} := \{ g : (0, \infty) \to (0, \infty) : g \text{ is a } C^1 \text{ function } \}$$
$$\mathbb{M} := \{ f \in \mathbb{Q} : f'(y) \ge (p-1)[f(y)]^{\frac{p-2}{p-1}} \text{ for all } y > 0 \}.$$

Clearly $f(x) = (x + c)^{p-1}$ belongs to $\mathbb{M}$ for any $c \ge 0$ and the equality is achieved. Other example of functions in $\mathbb{M}$ include $g(x) = e^{(p-1)x}$.

Lemma 3.1 (Picone Identity)**.** *Let Ω be any domain in $\mathbb{R}^N$ and $u \geq 0$, $v > 0$ in Ω be differentiable functions and w be a weight function defined in Ω. Define*

$$L(u,v) = w(x)\left\{|\nabla u|^p - \frac{pu^{p-1}}{f(v)}|\nabla v|^{p-2}\nabla u \cdot \nabla v + \frac{u^p f'(v)}{\big(f(v)\big)^2}|\nabla v|^p\right\},$$

$$R(u,v) = w(x)\left\{|\nabla u|^p - \nabla\Big(\frac{u^p}{f(v)}\Big) \cdot |\nabla v|^{p-2}\nabla v\right\}.$$

Then, we have

$$L(u,v) = R(u,v) \geq 0 \text{ in } \Omega. \tag{3.1}$$

Moreover, we have

$$L(u,v) = 0 \text{ in } \Omega \text{ if and only if } u = cv + d \text{ for some constants } c, d. \tag{3.2}$$

Proof. The conclusion (3.1) follows arguing similarly as in [1,2]. Moreover, proceeding similarly as in [1,2], we obtain $L(u,v) = 0$ in Ω if and only if the following three equality (3.3), (3.4) and (3.5) holds simultaneously for $q = \frac{p}{p-1}$.

$$f'(v) = (p-1)[f(v)]^{\frac{p-2}{p-1}}, \tag{3.3}$$

$$|\nabla u| = \frac{u}{[f(v)]^{\frac{q}{p}}}|\nabla v|, \tag{3.4}$$

and

$$|\nabla u||\nabla v| = \nabla u \cdot \nabla v. \tag{3.5}$$

We observe that the equality (3.3) gives $f(v) = (v+k)^{p-1}$ for some constant k. Now proceeding similarly as in [1,2], we obtain from (3.4) and (3.5) $u = cv + d$ in Ω for some constants c, d. Hence, (3.2) follows. □

Remark 3.2. For $p = 2$ and $w = 1$, we get back Theorem 1.1 of Tyagi [20] and (3.1) rectifies Theorem 2.1 of Bal [2] for $p > 1$.

Lemma 3.3. *For $\phi \in [C_c^\infty(\Omega)]^+$ and $v \in C(\Omega) \cap W^{1,p}(\Omega, w)$ such that $v > 0$ in Ω, we have $\frac{\phi^p}{h(v)} \in [W_0^{1,p}(\Omega, w)]^+$, provided $w \in \mathcal{A}_p$.*

Proof. Since $h \in C^1(0,\infty)$ and v is continuous on any compact subset K of Ω we have the functions

$$F_1 : K \to \mathbb{R} \text{ defined by } F_1(x) = h'(v)(x)$$

and

$$F_2 : K \to \mathbb{R} \text{ defined by } F_2(x) = h(v)(x)$$

are continuous. Therefore, on any compact subset $K := \operatorname{supp} \phi$ we have

$$|F_1(x)| \leq c, \tag{3.6}$$

where c is a constant independent of x and

$$|F_2(x)| \geq \delta > 0, \tag{3.7}$$

where δ is a constant independent of x. Now using (3.7), we obtain

$$\int_\Omega \Big|\frac{\phi^p}{h(v)}\Big|^p\,dx \le \frac{\|\phi\|_\infty^{p^2}}{\delta^p}|\Omega| < +\infty.$$

From (3.6) and (3.7), we get

$$\begin{aligned}
\int_\Omega w(x)\Big|\nabla\Big(\frac{\phi^p}{h(v)}\Big)\Big|^p\,dx &= \int_K w(x)\Big|\frac{p\phi^{p-1}\nabla\phi}{h(v)} - \frac{h'(v)\phi^p\nabla v}{(h(v))^2}\Big|^p\,dx\\
&\le 2^p\int_K w(x)\bigg\{p^p\frac{\phi^{p(p-1)}|\nabla\phi|^p}{\big(h(v)\big)^p} + \Big(\frac{h'(v)}{\big(h(v)\big)^2}\Big)^p\phi^{p^2}|\nabla v|^p\bigg\}\,dx\\
&\le 2^p\int_K w(x)\bigg\{p^p\frac{\|\phi\|_\infty^{p(p-1)}}{\delta^p}|\nabla\phi|^p + \frac{c^p\|\phi\|_\infty^{p^2}}{\delta^{2p}}|\nabla v|^p\bigg\}\,dx\\
&\le C\Big(\|\phi\|_{1,p,w}^p + \|v\|_{1,p,w}^p\Big)\\
&< +\infty,
\end{aligned}$$

for some positive constant C. Hence the lemma follows. □

For the next lemma, we assume $h \in \mathbb{Q}$ and is monotone increasing satisfying the following property:

$$q_\epsilon(x) := \bigg|\frac{h'(x+\epsilon)}{\big(h(x+\epsilon)\big)^2}\bigg|_\infty \le M(\epsilon) \text{ for any } \epsilon > 0,\ x \ge 0, \tag{3.8}$$

where $M(\epsilon)$ is a positive constant may depend on ϵ but independent of x. Clearly $e^{(p-1)x}$ and $(x+c)^{p-1}$ is in the above class for any $c > 0$ and $p > 1$.

Lemma 3.4. *Let $w \in \mathcal{A}_p$. Then for any $\epsilon > 0$, we have $\frac{u^p}{h(v+\epsilon)} \in [W_0^{1,p}(\Omega,w)]^+$, provided $u \in C(\bar\Omega) \cap [W_0^{1,p}(\Omega,w)]^+$ and $v \in C(\bar\Omega) \cap [W^{1,p}(\Omega,w)]^+$.*

Proof. Since $u, v \in C(\bar\Omega)$, we have

$$\int_\Omega \Big|\frac{u^p}{h(v+\epsilon)}\Big|^p\,dx \le \frac{\|u\|_\infty^{p^2}}{\big(h(\epsilon)\big)^p}|\Omega| < +\infty,$$

and from (3.8), we obtain

$$\begin{aligned}
&\int_\Omega w(x)\left|\nabla\Big(\frac{u^p}{h(v+\epsilon)}\Big)\right|^p dx\\
&\leq \int_\Omega w(x)\left|\frac{pu^{p-1}|\nabla u|}{h(v+\epsilon)}+q_\epsilon(v)u^p|\nabla v|\right|^p dx\\
&\leq 2^p\int_\Omega w(x)\left\{\frac{p^p\|u\|_\infty^{p(p-1)}|\nabla u|^p}{\big(h(\epsilon)\big)^p}+\big(M(\epsilon)\big)^p\|u\|_\infty^{p^2}|\nabla v|^p\right\}dx\\
&\leq C(\|u\|_{1,p,w}^p+\|v\|_{1,p,w}^p)\\
&< +\infty,
\end{aligned}$$

for some positive constant C. Hence the lemma follows. □

4. MAIN RESULTS

We start this section by stating our main results.

Let $f \in \mathbb{M}$ satisfies the hypothesis (3.8) and $0 \leq g_1, g_2 \in L^q(\Omega)$, $q = \frac{p}{p-1}$. Then we say $u \in C(\bar{\Omega}) \cap W^{1,p}(\Omega, w)$ is a positive weak supersolution of the following equation

$$-\Delta_{p,w}u - g_1(x)f(u) = g_2(x), \quad w \in \mathcal{A}_p, \tag{4.1}$$

if $u > 0$ in Ω and for all $\varphi \in [W_0^{1,p}(\Omega, w)]^+$, we have

$$\int_\Omega w(x)|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,dx - \int_\Omega g_1(x)f(u)\varphi\,dx \geq \int_\Omega g_2(x)\phi\,dx. \tag{4.2}$$

Theorem 4.1 (Nonexistence of positive supersolutions). *Given $w \in \mathcal{A}_p$ if there exists $u \in C(\bar{\Omega}) \cap [W_0^{1,p}(\Omega, w)]^+$ such that*

$$J(u) = \int_\Omega w(x)|\nabla u|^p\,dx - \int_\Omega g_1(x)u^p\,dx < 0,$$

then (4.1) has no positive weak supersolution in $C(\bar{\Omega}) \cap W^{1,p}(\Omega, w)$.

Theorem 4.2 (Hardy-type Inequality). *Let $w \in \mathcal{A}_p$ and assume that there exists $v \in C(\bar{\Omega}) \cap W^{1,p}(\Omega, w)$ satisfying*

$$-\Delta_{p,w}v \geq g(x)f(v), \quad v > 0 \text{ in } \Omega, \quad f \in \mathbb{M}, \tag{4.3}$$

for some nonnegative function $g \in L^q(\Omega)$, $q = \frac{p}{p-1}$. Then for any $u \in [C_c^\infty(\Omega)]^+$ it holds that

$$\int_\Omega w(x)|\nabla u|^p\,dx \geq \int_\Omega g(x)u^p\,dx. \tag{4.4}$$

Theorem 4.3 (Comparison Theorem). *Let $w \in \mathcal{A}_p$ and assume $u \in C(\bar{\Omega}) \cap [W^{1,p}(\Omega, w)]^+$ be such that*

$$-\Delta_{p,w} u = f_1(x)|u|^{p-2}u,\ u \geq 0\,(\not\equiv 0) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4.5}$$

Then any nontrivial solution $v \in C(\bar{\Omega}) \cap W^{1,p}(\Omega, w)$ of

$$-\Delta_{p,w} v = f_2(x)|v|^{p-2}v \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega, \tag{4.6}$$

where $0 \leq f_1(x) < f_2(x)$ a.e. $x \in \Omega$ and $f_1, f_2 \in L^q(\Omega)$, $q = \frac{p}{p-1}$ must change sign.

We conclude this section with a brief discussion of the following eigenvalue problem,

$$\begin{aligned} -\Delta_{p,w} u &= \lambda\beta(x)|u|^{p-2}u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4.7}$$

where $w \in \mathcal{A}_t$ and $\beta\,(\geq 0) \in L^\infty(\Omega)$. Moreover, let

$$\left|\left\{x \in \Omega : \beta(x) > 0\right\}\right| > 0.$$

Here we state a result, proof of which can be found in Chapter 1, of Drábek *et al.* [8], which in turn will give a new norm on the space $W_0^{1,p}(\Omega, w)$.

Lemma 4.4 (The weighted Friedrich inequality). *For any $w \in \mathcal{A}_t$, the following inequality*

$$\int_\Omega |u|^p\,dx \leq C \int_\Omega w(x)|\nabla u|^p\,dx, \tag{4.8}$$

holds for every $u \in W_0^{1,p}(\Omega, w)$ for some constant $C > 0$ independent of u.

Definition 4.5. A real number λ such that the eigenvalue problem (4.7) admits a solution u is called an eigenvalue of the operator $-\Delta_{p,w}$ and u is called the corresponding eigenfunction.

We denote the first eigenvalue of the operator $-\Delta_{p,w}$ as λ_1 and is defined as:

$$\lambda_1 := \inf\left\{\int_\Omega w(x)|\nabla v|^p\,dx : \int_\Omega \beta(x)|v|^p\,dx = 1\right\}. \tag{4.9}$$

Observe, that due to Lemma 4.4, the norm

$$\|u\|_{W_0^{1,p}(\Omega,w)} := \left(\int_\Omega w(x)|\nabla u|^p\,dx\right)^{\frac{1}{p}}$$

on the space $W_0^{1,p}(\Omega, w)$ is equivalent to the norm $\|\cdot\|_{1,p,w}$ defined by (2.1). We now turn to results related to the eigenvalue problem (4.7). From [8], it follows that, the infimum in (4.9) is attained. Moreover, the simplicity of λ_1 is proved in a different way by Drábek *et al.* [8] but here we provide a simpler proof using ideas from Belloni–Kawohl [5].

Theorem 4.6 (Simplicity of the first eigenvalue). *The first eigenvalue* λ_1 *of the operator* $-\Delta_{p,w}$ *is simple for any* $w \in \mathcal{A}_t$.

Theorem 4.7 (Monotonicity property of the first eigenvalue). *Let* $w \in \mathcal{A}_t$ *and suppose* $\Omega_1 \subset \Omega_2$ *such that* $\Omega_1 \neq \Omega_2$. *Then* $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$.

5. PROOF OF MAIN RESULTS

Proof of Theorem 4.1. Suppose v is a positive weak supersolution of (4.1) and let $\epsilon > 0$. Choosing $\varphi = \frac{u^p}{f(v+\epsilon)} \in [W_0^{1,p}(\Omega, w)]^+$ as a test function in (4.2) admissible by Lemma 3.4 and using Lemma 3.1, we get

$$\begin{aligned}
\int_\Omega g_1(x)\frac{f(v)}{f(v+\epsilon)}u^p\,dx &\leq \int_\Omega \left\{w(x)|\nabla v|^{p-2}\nabla v\cdot\nabla\Big(\frac{u^p}{f(v+\epsilon)}\Big) - g_2(x)\frac{u^p}{f(v+\epsilon)}\right\}dx \\
&= \int_\Omega \left\{w(x)|\nabla u|^p - w(x)|\nabla u|^p + w(x)|\nabla v|^{p-2}\nabla v\cdot\nabla\Big(\frac{u^p}{f(v+\epsilon)}\Big)\right. \\
&\qquad \left. - g_2(x)\frac{u^p}{f(v+\epsilon)}\right\}dx \\
&= \int_\Omega \left\{w(x)|\nabla u|^p - \int_\Omega R(u, v+\epsilon) - \int_\Omega g_2(x)\frac{u^p}{f(v+\epsilon)}\right\}dx \\
&\leq \int_\Omega w(x)|\nabla u|^p\,dx.
\end{aligned}$$

Then Fatou's lemma yields $J(u) \geq 0$. Hence the theorem. □

Proof of Theorem 4.2. Using $\frac{u^p}{f(v)} \in [W_0^{1,p}(\Omega, w)]^+$ as a test function in (4.3) which is admissible by Lemma 3.3, we have

$$\int_\Omega g(x)u^p dx \leq \int_\Omega w(x)|\nabla v|^{p-2}\nabla v\cdot\nabla\Big(\frac{u^p}{f(v)}\Big)\,dx. \tag{5.1}$$

By Lemma 3.1, we have

$$\int_\Omega R(u,v)dx = \int_\Omega w(x)\left\{|\nabla u|^p - |\nabla v|^{p-2}\nabla v\cdot\nabla\Big(\frac{u^p}{f(v)}\Big)\right\}dx \geq 0. \tag{5.2}$$

Using the inequality (5.1) in (5.2) we get the inequality (4.4). Hence, the theorem follows. □

Proof of Theorem 4.3. Without loss of generality let $v > 0$ in Ω. The case $v < 0$ can be dealt similarly by considering the function $-v$. For $\epsilon > 0$, using $\frac{u^p}{(v+\epsilon)^{p-1}} \in [W_0^{1,p}(\Omega, w)]^+$ as a test function in (4.6) admissible by Lemma 3.4 we get

$$\int_\Omega w|\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{u^p}{(v+\epsilon)^{p-1}}\Big)\,dx = \int_\Omega f_2(x)\Big(\frac{v}{v+\epsilon}\Big)^{p-1} u^p\,dx. \tag{5.3}$$

Taking u as a test function in (4.5) we get

$$\int_\Omega w|\nabla u|^p\,dx = \int_\Omega f_1(x)u^p\,dx. \tag{5.4}$$

By Lemma 3.1 with $f(v+\epsilon) = (v+\epsilon)^{p-1}$ and using (5.3), (5.4), we get

$$\begin{aligned} 0 &\le \int_\Omega L(u, v+\epsilon)\,dx \\ &= \int_\Omega R(u, v+\epsilon)\,dx \\ &= \int_\Omega w(x)|\nabla u|^p\,dx - \int_\Omega w(x)|\nabla v|^{p-2}\nabla v \cdot \nabla\Big(\frac{u^p}{(v+\epsilon)^{p-1}}\Big)\,dx \\ &= \int_\Omega f_1(x)u^p\,dx - \int_\Omega f_2(x)\Big(\frac{v}{v+\epsilon}\Big)^{p-1} u^p\,dx. \end{aligned}$$

Using Fatou's lemma we get

$$\int_\Omega \big\{f_1(x) - f_2(x)\big\}u^p\,dx \ge 0,$$

which gives a contradiction since $f_1 < f_2$ and $u \neq 0$. Hence, v must change sign. □

Proof of Theorem 4.6. Note that the first eigenvalue λ_1 of the operator $-\Delta_{p,w}$ is given by the minimizer of the functional

$$J_{p,w}(v) = \int_\Omega w(x)|\nabla v|^p\,dx \quad \text{on} \quad K = \left\{v \in W_0^{1,p}(\Omega, w) : \int_\Omega \beta(x)|v|^p\,dx = 1\right\}.$$

Let us suppose u_1 and u_2 be two positive eigenfunctions corresponding to the first eigenvalue λ_1. Therefore, we have

$$\int_\Omega \beta(x)|u_1(x)|^p dx = 1$$

and

$$\int_{\Omega} \beta(x)|u_2(x)|^p dx = 1.$$

Choosing $u_3(x) = U^{\frac{1}{p}}(x)$ for $U(x) = \frac{u_1^p(x)+u_2^p(x)}{2}$ we get $\int_{\Omega} \beta(x)|u_3(x)|^p dx = 1$. Now for $t(x) = \frac{u_1^p}{u_1^p+u_2^p}(x) \in (0,1)$, we have

$$\begin{aligned} w(x)|\nabla u_3|^p &= w(x)U^{1-p}\left|\frac{1}{2}\Big(u_1^{p-1}\nabla u_1 + u_2^{p-1}\nabla u_2\Big)\right|^p \\ &= w(x)U\left|\frac{1}{2}\Big(\frac{u_1^p}{U}\frac{\nabla u_1}{u_1} + \frac{u_2^p}{U}\frac{\nabla u_2}{u_2}\Big)\right|^p \\ &= w(x)U\left|t(x)\frac{\nabla u_1}{u_1} + \big(1-t(x)\big)\frac{\nabla u_2}{u_2}\right|^p \\ &\leq w(x)U\left\{t(x)\left|\frac{\nabla u_1}{u_1}\right|^p + \big(1-t(x)\big)\left|\frac{\nabla u_2}{u_2}\right|^p\right\} \\ &= \frac{w(x)}{2}\left(|\nabla u_1|^p + |\nabla u_2|^p\right) \\ &= \frac{1}{2}\left\{w(x)|\nabla u_1|^p + w(x)|\nabla u_2|^p\right\}. \end{aligned}$$

Therefore, we obtain

$$\int_{\Omega} w(x)|\nabla u_3(x)|^p\,dx \leq \frac{1}{2}\Bigg\{\int_{\Omega} w(x)|\nabla u_1(x)|^p\,dx + \int_{\Omega} w(x)|\nabla u_2(x)|^p\,dx\Bigg\}. \tag{5.5}$$

Note that since both u_1 and u_2 are minimizers of the functional $J_{p,w}$, the equality in (5.5) must hold i.e.,

$$\frac{\nabla u_1}{u_1} = \frac{\nabla u_2}{u_2} \quad \text{a.e. in} \quad \Omega,$$

which implies

$$u_1 = cu_2 \quad \text{a.e. in} \quad \Omega.$$

Hence, λ_1 is simple. □

Proof of Theorem 4.7. Note that $C_c^\infty(\Omega_1)$ is dense in $W_0^{1,p}(w,\Omega_1)$ provided $w \in \mathcal{A}_t$. The assumption $w \in \mathcal{A}_t$ is required to guarantee the existence of first eigenfunction. Using the above information, putting $f(x) = x^{p-1}$ in Lemma 3.1 and proceeding similarly as in [1, 9] we have our result. □

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