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ON SOME EXTENSIONS OF THE A-MODEL

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Abstract. The A-model for finite rank singular perturbations of class $\mathfrak{H}_{-m-2}\setminus\mathfrak{H}_{-m-1}$, $m\in\mathbb{N}$, is considered from the perspective of boundary relations. Assuming further that the Hilbert spaces $(\mathfrak{H}_n)_{n\in\mathbb{Z}}$ admit an orthogonal decomposition $\mathfrak{H}_n^-\oplus\mathfrak{H}_n^+$, with the corresponding projections satisfying $P_{n+1}^{\pm}\subseteq P_n^{\pm}$, nontrivial extensions in the A-model are constructed for the symmetric restrictions in the subspaces.

Keywords: finite rank higher order singular perturbation, cascade (A) model, peak model, Hilbert space, scale of Hilbert spaces, Pontryagin space, ordinary boundary triple, Krein Q-function, Weyl function, gamma field, symmetric operator, proper extension, resolvent.

Mathematics Subject Classification: 47A56, 47B25, 47B50, 35P05.

1. INTRODUCTION

Consider a lower semibounded self-adjoint operator L in a Hilbert space $\mathfrak{H}_0$. Let $\mathfrak{H}_{n+1}\subseteq\mathfrak{H}_n$, $n\in\mathbb{Z}$, be the scale of Hilbert spaces associated with L. Let also $\{\varphi_\sigma\}$ be the family of linearly independent functionals of class $\mathfrak{H}_{-m-2}\setminus\mathfrak{H}_{-m-1}$, $m\in\mathbb{N}$, where σ ranges over an index set $\mathcal{S}$ of dimension $d\in\mathbb{N}$. Then, the symmetric restriction $L_{\min}\subseteq L$ to the domain of $f\in\mathfrak{H}_{m+2}$ such that $\langle\varphi_\sigma,f\rangle=0$, for all σ, is an essentially self-adjoint operator in $\mathfrak{H}_0$. Sequentially, traditional methods, see e.g. [2, 20], for describing nontrivial extensions of $L_{\min}$ (i.e. perturbations of L) in $\mathfrak{H}_0$ are insufficient. The classical examples of higher order singular perturbations are the point-interactions modeled by the Dirac distribution and its derivatives.

To construct nontrivial realizations of $L_{\min}$ in Hilbert or Pontryagin spaces, one considers instead the so-called cascade (A or B) models [15–17, 25, 27] and the peak model [24, 26]. In these models the Weyl (or Krein Q-) function is the sum of a Nevanlinna function associated with $L_{\min}$ in $\mathfrak{H}_m$ and a generalized Nevanlinna function associated with a certain multiplication operator in a reproducing kernel Pontryagin space [5, Theorem 4.10]; more on reproducing kernel spaces can be found in [3, 6, 7, 10]. Successively, singular perturbations are interpreted by means of the compression to the reference space $\mathfrak{H}_0$ of the resolvent of an appropriate extension in the model space.

Here we study the cascade A-model for rank-d higher order singular perturbations. More precisely, for a specific choice of model parameters, we extend the main results obtained in [15] to the case of an arbitrary $d \in \mathbb{N}$ (see Theorem 3.2). The exposition utilizes the techniques based on the notion of boundary triples [11–14]. Then, by assuming that the Hilbert space $\mathfrak{H}_n$ is expressed as the Hilbert sum $\mathfrak{H}_n^- \oplus \mathfrak{H}_n^+$ of its subspaces $\mathfrak{H}_n^\pm$, we examine nontrivial realizations that account for the above described Hilbert space decomposition (Theorem 7.3). We assume that the corresponding orthogonal projections $P_n^\pm$ from $\mathfrak{H}_n$ onto $\mathfrak{H}_n^\pm$ satisfy the inclusions $P_{n+1}^\pm \subseteq P_n^\pm$. This further implies that the subspaces $\mathfrak{H}_n^\pm$ reduce the self-adjoint restriction to $\mathfrak{H}_{n+2}$ of L (Theorem 5.6). As a natural consequence of our hypothesis is that the Weyl function associated with the symmetric operator $L_{\min}$ in $\mathfrak{H}_m$ is the sum of the Weyl functions associated with the symmetric restrictions to $\mathfrak{H}_m^\pm$ of $L_{\min}$.

The projection of the model to the subspaces just described has a natural application in quantum mechanics when, for example, one wishes to account for the contribution to the eigenvalues of antisymmetric (resp. symmetric) eigenfunctions. For instance, if one takes L such that $\mathfrak{H}_n = W_2^n \otimes \mathbb{C}^4$, where W_2^n is the Sobolev space (Example 4.3), then the projections P_n^- and P_n^+ onto the spaces of antisymmetric spin states, $W_2^n \otimes \mathbb{C}^1$, and onto the spaces of symmetric spin states, $W_2^n \otimes \mathbb{C}^3$, satisfy our hypothesis. However, a concrete application of the present model will be demonstrated elsewhere.

Another motivation for considering the A-model, as opposed to the peak model, arises from an attempt to elude a too restrictive condition imposed on the Gram matrix $\mathcal{G} = (\mathcal{G}_{\sigma j,\sigma' j'}) \in [\mathbb{C}^{md}]$ of the peak model; namely, $\mathcal{G}$ must be diagonal in $j \in \{1, \ldots, m\}$. Although initially contemplated as an advantageous feature [26], this restriction is not satisfied for some operators L, for $m > 1$, for a simple reason that the eigenvectors of the triplet adjoint of $L_{\min}$ for the Hilbert triple $\mathfrak{H}_m \subseteq \mathfrak{H}_0 \subseteq \mathfrak{H}_{-m}$ are not necessarily orthogonal for distinct eigenvalues (Example 3.4).

2. PRELIMINARIES

Let A be a densely defined, closed, symmetric operator in a Pontryagin space $\mathfrak{H}$ (see e.g. [4, Sec. 1.9]) with an indefinite metric $[\cdot,\cdot]_{\mathfrak{H}}$. Let A^* be the adjoint in $\mathfrak{H}$ of A. A triple $(\mathcal{H}, \Gamma_0, \Gamma_1)$, where $\mathcal{H} = (\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a Hilbert space and $\Gamma\colon f \mapsto (\Gamma_0 f, \Gamma_1 f)$ is the operator from $\operatorname{dom} A^*$ to $\mathcal{H}^2 (:= \mathcal{H} \times \mathcal{H})$, is called an ordinary boundary triple (OBT) for A^* if Γ is surjective and the Green identity holds:

$$[f,g]_{A^*} := [f, A^* g]_{\mathfrak{H}} - [A^* f, g]_{\mathfrak{H}} = \langle \Gamma_0 f, \Gamma_1 g\rangle_{\mathcal{H}} - \langle \Gamma_1 f, \Gamma_0 g\rangle_{\mathcal{H}}$$

for all $f, g \in \operatorname{dom} A^*$; see e.g. [8, Definition 2.1]. It is shown that an OBT for A^* in a Pontryagin space (or more generally in a Krein space) exists iff A admits a self-adjoint extension in $\mathfrak{H}$ (cf. [5, Proposition 3.4], [9, p. 192]).

If the assumption on the density of dom A is dropped off, that is, if A^* is a linear relation [18,22], then an OBT $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for A^* is defined by considering Γ_i, $i \in \{0,1\}$, as a mapping from A^* onto $\mathcal{H}$. Sequentially, the Green identity reads

$$[f, g']_{\mathfrak{H}} - [f', g]_{\mathfrak{H}} = \langle \Gamma_0 \widehat{f}, \Gamma_1 \widehat{g} \rangle_{\mathcal{H}} - \langle \Gamma_1 \widehat{f}, \Gamma_0 \widehat{g} \rangle_{\mathcal{H}}$$

for $\widehat{f} = (f, f')$, $\widehat{g} = (g, g') \in A^*$. The reader may also consult [9, Definition 6], as well as [21, Definition 2.3], [14, Definition 7.11] in the Hilbert space case. In what follows we frequently identify operators with their graphs. Then the present definition of an OBT reduces to the previous definition as long as A becomes densely defined.

A proper extension A_Θ of A, i.e. such that $A \subseteq A_\Theta \subseteq A^*$, is uniquely determined by a linear relation Θ in $\mathcal{H}$ via $\Theta = \Gamma A_\Theta$ with $A_\Theta = \{\widehat{f} \in A^* \,|\, \Gamma \widehat{f} \in \Theta\}$; see e.g. [9, Proposition 2], [21, Proposition 2.5], [14, Proposition 7.12], [8, Proposition 2.1]. In particular, a distinguished self-adjoint extension $A_0 := A^* \,|_{\ker \Gamma_0}$ corresponds to a self-adjoint linear relation $\Theta = \{0\} \times \mathcal{H}$ (and similarly for the transversal one, corresponding to $\Theta = \mathcal{H} \times \{0\}$). A self-adjoint linear relation in a Krein (or Pontryagin) space may have an empty resolvent set (see e.g. [5, Example 3.7]). However, if there exists at least one self-adjoint extension of A, say $\widetilde{A}$, whose resolvent set res $\widetilde{A}$ is nonempty, then there exists an OBT for A^* such that $\widetilde{A} = A_0$.

Let A be a closed symmetric operator as above. Let $\mathfrak{N}_z(A^*) := \ker(A^* - z)$, $z \in \mathbb{C}$, denote the eigenspace of a linear relation A^* (and similarly for other linear relations and operators). Let $\widehat{\mathfrak{N}}_z(A^*)$ be the set of the pairs $(f_z, z f_z)$ with $f_z \in \mathfrak{N}_z(A^*)$. Let also π_1 denote the orthogonal projection in the Hilbert sum of a Hilbert space with itself onto the first factor. Assume that the resolvent set res $A_0 \neq \emptyset$. The γ-field γ and the Weyl function M corresponding to the OBT $(\mathcal{H}, \Gamma_0, \Gamma_1)$ for A^* are bounded operator valued functions defined by [9, Definition 7], [21, Definition 2.6]

$$\gamma(z) := \pi_1 \widehat{\gamma}(z), \quad \widehat{\gamma}(z) := (\Gamma_0 \,|_{\widehat{\mathfrak{N}}_z(A^*)})^{-1}, \quad M(z) := \Gamma_1 \widehat{\gamma}(z)$$

for $z \in \operatorname{res} A_0$. Then the resolvent of a closed proper extension A_Θ, i.e. such that Θ is closed, is represented by the Krein–Naimark resolvent formula (see e.g. [9, Theorem 4], [8, Theorem 2.1])

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \gamma(z)(\Theta - M(z))^{-1} \gamma(\overline{z})^*$$

for $z \in \operatorname{res} A_0 \cap \operatorname{res} A_\Theta$. Moreover, $z \in \operatorname{res} A_\Theta$ iff $0 \in \operatorname{res}(\Theta - M(z))$.

Let $\mathfrak{H} = (\mathfrak{H}, [\cdot,\cdot]_{\mathfrak{H}})$ be a Krein (or in particular Pontryagin) space, let $\mathcal{H} = (\mathcal{H}, \langle \cdot,\cdot \rangle_{\mathcal{H}})$ be a Hilbert space. Consider a linear relation $\Gamma \subseteq \mathfrak{H}^2 \times \mathcal{H}^2$. Let $\Gamma^{[+]}$ be its Krein space adjoint:

$$\Gamma^{[+]} := \{((h_\circ, h'_\circ), (g, g')) \in \mathcal{H}^2 \times \mathfrak{H}^2 \,| \\ \forall ((f, f'), (h, h')) \in \Gamma : [f, g']_{\mathfrak{H}} - [f', g]_{\mathfrak{H}} = \langle h, h'_\circ \rangle_{\mathcal{H}} - \langle h', h_\circ \rangle_{\mathcal{H}}\}.$$

Then Γ is said to be an isometric (resp. unitary) linear relation if the inverse linear relation $\Gamma^{-1} \subseteq \Gamma^{[+]}$ (resp. $\Gamma^{-1} = \Gamma^{[+]}$). If Γ is unitary and additionally single-valued (i.e. an operator identified with its graph), then by [12, Corollary 2.4(i)] $\overline{\operatorname{ran}} \Gamma = \mathcal{H}^2$

(the closure of the range). If, moreover, $\operatorname{dom}\Gamma$ is closed, then also $\operatorname{ran}\Gamma$ is closed, and is given by $\operatorname{ran}\Gamma = \mathcal{H}^2$ ([12, Corollary 2.4(iii)]).

Throughout we use quite standard notation for the domain $\operatorname{dom} A$, the range $\operatorname{ran} A$, the kernel $\ker A$, and the multivalued part $\operatorname{mul} A$ of a linear relation A. The resolvent set of A is denoted by $\operatorname{res} A$, the point spectrum by $\sigma_p(A)$.

3. THE A-MODEL FOR FINITE RANK PERTURBATIONS

Let $\mathfrak{H}_{n+1} \subseteq \mathfrak{H}_n$, $n \in \mathbb{Z}$, be the scale of Hilbert spaces associated with a lower semibounded self-adjoint operator L defined in the reference Hilbert space $\mathfrak{H}_0$ with domain $\operatorname{dom} L = \mathfrak{H}_2$. The scalar product in $\mathfrak{H}_n$ is defined via the scalar product $\langle\cdot,\cdot\rangle_0$ in $\mathfrak{H}_0$ by scaling according to

$$\langle\cdot,\cdot\rangle_n := \langle b_n(L)^{1/2}\cdot, b_n(L)^{1/2}\cdot\rangle_0\,, \quad b_n(L) := (L - z_1)^n.$$

The number $z_1 \in \operatorname{res} L \cap \mathbb{R}$ is fixed and referred to as the model parameter. Let us mention that the above definition of the $\mathfrak{H}_n$-scalar product allows us to avoid extra technicalities arising when, for example, one chooses $b_n(L)$ as the product of $(L - z_j)$ for $j \in \{1, \ldots, n\}$ for not necessarily identical model parameters z_j, as is done in [15] (where $z_j = -a_j$), or when, on top of that, one assumes L not necessarily semibounded, in which case one should put $|L|$ in $b_n(L)$ instead of L. On the other hand, our definition of the scalar product predefines the inner structure of the model space (to be defined later); namely, it is shown in [15, Theorem 3.2(iii)] for $d = 1$ that the present choice of the model parameters (i.e. $a_j = -z_1$ for all j) leads to an indefinite inner product space, as the model space. Let us moreover advertise that the current definition of the unitary operator $b_n(L)^{1/2}$ (from $\mathfrak{H}_n$ to $\mathfrak{H}_0$) is not allowed in the peak model [26], which is a purely Hilbert space model (cf. [15, Theorem 3.2(ii)]).

To $L = L_0$ one associates an operator $L_n := L\,|_{\,\mathfrak{H}_{n+2}}$ in $\mathfrak{H}_n$. Then L_n is self-adjoint in $\mathfrak{H}_n$, and moreover $L_{n+1} \subset L_n$ and $\operatorname{res} L_n = \operatorname{res} L$ (cf. Section 5). For notational simplicity we drop-off the subscript when no confusion can arise.

Let us fix $m \in \mathbb{N}$. Let $L_{\max}$ denote the triplet adjoint of $L_{\min}$ for the Hilbert triple $\mathfrak{H}_m \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-m}$; see also [15, Theorem 2.1], [26, Definition 3.1], [24, Proposition 4.2]. The operator $L_{\max}$ extends L_{-m+2} to (the direct sum)

$$\operatorname{dom}(L_{\max}) = \mathfrak{H}_{-m+2} \dotplus \mathfrak{N}_z(L_{\max}), \quad z \in \operatorname{res} L.$$

$\mathfrak{N}_z(L_{\max})$ is the linear span of the singular elements $\{g_\sigma(z) \in \mathfrak{H}_{-m} \smallsetminus \mathfrak{H}_{-m+1}\}$, each being defined so that $b_m(L)^{-1}g_\sigma(z) \in \mathfrak{H}_m \smallsetminus \mathfrak{H}_{m+1}$ is a deficiency element of the adjoint $L^*_{\min}$ in $\mathfrak{H}_m$ of a densely defined, closed, symmetric operator $L_{\min}$ in $\mathfrak{H}_m$ with defect numbers (d, d). Let us recall that the domain of $L_{\min}$ is parametrized via the family of linearly independent functionals $\{\varphi_\sigma \in \mathfrak{H}_{-m-2} \smallsetminus \mathfrak{H}_{-m-1}\}$ according to $\langle\varphi_\sigma, f\rangle = 0$ for $f \in \mathfrak{H}_{m+2}$; the duality pairing $\langle\cdot,\cdot\rangle$ between $\mathfrak{H}_{-m-2}$ and $\mathfrak{H}_{m+2}$ is defined via the $\mathfrak{H}_0$-scalar product in a usual way (cf. [2, Eq. (1.17)]). In the sequel we also use the vector notation $\langle\varphi,\cdot\rangle = (\langle\varphi_\sigma,\cdot\rangle)\colon \mathfrak{H}_{m+2} \to \mathbb{C}^d$, and similarly for other duality pairings. In terms of the functionals $\{\varphi_\sigma\}$ the eigenvectors of $L_{\max}$ are then given (in the generalized sense) by $g_\sigma(z) := (L - z)^{-1}\varphi_\sigma$.

As the space $\mathfrak{H}_{-m}$ in which $L_{\max}$ acts is too large, following the lines of [15] one further considers $L_{\max}$ in a finite-dimensional extension of $\mathfrak{H}_m$, referred to as an intermediate (or model) space. We now discuss the construction of the space in more detail.

Consider an md-dimensional linear space

$$\mathfrak{K}_{\mathrm{A}} := \operatorname{span}\{h_\alpha \,|\, \alpha = (\sigma, j) \in \mathcal{S} \times J\}, \quad J := \{1, 2, \dots, m\}$$

($\mathcal{S}$ is an index set of dimension d) spanned by the elements

$$h_{\sigma j} := (L - z_1)^{-j} \varphi_\sigma \in \mathfrak{H}_{-m-2+2j} \setminus \mathfrak{H}_{-m-1+2j}.$$

Note that $h_{\sigma 1} = g_\sigma(z_1) \in \mathfrak{N}_{z_1}(L_{\max})$. An element $k \in \mathfrak{K}_{\mathrm{A}} \subseteq \mathfrak{H}_{-m}$ is thus of the form

$$k = \sum_\alpha d_\alpha(k) h_\alpha, \quad d_\alpha(k) \in \mathbb{C}.$$

Since the system $\{h_\alpha\}$ is linearly independent, the Gram matrix

$$\widetilde{\mathcal{G}}_{\mathrm{A}} = ([\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\alpha\alpha'}) \in [\mathbb{C}^{md}], \quad [\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\alpha\alpha'} := \langle h_\alpha, h_{\alpha'} \rangle_{-m}$$

is positive definite, and one establishes a bijective correspondence

$$\mathfrak{K}_{\mathrm{A}} \ni k \leftrightarrow d(k) = (d_\alpha(k)) \in \mathbb{C}^{md}.$$

Observe that $\mathfrak{K}_{\mathrm{A}} \cap \mathfrak{H}_{m-1} = \{0\}$.

Define a linear space

$$\mathcal{H}_{\mathrm{A}} := (\mathfrak{H}_m \dotplus \mathfrak{K}_{\mathrm{A}}, [\cdot,\cdot]_{\mathrm{A}})$$

with an indefinite metric

$$[f + k, f' + k']_{\mathrm{A}} := \langle f, f' \rangle_m + \langle d(k), \mathcal{G}_{\mathrm{A}} d(k') \rangle_{\mathbb{C}^{md}}$$

for $f, f' \in \mathfrak{H}_m$; $k, k' \in \mathfrak{K}_{\mathrm{A}}$. An Hermitian matrix $\mathcal{G}_{\mathrm{A}} = ([\mathcal{G}_{\mathrm{A}}]_{\alpha\alpha'}) \in [\mathbb{C}^{md}]$ is referred to as the Gram matrix of the A-model. The model space $\mathcal{H}_{\mathrm{A}}$ is a Hilbert space if $\mathcal{G}_{\mathrm{A}} \geq 0$ and a Pontryagin space otherwise. Let also

$$\mathcal{H}'_{\mathrm{A}} := (\mathfrak{H}_m \oplus \mathbb{C}^{md}, [\cdot,\cdot]'_{\mathrm{A}})$$

with an indefinite metric

$$[(f, \xi), (f', \xi')]'_{\mathrm{A}} := \langle f, f' \rangle_m + \langle \xi, \mathcal{G}_{\mathrm{A}} \xi' \rangle_{\mathbb{C}^{md}}$$

for (f, ξ), $(f', \xi') \in \mathfrak{H}_m \oplus \mathbb{C}^{md}$. The isometric isomorphism (unitary operator) from $\mathcal{H}_{\mathrm{A}}$ onto $\mathcal{H}'_{\mathrm{A}}$, realized via the above established bijective correspondence $\mathfrak{K}_{\mathrm{A}} \leftrightarrow \mathbb{C}^{md}$, is denoted by U_{A}.

The construction of nontrivial extensions to $\mathcal{H}_{\mathrm{A}}$ of $L_{\min}$ relies upon the following lemma; cf. [15, Eq. (2.3)].

Lemma 3.1. *The restriction to $\mathcal{H}_{\mathrm{A}}$ of $L_{\max}$ is the operator $A_{\max}$ given by*

$$\operatorname{dom} A_{\max} = \Big\{ f^{\#} + h_{m+1}(c) + k \mid f^{\#} \in \mathfrak{H}_{m+2}\,,\, k \in \mathfrak{K}_{\mathrm{A}}\,, h_{m+1}(c) := \sum_{\sigma} c_\sigma h_{\sigma,m+1}, \\ c = (c_\sigma) \in \mathbb{C}^d, h_{\sigma,m+1} := b_{m+1}(L)^{-1}\varphi_\sigma \in \mathfrak{H}_m \smallsetminus \mathfrak{H}_{m+1} \Big\},$$

$$A_{\max}(f^{\#} + h_{m+1}(c) + k) = Lf^{\#} + z_1 h_{m+1}(c) + \widetilde{k}, \quad \widetilde{k} \in \mathfrak{K}_{\mathrm{A}}, \\ d(\widetilde{k}) := \mathfrak{M}_d d(k) + \eta(c), \quad \eta(c) := (\delta_{jm} c_\sigma) \in \mathbb{C}^{md},$$

where the matrix $\mathfrak{M}_d := \mathfrak{M} \oplus \ldots \oplus \mathfrak{M}$ (d times) is the matrix direct sum of d matrices $\mathfrak{M} = (\mathfrak{M}_{jj'}) \in [\mathbb{C}^m]$ defined by

$$\mathfrak{M}_{jj'} := \delta_{jj'} z_1 + \delta_{j+1,j'}, \quad j \in J \smallsetminus \{m\}, j' \in J$$

and $\mathfrak{M}_{mj'} := \delta_{j'm} z_1$, $j' \in J$. For $m = 1$, one puts $\mathfrak{M} := z_1$.

Proof. By definition, the action of $L_{\max}$ on $f + k \in \mathfrak{H}_m \dotplus \mathfrak{K}_{\mathrm{A}}$ is given (in the generalized sense) by

$$\begin{aligned} L_{\max}(f + k) &= Lf + \sum_{\sigma} z_1 d_{\sigma 1}(k) h_{\sigma 1} + \sum_{\sigma} \sum_{j=2}^{m} d_{\sigma j}(k) L(L - z_1)^{-j} \varphi_\sigma \\ &= Lf + z_1 k + \sum_{\sigma} \sum_{j=1}^{m-1} d_{\sigma, j+1}(k) h_{\sigma j}. \end{aligned}$$

Now $Lf \in \mathfrak{H}_{m-2}$, thus the range restriction $L_{\max}(f + k) \in \mathfrak{H}_m \dotplus \mathfrak{K}_{\mathrm{A}}$ implies that f is of the form $f^{\#} + g$ for some $f^{\#} \in \mathfrak{H}_{m+2}$ and $g \in \mathfrak{H}_m$ such that $Lg \in \mathcal{H}_{\mathrm{A}}$. By noting that $Lh_{m+1}(c) = z_1 h_{m+1}(c) + h_m(c)$ ($h_m(c) \in \mathfrak{K}_{\mathrm{A}}$ is defined similar to $h_{m+1}(c)$) for an arbitrary $c \in \mathbb{C}^d$, one concludes that $g = h_{m+1}(c)$, and the required result follows. ☐

Now we state the main realization theorem in the A-model.

Theorem 3.2. *Assume that an invertible Hermitian matrix $\mathcal{G}_{\mathrm{A}}$ satisfies the commutation relation*

$$\mathcal{G}_{\mathrm{A}} \mathfrak{M}_d = \mathfrak{M}_d^* \mathcal{G}_{\mathrm{A}}. \tag{3.1}$$

Then the triple $(\mathbb{C}^d, \Gamma_0^{\mathrm{A}}, \Gamma_1^{\mathrm{A}})$, where $\Gamma^{\mathrm{A}} \colon f \mapsto (\Gamma_0^{\mathrm{A}} f, \Gamma_1^{\mathrm{A}} f)$ from $\operatorname{dom} A_{\max}$ to $\mathbb{C}^d \times \mathbb{C}^d$ is defined by

$$\begin{aligned} \Gamma_0^{\mathrm{A}}(f^{\#} + h_{m+1}(c) + k) &:= c, \\ \Gamma_1^{\mathrm{A}}(f^{\#} + h_{m+1}(c) + k) &:= \langle \varphi, f^{\#} \rangle - [\mathcal{G}_{\mathrm{A}} d(k)]_m \end{aligned}$$

with

$$[\mathcal{G}_{\mathrm{A}} d(k)]_m := ([\mathcal{G}_{\mathrm{A}} d(k)]_{\sigma m}) \in \mathbb{C}^d$$

and $f^{\#} \in \mathfrak{H}_{m+2}$, $k \in \mathfrak{K}_{\mathrm{A}}$, $c \in \mathbb{C}^d$, is an OBT for the adjoint $A_{\min}^* = A_{\max}$ of a densely defined, closed, symmetric operator $A_{\min} = A_{\max}|_{\ker \Gamma^{\mathrm{A}}}$ in $\mathcal{H}_{\mathrm{A}}$.

Moreover, for a (closed) linear relation Θ *in* $\mathbb{C}^d$*, a proper extension* A_Θ *of* $A_{\min}$ *is the restriction of* $A_{\max}$ *to the set of* $f \in \operatorname{dom} A_{\max}$ *such that* $\Gamma^{\mathrm{A}} f \in \Theta$*. The Krein–Naimark resolvent formula reads*

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \gamma_{\mathrm{A}}(z)(\Theta - M_{\mathrm{A}}(z))^{-1}\gamma_{\mathrm{A}}(\overline{z})^*$$

for $z \in \operatorname{res} A_0 \cap \operatorname{res} A_\Theta$*. The resolvent of a distinguished self-adjoint extension* $A_0 := A_{\{0\}\times\mathbb{C}^d}$ *is given by*

$$(A_0 - z)^{-1} = U_{\mathrm{A}}^*[(L - z)^{-1} \oplus (\mathfrak{M}_d - z)^{-1}]U_{\mathrm{A}}$$

for $z \in \operatorname{res} A_0 = \operatorname{res} L \setminus \{z_1\}$*. The* γ*-field* γ_{A} *and the Weyl function* M_{A} *corresponding to* $(\mathbb{C}^d, \Gamma_0^{\mathrm{A}}, \Gamma_1^{\mathrm{A}})$ *are given by*

$$\gamma_{\mathrm{A}}(z)\mathbb{C}^d = \mathfrak{N}_z(A_{\max}) = \Big\{ \sum_\sigma c_\sigma F_\sigma(z) \,|\, c_\sigma \in \mathbb{C} \Big\}, \quad F_\sigma(z) := \frac{g_\sigma(z)}{(z - z_1)^m}$$

and

$$M_{\mathrm{A}}(z) = q(z) + r(z) \quad \textit{on } \mathbb{C}^d$$

for $z \in \operatorname{res} A_0$*. The Krein Q-function* q *of* $L_{\min}$ *is defined by*

$$q(z) = ([q(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [q(z)]_{\sigma\sigma'} := (z - z_1)\,\langle \varphi_\sigma, (L - z)^{-1} h_{\sigma',m+1} \rangle$$

for $z \in \operatorname{res} L$*, and the generalized Nevanlinna function* r *is defined by*

$$r(z) = ([r(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [r(z)]_{\sigma\sigma'} := -\sum_j \frac{[\mathcal{G}_{\mathrm{A}}]_{\sigma m,\sigma' j}}{(z - z_1)^{m-j+1}}$$

for $z \in \mathbb{C} \setminus \{z_1\}$*.*

Proof. By Lemma 3.1, the boundary form of $A_{\max}$ is given by

$$[f, g]_{A_{\max}} = \langle d(k), (\mathcal{G}_{\mathfrak{M}} - \mathcal{G}_{\mathfrak{M}}^*) d(k') \rangle_{\mathbb{C}^{md}} + \langle \Gamma_0^{\mathrm{A}} f, \Gamma_1^{\mathrm{A}} g \rangle_{\mathbb{C}^d} - \langle \Gamma_1^{\mathrm{A}} f, \Gamma_0^{\mathrm{A}} g \rangle_{\mathbb{C}^d}$$

with $\mathcal{G}_{\mathfrak{M}} := \mathcal{G}_{\mathrm{A}}\mathfrak{M}_d$, where $f = f^\# + h_{m+1}(c) + k \in \operatorname{dom} A_{\max}$, $g = g^\# + h_{m+1}(c') + k' \in \operatorname{dom} A_{\max}$, $f^\#, g^\# \in \mathfrak{H}_{m+2}$, $c, c' \in \mathbb{C}^d$, $k, k' \in \mathfrak{K}_{\mathrm{A}}$. Assuming that

$$\ker \mathcal{G}_{\mathrm{A}} = \{0\} \quad \text{and} \quad \mathfrak{M}_d^* \mathcal{G}_{\mathrm{A}} \mathbb{C}^{md} \subseteq \operatorname{ran} \mathcal{G}_{\mathrm{A}}$$

the adjoint $A_{\min} := A_{\max}^*$ in $\mathcal{H}_{\mathrm{A}}$ is given by

$$\operatorname{dom} A_{\min} = \ker \Gamma^{\mathrm{A}},$$
$$A_{\min}(f^\# + k) = L f^\# + \sum_\alpha [\mathcal{G}_{\mathrm{A}}^{-1} \mathfrak{M}_d^* \mathcal{G}_{\mathrm{A}} d(k)]_\alpha h_\alpha$$

and hence the boundary form of $A_{\min}$ reads

$$[f, g]_{A_{\min}} = \langle d(k), (\mathcal{G}_{\mathfrak{M}}^* - \mathcal{G}_{\mathfrak{M}}) d(k') \rangle_{\mathbb{C}^{md}}$$

with $f=f^{\#}+k\in\operatorname{dom}A_{\min}$ and $g=g^{\#}+k'\in\operatorname{dom}A_{\min}$ as above. One verifies that the adjoint $A^*_{\min}=A_{\max}$, and hence $A_{\max}$ is closed in $\mathcal{H}_{\mathrm{A}}$.

If (3.1) holds, the boundary form of $A^*_{\min}$ satisfies an abstract Green identity. Thus, since Γ^{A} is single-valued and surjective, the triple $(\mathbb{C}^d,\Gamma^{\mathrm{A}}_0,\Gamma^{\mathrm{A}}_1)$ is an OBT for $A^*_{\min}$.

The eigenvalue equation for $A_{\max}$ yields

$$f^{\#}=(z-z_1)(L-z)^{-1}h_{m+1}(c),\quad d(k)=-(\mathfrak{M}_d-z)^{-1}\eta(c) \tag{3.2}$$

for $f^{\#}+h_{m+1}(c)+k\in\operatorname{dom}A_{\max}$ as above. Now

$$[(\mathfrak{M}_d-z)^{-1}\eta(c)]_{\sigma j}=\sum_{\sigma'}[(\mathfrak{M}_d-z)^{-1}]_{\sigma j,\sigma' m}c_{\sigma'}$$

with $c=(c_\sigma)\in\mathbb{C}^d$ and with

$$[(\mathfrak{M}_d-z)^{-1}]_{\sigma j,\sigma' m}=\delta_{\sigma\sigma'}[(\mathfrak{M}-z)^{-1}]_{jm},\quad [(\mathfrak{M}-z)^{-1}]_{jm}=\frac{-1}{(z-z_1)^{m-j+1}}.$$

Thus, by noting that

$$(L-z)^{-1}(L-z_1)^{-m}+\sum_j(L-z_1)^{-j}(z-z_1)^{-m+j-1}=(L-z)^{-1}(z-z_1)^{-m}$$

one concludes that the eigenvector $f^{\#}+h_{m+1}(c)+k\in\mathfrak{N}_z(A_{\max})$ is given as stated in the theorem.

Finally, the Weyl function

$$M_{\mathrm{A}}(z)c=\langle\varphi,f^{\#}\rangle-[\mathcal{G}_{\mathrm{A}}d(k)]_m$$

for $f^{\#}$ and k as in (3.2). The first term on the right-hand side defines $q(z)c$ and the second term defines $r(z)c$. □

Let us mention that the Q-function q is actually the Weyl function associated with a certain boundary triple for the adjoint $L^*_{\min}$ in $\mathfrak{H}_m$; see Corollary 7.4 below. While q is a Nevanlinna function, r is a generalized Nevanlinna function, and the Nevanlinna class [3, 7] depends on the particular choice of the Gram matrix $\mathcal{G}_{\mathrm{A}}$.

The matrix $\mathcal{G}_{\mathfrak{M}}:=\mathcal{G}_{\mathrm{A}}\mathfrak{M}_d$ is Hermitian iff

$$[\mathcal{G}_{\mathrm{A}}]_{\sigma j,\sigma' j'}=0,\quad [\mathcal{G}_{\mathrm{A}}]_{\sigma j,\sigma' m}=\overline{[\mathcal{G}_{\mathrm{A}}]_{\sigma' m,\sigma j}}=[\mathcal{G}_{\mathrm{A}}]_{\sigma,j+1;\sigma',m-1} \tag{3.3}$$

for $j\in J\setminus\{m\}$, $j'\in\{1,\dots,m-j\}$ and $m\geq 2$. For $m=1$, however, the matrix $\mathcal{G}_{\mathfrak{M}}=z_1\mathcal{G}_{\mathrm{A}}$ is automatically Hermitian.

Due to (3.3), several remarks are in order. First one verifies that r is symmetric with respect to the real axis, that is, $r(z)^*=r(\overline{z})$, because $[\mathcal{G}_{\mathrm{A}}]_{\sigma m,\sigma' j}=[\mathcal{G}_{\mathrm{A}}]_{\sigma j,\sigma' m}$ ($j\in J$) by (3.3). Note that $q(z)^*=q(\overline{z})$ is clear from the definition. Next, one observes that the Gram matrix $\widetilde{\mathcal{G}}_{\mathrm{A}}$ does not satisfy (3.1) for $m\geq 2$, because $[\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\sigma 1,\sigma 1}>0$. This shows that, in order use Theorem 3.2 for $m\geq 2$, one cannot define the Gram matrix of the A-model in a way that is done in the peak model.

Remark 3.3. Let us recall that in the peak model the parameters $\{a_j\}$ are all necessarily distinct. However, putting $a_j = -z_1 + \delta_{j-1}$ for $\delta_j \neq 0$ and $j \in J \setminus \{1\}$ and $m \geq 2$, and formally taking the limits $\delta_j \to \delta_{j-1}$, as well as $\delta_1 \to 0$, one can show by induction that the Q-function associated with the Gram matrix $\mathcal{G}$ of the peak model approaches r, up to $O(\delta_1)$, with $[\mathcal{G}_{\mathrm{A}}]_{\sigma m,\sigma' j} = [\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\sigma m,\sigma' j}$. Notice that $[\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\sigma m,\sigma' j}$, with $m \geq 2$, satisfies the second relation in (3.3). On the other hand, taking the above described limits, the matrix element $\mathcal{G}_{\sigma 1,\sigma' 2} = [\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\sigma 1,\sigma' 1} + O(\delta_1)$, so the requirement that $\mathcal{G}$ must be diagonal in j – which is essential in applying the extension theory of symmetric operators in the peak model – fails for $m \geq 2$. For $m = 1$, both models produce the same Nevanlinna function $r(z) = \mathcal{G}_{\mathrm{A}}/(z_1 - z)$, provided that $\mathcal{G}_{\mathrm{A}} = \widetilde{\mathcal{G}}_{\mathrm{A}} (\in [\mathbb{C}^d])$.

Example 3.4. We briefly demonstrate by a concrete example the case when the eigenvectors $\{g_\sigma(z)\}$ of $L_{\max}$ are not orthogonal for distinct z, that is, the example when the peak model cannot be applied. We consider the two-particle Rashba spin-orbit-coupled operator L in $\mathfrak{H}_0 = \mathrm{L}^2(\mathbb{R}^6) \otimes \mathbb{C}^4$ with point-interaction between the two cold atoms [23]. The operator is nonseparable in the center-of-mass coordinate system $(x, X) \in \mathbb{R}^3 \times \mathbb{R}^3$ (x is the distance between the two atoms, X is the center-of-mass coordinate) for a nonzero spin-orbit-coupling strength ε. The interaction is modeled by the Dirac distribution $\varphi_\sigma \in \mathfrak{H}_{-4} \setminus \mathfrak{H}_{-3}$ concentrated at $x = 0$: $\langle \varphi_\sigma, f \rangle = N_\sigma f_\sigma(0, X)$, $f = \sum_\sigma f_\sigma \otimes |\sigma\rangle \in \mathfrak{H}_4$, $N_\sigma > 0$ is the normalization constant, $\{|\sigma\rangle\}$ is an orthonormal basis of $\mathbb{C}^4$. Thus we have $m = 2$ and $d = 4$. For simplicity, we assume that ε is negligibly small. In this regime L approximates, up to $O(\varepsilon)$, the operator $(-2\Delta_x - \frac{1}{2}\Delta_X) \otimes I_{\mathbb{C}^4}$ (cf. [1, Eq. (8)]), where Δ_x (resp. Δ_X) is the Laplacian in $x \in \mathbb{R}^3$ (resp. $X \in \mathbb{R}^3$). Then the distribution $g_\sigma(z) \in \mathfrak{H}_{-2} \setminus \mathfrak{H}_{-1}$ admits a relatively simple form

$$g_\sigma(z) = -\frac{N_\sigma}{(2\pi)^3} \frac{z K_2(|\cdot - W_0|\sqrt{-z})}{|\cdot - W_0|^2} \otimes |\sigma\rangle \,, \quad W_0 = (0, X), \quad z \in \mathbb{C} \setminus [0, \infty]$$

where K_2 is the Macdonald function of second order. Because $m = 2$, it suffices to have in the (peak) model two distinct model parameters z_1, $z_2 < 0$ (or else a_1, $a_2 > 0$). Because now $b_2(L) = (L - z_1)(L - z_2)$, the Gram matrix element $\mathcal{G}_{\sigma 1,\sigma 2}$ reads

$$\begin{aligned}
\mathcal{G}_{\sigma 1,\sigma 2} &:= \langle g_\sigma(z_1), g_\sigma(z_2) \rangle_{-2} = \langle g_\sigma(z_1), b_2(L)^{-1} g_\sigma(z_2) \rangle_0 \\
&= \Big\langle \varphi_\sigma, [(L - z_1)(L - z_2)]^{-2} \varphi_\sigma \Big\rangle = \Big\langle \varphi_\sigma, \frac{\partial^2}{\partial u\, \partial v} [(L - u)(L - v)]^{-1} \varphi_\sigma \,|_{\substack{u=z_1 \\ v=z_2}} \Big\rangle \\
&= \Big\langle \varphi_\sigma, \frac{\partial^2}{\partial u\, \partial v} \frac{g_\sigma(u) - g_\sigma(v)}{u - v} \,|_{\substack{u=z_1 \\ v=z_2}} \Big\rangle \\
&= -\frac{N_\sigma^2}{(2\pi)^3} \lim_{r \to 0} \frac{1}{r^2} \frac{\partial^2}{\partial u\, \partial v} \frac{u K_2(r\sqrt{-u}) - v K_2(r\sqrt{-v})}{u - v} \,|_{\substack{u=-a_1 \\ v=-a_2}} \\
&= \frac{N_\sigma^2}{(2\pi)^3 2^4} \frac{2 a_1 a_2 \log(a_1/a_2) - a_1^2 + a_2^2}{(a_2 - a_1)^3}
\end{aligned}$$

up to $O(\varepsilon^2)$ (a more accurate computation of $\mathcal{G}_{\sigma 1,\sigma 2}$ shows that the term $O(\varepsilon)$ vanishes).

4. PROJECTIONS

In the remaining part of the present paper we develop the A-model in the subspaces

$$\mathcal{H}'^{-}_{\mathrm{A}} := (\mathfrak{H}^{-}_{m} \oplus \mathbb{C}^{md}, [\cdot,\cdot]'_{\mathrm{A}}), \quad \mathcal{H}'^{+}_{\mathrm{A}} := (\mathfrak{H}^{+}_{m} \oplus \mathbb{C}^{md}, [\cdot,\cdot]'_{\mathrm{A}})$$

of $\mathcal{H}'_{\mathrm{A}}$, by assuming that the Hilbert space $\mathfrak{H}_m = \mathfrak{H}^{-}_{m} \oplus \mathfrak{H}^{+}_{m}$ is the Hilbert (orthogonal) sum of its subspaces $\mathfrak{H}^{\pm}_{m}$. The analogue of Theorem 3.2, in the case when $\mathfrak{H}^{\pm}_{n+1} \subseteq \mathfrak{H}^{\pm}_{n}$ ($\forall n \in \mathbb{Z}$) densely, is stated in Theorem 7.3. First we discuss the properties of the projections that we use later on, then we consider the restrictions to $\mathfrak{H}^{\pm}_{n}$ of L_n, and then finally we describe the min-max operators defined in $\mathcal{H}'^{\pm}_{\mathrm{A}}$. The principal difference between the case of the minimal operator $A_{\min}$ considered in $\mathcal{H}_{\mathrm{A}}$ and its analogue $A^{-}_{\min}$ (resp. $A^{+}_{\min}$) considered in $\mathcal{H}'^{-}_{\mathrm{A}}$ (resp. $\mathcal{H}'^{+}_{\mathrm{A}}$) is that $A^{-}_{\min}$ (resp. $A^{+}_{\min}$) becomes nondensely defined in general, that is, the corresponding maximal operator $A^{-}_{\max}$ (resp. $A^{+}_{\max}$) is a linear relation.

Let P^{-}_{n} be an orthogonal projection in $\mathfrak{H}_n$ onto a subspace $\mathfrak{H}^{-}_{n} \subseteq \mathfrak{H}_n$ and let $P^{+}_{n} := I_{\mathfrak{H}_n} - P^{-}_{n}$, an orthogonal projection in $\mathfrak{H}_n$ onto $\mathfrak{H}^{+}_{n} := (\mathfrak{H}^{-}_{n})^{\perp_{\mathfrak{H}_n}}$. Here and elsewhere the subscript in $\perp_{\mathfrak{H}_n}$ indicates with respect to which Hilbert space one takes the orthogonal complement.

Lemma 4.1. *P^{-}_{n} is an orthogonal projection in $\mathfrak{H}_n$ onto a subspace $\mathfrak{H}^{-}_{n}$ iff*

$$P^{-}_{0}(n) := b_n(L)^{1/2} P^{-}_{n} b_n(L)^{-1/2}$$

is an orthogonal projection in $\mathfrak{H}_0$ onto a subspace

$$\mathfrak{H}^{-}_{0}(n) := P^{-}_{0}(n)\mathfrak{H}_0 = b_n(L)^{1/2}\mathfrak{H}^{-}_{n}.$$

If this is the case, then

$$P^{+}_{0}(n) := I_{\mathfrak{H}_0} - P^{-}_{0}(n) = b_n(L)^{1/2} P^{+}_{n} b_n(L)^{-1/2}$$

is an orthogonal projection in $\mathfrak{H}_0$ onto a subspace

$$\mathfrak{H}^{+}_{0}(n) := \mathfrak{H}^{-}_{0}(n)^{\perp_{\mathfrak{H}_0}} = P^{+}_{0}(n)\mathfrak{H}_0 = b_n(L)^{1/2}\mathfrak{H}^{+}_{n}.$$

Proof. Because

$$P^{-}_{0}(n)^2 = b_n(L)^{1/2}(P^{-}_{n})^2 b_n(L)^{-1/2}$$

$P^{-}_{0}(n)$ is a projection iff so is P^{-}_{n}.

We show that the adjoint $P^{-}_{0}(n)^*$ of $P^{-}_{0}(n)$ in $\mathfrak{H}_0$ is given by

$$P^{-}_{0}(n)^* = b_n(L)^{1/2} P^{-\,*}_{n} b_n(L)^{-1/2} \tag{4.1}$$

on $\mathfrak{H}_0$, where $P^{-\,*}_{n}$ is the adjoint of P^{-}_{n} in $\mathfrak{H}_n$; then it follows that $P^{-}_{0}(n)$ is self-adjoint in $\mathfrak{H}_0$ iff so is P^{-}_{n} in $\mathfrak{H}_n$: The graph of the adjoint $P^{-}_{0}(n)^*$ in $\mathfrak{H}_0$ consists of $(y,x) \in \mathfrak{H}^2_0$ such that ($\forall u \in \mathfrak{H}_0$)

$$\langle u, x\rangle_0 = \langle P^{-}_{0}(n)u, y\rangle_0 .$$

Every u is of the form $u = b_n(L)^{1/2} f$ with some $f \in \mathfrak{H}_n$. Then

$$\langle u, x \rangle_0 = \langle f, b_n(L)^{-1/2} x \rangle_n$$

and

$$\langle P_0^-(n) u, y \rangle_0 = \langle P_n^- f, b_n(L)^{-1/2} y \rangle_n = \langle f, P_n^{-\,*} b_n(L)^{-1/2} y \rangle_n$$

from which the claim follows. The remaining statements are verified straightforwardly. □

The present lemma allows us to freely transfer between the $\mathfrak{H}_n$-space representation and the $\mathfrak{H}_0$-space representation. In particular $\mathfrak{H}_0^-(0) = \mathfrak{H}_0^-$, but in general $\mathfrak{H}_0^-(n) \neq \mathfrak{H}_0^-$ for $n \neq 0$. The equality holds for all n iff

$$P_n^- = b_n(L)^{-1/2} P_0^- b_n(L)^{1/2} \tag{4.2}$$

on $\mathfrak{H}_n$; in this case one would have $\mathfrak{H}_{n+l}^- = b_l(L)^{-1/2} \mathfrak{H}_n^-$ for $l \in \mathbb{N}_0$ (cf. Example 4.5). Moreover, $P_0^{\pm}(n) P_0^{\mp}(n+l) \neq 0$ in general. However, the product of projections vanishes for $l \in 2\mathbb{Z}$, provided that $P_{n+1}^- \subseteq P_n^-$; see Lemma 4.4 below.

Let $n \in \mathbb{Z}$, $l \in \mathbb{N}_0$ as above and let

$$\mathfrak{H}_{n,l}^- := P_n^- \mathfrak{H}_{n+l} .$$

Throughout we assume that

$$\mathfrak{H}_{n,l}^- \subseteq \mathfrak{H}_{n+l} .$$

Then

$$\mathfrak{H}_{n,l}^- = \mathfrak{H}_n^- \cap \mathfrak{H}_{n+l} .$$

The latter equality follows from the following observations. The set $\mathfrak{H}_n^- \cap \mathfrak{H}_{n+l}$ consists of $f \in \mathfrak{H}_n^-$ such that $f \in \mathfrak{H}_{n+l}$. Then $P_n^- f = f \in \mathfrak{H}_{n,l}^-$, and therefore $\mathfrak{H}_n^- \cap \mathfrak{H}_{n+l}$ is the set of $f \in \mathfrak{H}_{n,l}^-$ such that $f \in \mathfrak{H}_{n+l}$. By the above assumption this yields $\mathfrak{H}_n^- \cap \mathfrak{H}_{n+l} = \mathfrak{H}_{n,l}^-$.

Using the definition of the projection $P_0^-(n)$ it follows that

$$\mathfrak{H}_{n,l}^- = b_n(L)^{-1/2} \mathfrak{H}_l^-(n), \quad \mathfrak{H}_l^-(n) := P_0^-(n) \mathfrak{H}_l = \mathfrak{H}_0^-(n) \cap \mathfrak{H}_l$$

and hence $\mathfrak{H}_l^-(n)$ is a subset of $\mathfrak{H}_l$. Similarly, one defines $\mathfrak{H}_{n,l}^+ := P_n^+ \mathfrak{H}_{n+l}$ and $\mathfrak{H}_l^+(n) := P_0^+(n) \mathfrak{H}_l$, with the assumption $\mathfrak{H}_{n,l}^+ \subseteq \mathfrak{H}_{n+l}$. We note that

$$P_0^s(n) P_0^{s'}(n') = P_0^{s'}(n') P_0^s(n), \quad s, s' \in \{-,+\}, \quad n, n' \in \mathbb{Z}$$

and that

$$\mathfrak{H}_l^s(n) \cap \mathfrak{H}_0^{s'}(n') = \mathfrak{H}_l^s(n) \cap \mathfrak{H}_l^{s'}(n')$$

for $l \in \mathbb{N}_0$.

In general $\mathfrak{H}^-_{n,l} \neq \mathfrak{H}^-_{n+l}$, but the following holds.

Lemma 4.2. *Let $n \in \mathbb{Z}$, $l \in \mathbb{N}_0$. $\mathfrak{H}^-_{n,l}$ (resp. $\mathfrak{H}^-_l(n)$) is dense in $\mathfrak{H}^-_n$ (resp. $\mathfrak{H}^-_0(n)$). Moreover $\mathfrak{H}^-_{n,l} = \mathfrak{H}^-_{n+l}$ iff $\mathfrak{H}^-_{n+l}$ is dense in $\mathfrak{H}^-_n$, or equivalently, iff $P^-_{n+l} \subseteq P^-_n$ (in fact, if $\mathfrak{H}^-_{n+l} \subseteq \mathfrak{H}^-_n$ densely, then $\mathfrak{H}^+_{n+l} \subseteq \mathfrak{H}^+_n$ densely and $P^\pm_{n+l} \subseteq P^\pm_n$; conversely, if $P^-_{n+l} \subseteq P^-_n$, then $P^+_{n+l} \subseteq P^+_n$ and $\mathfrak{H}^\pm_{n+l} \subseteq \mathfrak{H}^\pm_n$ densely); if this is the case then*

$$P^-_0(n+l) \subseteq b_l(L)^{1/2} P^-_0(n) b_l(L)^{-1/2}$$

and hence $\mathfrak{H}^-_0(n+l) = b_l(L)^{1/2}\mathfrak{H}^-_l(n)$ (and similarly for $P^+_0(n+l)$ and $\mathfrak{H}^+_0(n+l)$).

Proof. The orthogonal complement $(\mathfrak{H}^-_{n,l})^{\perp_{\mathfrak{H}_n}}$ in $\mathfrak{H}_n$ of $\mathfrak{H}^-_{n,l}$ consists of all $g \in \mathfrak{H}_n$ such that $(\forall f \in \mathfrak{H}_{n+l})$

$$0 = \langle P^-_n f, g\rangle_n = \langle f, P^-_n g\rangle_n .$$

Because $\mathfrak{H}_{n+l}$ is dense in $\mathfrak{H}_n$, this implies $P^-_n g = 0$; hence $(\mathfrak{H}^-_{n,l})^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}^+_n$. This shows that $\mathfrak{H}^-_{n,l} \subseteq \mathfrak{H}^-_n$ densely in $\|\cdot\|_n$-norm. Similarly, the orthogonal complement $\mathfrak{H}^-_l(n)^{\perp_{\mathfrak{H}_0}}$ in $\mathfrak{H}_0$ of $\mathfrak{H}^-_l(n)$ consists of all $v \in \mathfrak{H}_0$ such that $(\forall u^- \in \mathfrak{H}^-_l(n))$ $0 = \langle u^-, v\rangle_0$. Now u^- is of the form $u^- = b_n(L)^{1/2} P^-_n f$ with some $f \in \mathfrak{H}_{n+l}$, so

$$\langle u, v\rangle_0 = \langle b_n(L)^{1/2} P^-_n f, v\rangle_0 = \langle P^-_n f, b_n(L)^{-1/2} v\rangle_n = \langle f, P^-_n b_n(L)^{-1/2} v\rangle_n .$$

This implies $P^-_n b_n(L)^{-1/2} v = 0$, and hence

$$\mathfrak{H}^-_l(n)^{\perp_{\mathfrak{H}_0}} = b_n(L)^{1/2}\mathfrak{H}^+_n = \mathfrak{H}^+_0(n).$$

One concludes that $\mathfrak{H}^-_l(n) \subseteq \mathfrak{H}^-_0(n)$ densely in $\|\cdot\|_0$-norm.

Next one shows that $\mathfrak{H}^-_{n+l} \subseteq \mathfrak{H}^-_n$ densely iff $P^-_{n+l} \subseteq P^-_n$. The orthogonal complement $(\mathfrak{H}^-_{n+l})^{\perp_{\mathfrak{H}_n}}$ in $\mathfrak{H}_n$ of $\mathfrak{H}^-_{n+l}$ is the set of all $g \in \mathfrak{H}_n$ such that $(\forall f \in \mathfrak{H}_{n+l})$ $0 = \langle P^-_{n+l} f, g\rangle_n$. If $P^-_{n+l} \subseteq P^-_n$, then one arrives at the previously considered case, namely, $(\mathfrak{H}^-_{n+l})^{\perp_{\mathfrak{H}_n}} = (\mathfrak{H}^-_{n,l})^{\perp_{\mathfrak{H}_n}}$; hence $\mathfrak{H}^-_{n+l} \subseteq \mathfrak{H}^-_n$ densely. Moreover, $P^-_{n+l} \subseteq P^-_n$ implies that also $\mathfrak{H}^+_{n+l} \subseteq \mathfrak{H}^+_n$ densely: $(\mathfrak{H}^+_{n+l})^{\perp_{\mathfrak{H}_n}}$ is the set of all $g \in \mathfrak{H}_n$ such that $(\forall f \in \mathfrak{H}_{n+l})$

$$0 = \langle P^+_{n+l} f, g\rangle_n = \langle f, g\rangle_n - \langle P^-_{n+l} f, g\rangle_n ,$$

but

$$\langle P^-_{n+l} f, g\rangle_n = \langle P^-_n f, g\rangle_n = \langle f, P^-_n g\rangle_n ,$$

so

$$0 = \langle P^+_{n+l} f, g\rangle_n = \langle f, P^+_n g\rangle_n .$$

This shows $(\mathfrak{H}^+_{n+l})^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}^-_n$. Conversely, $(\mathfrak{H}^-_{n+l})^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}^+_n$ implies that $(\forall f \in \mathfrak{H}_{n+l})$ $(\forall g \in \mathfrak{H}_n)$

$$0 = \langle P^-_{n+l} f, P^+_n g\rangle_n = \langle P^+_n P^-_{n+l} f, g\rangle_n$$

hence $P^+_n P^-_{n+l} = 0$. On the other hand, $(\mathfrak{H}^-_{n+l})^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}^+_n$ also implies that $(\mathfrak{H}^+_{n+l})^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}^-_n$: $(\mathfrak{H}^+_{n+l})^{\perp_{\mathfrak{H}_n}}$ is the set of all $g \in \mathfrak{H}_n$ such that $(\forall f \in \mathfrak{H}_{n+l})$

$$0 = \langle P^+_{n+l} f, g\rangle_n = \langle f, g\rangle_n - \langle P^-_{n+l} f, g\rangle_n .$$

Now

$$\langle P_{n+l}^- f, g\rangle_n = \langle P_{n+l}^- f, P_n^- g\rangle_n + \langle P_{n+l}^- f, P_n^+ g\rangle_n$$

and

$$\langle P_{n+l}^- f, P_n^+ g\rangle_n = \langle P_n^+ P_{n+l}^- f, g\rangle_n = 0,$$

so

$$0 = \langle P_{n+l}^+ f, g\rangle_n = \langle f, g\rangle_n - \langle P_{n+l}^- f, P_n^- g\rangle_n = \langle P_{n+l}^- f, g\rangle - \langle P_{n+l}^- f, P_n^- g\rangle_n .$$

As a result $(\mathfrak{H}_{n+l}^+)^{\perp_{\mathfrak{H}_n}}$ is the set of all $g \in \mathfrak{H}_n$ such that $(\forall f^- \in \mathfrak{H}_{n+l}^-)\ 0 = \langle f^-, P_n^+ g\rangle_n$. Because by hypothesis $\mathfrak{H}_{n+l}^-$ is dense in $\mathfrak{H}_n^-$, this shows $(\mathfrak{H}_{n+l}^+)^{\perp_{\mathfrak{H}_n}} = \mathfrak{H}_n^-$, as claimed. Sequentially, $(\forall f \in \mathfrak{H}_{n+l}\, \forall g \in \mathfrak{H}_n)$

$$0 = \langle P_{n+l}^+ f, P_n^- g\rangle_n = \langle P_n^- P_{n+l}^+ f, g\rangle_n$$

and hence $P_n^- P_{n+l}^+ = 0$. This together with $P_n^+ P_{n+l}^- = 0$ implies that $P_{n+l}^\pm \subseteq P_n^\pm$.

If $P_{n+l}^- \subseteq P_n^-$ then $\mathfrak{H}_{n,l}^- = \mathfrak{H}_{n+l}^-$ by definition. Assuming the converse, again by definition one gets that $P_n^- \mathfrak{H}_{n+l} = P_{n+l}^- \mathfrak{H}_{n+l}$, i.e. $P_n^- \mid_{\mathfrak{H}_{n+l}} = P_{n+l}^-$. This shows that $\mathfrak{H}_{n,l}^- = \mathfrak{H}_{n+l}^-$ iff $\mathfrak{H}_{n+l}^-$ is dense in $\mathfrak{H}_n^-$, or equivalently, iff $P_{n+l}^- \subseteq P_n^-$.

Using $P_{n+l}^- \subseteq P_n^-$, for $u \in \mathfrak{H}_0$

$$\begin{aligned} P_0^-(n+l)u &= b_{n+l}(L)^{1/2} P_{n+l}^- b_{n+l}(L)^{-1/2} u \\ &= b_{n+l}(L)^{1/2} P_n^- b_{n+l}(L)^{-1/2} u \\ &= b_l(L)^{1/2} P_0^-(n) b_l(L)^{-1/2} u \end{aligned}$$

and this completes the proof of the lemma. □

Example 4.3. Let $\mathrm{H}^n = W_2^n(\mathbb{R}^\nu)$, $\nu \in \mathbb{N}$, be the Sobolev space. Then we have $\mathrm{L}^2 = \mathrm{L}^2(\mathbb{R}^\nu) = \mathrm{H}^0$. Let L be such that

$$\mathfrak{H}_n = b_n(L)^{-1/2}(\mathrm{L}^2 \otimes \mathbb{C}^4) = \mathrm{H}^n \otimes \mathbb{C}^4, \quad n \in \mathbb{Z}$$

and

$$P_n^-(\mathrm{H}^n \otimes \mathbb{C}^4) = \mathrm{H}^n \otimes \mathbb{C}^1 = \mathfrak{H}_n^-, \quad P_n^+(\mathrm{H}^n \otimes \mathbb{C}^4) = \mathrm{H}^n \otimes \mathbb{C}^3 = \mathfrak{H}_n^+.$$

Then $P_{n+1}^- \subseteq P_n^-$, and similarly for P_n^+. The subspaces

$$\mathfrak{H}_0^-(n) = b_n(L)^{1/2}(\mathrm{H}^n \otimes \mathbb{C}^1), \quad \mathfrak{H}_0^+(n) = b_n(L)^{1/2}(\mathrm{H}^n \otimes \mathbb{C}^3).$$

For $l \in \mathbb{N}_0$, the subset

$$\mathfrak{H}_{n,l}^- = (\mathrm{H}^n \otimes \mathbb{C}^1) \cap (\mathrm{H}^{n+l} \otimes \mathbb{C}^4) = \mathrm{H}^{n+l} \otimes \mathbb{C}^1 = \mathfrak{H}_{n+l}^-$$

is dense in $\mathfrak{H}_n^-$; and similarly for $\mathfrak{H}_{n,l}^+ = \mathfrak{H}_{n+l}^+ \subseteq \mathfrak{H}_n^+$. Likewise, the subset

$$\begin{aligned} \mathfrak{H}_l^-(n) &= [b_n(L)^{1/2}(\mathrm{H}^n \otimes \mathbb{C}^1)] \cap (\mathrm{H}^l \otimes \mathbb{C}^4) \\ &= b_n(L)^{1/2}[(\mathrm{H}^n \otimes \mathbb{C}^1) \cap (\mathrm{H}^{n+l} \otimes \mathbb{C}^4)] \\ &= b_n(L)^{1/2}(\mathrm{H}^{n+l} \otimes \mathbb{C}^1) = b_l(L)^{-1/2} \mathfrak{H}_0^-(n+l) \end{aligned}$$

is dense in $\mathfrak{H}_0^-(n)$, and similarly for $\mathfrak{H}_l^+(n) \subseteq \mathfrak{H}_0^+(n)$.

Due to the dense inclusion $\mathfrak{H}_{n+1} \subseteq \mathfrak{H}_n$ one also has the following result.

Lemma 4.4. *Assume that $P^-_{n+1} \subseteq P^-_n$ for all $n \in \mathbb{Z}$. Then*

$$\mathfrak{H}^-_0 = \mathfrak{H}^-_0(2n), \quad \mathfrak{H}^-_0(1) = \mathfrak{H}^-_0(2n+1).$$

Proof. We show that $\mathfrak{H}^+_0(n) = \mathfrak{H}^+_0(n-2l)$ for $n \in \mathbb{Z}$, $l \in \mathbb{N}_0$; by relabeling $n-2l$ by n, the result extends to all $l \in \mathbb{Z}$. Taking the orthogonal complements one deduces an analogous result for $\mathfrak{H}^-_0(n)$.

We use two facts: that $\mathfrak{H}^+_0(n) = \ker P^-_0(n)$ and that $\mathfrak{H}_l \subseteq \mathfrak{H}_0$ densely for $l \in \mathbb{N}_0$. The kernel of $P^-_0(n)$ consists of $u \in \mathfrak{H}_0$ such that $P^-_0(n)u = 0$; this is equivalent to saying that $(\forall v \in \mathfrak{H}_0)$ $\langle v, P^-_0(n)u\rangle_0 = 0$. By Lemma 4.2,

$$P^-_0(n)u = b_l(L)^{1/2} P^-_0(n-l) b_l(L)^{-1/2} u \Rightarrow P^-_0(n-l) b_l(L)^{-1/2} u = 0.$$

Thus for all $v \in \mathfrak{H}_0$

$$\begin{aligned} 0 = \langle v, P^-_0(n)u\rangle_0 &= \langle v, P^-_0(n-l) b_l(L)^{-1/2} u\rangle_0 \\ &= \langle b_l(L)^{-1/2} P^-_0(n-l) v, u\rangle_0 \\ &= \langle P^-_0(n-2l) b_l(L)^{-1/2} v, u\rangle_0 \quad \text{(by Lemma 4.2)}. \end{aligned}$$

Since every v is of the form $v = b_l(L)^{1/2} w$ with some $w \in \mathfrak{H}_l$, it follows that $(\forall w \in \mathfrak{H}_l)$

$$0 = \langle P^-_0(n-2l) w, u\rangle_0 = \langle w, P^-_0(n-2l) u\rangle_0 .$$

Since $\mathfrak{H}_l \subseteq \mathfrak{H}_0$ densely, the latter implies that $P^-_0(n-2l)u = 0$; hence

$$\mathfrak{H}^+_0(n) = \ker P^-_0(n) = \ker P^-_0(n-2l) = \mathfrak{H}^+_0(n-2l)$$

as claimed. □

Thus, if the hypothesis of Lemma 4.4 holds, then the projections $P^-_0(n)$, $n \in \mathbb{Z}$, are in fact characterized by only two projections: $P^-_0 = P^-_0(2n)$ and $P^-_0(1) = P^-_0(2n+1)$; in this case P^-_n is as in (4.2) for $n \in 2\mathbb{Z}$, and

$$P^-_n = b_n(L)^{-1/2} P^-_0(1) b_n(L)^{1/2}$$

for $n \in 2\mathbb{Z}+1$. But the converse is not necessarily true in general.

Example 4.5. Let P^-_n be as in (4.2). Then $P^-_0(n) = P^-_0$ for all $n \in \mathbb{Z}$. Let $l \in \mathbb{N}_0$; then

$$\mathfrak{H}^-_{n,l} := P^-_n \mathfrak{H}_{n+l} = b_n(L)^{-1/2} P^-_0 b_n(L)^{1/2} \mathfrak{H}_{n+l} = b_n(L)^{-1/2} \mathfrak{H}^-_{0,l}$$

while

$$\mathfrak{H}^-_{n+l} := P^-_{n+l} \mathfrak{H}_{n+l} = b_n(L)^{-1/2} \mathfrak{H}^-_l .$$

Thus $\mathfrak{H}^-_{n,l} = \mathfrak{H}^-_{n+l}$ iff

$$\mathfrak{H}^-_{0,l} (= P^-_0 \mathfrak{H}_l) = \mathfrak{H}^-_l (= P^-_l \mathfrak{H}_l)$$

or what is the same, iff $P^-_0 \supseteq P^-_l$.

5. PROJECTED OPERATORS

Let $n \in \mathbb{Z}$. By scaling every self-adjoint operator L_n in $\mathfrak{H}_n$ admits the form

$$L_n = b_n(L)^{-1/2} L b_n(L)^{1/2}, \quad L = L_0 \tag{5.1}$$

on $\operatorname{dom} L_n = \mathfrak{H}_{n+2}$. To every L_n one associates densely defined (Lemma 4.2) projected operators

$$L_n^- := P_n^- L_n \mid_{\mathfrak{H}_{n,2}^-}, \quad L_n^+ := P_n^+ L_n \mid_{\mathfrak{H}_{n,2}^+}$$

in $\mathfrak{H}_n^-$ and $\mathfrak{H}_n^+$, respectively. In analogy to (5.1), every operator L_n^- admits the form

$$L_n^- = b_n(L)^{-1/2} L_0^-(n) b_n(L)^{1/2}, \quad L_0^-(n) := P_0^-(n) L \mid_{\mathfrak{H}_2^-(n)}$$

and similarly for L_n^+. The operators $L_0^\pm(n)$ are considered in $\mathfrak{H}_0^\pm(n)$, and hence they are densely defined.

Using $\mathfrak{H}_0^-(n) := P_0^-(n)\mathfrak{H}_0$ and $\mathfrak{H}_0 = (L - z_1)\mathfrak{H}_2$, $\mathfrak{H}_0^-(n)$ is the sum of sets

$$\mathfrak{H}_0^-(n) = \operatorname{ran}(L_0^-(n) - z_1) + P_0^-(n) L \mathfrak{H}_2^+(n). \tag{5.2}$$

Thus in general the operator $L_0^-(n) - z_1$ is not surjective (unlike $L - z_1$). But the following holds.

Theorem 5.1. *Under hypothesis of Lemma 4.4 the operator $L_0^-(n) - z_1$, $n \in \mathbb{Z}$, is surjective.*

Proof. By Lemma 4.2,

$$\operatorname{ran}(L_0^-(n) - z_1) = P_0^-(n) b_1(L) \mathfrak{H}_2^-(n) = P_0^-(n) \mathfrak{H}_0^-(n+2).$$

Now apply Lemma 4.4. □

The statement of the theorem is therefore equivalent to the statement

$$P_0^-(n) L \mathfrak{H}_2^+(n) = \{0\}. \tag{5.3}$$

Indeed, by Lemmas 4.2 and 4.4,

$$P_0^-(n) L \mathfrak{H}_2^+(n) = P_0^-(n) b_1(L) \mathfrak{H}_2^+(n) = P_0^-(n) \mathfrak{H}_0^+(n+2) = P_0^-(n) \mathfrak{H}_0^+(n) = \{0\},$$

so the sum in (5.2) implies that the operator $L_0^-(n) - z_1$ is surjective, and vice verse. In this case the operators $L_0^-(n)$ satisfy $L_0^- = L_0^-(2n)$ and $L_0^-(1) = L_0^-(2n+1)$. Analogous results hold for $L_0^+(n)$ and $L_n^\pm$.

If $L_n^{-\,*}$ is the adjoint in $\mathfrak{H}_n^-$ of L_n^- and if $L_0^-(n)^*$ is the adjoint in $\mathfrak{H}_0^-(n)$ of $L_0^-(n)$, then the following result holds.

Lemma 5.2. $L_n^{-\,*} = b_n(L)^{-1/2} L_0^-(n)^* b_n(L)^{1/2}$.

Proof. The basic arguments are as in the proof of (4.1). □

Theorem 5.3. *Under hypothesis of Lemma 4.4 the operator $L_0^-(n)$, $n \in \mathbb{Z}$, is self-adjoint in $\mathfrak{H}_0^-(n)$.*

Proof. Consider the adjoint $L_0^-(n)^*$ as a linear relation in $\mathfrak{H}_0^-(n)$. Then $L_0^-(n)^*$ consists of $(y^-, x^-) \in \mathfrak{H}_0^-(n)^2$ such that $(\forall w^- \in \mathfrak{H}_2^-(n))$

$$\langle w^-, x^- \rangle_0 = \langle L_0^-(n) w^-, y^- \rangle_0 .$$

Every $w^- \in \mathfrak{H}_2^-(n)$ is of the form $w^- = P_0^-(n) b_1(L)^{-1} v$ with some $v \in \mathfrak{H}_0$. Then

$$\begin{aligned}\langle L_0^-(n) w^-, y^- \rangle_0 &= \langle L P_0^-(n) b_1(L)^{-1} v, y^- \rangle_0 \\ &= \langle b_1(L) P_0^-(n) b_1(L)^{-1} v, y^- \rangle_0 + \langle P_0^-(n) b_1(L)^{-1} v, z_1 y^- \rangle_0 \\ &= \langle b_1(L) P_0^-(n) b_1(L)^{-1} v, y^- \rangle_0 + \langle v, b_1(L)^{-1} z_1 y^- \rangle_0 .\end{aligned}$$

By applying Lemma 4.2

$$\langle b_1(L) P_0^-(n) b_1(L)^{-1} v, y^- \rangle_0 = \langle P_0^-(n+2) v, y^- \rangle_0 = \langle v, P_0^-(n+2) y^- \rangle_0 .$$

On the other hand,

$$\langle w^-, x^- \rangle_0 = \langle b_1(L)^{-1} v, x^- \rangle_0 = \langle v, b_1(L)^{-1} x^- \rangle_0 .$$

Therefore $(y^-, x^-) \in \mathfrak{H}_0^-(n)^2$ such that

$$b_1(L)^{-1} x^- = P_0^-(n+2) y^- + b_1(L)^{-1} z_1 y^- .$$

Because $y^- = u^- + u^+$ is the sum of disjoint elements $u^\pm \in P_0^\pm(n+2)\mathfrak{H}_0^-(n)$, it follows from the above that

$$b_1(L)^{-1} x^- = u^- + b_1(L)^{-1} z_1 (u^- + u^+).$$

Because $b_1(L)^{-1}\mathfrak{H}_0^-(n) = \mathfrak{H}_2^-(n-2)$ by Lemma 4.2, from here one concludes that

$$u^- \in \mathfrak{H}_2^-(n-2) \cap P_0^-(n+2)\mathfrak{H}_0^-(n) = \mathfrak{H}_2^-(n-2) \cap \mathfrak{H}_2^-(n) \cap \mathfrak{H}_2^-(n+2).$$

Sequentially

$$x^- = b_1(L) u^- + z_1(u^- + u^+) = P_0^-(n) b_1(L) u^- + z_1 (u^- + u^+) = L_0^-(n) u^- + z_1 u^+ .$$

Finally, by applying Lemma 4.4 one gets that $u^- \in \mathfrak{H}_2^-(n)$ and $u^+ = 0$. □

Corollary 5.4. $z_1 \in \operatorname{res} L_0^-(n)$.

Proof. This follows from Theorems 5.1 and 5.3. □

Under hypothesis of Lemma 4.4 and applying Lemma 5.2, the operator L_n^- is therefore self-adjoint in $\mathfrak{H}_n^-$. Moreover, $z_1 \in \operatorname{res} L_n^- = \operatorname{res} L_0^-(n)$ or, what is equivalent, $P_n^- L_n \mathfrak{H}_{n+2}^+ = \{0\}$. Similar conclusions apply to operators $L_0^+(n)$ and L_n^+.

Lemma 5.5. *Under hypothesis of Lemma 4.4 the resolvent*

$$(L_0^-(n) - z)^{-1} = P_0^-(n)(L-z)^{-1}P_0^-(n) \quad \text{on } \mathfrak{H}_0^-(n)$$

for $z \in \operatorname{res} L \subseteq \operatorname{res} L_0^-(n)$ *(and similarly for* $L_0^+(n)$*).*

Proof. First we derive the resolvent formula for $z \in \operatorname{res} L \cap \operatorname{res} L_0^-(n)$ and then we show that $\operatorname{res} L \subseteq \operatorname{res} L_0^-(n)$. Consider an arbitrary $v \in \mathfrak{H}_0$. Then, for $z \in \operatorname{res} L$, $(\exists u \in \mathfrak{H}_2)$ $v = (L-z)u$. Projecting the latter onto $\mathfrak{H}_0^-(n)$ and applying (5.3) yields

$$P_0^-(n)v = (L_0^-(n) - z)P_0^-(n)u$$

and the resolvent formula follows for $z \in \operatorname{res} L \cap \operatorname{res} L_0^-(n)$.

The eigenspace

$$\mathfrak{N}_z(L_0^-(n)) = \{u^- \in \mathfrak{H}_2^-(n) \mid P_0^-(n)(L-z)u^- = 0\}$$

is nontrivial for some $z \in \mathbb{R}$ (cf. Theorem 5.3). From here and (5.3) one gets that

$$(L-z)u^- = P_0^+(n)(L-z)u^- = 0\,;$$

hence

$$\mathfrak{N}_z(L_0^-(n)) = \mathfrak{H}_2^-(n) \cap \mathfrak{N}_z(L).$$

If $z \notin \sigma_p(L)$ then also $z \notin \sigma_p(L_0^-(n))$, but the converse $z \notin \sigma_p(L_0^-(n))$ implies only that $\mathfrak{N}_z(L) = \mathfrak{H}_2^+(n) \cap \mathfrak{N}_z(L)$ in this case. Therefore $\sigma_p(L_0^-(n)) \subseteq \sigma_p(L)$.

Now let $z \in \operatorname{res} L$, that is, $z \notin \sigma_p(L)$ and $\operatorname{ran}(L-z) = \mathfrak{H}_0$. Because by (5.3)

$$\operatorname{ran}(L-z) = \operatorname{ran}(L_0^-(n) - z) \dotplus \operatorname{ran}(L_0^+(n) - z)$$

it follows that

$$\operatorname{ran}(L_0^-(n) - z) = \mathfrak{H}_0^-(n), \quad \operatorname{ran}(L_0^+(n) - z) = \mathfrak{H}_0^+(n)$$

so $z \in \operatorname{res} L_0^-(n)$. □

Under the same hypothesis the resolvent of L_n^- is given by

$$(L_n^- - z)^{-1} = P_n^-(L_n - z)^{-1}P_n^- \quad \text{on } \mathfrak{H}_n^-$$

for $z \in \operatorname{res} L_n = \operatorname{res} L$ (and similarly for L_n^+).

We summarize the main results obtained so far in the following theorem.

Theorem 5.6. *Let* $\mathfrak{H}_{n+1} \subseteq \mathfrak{H}_n$ *be the scale of Hilbert spaces associated with a self-adjoint operator* L *in* $\mathfrak{H}_0$*. For each* $n \in \mathbb{Z}$*, let* P_n^- *be an orthogonal projection in* $\mathfrak{H}_n$ *onto a subspace* $\mathfrak{H}_n^- \subseteq \mathfrak{H}_n$*;* $\mathfrak{H}_n^+$ *is the orthogonal complement in* $\mathfrak{H}_n$ *of* $\mathfrak{H}_n^-$*. Assume that* $P_{n+1}^- \subseteq P_n^-$*. Then the projections* $(P_n^-)_{n\in\mathbb{Z}}$ *are characterized, by scaling, by any two adjacent projections, say* P_0^- *and* P_1^-*, according to*

$$P_{2n}^- = b_n(L)^{-1}P_0^-b_n(L), \quad P_{2n+1}^- = b_n(L)^{-1}P_1^-b_n(L).$$

For each n*, the subspace* $\mathfrak{H}_n^-$ *(resp.* $\mathfrak{H}_n^+$*) is therefore a reducing subspace for the restriction* L_n *to* $\mathfrak{H}_{n+2}$ *of* L*. The part of* L_n *in* $\mathfrak{H}_n^-$ *(resp.* $\mathfrak{H}_n^+$*) is a self-adjoint operator.*

Proof. This follows from Lemmas 4.4, 5.2, and Theorem 5.3. □

6. MIN-MAX OPERATORS IN A SUBSPACE

In the present and subsequent paragraphs $\mathfrak{M}_d^*\mathcal{G}_{\mathrm{A}}=\mathcal{G}_{\mathrm{A}}\mathfrak{M}_d$, as in (3.1), for an invertible Hermitian $\mathcal{G}_{\mathrm{A}}$, and $P_{n+1}^-\subseteq P_n^-$, $n\in\mathbb{Z}$, as in Theorem 5.6. Let

$$\begin{aligned}A'_{\min}&:=U_{\mathrm{A}}A_{\min}U_{\mathrm{A}}^{-1}\\&=\{((f^\#,\xi),(Lf^\#,\mathfrak{M}_d\xi))\mid f^\#\in\mathfrak{H}_{m+2}\,,\,\xi\in\mathbb{C}^{md}\,,\,\langle\varphi,f^\#\rangle=[\mathcal{G}_{\mathrm{A}}\xi]_m\}.\end{aligned}$$

Then $A'_{\min}$ is a closed, densely defined, symmetric operator in $\mathcal{H}'_{\mathrm{A}}$, whose adjoint $A'^*_{\min}$ is given by

$$\begin{aligned}A'_{\max}&:=A'^*_{\min}=U_{\mathrm{A}}A_{\max}U_{\mathrm{A}}^{-1}\\&=\{((f^\#+h_{m+1}(c),\xi),(Lf^\#+z_1h_{m+1}(c),\mathfrak{M}_d\xi+\eta(c)))\mid f^\#\in\mathfrak{H}_{m+2},\\&\qquad c\in\mathbb{C}^d\,,\,\xi\in\mathbb{C}^{md}\}.\end{aligned}$$

If $(\mathbb{C}^d,\Gamma_0^{\mathrm{A}},\Gamma_1^{\mathrm{A}})$ is an OBT for $A_{\max}$ then the triple $(\mathbb{C}^d,\Gamma_0'^{\mathrm{A}},\Gamma_1'^{\mathrm{A}})$, with $\Gamma_i'^{\mathrm{A}}:=\Gamma_i^{\mathrm{A}}U_{\mathrm{A}}^{-1}$, $i\in\{0,1\}$, is an OBT for $A'_{\max}$.

Let

$$\Pi^\pm:=P_m^\pm\oplus I_{\mathbb{C}^{md}}\quad\text{in }\mathfrak{H}_m\oplus\mathbb{C}^{md}.$$

Then Π^- (resp. Π^+) is an orthogonal (with respect to the $\mathfrak{H}_m\oplus\mathbb{C}^{md}$-metric) projection onto a subspace $\mathfrak{H}_m^-\oplus\mathbb{C}^{md}$ (resp. $\mathfrak{H}_m^+\oplus\mathbb{C}^{md}$). Note that

$$\Pi^-\Pi^+\neq0,\quad\Pi^+\Pi^-\neq0,\quad\Pi^-+\Pi^+\neq I_{\mathfrak{H}_m\oplus\mathbb{C}^{md}}.$$

However, given Π^-, the above inequalities become the equalities with Π^+ replaced by the orthogonal projection $\Pi'^+:=I_{\mathfrak{H}_m\oplus\mathbb{C}^{md}}-\Pi^-$ onto

$$(\mathfrak{H}_m^-\oplus\mathbb{C}^{md})^{\perp_{\mathfrak{H}_m\oplus\mathbb{C}^{md}}}=\mathfrak{H}_m^+\oplus\{0\}.$$

Likewise, given Π^+, the above inequalities become the equalities with Π^- replaced by the orthogonal projection $\Pi'^-:=I_{\mathfrak{H}_m\oplus\mathbb{C}^{md}}-\Pi^+$ onto

$$(\mathfrak{H}_m^+\oplus\mathbb{C}^{md})^{\perp_{\mathfrak{H}_m\oplus\mathbb{C}^{md}}}=\mathfrak{H}_m^-\oplus\{0\}.$$

By Theorem 5.6, $A'_{\min}$ maps

$$\operatorname{dom}A'_{\min}\cap(\mathfrak{H}_m^-\oplus\mathbb{C}^{md})=\Pi^-\operatorname{dom}A'_{\min}$$

into $\mathfrak{H}_m^-\oplus\mathbb{C}^{md}$; therefore $\mathfrak{H}_m^-\oplus\mathbb{C}^{md}$ is an invariant ([28, Definition 1.7]) subspace for $A'_{\min}$. Let $A^-_{\min}$ denote the part of $A'_{\min}$ in $\mathfrak{H}_m^-\oplus\mathbb{C}^{md}$, that is

$$\begin{aligned}A^-_{\min}&:=A'_{\min}\restriction_{\Pi^-\operatorname{dom}A'_{\min}}=\Pi^-A'_{\min}\restriction_{\Pi^-\operatorname{dom}A'_{\min}}\\&=\{((f^{\#-},\xi),(L_m^-f^{\#-},\mathfrak{M}_d\xi))\mid f^{\#-}\in\mathfrak{H}_{m+2}^-\,,\,\xi\in\mathbb{C}^{md},\langle\varphi,f^{\#-}\rangle=[\mathcal{G}_{\mathrm{A}}\xi]_m\}.\end{aligned}$$

Similarly one defines the part $A^+_{\min}$ of $A'_{\min}$ in $\mathfrak{H}_m^+\oplus\mathbb{C}^{md}$. Because $\mathfrak{H}_m^+\oplus\{0\}$ (resp. $\mathfrak{H}_m^-\oplus\{0\}$) is also an invariant subspace for $A'_{\min}$, the operator $A'_{\min}$ is represented by the orthogonal sum of its part $A^-_{\min}$ in $\mathfrak{H}^-_m \oplus \mathbb{C}^{md}$ (resp. $A^+_{\min}$ in $\mathfrak{H}^+_m \oplus \mathbb{C}^{md}$) and its part $L^+_{\min} \oplus 0$ in $\mathfrak{H}^+_m \oplus \{0\}$ (resp. $L^-_{\min} \oplus 0$ in $\mathfrak{H}^-_m \oplus \{0\}$), where the operator

$$L^+_{\min} := L^+_m \restriction_{\{f^+ \in \mathfrak{H}^+_{m+2} \,|\, \langle \varphi, f^+ \rangle = 0\}} \quad (\text{resp. } L^-_{\min} := L^-_m \restriction_{\{f^- \in \mathfrak{H}^-_{m+2} \,|\, \langle \varphi, f^- \rangle = 0\}})\,;$$

symbolically ($[\oplus]$ indicates both $\mathfrak{H}_m \oplus \mathbb{C}^{md}$-orthogonal and $\mathcal{H}'_{\mathrm{A}}$-orthogonal sum)

$$A'_{\min} = A^-_{\min}[\oplus](L^+_{\min} \oplus 0) = (L^-_{\min} \oplus 0)[\oplus]A^+_{\min}. \tag{6.1}$$

Let φ^- (resp. φ^+) denote the vector valued functional whose components φ^-_σ (resp. φ^+_σ) are defined by

$$\varphi^-_\sigma := b_{m+2}(L)^{1/2} P^-_0(m) b_{m+2}(L)^{-1/2} \varphi_\sigma \in b_{m+2}(L)^{1/2}(\mathfrak{H}^-_0 \setminus \mathfrak{H}^-_1)$$
$$(\text{resp. } \varphi^+_\sigma := b_{m+2}(L)^{1/2} P^+_0(m) b_{m+2}(L)^{-1/2} \varphi_\sigma \in b_{m+2}(L)^{1/2}(\mathfrak{H}^+_0 \setminus \mathfrak{H}^+_1)).$$

The duality pairing $\langle \varphi^-_\sigma, \cdot \rangle$ (resp. $\langle \varphi^+_\sigma, \cdot \rangle$) is defined via the $\mathfrak{H}_0$-scalar product in a usual way. $\langle \varphi^-, \cdot \rangle = (\langle \varphi^-_\sigma, \cdot \rangle) \colon \mathfrak{H}^-_{m+2} \to \mathbb{C}^d$ denotes the action of the vector valued functional φ^-, and similarly for φ^+.

Lemma 6.1. *For $f^{\#\,-} \in \mathfrak{H}^-_{m+2}$*

$$\langle \varphi, f^{\#\,-} \rangle = \langle \varphi^-, f^{\#\,-} \rangle = \langle h^-_{m+1}, (L^-_m - z_1) f^{\#\,-} \rangle_m$$

and similarly for the action of φ on $\mathfrak{H}^+_{m+2}$.

Proof. By the definition of the duality pairing and that of φ^-_σ,

$$\begin{aligned}\langle \varphi^-_\sigma, f^{\#\,-} \rangle &= \langle b_{m+2}(L)^{-1/2} \varphi^-_\sigma, b_{m+2}(L)^{1/2} f^{\#\,-} \rangle_0 \\ &= \langle P^-_0(m) b_{m+2}(L)^{-1/2} \varphi_\sigma, b_{m+2}(L)^{1/2} f^{\#\,-} \rangle_0 \\ &= \langle b_{m+2}(L)^{-1/2} \varphi_\sigma, P^-_0(m) b_{m+2}(L)^{1/2} f^{\#\,-} \rangle_0 \,.\end{aligned}$$

But

$$b_{m+2}(L)^{1/2} f^{\#\,-} \in b_{m+2}(L)^{1/2} \mathfrak{H}^-_{m+2} = b_{m+2}(L)^{1/2} P^-_{m+2} \mathfrak{H}_{m+2} = \mathfrak{H}^-_0(m+2)$$

and hence by Lemma 4.4 $b_{m+2}(L)^{1/2} f^{\#\,-} \in \mathfrak{H}^-_0(m)$; therefore

$$\begin{aligned}\langle b_{m+2}(L)^{-1/2} \varphi_\sigma, P^-_0(m) b_{m+2}(L)^{1/2} f^{\#\,-} \rangle_0 &= \langle b_{m+2}(L)^{-1/2} \varphi_\sigma, b_{m+2}(L)^{1/2} f^{\#\,-} \rangle_0 \\ &= \langle \varphi_\sigma, f^{\#\,-} \rangle \,.\end{aligned}$$

This proves the first equality. Using that $b_{m+2}(L)^{1/2} f^{\#\,-} \in \mathfrak{H}^-_0(m)$, the second equality is due to

$$\begin{aligned}\langle h^-_{\sigma,m+1}, (L^-_m - z_1) f^{\#\,-} \rangle_m &= \langle h^-_{\sigma,m+1}, b_1(L) f^{\#\,-} \rangle_m \\ &= \langle b_m(L)^{1/2} P^-_m h_{\sigma,m+1}, b_{m+2}(L)^{1/2} f^{\#\,-} \rangle_0 \\ &= \langle P^-_0(m) b_{m+2}(L)^{-1/2} \varphi_\sigma, b_{m+2}(L)^{1/2} f^{\#\,-} \rangle_0 \\ &= \langle b_{m+2}(L)^{-1/2} \varphi_\sigma, b_{m+2}(L)^{1/2} f^{\#\,-} \rangle_0 = \langle \varphi_\sigma, f^{\#\,-} \rangle \,.\end{aligned}$$

The proof of $\langle \varphi, \cdot \rangle$ on $\mathfrak{H}^+_{m+2}$ is analogous. □

By the lemma the boundary conditions defining the operators $L^{\pm}_{\min}$ are therefore reduced to $\langle\varphi^{\pm}, f^{\pm}\rangle = 0$, $f^{\pm} \in \mathfrak{H}_{m+2}$, where $\varphi^{-} + \varphi^{+} = \varphi$. Explicitly

$$L^{-}_{\min} := L^{-}_{m}\,|_{\{f^{-}\in\mathfrak{H}^{-}_{m+2}\,|\,\langle\varphi^{-},f^{-}\rangle=0\}}, \quad L^{+}_{\min} := L^{+}_{m}\,|_{\{f^{+}\in\mathfrak{H}^{+}_{m+2}\,|\,\langle\varphi^{+},f^{+}\rangle=0\}}.$$

Just like the functionals φ_σ define the elements $h_{\sigma j} := b_j(L)^{-1}\varphi_\sigma$, $j \in J$, that generate the linear space $\mathfrak{K}_{\mathrm{A}}$, the functionals $\varphi^{\pm}_{\sigma}$ define the elements

$$h^{\pm}_{\sigma j} := b_j(L)^{-1}\varphi^{\pm}_{\sigma} = P^{\pm}_{-m-2+2j} h_{\sigma j} \tag{6.2}$$

that generate (span) the linear subspaces $\mathfrak{K}^{\pm}_{\mathrm{A}}$ of $\mathfrak{K}_{\mathrm{A}}$; that is, $\mathfrak{K}_{\mathrm{A}} = \mathfrak{K}^{-}_{\mathrm{A}} \dotplus \mathfrak{K}^{+}_{\mathrm{A}}$. The proof of the second equality in (6.2) uses the definition of $P^{\pm}_{0}(\cdot)$ and then Lemma 4.4, in the same spirit as in the proof of Lemma 6.1.

Unlike the case of $A'_{\min}$, the operator $A'_{\max}$ does not commute with the projection Π^{-} (resp. Π^{+}). The reason is that now the projection of $h_{m+1}(c)$ onto $\mathfrak{H}^{-}_{m}$ affects the value of the extra term $\eta(c) \in \mathbb{C}^{md}$. This seems to be better seen in the representation of the operator $A'_{\max}$ in the space $\mathfrak{H}_m \dotplus \mathfrak{K}_{\mathrm{A}}$, i.e. in analyzing the operator $A_{\max}$. Thus we have by Lemma 3.1 (here $k \in \mathfrak{K}_{\mathrm{A}}$)

$$\begin{gathered} A_{\max}(f^{\#} + h_{m+1}(c) + k) = L_{m-2}(f^{\#} + h_{m+1}(c)) + k', \\ k' \in \mathfrak{K}_{\mathrm{A}}, \quad d(k') = \mathfrak{M}_d d(k) \end{gathered}$$

and

$$L_{m-2} h_{m+1}(c) = z_1 h_{m+1}(c) + h_m(c),$$

where

$$h_m(c) = b_1(L) h_{m+1}(c) = \sum_{\alpha} [\eta(c)]_\alpha h_\alpha = \sum_{\sigma} c_\sigma h_{\sigma m} \in \mathfrak{K}_{\mathrm{A}}.$$

Now projecting $f^{\#} + h_{m+1}(c) + k$ onto $\mathfrak{H}^{-}_{m} \dotplus \mathfrak{K}_{\mathrm{A}}$ one gets that

$$\begin{aligned} A_{\max} U^{-1}_{\mathrm{A}} \Pi^{-} U_{\mathrm{A}}(f^{\#} + h_{m+1}(c) + k) &= L^{-}_{m-2}(f^{\#\,-} + h^{-}_{m+1}(c)) + k' \\ &= L^{-}_{m} f^{\#\,-} + z_1 h^{-}_{m+1}(c) + k' + h^{-}_{m}(c) \end{aligned}$$

with

$$h^{-}_{m}(c) := b_1(L) h^{-}_{m+1}(c) = \sum_{\alpha} [\eta(c)]_\alpha h^{-}_{\alpha} = \sum_{\sigma} c_\sigma h^{-}_{\sigma m} \in \mathfrak{K}^{-}_{\mathrm{A}}$$

(it is precisely for this reason why $\eta(c)$ changes to $\eta^{-}(c) \neq \eta(c)$; see below), while

$$\begin{aligned} U^{-1}_{\mathrm{A}} \Pi^{-} U_{\mathrm{A}} A_{\max}(f^{\#} + h_{m+1}(c) + k) &= L^{-}_{m} f^{\#\,-} + z_1 h^{-}_{m+1}(c) + k' + h_m(c) \\ &= A_{\max} U^{-1}_{\mathrm{A}} \Pi^{-} U_{\mathrm{A}}(f^{\#} + h_{m+1}(c) + k) + h^{+}_{m}(c) \end{aligned}$$

with $h^{+}_{m}(c) \in \mathfrak{K}^{+}_{\mathrm{A}}$ defined similarly as $h^{-}_{m}(c)$. Because $h^{\pm}_{m}(c) \in \mathfrak{K}^{\pm}_{\mathrm{A}}$ and $\mathfrak{K}^{\pm}_{\mathrm{A}} \subseteq \mathfrak{K}_{\mathrm{A}}$, it follows that

$$h^{\pm}_{m}(c) = \sum_{\alpha} [\eta(c)]_\alpha h^{\pm}_{\alpha} = \sum_{\alpha} [\eta^{\pm}(c)]_\alpha h_\alpha$$

for $\eta^{\pm}(c) \in \mathbb{C}^{md}$ given by

$$\eta^{\pm}(c) := \widetilde{\mathcal{G}}_{\mathrm{A}}^{-1} \langle h, h_m^{\pm}(c) \rangle_{-m} = \widetilde{\mathcal{G}}_{\mathrm{A}}^{-1} \widetilde{\mathcal{G}}_{\mathrm{A}}^{\pm} \eta(c)$$

with the matrix

$$\widetilde{\mathcal{G}}_{\mathrm{A}}^{\pm} = ([\widetilde{\mathcal{G}}_{\mathrm{A}}^{\pm}]_{\alpha\alpha'}), \quad [\widetilde{\mathcal{G}}_{\mathrm{A}}^{\pm}]_{\alpha\alpha'} := \langle h_\alpha, h_{\alpha'}^{\pm} \rangle_{-m} .$$

With this notation, and going back to the representation of $A_{\max}$ in $\mathfrak{H}_m \oplus \mathbb{C}^{md}$, one gets that

$$A'_{\max} \Pi^{-}(f^{\#} + h_{m+1}(c), \xi) = (L_m^{-} f^{\# -} + z_1 h_{m+1}^{-}(c), \mathfrak{M}_d \xi + \eta^{-}(c))$$

while

$$\Pi^{-} A'_{\max}(f^{\#} + h_{m+1}(c), \xi) = (L_m^{-} f^{\# -} + z_1 h_{m+1}^{-}(c), \mathfrak{M}_d \xi + \eta(c)).$$

Similarly, projecting $(f^{\#} + h_{m+1}(c), \xi)$ onto $\mathfrak{H}_m^{+} \oplus \{0\}$ gives

$$A'_{\max} \Pi'^{+}(f^{\#} + h_{m+1}(c), \xi) = (L_m^{+} f^{\# +} + z_1 h_{m+1}^{+}(c), \eta^{+}(c))$$

while

$$\Pi'^{+} A'_{\max}(f^{\#} + h_{m+1}(c), \xi) = (L_m^{+} f^{\# +} + z_1 h_{m+1}^{+}(c), 0).$$

From these formulas one observes that one still is able to represent the extension of the operator $A'_{\max}$ (but not the operator $A'_{\max}$ itself) as the orthogonal sum of its parts in subspaces $\mathfrak{H}_m^{-} \oplus \mathbb{C}^{md}$ (resp. $\mathfrak{H}_m^{+} \oplus \mathbb{C}^{md}$) and $\mathfrak{H}_m^{+} \oplus \{0\}$ (resp. $\mathfrak{H}_m^{-} \oplus \{0\}$), similarly as in (6.1), by moving an element $(0, \eta^{+}(c))$ from $A'_{\max}\Pi'^{+}$ to $A'_{\max}\Pi^{-}$.

To make this precise, one therefore introduces the linear relation

$$\begin{aligned} A_{\max}^{-} := \{ \big((f^{\# -} + h_{m+1}^{-}(c), \xi), (L_m^{-} f^{\# -} + z_1 h_{m+1}^{-}(c), \mathfrak{M}_d \xi + \eta(c)) \big) \mid \\ f^{\# -} \in \mathfrak{H}_{m+2}^{-}, \, c \in \mathbb{C}^d, \, \xi \in \mathbb{C}^{md} \} \end{aligned}$$

in $\mathfrak{H}_m^{-} \oplus \mathbb{C}^{md}$ with the multivalued part

$$\operatorname{mul} A_{\max}^{-} = \{0\} \times \eta^{+}(\Sigma^{-}), \quad \Sigma^{-} := \Big\{ c \in \mathbb{C}^d \mid \sum_{\sigma} c_\sigma \varphi_\sigma^{-} = 0 \Big\}$$

(the multivalued part is exactly the orthogonal complement in $\mathcal{H}_{\mathrm{A}}'^{-}$ of $\operatorname{dom} A_{\min}^{-}$) and the operator

$$L_{\max}^{-} \oplus 0 = \Pi'^{-} A'_{\max} \,|\, _{\Pi'^{-} \operatorname{dom} A'_{\max}}$$

in $\mathfrak{H}_m^{-} \oplus \{0\}$ with

$$L_{\max}^{-} := \{ (f^{\# -} + h_{m+1}^{-}(c), L_m^{-} f^{\# -} + z_1 h_{m+1}^{-}(c)) \mid f^{\# -} \in \mathfrak{H}_{m+2}^{-}, \, c \in \mathbb{C}^d \}.$$

Analogously one defines the linear relation $A_{\max}^{+}$ in $\mathfrak{H}_m^{+} \oplus \mathbb{C}^{md}$, with the multivalued part $\{0\} \times \eta^{-}(\Sigma^{+})$, and the operator $L_{\max}^{+}$ in $\mathfrak{H}_m^{+}$. Note that the domain of the operator $L_{\max}^{-}$ in $\mathfrak{H}_m^{-}$ can be also written thus

$$\operatorname{dom} L_{\max}^{-} = \mathfrak{H}_{m+2}^{-} \dotplus \mathfrak{N}_z(L_{\max}^{-}), \quad z \in \operatorname{res} L_m^{-}$$

with the eigenspace

$$\mathfrak{N}_z(L^-_{\max}) = (L^-_m - z_1)(L^-_m - z)^{-1} h^-_{m+1}(\mathbb{C}^d)$$

and similarly for $L^+_{\max}$. (The operators $L^\pm_{\max}$ should not be confused with the triplet adjoint $L_{\max}$; as we show below, $L^-_{\max}$ is the adjoint in $\mathfrak{H}^-_m$ of $L^-_{\min}$, and similarly for $L^+_{\max}$.)

It follows from the above constructions that the orthogonal (both in $\mathfrak{H}_m \oplus \mathbb{C}^{md}$-metric and in $\mathcal{H}'_{\mathrm{A}}$-metric) componentwise sum of linear relations (cf. [18, 19, 22] for the notation)

$$A^-_{\max}[\,\widehat{\oplus}\,](L^+_{\max} \oplus 0) = (L^-_{\max} \oplus 0)[\,\widehat{\oplus}\,]A^+_{\max} \tag{6.3}$$

is an extension in $\mathfrak{H}_m \oplus \mathbb{C}^{md}$ of the operator $A'_{\max}$. By comparing (6.1) with (6.3) one concludes that $A^-_{\min} \subseteq A^-_{\max}$ and $L^-_{\min} \subseteq L^-_{\max}$, and similarly for $A^+_{\min}$ and $L^+_{\min}$. In fact, one can say more.

Theorem 6.2. *The linear relation $A^-_{\max} = A^{-\,*}_{\min}$ is the adjoint in $\mathcal{H}'^{\,-}_{\mathrm{A}}$ of a nondensely defined (in general), closed, symmetric operator $A^-_{\min}$.*

Proof. The main arguments are as in the proof of the self-adjointness of L^-_m (Theorem 5.3) by using in addition that the boundary condition for $(f^{\#\,-}, \xi) \in \operatorname{dom} A^-_{\min}$ implies that $(\forall c \in \mathbb{C}^d)$

$$\langle w, b_m(L)^{1/2} h^-_{m+1}(c)\rangle_0 = \langle \xi, \mathcal{G}_{\mathrm{A}}\eta(c)\rangle_{\mathbb{C}^{md}}\,, \quad f^{\#\,-} = b_{m+2}(L)^{-1/2} P^-_0(m) w, \tag{6.4}$$

$w \in \mathfrak{H}_0$; note that

$$b_m(L)^{1/2} h^-_{m+1}(c) = b_{m+2}(L)^{-1/2} \sum_\sigma c_\sigma \varphi^-_\sigma$$

and the representation of $f^{\#\,-}$ is shown in the proof of Lemma 6.1. The duality pairing then reads

$$\begin{aligned}\langle \varphi^-, f^{\#\,-}\rangle &= \langle b_{m+2}(L)^{-1/2}\varphi^-, b_{m+2}(L)^{1/2} f^{\#\,-}\rangle_0 \\ &= \langle b_{m+2}(L)^{-1/2}\varphi^-, P^-_0(m) w\rangle_0\,,\end{aligned}$$

but $b_{m+2}(L)^{-1/2}\varphi^- \in \mathfrak{H}^-_0(m)$, so the boundary condition reads

$$\langle \varphi^-, f^{\#\,-}\rangle = \langle b_{m+2}(L)^{-1/2}\varphi^-, w\rangle_0 = [\mathcal{G}_{\mathrm{A}}\xi]_m$$

from which (6.4) follows.

Now one computes $A^{-\,*}_{\min}$; as a linear relation, it is the set of $((y^-, \xi_y), (x^-, \xi_x)) \in (\mathfrak{H}^-_m \oplus \mathbb{C}^{md})^2$ such that $(\forall (f^{\#\,-}, \xi) \in \operatorname{dom} A^-_{\min})$

$$\langle f^{\#\,-}, x^-\rangle_m + \langle \xi, \mathcal{G}_{\mathrm{A}}\xi_x\rangle_{\mathbb{C}^{md}} = \langle L^-_m f^{\#\,-}, y^-\rangle_m + \langle \mathfrak{M}_d \xi, \mathcal{G}_{\mathrm{A}}\xi_y\rangle_{\mathbb{C}^{md}}\,. \tag{6.5}$$

Applying the representation

$$\begin{aligned} x^- &= b_m(L)^{-1/2} u^-, \quad u^- \in \mathfrak{H}^-_0(m), \\ y^- &= b_m(L)^{-1/2} v^-, \quad v^- \in \mathfrak{H}^-_0(m) \end{aligned}$$

and using that $b_1(L)^{-1}\mathfrak{H}_0^-(m) = \mathfrak{H}_2^-(m)$ one gets that

$$\langle f^{\#\,-}, x^- \rangle_m = \langle w, b_1(L)^{-1} u^- \rangle_0$$

and

$$\begin{aligned}\langle L_m^- f^{\#\,-}, y^- \rangle_m &= \langle b_1(L) f^{\#\,-}, y^- \rangle_m + \langle f^{\#\,-}, z_1 y^- \rangle_m \\ &= \langle w, v^- \rangle_0 + \langle w, b_1(L)^{-1} z_1 v^- \rangle_0 .\end{aligned}$$

Therefore (6.5) reads

$$\langle w, v^- - b_1(L)^{-1}(u^- - z_1 v^-) \rangle_0 = \langle \xi, \mathcal{G}_{\mathrm{A}}(\xi_x - \mathfrak{M}_d \xi_y) \rangle_{\mathbb{C}^{md}} .$$

Comparing the latter with (6.4) yields

$$\begin{aligned} v^- - b_1(L)^{-1}(u^- - z_1 v^-) =& b_m(L)^{1/2} h_{m+1}^-(c), \\ \xi_x =& \mathfrak{M}_d \xi_y + \eta(c). \end{aligned}$$

The first equation above implies that

$$v^- - b_m(L)^{1/2} h_{m+1}^-(c) \in b_1(L)^{-1}\mathfrak{H}_0^-(m) = \mathfrak{H}_2^-(m),$$

that is,

$$y^- = f^- + h_{m+1}^-(c), \quad f^- \in \mathfrak{H}_{m+2}^- .$$

Then

$$x^- = z_1 y^- + b_1(L) f^- = L_m^- f^- + z_1 h_{m+1}^-(c).$$

This proves $A_{\min}^{-\,*} = A_{\max}^-$. It remains to verify that $A_{\min}^-$ is closed. The adjoint $A_{\max}^{-\,*}$ consists of $((y^-, \xi_y), (x^-, \xi_x)) \in (\mathfrak{H}_m^- \oplus \mathbb{C}^{md})^2$ such that $(\forall f^{\#\,-} \in \mathfrak{H}_{m+2}^-$ $\forall c \in \mathbb{C}^d$ $\forall \xi \in \mathbb{C}^{md})$

$$\begin{aligned}\langle f^{\#\,-} + h_{m+1}^-(c), x^- \rangle_m + \langle \xi, \mathcal{G}_{\mathrm{A}} \xi_x \rangle_{\mathbb{C}^{md}} =& \langle L_m^- f^{\#\,-} + z_1 h_{m+1}^-(c), y^- \rangle_m \\ &+ \langle \mathfrak{M}_d \xi + \eta(c), \mathcal{G}_{\mathrm{A}} \xi_y \rangle_{\mathbb{C}^{md}} .\end{aligned}$$

Using the representation of $f^{\#\,-}$, x^-, y^- as above, and noting that

$$\langle h_{m+1}^-(c), x^- \rangle_m = \langle c, \langle h_{m+1}^-, x^- \rangle_m \rangle_{\mathbb{C}^d} , \quad \langle \eta(c), \mathcal{G}_{\mathrm{A}} \xi_y \rangle_{\mathbb{C}^{md}} = \langle c, [\mathcal{G}_{\mathrm{A}} \xi_y]_m \rangle_{\mathbb{C}^d}$$

one gets that

$$\begin{aligned} 0 =& \langle w, v^- - b_1(L)^{-1}(u^- - z_1 v^-) \rangle_0 + \langle c, \langle h_{m+1}^-, z_1 y^- - x^- \rangle_m + [\mathcal{G}_{\mathrm{A}} \xi_y]_m \rangle_{\mathbb{C}^d} \\ &+ \langle \xi, \mathcal{G}_{\mathrm{A}}(\mathfrak{M}_d \xi_y - \xi_x) \rangle_{\mathbb{C}^{md}} \end{aligned}$$

and from which one concludes that

$$v^- = b_1(L)^{-1}(u^- - z_1 v^-) \in \mathfrak{H}_2^-(m) \Rightarrow x^- = L_m^- y^- , \quad y^- \in \mathfrak{H}_{m+2}^-$$

and

$$\langle h_{m+1}^-, x^- - z_1 y^- \rangle_m = \langle h_{m+1}^-, (L_m^- - z_1) y^- \rangle_m = \langle \varphi, y^- \rangle = [\mathcal{G}_{\mathrm{A}} \xi_y]_m$$

(cf. Lemma 6.1) and $\xi_x = \mathfrak{M}_d \xi_y$. Thus $A_{\min}^-$ is closed, and this completes the proof. □

The above proof also shows that:

Corollary 6.3. *The operator* $L^-_{\max} = L^{-*}_{\min}$ *is the adjoint in* $\mathfrak{H}^-_m$ *of a densely defined, closed, symmetric operator* $L^-_{\min}$.

From here one concludes that $L^-_{\min}$ (resp. $L^+_{\min}$) is an essentially self-adjoint operator in $\mathfrak{H}^-_0$ (resp. $\mathfrak{H}^+_0$). Since $A^-_{\min}$ extends $L^-_{\min}$ to $\mathcal{H}'^{\,-}_{\mathrm{A}}$ just like $A_{\min}$ extends $L_{\min}$ to $\mathcal{H}_{\mathrm{A}}$ it is therefore a subject of interest to formulate a similar realization theorem in the A-model for the symmetric operator $L^-_{\min}$. This is done in the next (the last) paragraph.

7. REALIZATION THEOREM IN A SUBSPACE

By a straightforward computation and applying Lemma 6.1, the boundary form of the linear relation $A^-_{\max}$ is given by

$$\begin{aligned}&[(f^{\#-}+h^-_{m+1}(c),\xi),(L^-_m g^{\#-}+z_1h^-_{m+1}(c'),\mathfrak{M}_d\xi'+\eta(c'))]'_{\mathrm{A}}\\ &-[(L^-_m f^{\#-}+z_1h^-_{m+1}(c),\mathfrak{M}_d\xi+\eta(c)),(g^{\#-}+h^-_{m+1}(c'),\xi')]'_{\mathrm{A}}\\ &=\langle c,\langle\varphi^-,g^{\#-}\rangle-[\mathcal{G}_{\mathrm{A}}\xi']_m\rangle_{\mathbb{C}^d}-\langle\langle\varphi^-,f^{\#-}\rangle-[\mathcal{G}_{\mathrm{A}}\xi]_m,c'\rangle_{\mathbb{C}^d}\end{aligned}$$

for $f^{\#-},g^{\#-}\in\mathfrak{H}^-_{m+2}$; $c,c'\in\mathbb{C}^d$; $\xi,\xi'\in\mathbb{C}^{md}$. By introducing the mappings from $A^-_{\max}$ to $\mathbb{C}^d$ by

$$\Gamma^{\mathrm{A}\,-}_0\widehat{f}^-:=c,\quad \Gamma^{\mathrm{A}\,-}_1\widehat{f}^-:=\langle\varphi^-,f^{\#-}\rangle-[\mathcal{G}_{\mathrm{A}}\xi]_m, \tag{7.1}$$

$$\widehat{f}^-=\big((f^{\#-}+h^-_{m+1}(c),\xi),(L^-_m f^{\#-}+z_1h^-_{m+1}(c),\mathfrak{M}_d\xi+\eta(c))\big)\in A^-_{\max}$$

the above boundary form simplifies thus

$$[f^-,g'^{\,-}]'_{\mathrm{A}}-[f'^{\,-},g^-]'_{\mathrm{A}}=\langle\Gamma^{\mathrm{A}\,-}_0\widehat{f}^-,\Gamma^{\mathrm{A}\,-}_1\widehat{g}^-\rangle_{\mathbb{C}^d}-\langle\Gamma^{\mathrm{A}\,-}_1\widehat{f}^-,\Gamma^{\mathrm{A}\,-}_0\widehat{g}^-\rangle_{\mathbb{C}^d},$$

$$\widehat{f}^-=(f^-,f'^{\,-})\in A^-_{\max},\quad \widehat{g}^-=(g^-,g'^{\,-})\in A^-_{\max}$$

and it therefore represents the Green identity. Consider $\Gamma^{\mathrm{A}\,-}\colon\widehat{f}^-\mapsto(\Gamma^{\mathrm{A}\,-}_0\widehat{f}^-,\Gamma^{\mathrm{A}\,-}_1\widehat{f}^-)$ from $A^-_{\max}$ to $\mathbb{C}^d\times\mathbb{C}^d$ as an (isometric) linear relation from $(\mathcal{H}'^{\,-}_{\mathrm{A}})^2$ to $\mathbb{C}^d\times\mathbb{C}^d$. Thus by definition $\operatorname{dom}\Gamma^{\mathrm{A}\,-}=A^-_{\max}$ and $\ker\Gamma^{\mathrm{A}\,-}=A^-_{\min}$. Moreover, the multivalued part $\operatorname{mul}\Gamma^{\mathrm{A}\,-}$ consists of $(c,0)$ such that $c\in\Sigma^-\cap\Sigma^+=\{0\}$; hence $\Gamma^{\mathrm{A}\,-}$ is an operator. Below we show that $\Gamma^{\mathrm{A}\,-}$ is a unitary relation from $(\mathcal{H}'^{\,-}_{\mathrm{A}})^2$ to $\mathbb{C}^d\times\mathbb{C}^d$ (by the above, it would actually suffice to show that $\operatorname{dom}(\Gamma^{\mathrm{A}\,-})^{[+]}=\operatorname{ran}\Gamma^{\mathrm{A}\,-}$). By [12, Corollary 2.4(iii)] this would imply that $\Gamma^{\mathrm{A}\,-}$ is surjective, and that therefore the triple $(\mathbb{C}^d,\Gamma^{\mathrm{A}\,-}_0,\Gamma^{\mathrm{A}\,-}_1)$ is an OBT for $A^-_{\max}$.

Lemma 7.1. $(\mathbb{C}^d,\Gamma^{\mathrm{A}\,-}_0,\Gamma^{\mathrm{A}\,-}_1)$ *is an OBT for* $A^-_{\max}$.

Proof. By definition, the Krein space adjoint $(\Gamma^{\mathrm{A}\,-})^{[+]}$ is a linear relation consisting of

$$\big((\chi,\chi'),\big((y^-,\xi_y),(x^-,\xi_x)\big)\big)\in\mathbb{C}^{2d}\times(\mathfrak{H}^-_m\oplus\mathbb{C}^{md})^2$$

such that $(\forall f^{\#-}\in\mathfrak{H}^-_{m+2}\,\forall c\in\mathbb{C}^d\,\forall\xi\in\mathbb{C}^{md})$

$$\begin{aligned}&\langle f^{\#-}+h^-_{m+1}(c),x^-\rangle_m+\langle\xi,\mathcal{G}_{\mathrm{A}}\xi_x\rangle_{\mathbb{C}^{md}}\\&-\langle L^-_m f^{\#-}+z_1h^-_{m+1}(c),y^-\rangle_m-\langle\mathfrak{M}_d\xi+\eta(c),\mathcal{G}_{\mathrm{A}}\xi_y\rangle_{\mathbb{C}^{md}}\\&=\langle c,\chi'\rangle_{\mathbb{C}^d}-\langle\langle h^-_{m+1},(L^-_m-z_1)f^{\#-}\rangle_m-[\mathcal{G}_{\mathrm{A}}\xi]_m,\chi\rangle_{\mathbb{C}^d}\,.\end{aligned}$$

The above equation splits into three equations

$$(\forall f^{\#-})\ \langle f^{\#-},x^- - z_1h^-_{m+1}(\chi)\rangle_m=\langle L^-_m f^{\#-},y^- - h^-_{m+1}(\chi)\rangle_m\,,$$

$$(\forall c)\ 0=\langle c,\langle h^-_{m+1},x^- - z_1y^-\rangle_m-[\mathcal{G}_{\mathrm{A}}\xi_y]_m-\chi'\rangle_{\mathbb{C}^d}\,,$$

$$(\forall\xi)\ 0=\langle\xi,\mathcal{G}_{\mathrm{A}}(\xi_x-\mathfrak{M}_d\xi_y-\eta(\chi))\rangle_{\mathbb{C}^{md}}\,.$$

Because L^-_m is self-adjoint in $\mathfrak{H}^-_m$, the first equation gives

$$y^-=f^-+h^-_{m+1}(\chi),\quad f^-\in\mathfrak{H}^-_{m+2},\quad x^-=L^-_m f^-+z_1h^-_{m+1}(\chi).$$

Then the second equation yields

$$\chi'=\langle\varphi^-,f^-\rangle-[\mathcal{G}_{\mathrm{A}}\xi_y]_m\quad\text{(Lemma 6.1)}.$$

Finally, by the third equation

$$\xi_x=\mathfrak{M}_d\xi_y+\eta(\chi).$$

As a result $(\Gamma^{\mathrm{A}\,-})^{[+]}=(\Gamma^{\mathrm{A}\,-})^{-1}$. □

Let

$$\Gamma^-_0(f^{\#-}+h^-_{m+1}(c)):=c,\quad\Gamma^-_1(f^{\#-}+h^-_{m+1}(c)):=\langle\varphi^-,f^{\#-}\rangle\tag{7.2}$$

for $f^{\#-}+h^-_{m+1}(c)\in\operatorname{dom}L^-_{\max}$. The above proof also shows that:

Corollary 7.2. $(\mathbb{C}^d,\Gamma^-_0,\Gamma^-_1)$ *is an OBT for* $L^-_{\max}$.

We are now ready to state the main realization theorem in the A-model for the symmetric operator $L^-_{\min}$, by assuming (3.1) and $P^-_{n+1}\subseteq P^-_n$, $n\in\mathbb{Z}$.

Theorem 7.3. *The extensions to* $\mathcal{H}'^-_{\mathrm{A}}$ *of a densely defined, closed, symmetric operator* $L^-_{\min}=L_{\min}\cap(\mathfrak{H}^-_m)^2$ *in* $\mathfrak{H}^-_m$*, which has defect numbers* (d,d) *and which is essentially self-adjoint in* $\mathfrak{H}^-_0$*, are described by the proper extensions in* $\mathcal{H}'^-_{\mathrm{A}}$ *of a nondensely defined (in general), closed, symmetric operator* $A^-_{\min}=A'_{\min}\cap(\mathfrak{H}^-_m\oplus\mathbb{C}^d)^2$*. A proper extension* A^-_Θ *is characterized by restricting the adjoint linear relation* $A^-_{\max}=A^{-\,*}_{\min}$ *in* $\mathcal{H}'^-_{\mathrm{A}}$ *to the set of* $\widehat{f}^-\in A^-_{\max}$ *such that the pair* $(\Gamma^{\mathrm{A}\,-}_0\widehat{f}^-,\Gamma^{\mathrm{A}\,-}_1\widehat{f}^-)$ *is an element of a linear relation* Θ *in* $\mathbb{C}^d$*; an OBT* $(\mathbb{C}^d,\Gamma^{\mathrm{A}\,-}_0,\Gamma^{\mathrm{A}\,-}_1)$ *for* $A^-_{\max}$ *is as in (7.1). The Krein–Naimark resolvent formula for a (closed) proper extension* A^-_Θ *reads*

$$(A^-_\Theta-z)^{-1}=(A^-_0-z)^{-1}+\gamma^-_{\mathrm{A}}(z)(\Theta-M^-_{\mathrm{A}}(z))^{-1}\gamma^-_{\mathrm{A}}(\overline{z})^*$$

for $z \in \operatorname{res} A_0^- \cap \operatorname{res} A_\Theta^-$. *A distinguished self-adjoint extension* A_0^- *of* $A_{\min}^-$ *is a self-adjoint operator* $A_0^- := A_{\{0\}\times\mathbb{C}^d}^-$ *whose resolvent is given by*

$$(A_0^- - z)^{-1} = (L_m^- - z)^{-1} \oplus (\mathfrak{M}_d - z)^{-1}$$

for $z \in \operatorname{res} A_0^- = \operatorname{res} L_m^- \setminus \{z_1\}$. *The* γ*-field* γ_{A}^- *and the Weyl function* M_{A}^- *corresponding to* $(\mathbb{C}^d, \Gamma_0^{\mathrm{A}\,-}, \Gamma_1^{\mathrm{A}\,-})$ *are described by*

$$\gamma_{\mathrm{A}}^-(z) = \big((L_m^- - z_1)(L_m^- - z)^{-1} h_{m+1}^-(\cdot), -(\mathfrak{M}_d - z)^{-1}\eta(\cdot)\big) \quad \text{on } \mathbb{C}^d,$$

$$M_{\mathrm{A}}^-(z) = q^-(z) + r(z) \quad \text{on } \mathbb{C}^d$$

for $z \in \operatorname{res} A_0^-$. *The matrix valued function* q^- *given by*

$$q^-(z) = ([q^-(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad z \in \operatorname{res} L_m^-,$$
$$[q^-(z)]_{\sigma\sigma'} := (z - z_1)\,\langle \varphi_\sigma^-, (L_m^- - z)^{-1} h_{\sigma',m+1}^- \rangle$$

is the Weyl function which corresponds to the OBT $(\mathbb{C}^d, \Gamma_0^-, \Gamma_1^-)$, (7.2), *for the adjoint operator* $L_{\max}^- = L_{\min}^{-\,*}$ *in* $\mathfrak{H}_m^-$.

Proof. In view of what has been achieved so far, it remains to compute the γ-field and the Weyl function. But these functions follow straightforwardly from their definitions as long as one notices that the eigenspace of $A_{\max}^-$ for the eigenvalue $z \in \operatorname{res} L_m^- \setminus \{z_1\}$ consists of $(f^{\#\,-} + h_{m+1}^-(c), \xi) \in \operatorname{dom} A_{\max}^-$ such that

$$f^{\#\,-} = (z - z_1)(L_m^- - z)^{-1} h_{m+1}^-(c), \quad \xi = -(\mathfrak{M}_d - z)^{-1}\eta(c).$$

Because $L_{\max}^- = A_{\max}^- \cap (\mathfrak{H}_m^- \oplus \{0\})^2$, the results for $L_{\max}^-$ are derived analogously. $\square$

In particular, putting $P_n^- = I_{\mathfrak{H}_n}$ (hence $P_n^+ = 0$), $n \in \mathbb{Z}$, the part of the theorem concerning the Weyl function q^- yields the following:

Corollary 7.4. *The Krein Q-function* q *is the Weyl function associated with the OBT* $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$,

$$\Gamma_0(f^\# + h_{m+1}(c)) := c, \quad \Gamma_1(f^\# + h_{m+1}(c)) := \langle \varphi, f^\# \rangle$$

$(f^\# \in \mathfrak{H}_{m+2}, c \in \mathbb{C}^d)$, *for the adjoint* $L_{\min}^*$ *of* $L_{\min}$ *in* $\mathfrak{H}_m$. *The domain* $\operatorname{dom} L_{\min}^* = \mathfrak{H}_{m+2} \dotplus \mathfrak{N}_z(L_{\min}^*)$, *where the eigenspace* $\mathfrak{N}_z(L_{\min}^*) = (L - z)^{-1} h_m(\mathbb{C}^d)$, $z \in \operatorname{res} L$. $\square$

An analogous theorem can be formulated for $L_{\min}^+$ as well, where the corresponding Weyl function $M_{\mathrm{A}}^+ = q^+ + r$ is the sum of the Weyl function q^+ of $L_{\min}^+$ and the generalized Nevanlinna function r.

Let

$$\widehat{h}_\sigma := b_{m+2}(L)^{-1/2}\varphi_\sigma \in \mathfrak{H}_0 \setminus \mathfrak{H}_1.$$

Using this definition and the operator identity

$$(L - z_1)(L - z)^{-1} = I_{\mathfrak{H}_0} + (z - z_1)(L - z)^{-1}$$

the Weyl function q is rewritten in terms of the initial operator L and the reference $\mathfrak{H}_0$-scalar product according to

$$[q(z)]_{\sigma\sigma'} = (z - z_1)\,\langle\widehat{h}_\sigma, \widehat{h}_{\sigma'}\rangle_0 + (z - z_1)^2\,\langle\widehat{h}_\sigma, (L - z)^{-1}\widehat{h}_{\sigma'}\rangle_0\,,$$

$z \in \operatorname{res} L$. Using in addition (5.3) and applying [28, Proposition 5.26] and Lemma 5.5, the Weyl function q^- admits the form

$$[q^-(z)]_{\sigma\sigma'} = (z - z_1)\,\langle\widehat{h}_\sigma, P_0^-(m)\widehat{h}_{\sigma'}\rangle_0 + (z - z_1)^2\,\langle\widehat{h}_\sigma, P_0^-(m)(L - z)^{-1}\widehat{h}_{\sigma'}\rangle_0\,,$$

$z \in \operatorname{res} L$, and similarly for q^+. Thus the Weyl function $q = q^- + q^+$ of the symmetric operator $L_{\min}$ is the sum of the Weyl functions $q^\pm$ of the corresponding symmetric restrictions $L^\pm_{\min}$. The latter property of additivity is clearly a consequence of the initial hypothesis that the subspaces $\mathfrak{H}_0^\pm$ reduce the operator L (Theorem 5.6).

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