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ON 2-RAINBOW DOMINATION NUMBER OF FUNCTIGRAPH AND ITS COMPLEMENT

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Abstract. Let G be a graph and $f : V(G) \to P(\{1,2\})$ be a function where for every vertex $v \in V(G)$, with $f(v) = \emptyset$ we have $\bigcup_{u \in N_G(v)} f(u) = \{1,2\}$. Then f is a 2-rainbow dominating function or a $2RDF$ of G. The weight of f is $\omega(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of all 2-rainbow dominating functions is 2-rainbow domination number of G, denoted by $\gamma_{r2}(G)$. Let G_1 and G_2 be two copies of a graph G with disjoint vertex sets $V(G_1)$ and $V(G_2)$, and let σ be a function from $V(G_1)$ to $V(G_2)$. We define the functigraph $C(G,\sigma)$ to be the graph that has the vertex set $V(C(G,\sigma)) = V(G_1) \cup V(G_2)$, and the edge set $E(C(G,\sigma)) = E(G_1) \cup E(G_2) \cup \{uv; u \in V(G_1), v \in V(G_2), v = \sigma(u)\}$. In this paper, 2-rainbow domination number of the functigraph of $C(G,\sigma)$ and its complement are investigated. We obtain a general bound for $\gamma_{r2}(C(G,\sigma))$ and we show that this bound is sharp.

Keywords: 2-rainbow domination number, functigraph, complement, cubic graph.

Mathematics Subject Classification: 05C69, 05C75.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple, finite and undirected graph. The *open neighborhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of vertices adjacent to v in G. The *closed neighborhood* of a vertex v is G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) =| N_G(v) |$. The *maximum* degree and *minimum* degree are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A vertex is called *universal* vertex if its degree is $|V(G)| - 1$.

The *complement* of graph G is denoted by $\overline{G}$ is a graph with vertex set $V(G)$ which $e \in E(\overline{G})$ if and only if $e \notin E(G)$. For any $S \subseteq V(G)$, the *induced subgraph* on S, denoted by $G[S]$.

Let $f : V(G) \to P(\{1,2\})$ be a function where for every vertex $v \in V(G)$, with $f(v) = \emptyset$ we have $\bigcup_{u \in N_G(v)} f(u) = \{1,2\}$. Then f is a 2-rainbow dominating function or a $2RDF$ of G. The weight of f is $\omega(f) = \sum_{v\in V(G)} |f(v)|$. The minimum weight of all 2-rainbow dominating functions is 2-rainbow domination number of G, denoted by $\gamma_{r2}(G)$.

Let G_1 and G_2 be two disjoint copies of graph G and $\sigma : V(G_1) \to V(G_2)$ be a function, where $V(G_1) = \{v_1, v_2, \dots, v_n\}$ and $V(G_2) = \{u_1, u_2, \dots, u_n\}$. Then a *functigraph* of G with function σ is denoted by $C(G, \sigma)$, has vertex set

$$V(C(G,\sigma)) = V(G_1) \cup V(G_2)$$

and edge set

$$E(C(G,\sigma)) = E(G_1) \cup E(G_2) \cup \{vu; v \in V(G_1), u \in V(G_2), \sigma(v) = u\}.$$

For $u \in V(G_2)$,

$$R_u = \{v \in V(G_1); \sigma(v) = u\}$$

and for $\ell \in \{0, 1, 2, \dots, n = |V(G)|\}$, we define

$$B_\ell = \{u \in V(G_2); |R_u| = \ell\}.$$

For simplicity the open neighbourhood of x in $C(G,\sigma)$ (or in $\overline{C(G,\sigma)}$) is denoted by $N_C(x)$ (or $N_{\overline{C}}(x)$).

In recent years much attention drawn to the domination theory which is very interesting branch in graph theory. Recently, the concept of domination expanded to other parameters of domination such as signed domination, Roman domination and rainbow domination. For more details see [3, 8, 12]. In [17], Wu and Xing obtained sharp lower and upper bounds for $\gamma_{r2}(G) + \gamma_{r2}(\overline{G})$. In 2013, Wu and Jafari Rad proved that if G is a connected graph of order $n \geq 3$, then $\gamma_{r2}(G) \leq \frac{3n}{4}$ (see [15]). In [16], a conjecture was posted regarding generalized Peterson graphs and it was answered in [9].

These motivated us to consider the 2-rainbow domination number of the functigraph and its complement. For this aim we obtain a general bound of $\gamma_{r2}(C(G,\sigma))$ for any graph G and we discuss the tightness of this bound. Also we investigate $\gamma_{r2}(\overline{C(G,\sigma)})$.

2. PRELIMINARIES

For investigating the 2-rainbow domination number of functigraph, the following basic properties are useful.

Lemma 2.1 ([4]). $\gamma_{r2}(P_n) = \left\lfloor \dfrac{n}{2} \right\rfloor + 1.$

Lemma 2.2 ([4]). *For $n \geq 3$, $\gamma_{r2}(C_n) = \left\lfloor \dfrac{n}{2} \right\rfloor + \left\lceil \dfrac{n}{4} \right\rceil - \left\lfloor \dfrac{n}{4} \right\rfloor$.*

Lemma 2.3. *Let G be a graph of order n. Then $\gamma_{r2}(G) = 1$ if and only if $n = 1$.*

Proof. If $n = 1$, then the proof is straightforward. Conversely, let $n \geq 2$, $\gamma_{r2}(G) = 1$ and f be a $2RDF$ of G such that $|f(v)| = 1$ and $f(x) = \emptyset$ for every $x \in V(G) \setminus \{v\}$. Then $\bigcup_{y \in N_G(x)} f(y) \neq \{1, 2\}$, where $x \in V(G) \setminus \{v\}$. This is impossible. □

Lemma 2.4. *Let G be a graph and w be an universal vertex of G. Then $\gamma_{r2}(G) = 2$.*

Proof. Let $f : V(G) \to P(\{1,2\})$ be a function where $f(w) = \{1,2\}$ and $f(x) = \emptyset$, for every $x \in V(G) \setminus \{w\}$. Then f is a $2RDF$ of G. Hence, $\gamma_{r2}(G) \le 2$. Since $n \ge 2$, so by Lemma 2.3, $\gamma_{r2}(G) = 2$. □

Lemma 2.5. *Let G be a graph of order $n = 1$. Then $\gamma_{r2}(C(G,\sigma)) = \gamma_{r2}(\overline{C(G,\sigma)}) = 2$.*

Proof. It is clear that if $n = 1$, then $C(G,\sigma) \cong P_2$. By Lemma 2.1, $\gamma_{r2}(P_2) = 2$ and so $\gamma_{r2}(\overline{P_2}) = 2$. □

Lemma 2.6. *For any graph G, $2 \le \gamma_{r2}(\overline{C(G,\sigma)}) \le 5$.*

Proof. Let $B_1 \neq \emptyset$, $u \in B_1$ and $R_u = \{v\}$. Also let $g : V(\overline{C(G,\sigma)}) \to P(\{1,2\})$ be a function with $g(u) = g(v) = \{1,2\}$ and for every $x \in V(G_1) \cup V(G_2) \setminus \{u,v\}$, $g(x) = \emptyset$. Then g is a $2RDF$ of $\overline{C(G,\sigma)}$. Hence, $\gamma_{r2}(\overline{C(G,\sigma)}) \le \omega(g) = 4$.

Let $B_1 = \emptyset$. Then $B_0 \neq \emptyset$. Assume that $u \in B_0$, $v \in V(G_1)$ and let $g : V(\overline{C(G,\sigma)}) \to P(\{1,2\})$ be a function such that $g(u) = g(v) = \{1,2\}$, $g(\sigma(v)) = \{1\}$ and for every $x \in V(G_1) \cup V(G_2) \setminus \{u,v,\sigma(v)\}$, $g(x) = \emptyset$. Then g is a $2RDF$ for $\overline{C(G,\sigma)}$. Therefore, $\gamma_{r2}(\overline{C(G,\sigma)}) \le \omega(g) = 5$. By Lemma 2.3, $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{2,3,4,5\}$. □

Lemma 2.7. *Let G be a graph and there is $u \in V(G_2)$ such that $G[N_{G_2}(u)]$ has one isolated vertex. Then $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{2,3,4\}$.*

Proof. Let $u_0 \in V(G_2)$ be an isolated vertex of $G[N_{G_2}(u)]$. Assume that $f : V(\overline{C(G,\sigma)}) \to P(\{1,2\})$ be a function such that $f(u) = f(u_0) = \{1,2\}$ and $f(x) = \emptyset$, for every $x \in V(G_1) \cup V(G_2) \setminus \{u,u_0\}$. If $x \in (V(G_2) \setminus N_{G_2}(u)) \cup (V(G_1) \setminus R_u)$, then

$$\bigcup_{y \in N_{\overline{C}}(x)} f(y) = f(u) = \{1,2\}$$

and if $x \in N_{G_2}(u) \cup R_u$, then

$$\bigcup_{y \in N_{\overline{C}}(x)} f(y) = f(u_0) = \{1,2\}.$$

Therefore, f is a $2RDF$ of $\overline{C(G,\sigma)}$ and so $\gamma_{r2}(\overline{C(G,\sigma)}) \le \omega(f) = 4$. By Lemma 2.3, $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{2,3,4\}$. □

Lemma 2.8. *Let G be a bipartite graph. Then $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{2,3,4\}$.*

Proof. Let $V(G_2) = X \cup Y$, $a \in X$ and $b \in Y$. Also assume that $g : V(\overline{C(G,\sigma)}) \to P(\{1,2\})$ be a function such that $g(a) = g(b) = \{1,2\}$ and $g(x) = \emptyset$ for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b\}$. For every $x \in V(G_1)$, we have $x \in N_C(a)$ or $x \in N_C(b)$ or $x \in N_C(y)$, where $y \in V(G_2) \setminus \{a,b\}$. Thus, $x \in N_{\overline{C}}(b)$ or $x \in N_{\overline{C}}(a)$ or $x \in N_{\overline{C}}(a) \cap N_{\overline{C}}(b)$, respectively. So $\bigcup_{z \in N_{\overline{C}}(x)} g(z) = \{1,2\}$. On the other hand, for every $x \in V(G_2)$, $x \in X$ or $x \in Y$. So $x \in N_{\overline{C}}(a)$ or $x \in N_{\overline{C}}(b)$. Hence, $\bigcup_{z \in N_{\overline{C}}(x)} g(z) = \{1,2\}$. Therefore, g is a $2RDF$ of $\overline{C(G,\sigma)}$ and so $\gamma_{r2}(\overline{C(G,\sigma)}) \le \omega(g) = 4$. Lemma 2.6 completes the proof. □

3. 2-RAINBOW DOMINATION NUMBER OF FUNCTIGRAPH

In the following theorem we obtain a tight bound of $\gamma_{r2}(C(G,\sigma))$ for any graph G.

Theorem 3.1. *For any graph G,*

$$\gamma_{r2}(G) \leq \gamma_{r2}(C(G,\sigma)) \leq 2\gamma_{r2}(G).$$

Furthermore, these bounds are sharp.

Proof. Let $f_i : V(G_i) \to P(\{1,2\})$ be a $2RDF$ for G_i and $\gamma_{r2}(G_i) = \omega(f_i)$, where $i \in \{1,2\}$. Also let $f : V(G_1) \cup V(G_2) \to P(\{1,2\})$ where if $x \in V(G_i)$, then $f(x) = f_i(x)$ for $i \in \{1,2\}$. Clearly, f is a $2RDF$ of $C(G,\sigma)$. So

$$\gamma_{r2}(C(G,\sigma)) \leq \omega(f) = \omega(f_1) + \omega(f_2) = 2\gamma_{r2}(G).$$

Now, let g be a $2RDF$ of $C(G,\sigma)$ such that $\omega(g) = \gamma_{r2}(C(G,\sigma))$. Define

$$\begin{aligned} S_1 &= \Big\{u \in V(G_2); g(u) \neq \emptyset\Big\}, \\ S_2 &= \Big\{u \in V(G_2); g(u) = \emptyset, \bigcup_{u_k \in N_{G_2}(u)} g(u_k) = \{1,2\}\Big\}, \\ S_3 &= \Big\{u \in V(G_2); g(u) = \emptyset, \bigcup_{u_k \in N_{G_2}(u)} g(u_k) \neq \{1,2\}\Big\}. \end{aligned}$$

If $u \in S_3$, then there exists $v \in V(G_1)$ such that $\sigma(v) = u$ and $|g(v)| \geq 1$. Hence,

$$\sum_{v_i \in V(G_1)} |g(v_i)| \geq |S_3|.$$

Suppose that $f : V(G_2) \to P(\{1,2\})$ where $f(x) = g(x)$ for every $x \in S_1 \cup S_2$ and $f(x) = \{1\}$ for every $x \in S_3$. Clearly, f is a $2RDF$ of G_2. Then we have

$$\begin{aligned} \gamma_{r2}(C(G,\sigma)) = \omega(g) &= \sum_{v_i \in V(G_1)} |g(v_i)| + \sum_{u_i \in V(G_2)} |g(u_i)| \\ &= \sum_{v_i \in V(G_1)} |g(v_i)| + \sum_{u_i \in S_1 \cup S_2} |g(u_i)| + \sum_{u_i \in S_3} |g(u_i)| \\ &= \sum_{v_i \in V(G_1)} |g(v_i)| + \sum_{u_i \in S_1 \cup S_2} |g(u_i)| + 0 \\ &= \sum_{v_i \in V(G_1)} |g(v_i)| + \sum_{u_i \in S_1 \cup S_2} |f(u_i)| + |S_3| - |S_3| \\ &= \sum_{v_i \in V(G_1)} |g(v_i)| - |S_3| + \sum_{u_i \in S_1 \cup S_2} |f(u_i)| + \sum_{u_i \in S_3} |f(u_i)| \\ &= \sum_{v_i \in V(G_1)} |g(v_i)| - |S_3| + \omega(f) \geq \omega(f) \geq \gamma_{r2}(G_2) = \gamma_{r2}(G). \end{aligned}$$

Therefore,

$$\gamma_{r2}(G) \leq \gamma_{r2}(C(G,\sigma)) \leq 2\gamma_{r2}(G).$$

It is easy to see that if σ is a permutation, then $C(P_2,\sigma) \cong C_4$. Since $\gamma_{r2}(P_2) = 2$ and $\gamma_{r2}(C_4) = 2$, so $\gamma_{r2}(P_2) = \gamma_{r2}(C(P_2,\sigma))$. Also we know that $\gamma_{r2}(\overline{K_n}) = n$ and $C(\overline{K_n}, id) \cong nP_2$. Hence, $\gamma_{r2}(C(\overline{K_n}, id)) = 2\gamma_{r2}(\overline{K_n})$. Thus, the bounds are sharp. □

Theorem 3.2. *Let G be a graph of order n and $B_n = \{u\}$. Then*

$$\gamma_{r2}(G) \leq \gamma_{r2}(C(G,\sigma)) \leq \gamma_{r2}(G) + 2.$$

Proof. Let f be a $2RDF$ of G such that $\gamma_{r2}(G)=\omega(f)$. Define $g: V(C(G,\sigma)) \to P(\{1,2\})$ such that $g(u) = \{1,2\}$, $g(x) = \emptyset$ for every $x \in V(G_1)$ and $g(y) = f(y)$ for $y \in V(G_2) \setminus \{u\}$. Then for $y \in V(G_2) \setminus \{u\}$ we have

$$\bigcup_{u' \in N_C(y)} g(u') = \bigcup_{u' \in N_{G_2}(y)} f(u') = \{1,2\}$$

and for every $v \in V(G_1)$,

$$\bigcup_{x \in N_C(v)} g(x) = g(u) = \{1,2\}.$$

So g is a $2RDF$ of $C(G,\sigma)$. Hence, $\gamma_{r2}(C(G,\sigma)) \leq \omega(g)$. Now if $f(u) = \{1,2\}$ or $|f(u)| = 1$, then $\omega(g) = \omega(f)$ or $\omega(g) = 1 + \omega(f)$, respectively. Also $\omega(g) = 2 + \omega(f)$, if $f(u) = \emptyset$. Hence, by Theorem 3.1, we have

$$\gamma_{r2}(C(G,\sigma)) \in \{\gamma_{r2}(G), 1 + \gamma_{r2}(G), 2 + \gamma_{r2}(G)\}.$$ □

Theorem 3.3. *Let G be a graph of order $n \geq 3$ such that has a universal vertex. Then $\gamma_{r2}(G) = \gamma_{r2}(C(G,\sigma))$ if and only if $B_n = \{w\}$, where w is an universal vertex of G_2.*

Proof. Let w be an universal vertex of G_2 and $B_n = \{w\}$. Then w is an universal vertex of $C(G,\sigma)$. By Lemma 2.4, $\gamma_{r2}(G) = \gamma_{r2}(G_2) = \gamma_{r2}(C(G,\sigma)) = 2$.

Conversely, let $\gamma_{r2}(G) = \gamma_{r2}(C(G,\sigma))$. By Lemma 2.4, $\gamma_{r2}(G) = 2$ and so $\gamma_{r2}(C(G,\sigma)) = 2$. Assume that g be a $2RDF$ of $C(G,\sigma)$ such that $\omega(g) = 2$. Let a and b be two vertices in $V(G_1) \cup V(G_2)$ such that $|g(a)| = |g(b)| = 1$. Then $g(x) = \emptyset$, for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b\}$. Hence, every vertex in $V(G_1) \cup V(G_2) \setminus \{a,b\}$ is adjacent to a and b, which is impossible. Now let $a \in V(G_1) \cup V(G_2)$ and $g(a) = \{1,2\}$. Then $g(x) = \emptyset$, for every $x \in V(G_1) \cup V(G_2) \setminus \{a\}$. So every vertex in $V(G_1) \cup V(G_2) \setminus \{a\}$ is adjacent to a. Hence, a is a universal vertex of G_2 and $B_n = \{a\}$. □

Theorem 3.4. *Let G be a graph of order $n \geq 4$ and has an universal vertex. Then:*

(1) $\gamma_{r2}(C(G,\sigma)) \in \{2,3,4\}$,
(2) $\gamma_{r2}(C(G,\sigma)) = 2$ *if and only if* $B_n = \{w\}$, *where w is an universal vertex of* G_2,
(3) $\gamma_{r2}(C(G,\sigma)) = 3$ *if and only if* $B_n = \{a\}$ *and* $\deg_{G_2}(a) = n-2$ *or* $B_{n-1} = \{a\}$ *and* $\deg_{G_2}(a) = n-1$ *or* $B_{n-1} = \{a\}$, $R_a = G_1 \setminus \{b\}$, $N_{G_1}(b) = G_1 \setminus \{b\}$, $G_2 \setminus \{a, \sigma(b)\} \subseteq N_{G_2}(c)$ *and* $G_2 \setminus \{c\} \subseteq N_{G_2}(a)$, *for some* $c \in V(G_2)$.

Proof. (1) Let v and u be two universal vertices of G_1 and G_2, respectively. Define $g : V(C(G,\sigma)) \to P(\{1,2\})$ such that $g(v) = g(u) = \{1,2\}$ and $g(x) = \emptyset$ for every $x \in V(G_1) \cup V(G_2) \setminus \{u,v\}$. So g is a $2RDF$ of $C(G,\sigma)$ and so $\gamma_{r2}(C(G,\sigma)) \leq \omega(g) = 4$. By Lemma 2.3, $\gamma_{r2}(C(G,\sigma)) \in \{2,3,4\}$.

(2) This is the result of Lemma 2.4 and Theorem 3.3.

(3) Let $\gamma_{r2}(C(G,\sigma)) = 3$ and g be a $2RDF$ of $C(G,\sigma)$, such that $\gamma_{r2}(C(G,\sigma)) = \omega(g) = 3$. Then there are two following cases:

Case 1. Let $a,b \in V(G_1) \cup V(G_2)$, $g(a) = \{1,2\}$, $|g(b)| = 1$ and $g(x) = \emptyset$, for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b\}$. Then every vertex in $V(G_1) \cup V(G_2) \setminus \{a,b\}$, is adjacent to a. So $a \in V(G_2)$. If $b \in V(G_2)$, then $B_n = \{a\}$ and by item (2), a is not a universal vertex of G_2. Hence, $b \notin N_{G_2}(a)$. So $\deg_{G_2}(a) = n-2$. If $b \in V(G_1)$, then a is a universal vertex of G_2 and by item (2), $\sigma(b) \neq a$. Hence, $B_{n-1} = \{a\}$ and $\deg_{G_2}(a) = n-1$.

Case 2. Let $a,b,c \in V(G_1) \cup V(G_2)$ and $|g(a)| = |g(b)| = |g(c)| = 1$. Also assume that $g(a) = \{2\}$, $g(b) = g(c) = \{1\}$ and $g(x) = \emptyset$, for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b,c\}$ (or $g(a) = \{1\}$ and $g(b) = g(c) = \{2\}$). Then every vertex in $V(G_1) \cup V(G_2) \setminus \{a,b,c\}$ is adjacent to a. So $a \in V(G_2)$ and $G_2 \setminus \{b,c\} \subseteq N_{G_2}(a)$. Since g is a $2RDF$ of $C(G,\sigma)$, so $|\{b,c\} \cap V(G_1)| \geq 1$. If $\{b,c\} \subseteq V(G_1)$, then there exists $x \in V(G_2) \setminus \{a,\sigma(b),\sigma(c)\}$ such that $\bigcup_{y \in N_C(x)} g(y) = \{2\}$, which is a contradiction. Thus $|\{b,c\} \cap V(G_i)| = 1$, for $i \in \{1,2\}$. Without loss of generality, let $b \in V(G_1)$ and $c \in V(G_2)$. Then $G_2 \setminus \{c\} \subseteq N_{G_2}(a)$ and so $\deg_{G_2}(a) \geq n-2$. If $\sigma(b) = a$, then $B_n = \{a\}$ and by item (2), a is not an universal vertex of G_2. So $c \notin N_{G_2}(a)$. Hence, $\deg_{G_2}(a) = n-2$. If $\sigma(b) \neq a$, then $R_a = G_1 \setminus \{b\}$, $B_{n-1} = \{a\}$ and $\sigma(b) \in N_{G_2}(a)$ (when $\sigma(b) \neq c$). Since g is a $2RDF$ and $g(x) = \emptyset$, for $x \in V(G_1) \cup V(G_2) \setminus \{a,b,c\}$, so $V(G_2) \setminus \{\sigma(b),a\} \subseteq N_{G_2}(c)$ and $N_{G_1}(b) = G_1 \setminus \{b\}$.

Conversely, let $B_n = \{a\}$, $\deg_{G_2}(a) = n-2$ and $b \in V(G_2) \setminus N_{G_2}(a)$. Then $g : V(G_1) \cup V(G_2) \to P(\{1,2\})$ with $g(a) = \{1,2\}$, $g(b) = \{1\}$ and $g(x) = \emptyset$, for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b\}$, is a $2RDF$ of $C(G,\sigma)$. So $\gamma_{r2}(C(G,\sigma)) \leq \omega(g) = 3$. By item (2) and Lemma 2.3, $\gamma_{r2}(C(G,\sigma)) = 3$.

Now suppose that $B_{n-1} = \{a\}$ and $\deg_{G_2}(a) = n-1$. Also let $b \in V(G_1)$ and $\sigma(b) \neq a$. Then define $g(a) = \{1,2\}$, $g(b) = \{1\}$ and $g(x) = \emptyset$, for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b\}$. So g is a $2RDF$ of $C(G,\sigma)$ and so $\gamma_{r2}(C(G,\sigma)) \leq \omega(g) = 3$. By item (2) and Lemma 2.3, $\gamma_{r2}(C(G,\sigma)) = 3$.

Finally, let $B_{n-1} = \{a\}$, $R_a = G_1 \setminus \{b\}$, $N_{G_1}(b) = G_1 \setminus \{b\}$, $G_2 \setminus \{a,\sigma(b)\} \subseteq N_{G_2}(c)$ and $G_2 \setminus \{c\} \subseteq N_{G_2}(a)$, for some $c \in V(G_2)$. Define $g(a) = \{2\}$, $g(b) = g(c) = \{1\}$ and $g(x) = \emptyset$, for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b,c\}$, then g is a $2RDF$ of $C(G,\sigma)$. For this reason $\gamma_{r2}(C(G,\sigma)) \leq \omega(g) = 3$. Again, by item (2) and Lemma 2.3, give the result. □

Corollary 3.5. *Let $n \geq 4$ and $G \cong K_n$. Then*

(1) $\gamma_{r2}(C(G,\sigma)) \in \{2,3,4\}$,
(2) $\gamma_{r2}(C(G,\sigma)) = 2$ *if and only if* $|B_n| = 1$,
(3) $\gamma_{r2}(C(G,\sigma)) = 3$ *if and only if* $|B_{n-1}| = 1$.

Proof. By Theorem 3.4, the proof is straightforward. □

In the mathematical field of graph theory, the friendship graph (or Dutch windmill graph) K_3^m is a graph with $2m+1$ vertices and $3m$ edges. The friendship graph K_3^m can be constructed by joining m copies of the cycle graph C_3 with a common vertex. Also fan graph F_n is isomorphic to corona product $K_1 \circ P_n$ and wheel graph W_n is isomorphic to corona product $K_1 \circ C_n$.

Corollary 3.6. *Let $n \geq 5$, $m \geq 3$, $G \in \{F_n, W_n, K_{1,n}, K_3^m\}$ and w be an universal vertex of G. Then*

(1) $\gamma_{r2}(C(G,\sigma)) \in \{2,3,4\}$,
(2) $\gamma_{r2}(C(G,\sigma)) = 2$ *if and only if* $B_n = \{w\}$,
(3) $\gamma_{r2}(C(G,\sigma)) = 3$ *if and only if* $B_{n-1} = \{w\}$.

Proof. Since G does not have any vertex of degree $n-2$, by Theorem 3.4, the proof is straightforward. □

4. 2-RAINBOW DOMINATION NUMBER OF COMPLEMENT OF FUNCTIGRAPH

In this section, we investigate 2-rainbow domination number of complement of functigraph.

Theorem 4.1. *Let G be graph and $\delta(G) \geq 1$. Then $\gamma_{r2}(\overline{C(G,\sigma)}) = 2$ if and only if G has P_2 as a component and $V(P_2) \cap R(\sigma) = \emptyset$, where $R(\sigma)$ is the image of σ.*

Proof. Let $\gamma_{r2}(\overline{C(G,\sigma)}) = 2$ and g be a $2RDF$ of $\overline{C(G,\sigma)}$, where $\omega(g) = 2$. Then there is $a \in V(\overline{C(G,\sigma)})$ such that $g(a) = \{1,2\}$ or there are $a,b \in V(G_1) \cup V(G_2)$ such that $g(a) = \{1\}$ and $g(b) = \{2\}$ (or $g(a) = \{2\}$ and $g(b) = \{1\}$).

Let $g(a) = \{1,2\}$. Then every vertex in $V(G_1) \cup V(G_2) \setminus \{a\}$ is adjacent to a in $\overline{C(G,\sigma)}$. So a is an isolated vertex of G. This is contradiction by $\delta(G) \geq 1$.

Let $a,b \in V(G_1) \cup V(G_2)$, such that $g(a) = \{1\}$ and $g(b) = \{2\}$. If $a \in V(G_1)$ and $b \in V(G_2)$, then all of the vertices in $V(G_1) \cup V(G_2) \setminus \{a,b\}$ are adjacent to a and b in $\overline{C(G,\sigma)}$. It follows that a is an isolated vertex in G, which is impossible. If $a,b \in V(G_1)$, then all of the vertices $V(G_2)$ are adjacent to a and b in $\overline{C(G,\sigma)}$. That is impossible. Let $a,b \in V(G_2)$. Since

$$V(G_1) \cup V(G_2) \setminus \{a,b\} \subseteq N_{\overline{C}}(a) \cap N_{\overline{C}}(b),$$

so $a,b \in B_0$ of $C(G,\sigma)$. Since $\delta(G) \geq 1$, so $\deg_{G_2}(a) = \deg_{G_2}(b) = 1$ and a is adjacent to b. Thus, P_2 is a component of G and $V(P_2) \cap R(\sigma) = \emptyset$.

Conversely, let P_2 be a component of G, $V(P_2) = \{a,b\}$ and $V(P_2) \cap R(\sigma) = \emptyset$. Then all of the vertices $V(G_1) \cup V(G_2) \setminus \{a,b\}$ in $\overline{C(G,\sigma)}$ are adjacent to a and b. Suppose that $g : V(G_1) \cup V(G_2) \to P(\{1,2\})$ where $g(a) = \{1\}$, $g(b) = \{2\}$ and $g(x) = \emptyset$ for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b\}$. Then g is a $2RDF$ of $\overline{C(G,\sigma)}$ and so $\gamma_{r2}(\overline{C(G,\sigma)}) \leq \omega(g) = 2$. By Lemma 2.6, $\gamma_{r2}(\overline{C(G,\sigma)}) = 2$. □

Corollary 4.2. *If G is a tree of order $n \geq 4$, then $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{3,4\}$.*

Proof. By Lemma 2.8, $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{2,3,4\}$. Since G is a tree of order at least 4, so G does not have P_2 as a component and so by Theorem 4.1, $\gamma_{r2}(\overline{C(G,\sigma)}) \neq 2$. Therefore, $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{3,4\}$. □

Corollary 4.3. *For any connected graph G of order $n \geq 3$, $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{3,4,5\}$.*

Proof. Since G is a connected graph, so G does not have P_2 as a component. By Lemma 2.3 and Theorem 4.1, $\gamma_{r2}(\overline{C(G,\sigma)}) \in \{3,4,5\}$. □

Theorem 4.4. *Let G be a graph of order n with $\delta(G) \geq 1$ and P_2 is not a component of G. Then $\gamma_{r2}(\overline{C(G,\sigma)}) = 3$ if and only if one of the following items holds:*

(1) *there exists $a \in V(G_1)$ such that $N_{G_1}(a) = \{b\}$ and $N_C(b) \cap N_C(\sigma(a)) = \{a\}$,*
(2) *$B_0 \neq \emptyset$, $a \in B_0$ and $\deg_{G_2}(a) = 1$,*
(3) *$B_1 \neq \emptyset$, $a \in B_1$, $N_{G_2}(a) = b$ and $N_{G_1}(v) \cap R_b = \emptyset$, where $R_a = \{v\}$,*
(4) *$B_0 \neq \emptyset$, $a \in B_0$, $N_{G_2}(a) = \{b,c\}$ and $N_{G_2}(b) \cap N_{G_2}(c) = \{a\}$.*

Proof. Let $\gamma_{r2}(\overline{C(G,\sigma)}) = 3$ and g be a $2RDF$ of $\overline{C(G,\sigma)}$ with $\omega(g) = 3$. Then we have the following two cases:

Case 1. There are two vertices a and b in $V(G_1) \cup V(G_2)$ such that $g(a) = \{1,2\}$, $g(b) = \{1\}$ (or $g(b) = \{2\}$) and $g(x) = \emptyset$ for every $x \in V(G_1) \cup V(G_2) \setminus \{a,b\}$. Then all of the vertices in $V(G_1) \cup V(G_2) \setminus \{a,b\}$ are adjacent to a in $\overline{C(G,\sigma)}$. If $a \in V(G_1)$, then $b \in V(G_1)$ and $N_{G_1}(a) = \{b\}$, because $\delta(G_1) \geq 1$. So $\sigma(a)$ is adjacent to a in $\overline{C(G,\sigma)}$, which is impossible. Hence, $a \in V(G_2)$ and $N_{G_2}(a) = \{b\}$. Thus, $a \in B_0$ and $\deg_{G_2}(a) = 1$. This gives (2).

Case 2. There are three vertices a, b and c in $V(G_1) \cup V(G_2)$ such that $g(a) = \{2\}$ and $g(b) = g(c) = \{1\}$. Then all of the vertices in $V(G_1) \cup V(G_2) \setminus \{a,b,c\}$ are adjacent to a in $\overline{C(G,\sigma)}$. So $\deg_{C(G,\sigma)}(a) \in \{1,2\}$. If $\deg_{C(G,\sigma)}(a) = 1$, then $N_C(a) = \{b\}$ or $N_C(a) = \{c\}$. Without loss of generality, let $N_C(a) = \{b\}$. Then $a,b \in V(G_1)$ or $a,b \in V(G_2)$, because $\delta(G) \geq 1$. If $a,b \in V(G_1)$, then $\sigma(a) \in N_C(a)$ and so $b = \sigma(a) \in V(G_2)$. This is not true. So $a,b \in V(G_2)$ and $a \in B_0$. This gives (2). Now suppose that $\deg_{C(G,\sigma)}(a) = 2$. Then $N_C(a) = \{b,c\}$. If $a \in V(G_1)$, then $\sigma(a) = c$ (or $\sigma(a) = b$) and so $N_{G_1}(a) = \{b\}$ (or $N_{G_1}(a) = c$). Also since g is a $2RDF$ of $\overline{C(G,\sigma)}$, $N_C(b) \cap N_C(\sigma(a)) = \{a\}$. This gives (1).

Now let $a \in V(G_2)$. Since $N_C(a) = \{b,c\}$ and $\delta(G_2) \geq 1$, so $N_{G_2}(a) = b$ and $R_a = \{c\}$ or $N_{G_2}(a) = \{b,c\}$. If $N_{G_2}(a) = b$ and $R_a = \{c\}$, then $a \in B_1$, $\deg_{G_2}(a) = 1$ and $N_{G_1}(c) \cap R_b = \emptyset$. This gives (3). If $N_{G_2}(a) = \{b,c\}$, then $a \in B_0$. Furthermore, since g is a $2RDF$ of $\overline{C(G,\sigma)}$, $N_{G_2}(b) \cap N_{G_2}(c) = \{a\}$. This gives (4).

Conversely, let there exists $a \in V(G_1)$ such that $N_{G_1}(a) = \{b\}$ and $N_C(b) \cap N_C(\sigma(a)) = \{a\}$. Then function $g : V(\overline{C(G,\sigma)}) \to P(\{1,2\})$ with $g(a) = \{2\}$, $g(b) = \{1\}$ and $g(\sigma(a)) = \{1\}$ is a $2RDF$ of $\overline{C(G,\sigma)}$. Hence, $\gamma_{r2}(\overline{C(G,\sigma)}) \leq \omega(g) = 3$.

Let $a \in B_0$ and $\deg_{G_2}(a) = 1$. Define $g : V(\overline{C(G,\sigma)}) \to P(\{1,2\})$ such that $g(a) = \{1,2\}$ and $g(b) = \{1\}$. So g is a $2RDF$ of $\overline{C(G,\sigma)}$, where $N_{G_2}(a) = \{b\}$. Therefore, $\gamma_{r2}(\overline{C(G,\sigma)}) \leq \omega(g) = 3$.

Let $a \in B_1$, $N_{G_2}(a) = \{b\}$ and $N_{G_1}(v) \cap R_b = \emptyset$, where $R_a = \{v\}$. Then function $g : V(\overline{C(G,\sigma)}) \to P(\{1,2\})$ where $g(a) = \{2\}$ and $g(b) = g(v) = \{1\}$ is a $2RDF$ of $\overline{C(G,\sigma)}$. So $\gamma_{r2}(\overline{C(G,\sigma)}) \leq \omega(g) = 3$.

Let $a \in B_0$, $N_{G_2}(a) = \{b,c\}$ and $N_{G_2}(b) \cap N_{G_2}(c) = \{a\}$. Let $g(a) = \{2\}$ and $g(b) = g(c) = \{1\}$. Then g is a $2RDF$ of $\overline{C(G,\sigma)}$. It follows that $\gamma_{r2}(\overline{C(G,\sigma)}) \leq \omega(g) = 3$.

In all cases, by Lemma 2.6 and Theorem 4.1, $\gamma_{r2}(\overline{C(G,\sigma)}) = 3$. □

Corollary 4.5. *Let $n \geq 5$. Then $\gamma_{r2}(\overline{C(K_n,\sigma)}) \in \{4,5\}$.*

Proof. By Theorems 4.1 and 4.4, $\gamma_{r2}(\overline{C(K_n,\sigma)}) \in \{4,5\}$. □

Theorem 4.6. *Let $G \cong K_n$ be a graph of order $n \geq 5$. Then $\gamma_{r2}(\overline{C(G,\sigma)}) = 5$ if and only if $B_1 = B_2 = \emptyset$.*

Proof. Let $B_1 = B_2 = \emptyset$. On the contrary, suppose that $\gamma_{r2}(\overline{C(G,\sigma)}) \neq 5$. By Corollary 4.5, $\gamma_{r2}(\overline{C(G,\sigma)}) = 4$. Assume that g is a $2RDF$ of $\overline{C(G,\sigma)}$ such that $\omega(g) = 4$. Since $n \geq 5$, $\sum_{x \in V(G_2)} |g(x)| = 2$ and $\sum_{x \in V(G_1)} |g(x)| = 2$. We have two following cases.

Case 1. Let $v, v^{'} \in V(G_1)$ and $|g(v)| = |g(v^{'})| = 1$. If $g(\sigma(v)) = \emptyset$, then $\sum_{x \in N_{\overline{C}}(\sigma(v))} |g(x)| = 1$. That is not true. So $|g(\sigma(v))| = |g(\sigma(v^{'}))| = 1$ or $\sigma(v) = \sigma(v^{'})$ and $g(\sigma(v)) = \{1,2\}$. Let $|g(\sigma(v))| = |g(\sigma(v^{'}))| = 1$. Since $|B_1| = |B_2| = 0$, there exists an $v^{''} \in V(G_1) \setminus \{v, v^{'}\}$ such that $\sigma(v^{''}) = \sigma(v)$ (or $\sigma(v^{''}) = \sigma(v^{'})$). Hence, $\sum_{x \in N_{\overline{C}}(v^{''})} |g(x)| = 1$. Which is a contradiction. Let $\sigma(v) = \sigma(v^{'})$ and $g(\sigma(v)) = \{1,2\}$. Since $|B_1| = |B_2| = 0$, there exists an $v^{''} \in V(G_1) \setminus \{v, v^{'}\}$ such that $\sigma(v^{''}) = \sigma(v)$ (or $\sigma(v^{''}) = \sigma(v^{'})$). Hence, $\sum_{x \in N_{\overline{C}}(v^{''})} |g(x)| = 0$. That is not true.

Case 2. Let $v \in V(G_1)$ and $g(v) = \{1,2\}$. Since $B_1 = B_2 = \emptyset$, so there are $v_1, v_2 \in V(G_1)$, such that $\sigma(v_1) = \sigma(v_2) = \sigma(v)$. Since $g(v_1) = \emptyset$ and $\sum_{x \in N_{\overline{C}}(v_1)} g(x) = \{1,2\}$, so $g(\sigma(v)) = \emptyset$. It is clear that $\sum_{x \in N_{\overline{C}}(\sigma(v))} g(x) = \emptyset$, which is a contradiction.

Conversely, let $B_1 \neq \emptyset$, $u \in B_1$ and f be a function such that $f(u) = f(v) = \{1,2\}$, and $f(x) = \emptyset$ for every $x \in V(G_1) \cup V(G_2) \setminus \{v,u\}$, where $R_u = \{v\}$. Then f is a $2RDF$ of $\overline{C(G,\sigma)}$ and so $\gamma_{r2}(\overline{C(G,\sigma)}) \leq \omega(f) = 4$. Let $B_2 \neq \emptyset$, $u \in B_2$, $R_u = \{v_1, v_2\}$ and f be a function such that $f(v_1) = \{1\}$, $f(v_2) = \{2\}$, $f(u) = \{1,2\}$ and $f(x) = \emptyset$ for every $x \in V(G_1) \cup V(G_2) \setminus \{v_1, v_2, u\}$. Then f is a $2RDF$ of $\overline{C(G,\sigma)}$ and so $\gamma_{r2}(\overline{C(G,\sigma)}) \leq \omega(f) = 4$. However, $\gamma_{r2}(\overline{C(G,\sigma)}) \neq 5$. The proof is completed. □

Theorem 4.7. *Let $G \ncong K_4$ be a cubic graph. Then $\gamma_{r2}(\overline{C(G,\sigma)}) = 4$.*

Proof. By Lemma 2.6, Theorems 4.1 and 4.4, $\gamma_{r2}(\overline{C(G,\sigma)}) \geq 4$.

If $B_1 \neq \emptyset$, then $\gamma_{r2}(\overline{C(G,\sigma)}) \leq 4$ and so $\gamma_{r2}(\overline{C(G,\sigma)}) = 4$.

Let $B_1 = \emptyset$. Then $B_0 \neq \emptyset$. Assume that $u \in B_0$ and $N_{G_1}(u) = \{u_1, u_2, u_3\}$. If $G_2[\{u_1, u_2, u_3\}]$ has an isolated vertex, then by Lemma 2.7, $\gamma_{r2}(\overline{C(G,\sigma)}) \leq 4$ and so $\gamma_{r2}(\overline{C(G,\sigma)}) = 4$.

Let $G_2[\{u_1, u_2, u_3\}]$ does not have any isolated vertices. Then $G_2[\{u_1, u_2, u_3\}] \cong P_3$ or K_3. Since $G \ncong K_4$, so $G_2[\{u_1, u_2, u_3\}] \ncong K_3$. Thus, $G_2[N_{G_2}(u)]$ is isomorphic to H (see Figure 1). Let $u' \in N_{G_2}(u_3)$. Then $G_2[N_{G_2}(u')]$ has one isolated vertex. By Lemma 2.7, $\gamma_{r2}(\overline{C(G,\sigma)}) \leq 4$. Therefore, $\gamma_{r2}(\overline{C(G,\sigma)}) = 4$.

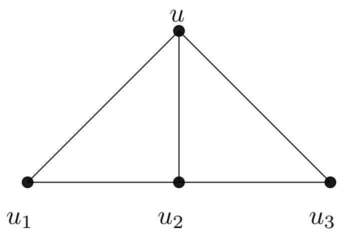

Fig. 1. The graph H

□

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