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OSCILLATORY BEHAVIOR OF SECOND-ORDER DAMPED DIFFERENTIAL EQUATIONS WITH A SUPERLINEAR NEUTRAL TERM

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Abstract. This article concerns the oscillatory behavior of solutions to second-order damped nonlinear differential equations with a superlinear neutral term. The results are obtained by a Riccati type transformation as well as by an integral criterion. Examples illustrating the results are provided and some suggestions for further research are indicated.

Keywords: oscillation, second-order, neutral differential equation, damping term.

Mathematics Subject Classification: 34C10, 34K11, 34K40.

1. INTRODUCTION

This paper deals with the oscillation of solutions to second-order nonlinear differential equation with a superlinear neutral term and a damping term

$$z''(t) + d(t)z'(t) + q(t)x^{\beta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x^{\alpha}(\tau(t))$. Throughout this paper, we always assume that the following conditions are satisfied:

(C_1) α and β are the ratios of odd positive integers with $\alpha \geq 1$;
(C_2) $p, q : [t_0, \infty) \to \mathbb{R}$ are continuous functions with $p(t) \geq 1$, $p(t) \not\equiv 1$ for large t, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t;
(C_3) $d : [t_0, \infty) \to (0, \infty)$ is a continuous function such that

$$\int_{t_0}^{\infty} \exp\left(-\int_{t_0}^{t} d(s)ds\right) dt = \infty; \tag{1.2}$$

(C_4) $\tau, \sigma : [t_0, \infty) \to \mathbb{R}$ are continuous functions such that $\sigma(t) \leq \tau(t) \leq t$, τ is strictly increasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$.

By a *solution* of equation (1.1), we mean a function $x \in C([t_x, \infty), \mathbb{R})$ for some $t_x \geq t_0$ such that $z \in C^2([t_x, \infty), \mathbb{R})$, and x satisfies (1.1) on $[t_x, \infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_x, \infty)$ and satisfy the condition

$$\sup\{|x(t)| : T \leq t < \infty\} > 0 \quad \text{for any} \quad T \geq t_x;$$

and moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x, \infty)$, i.e., for any $t_1 \in [t_x, \infty)$ there exists $t_2 \geq t_1$ such that $x(t_2) = 0$; otherwise it is called *nonoscillatory*, i.e., if it is eventually positive or eventually negative. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

The oscillatory behavior of solutions of various classes of second-order neutral differential equations without damping terms has been a very active area of research over the years; for recent contributions see, for example, [1–5, 8–14, 16, 19] and the references contained therein. However, in reviewing the literature, it becomes apparent that results on the oscillatory behavior of second-order neutral differential equations with damping terms are relatively scarce; see [6, 7, 15, 17, 18] for some typical results. It should be noted that although papers [6, 7, 15, 17, 18] deal with second-order neutral differential equations with a damping term, the results obtained in these papers except [17, 18] cannot be applied to the case where here $p(t) > 1$ and/or $p(t) \to \infty$ as $t \to \infty$. On the other hand, the results in [17, 18] were obtained for the second-order damped differential equations with a linear neutral term(i.e., $\alpha = 1$), and so the results in [17, 18] cannot be applied to the equations with a superlinear neutral term (i.e., $\alpha > 1$). To the best of our knowledge, there are no results for second-order differential equations with a superlinear neutral term and a damping term in the case where $p(t) \to \infty$ as $t \to \infty$, and so, the aim of the present paper is to initiate the study of the oscillation problem of (1.1) and to provide new results, which can easily be extended to more general second-order damped differential equations with a superlinear neutral term to derive more general oscillation results (see Remarks 2.8 and 2.9 below). It should be noted that the results of the present paper can be applied to the case where $p(t) \to \infty$ as $t \to \infty$ for $\alpha > 1$, and to the cases where $p(t)$ is a bounded function and/or $p(t) \to \infty$ as $t \to \infty$ for $\alpha = 1$. For these reasons, it is our belief that the present paper will contribute significantly to the study of oscillatory behavior of solutions of second-order damped differential equations with a superlinear neutral term.

2. MAIN RESULTS

In this section, we establish some new criteria for the oscillation of equation (1.1). It will be convenient to employ the following notations:

$$g(t) := \tau^{-1}(\sigma(t)), \quad \text{where } \tau^{-1} \text{ is the inverse function of } \tau;$$

$$\pi(t) := \begin{cases} 1, & \text{if } \frac{\beta}{\alpha} - 1 = 0, \\ k_1, & \text{if } \frac{\beta}{\alpha} - 1 > 0, \\ k_2 t^{\frac{\beta}{\alpha} - 1}, & \text{if } \frac{\beta}{\alpha} - 1 < 0, \end{cases}$$

where k_i $(i = 1,2)$ are positive real constants, and for any positive function $\xi \in C^1([t_0,\infty),\mathbb{R})$
$$\eta(t) := \frac{\xi'(t) - \xi(t)d(t)}{\xi(t)}.$$
For proving our results we use the additional condition:

(C_5) For every positive constant δ, we have
$$\varphi(t) := \frac{1}{p(\tau^{-1}(t))}\left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)}\right)^{2/\alpha} \frac{\delta^{\frac{1}{\alpha}-1}}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}\right] \geq 0$$
for all sufficiently large t.

Note that if $\alpha > 1$, this assumption requires $\lim_{t\to\infty} p(t) = \infty$.

Our first oscillation result is the following.

Theorem 2.1. *Let conditions* (C_1)–(C_5) *and* (1.2) *hold. If*
$$\int_{t_0}^{\infty} q(s)\varphi^{\beta/\alpha}(\sigma(s))ds = \infty, \tag{2.1}$$
then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. It follows from (1.1) that
$$z''(t) + d(t)z'(t) = -q(t)x^{\beta}(\sigma(t)) \leq 0,$$
i.e.,
$$z''(t) + d(t)z'(t) \leq 0 \quad \text{for } t \geq t_1,$$
which implies
$$\left(\exp\left(\int_{t_1}^{t} d(s)ds\right) z'(t)\right)' \leq 0 \quad \text{for } t \geq t_1.$$
Thus, $\exp\left(\int_{t_1}^{t} d(s)ds\right) z'(t)$ is nonincreasing and eventually does not change its sign, say on $[t_2,\infty)$ for some $t_2 \geq t_1$. Therefore, $z'(t)$ eventually has a fixed sign on $[t_2,\infty)$, and so we have one of the following cases:

Case (I): $z'(t) > 0$ for $t \geq t_2$,

Case (II): $z'(t) < 0$ for $t \geq t_2$.

First, we consider case (I). Since $z'(t) > 0$ for $t \geq t_2$, from (1.1) we have
$$z(t) > 0, \; z'(t) > 0, \text{ and } \; z''(t) \leq 0 \quad \text{for } t \geq t_2,$$

and so

$$z(t) = z(t_2) + \int_{t_2}^{t} z'(s)ds \geq (t - t_2)z'(t).$$

From this it follows that for all $t \geq t_3 := 2t_2$,

$$z(t) \geq \frac{t}{2} z'(t) \quad \text{for } t \geq t_3. \tag{2.2}$$

From (2.2) one can easily see that $z(t)/t^2$ is decreasing for $t \geq t_3$. It follows from the definition of z that

$$x^\alpha(\tau(t)) = \frac{1}{p(t)}(z(t) - x(t)) \leq \frac{z(t)}{p(t)},$$

from which and the fact that $\tau(t) \leq t$ is strictly increasing, it is easy to see that

$$x(\tau^{-1}(t)) \leq \frac{z^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}.$$

Using this in the definition of z, we obtain

$$\begin{aligned} x^\alpha(t) &= \frac{1}{p(\tau^{-1}(t))} \left[z(\tau^{-1}(t)) - x(\tau^{-1}(t)) \right] \\ &\geq \frac{1}{p(\tau^{-1}(t))} \left[z(\tau^{-1}(t)) - \frac{z^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))} \right]. \end{aligned} \tag{2.3}$$

Since $\tau(t) \leq t$ and τ is strictly increasing, so τ^{-1} is increasing and $t \leq \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \leq \tau^{-1}(\tau^{-1}(t)),$$

and since $z(t)/t^2$ is decreasing, we arrive at

$$\frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^2 z(\tau^{-1}(t))}{\left(\tau^{-1}(t)\right)^2} \geq z\left(\tau^{-1}(\tau^{-1}(t))\right). \tag{2.4}$$

Using (2.4) in (2.3), we obtain

$$\begin{aligned} x^\alpha(t) &\geq \frac{1}{p(\tau^{-1}(t))} \left[z(\tau^{-1}(t)) - \frac{\left(\tau^{-1}(\tau^{-1}(t))\right)^{2/\alpha}}{\left(\tau^{-1}(t)\right)^{2/\alpha}} \frac{z^{1/\alpha}(\tau^{-1}(t))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))} \right] \\ &= \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))} \left[1 - \left(\frac{\tau^{-1}(\tau^{-1}(t))}{\tau^{-1}(t)} \right)^{2/\alpha} \frac{z^{\frac{1}{\alpha}-1}(\tau^{-1}(t))}{p^{1/\alpha}(\tau^{-1}(\tau^{-1}(t)))} \right]. \end{aligned} \tag{2.5}$$

Since $z(t)$ is positive and increasing for $t \geq t_2$, there exist $t_3 \in [t_2, \infty)$ and a constant $c > 0$ such that

$$z(t) \geq c \quad \text{for } t \geq t_3. \tag{2.6}$$

Substituting (2.6) into (2.5) yields

$$x^\alpha(t) \geq \varphi(t) z(\tau^{-1}(t)) \quad \text{for } t \geq t_3. \tag{2.7}$$

Since $\lim_{t\to\infty} \sigma(t) = \infty$, we can choose $t_4 \geq t_3$ such that $\sigma(t) \geq t_3$ for all $t \geq t_4$. Thus, it follows from (2.7) that

$$x^\alpha(\sigma(t)) \geq \varphi(\sigma(t)) z(\tau^{-1}(\sigma(t))) \quad \text{for } t \geq t_4. \tag{2.8}$$

Using (2.8) in (1.1) gives

$$z''(t) + d(t)z'(t) + q(t)\varphi^{\beta/\alpha}(\sigma(t)) z^{\beta/\alpha}(g(t)) \leq 0 \quad \text{for } t \geq t_4. \tag{2.9}$$

In view of the fact that $d(t) > 0$ and $z'(t) > 0$, it follows from (2.9) that

$$z''(t) + q(t)\varphi^{\beta/\alpha}(\sigma(t)) z^{\beta/\alpha}(g(t)) \leq 0.$$

Integrating from t_4 to t yields

$$z'(t) \leq z'(t_4) - c^{\beta/\alpha} \int_{t_4}^{t} q(s)\varphi^{\beta/\alpha}(\sigma(s)) ds \to -\infty \ \text{ as } t \to \infty,$$

which contradicts the fact that $z'(t)$ is positive.

Next, we consider case (II). Letting $u(t) = -z'(t) > 0$, it follows from (1.1) that

$$u'(t) + d(t)u(t) \geq 0 \quad \text{for } t \geq t_2.$$

Integrating this inequality from t_2 to t, we obtain

$$u(t) \geq u(t_2) \exp\left(-\int_{t_2}^{t} d(s) ds\right),$$

from which we see that

$$z'(t) \leq z'(t_2) \exp\left(-\int_{t_2}^{t} d(s) ds\right). \tag{2.10}$$

Integrating (2.10) from t_2 to t and taking (1.2) into account, we obtain

$$z(t) \leq z(t_2) + z'(t_2) \int_{t_2}^{t} \exp\left(-\int_{t_2}^{s} d(u) du\right) ds \to -\infty \text{ as } t \to \infty,$$

which contradicts the positivity of z and completes the proof of the theorem. □

Theorem 2.2. *Let conditions* (C_1)–(C_5) *and* (1.2) *hold. If there exists a positive function* $\xi \in C^1([t_0,\infty),\mathbb{R})$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\xi(s)q(s)\varphi^{\beta/\alpha}(\sigma(s))\pi(g(s))\left(\frac{g(s)}{s}\right)^2 - \frac{\xi(s)\eta^2(s)}{4}\right] ds = \infty, \tag{2.11}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $t_1 \in [t_0,\infty)$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$. Then, from Theorem 2.1, $z(t)$ satisfies either case (I) or case (II) for $t \geq t_2$. If case (II) holds, proceeding exactly as in the proof of Theorem 2.1, we again obtain a contradiction to the positivity of z.

Next, we consider case (I). Proceeding as in the proof of Theorem 2.1, we again arrive at (2.9) for $t \geq t_4$, which can be written as

$$z''(t) + d(t)z'(t) + q(t)\varphi^{\beta/\alpha}(\sigma(t))z^{\beta/\alpha-1}(g(t))z(g(t)) \leq 0 \tag{2.12}$$

for $t \geq t_4$. Since $z'(t)$ is positive and decreasing on $[t_2,\infty)$, there exist a constant $c_1 > 0$ and $t_3 \geq t_2$ such that

$$z'(t) \leq c_1 \quad \text{for } t \geq t_3.$$

Integrating the last inequality from t_3 to t, we obtain

$$z(t) \leq bt \tag{2.13}$$

for $t \geq t_3$ and for some constant $b > 0$. In view of (2.6) and (2.13), inequality (2.12) takes the form

$$z''(t) + d(t)z'(t) + q(t)\varphi^{\beta/\alpha}(\sigma(t))\pi(g(t))z(g(t)) \leq 0 \tag{2.14}$$

for $t \geq t_4$. Define the function $w(t)$ by the Riccati substitution

$$w(t) := \xi(t)\frac{z'(t)}{z(t)} \quad \text{for } t \geq t_4. \tag{2.15}$$

Clearly, $w(t) > 0$, and from (2.14) and (2.15), we observe that

$$\begin{aligned} w'(t) &= \frac{\xi'(t)}{\xi(t)}w(t) + \xi(t)\left(\frac{z''(t)z(t) - (z'(t))^2}{z^2(t)}\right) \\ &\leq \frac{\xi'(t)}{\xi(t)}w(t) + \frac{\xi(t)}{z(t)}\left[-d(t)z'(t) - q(t)\varphi^{\beta/\alpha}(\sigma(t))\pi(g(t))z(g(t))\right] \\ &\quad - \frac{1}{\xi(t)}w^2(t) \\ &= \eta(t)w(t) - \xi(t)q(t)\varphi^{\beta/\alpha}(\sigma(t))\pi(g(t))\frac{z(g(t))}{z(t)} - \frac{1}{\xi(t)}w^2(t). \end{aligned} \tag{2.16}$$

Using the fact $z(t)/t^2$ is decreasing, and noting that $\sigma(t) \le \tau(t)$ implies $\tau^{-1}(\sigma(t)) \le t$, we obtain

$$\frac{z\left(\tau^{-1}(\sigma(t))\right)}{z(t)} \ge \left(\frac{\tau^{-1}(\sigma(t))}{t}\right)^2 = \left(\frac{g(t)}{t}\right)^2. \tag{2.17}$$

Substituting (2.17) into (2.16) gives

$$w'(t) \le \eta(t)w(t) - \xi(t)q(t)\varphi^{\beta/\alpha}(\sigma(t))\pi(g(t))\left(\frac{g(t)}{t}\right)^2 - \frac{1}{\xi(t)}w^2(t). \tag{2.18}$$

Completing the square with respect to w, it follows from (2.18) that

$$w'(t) \le -\xi(t)q(t)\varphi^{\beta/\alpha}(\sigma(t))\pi(g(t))\left(\frac{g(t)}{t}\right)^2 + \frac{\xi(t)\eta^2(s)}{4} \quad \text{for } t \ge t_4.$$

Integrating the last inequality from t_4 to t yields

$$\int_{t_4}^{t}\left[\xi(s)q(s)\varphi^{\beta/\alpha}(\sigma(s))\pi(g(s))\left(\frac{g(s)}{s}\right)^2 - \frac{\xi(s)\eta^2(s)}{4}\right]ds < w(t_4),$$

which contradicts (2.11) and completes the proof of the theorem. □

From Theorem 2.2, we can establish different conditions for the oscillation of (1.1) using different choices of $\xi(t)$. For example, letting $\xi(t) = 1$ and $\xi(t) = t^\gamma$ with $\gamma \ge 1$, we obtain the following corollaries, respectively.

Corollary 2.3. *Let conditions* (C_1)–(C_5) *and* (1.2) *hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[q(s)\varphi^{\beta/\alpha}(\sigma(s))\pi(g(s))\left(\frac{g(s)}{s}\right)^2 - \frac{d^2(s)}{4}\right]ds = \infty,$$

then equation (1.1) *is oscillatory.*

Corollary 2.4. *Let conditions* (C_1)–(C_5) *and* (1.2) *hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[s^{\gamma-2}q(s)\varphi^{\beta/\alpha}(\sigma(s))\pi(g(s))g^2(s) - \frac{\left[(s^\gamma)' - s^\gamma d(s)\right]^2}{4s^\gamma}\right]ds = \infty, \tag{2.19}$$

then equation (1.1) *is oscillatory.*

Next, we present a new oscillation result in which we assume that $\eta(t) \le 0$.

Theorem 2.5. *Let conditions* (C_1)–(C_5) *and* (1.2) *hold. If there exists a positive function* $\xi \in C^1([t_0,\infty),\mathbb{R})$ *such that* $\eta(t) \le 0$ *for* $t \ge t_0$, *and*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\xi(s)q(s)\varphi^{\beta/\alpha}(\sigma(s))\pi(g(s))\left(\frac{g(s)}{s}\right)^2 ds = \infty, \tag{2.20}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Then, from Theorem 2.1, $z(t)$ satisfies either case (I) or case (II) for $t \geq t_2$. If case (II) holds, proceeding exactly as in the proof of Theorem 2.1, we again obtain a contradiction to the positivity of z.

Next, we consider case (I). Proceeding as in the proof of Theorem 2.2, we again arrive at (2.18) for $t \geq t_4$. Since $\eta(t) \leq 0$ and $w(t) > 0$, inequality (2.18) can be written as

$$w'(t) \leq -\xi(t)q(t)\varphi^{\beta/\alpha}(\sigma(t))\pi(g(t))\left(\frac{g(t)}{t}\right)^2 \quad \text{for } t \geq t_4.$$

Integrating the last inequality from t_4 to t gives

$$\int_{t_4}^{t} \xi(s)q(s)\varphi^{\beta/\alpha}(\sigma(s))\pi(g(s))\left(\frac{g(s)}{s}\right)^2 ds < w(t_4),$$

which contradicts (2.20) and completes the proof of the theorem. □

We conclude this paper with two examples and remarks to illustrate our results. The first example is concerned with the equation with superlinear neutral term in the case where $p(t) \to \infty$ as $t \to \infty$, and the second example is concerned with the equation with linear neutral term in the case where p is a constant function.

Example 2.6. Consider the differential equation with a superlinear neutral term and a damping term

$$z''(t) + \frac{1}{t^3}z'(t) + \frac{t}{2}x^5\left(\frac{t}{4}\right) = 0, \quad t \geq 1, \tag{2.21}$$

with

$$z(t) = x(t) + tx^5\left(\frac{t}{2}\right).$$

Here $p(t) = t$, $d(t) = 1/t^3$, $q(t) = t/2$, $\tau(t) = t/2$, $\alpha = 5$, $\beta = 5$, and $\sigma(t) = t/4$. Then, it is easy to see that conditions (C_1)–(C_5) and (1.2) hold,

$$\tau^{-1}(t) = 2t,\ \tau^{-1}(\tau^{-1}(t)) = 4t,\ g(t) = t/2,$$

and

$$\varphi(t) = \frac{1}{2t}\left[1 - \frac{2^{2/5}}{\delta^{4/5}(4t)^{1/5}}\right].$$

Thus, it follows from (2.1) that

$$\int_{t_0}^{\infty} q(s)\varphi^{\beta/\alpha}(\sigma(s))ds = \int_{1}^{\infty}\left[1 - \frac{2^{2/5}}{\delta^{4/5}s^{1/5}}\right]ds = \infty,$$

i.e., condition (2.1) holds. Thus, all conditions of Theorem 2.1 hold. Therefore, by Theorem 2.1, equation (2.21) is oscillatory.

Example 2.7. Consider the differential equation with a linear neutral term and a damping term

$$z''(t) + \frac{1}{t^2} z'(t) + t^2 x^3 \left(\frac{t}{8}\right) = 0, \quad t \geq 1, \tag{2.22}$$

with

$$z(t) = x(t) + 32x\left(\frac{t}{4}\right).$$

Here $\alpha = 1$, $\beta = 3$, $p(t) = 32$, $d(t) = 1/t^2$, $q(t) = t^2$, $\tau(t) = t/4$, and $\sigma(t) = t/8$. Then, it is easy to see that conditions (C_1)–(C_5) and (1.2) hold,

$$\tau^{-1}(t) = 4t, \ \tau^{-1}(\tau^{-1}(t)) = 16t, \ g(t) = t/2 \ \text{ and } \ \varphi(t) = 1/64.$$

With $\xi(t) = t$, condition (2.19) becomes

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[s^{\gamma-2} q(s) \varphi^{\beta/\alpha}(\sigma(s)) \pi(g(s)) g^2(s) - \frac{[(s^\gamma)' - s^\gamma d(s)]^2}{4s^\gamma} \right] ds$$

$$= \limsup_{t\to\infty} \int_{1}^{t} \left[\frac{k_1}{2^{20}} s^3 - \frac{(s-1)^2}{4s^3} \right] ds = \infty,$$

i.e., condition (2.19) holds. Thus, all conditions of Corollary 2.4 hold. Therefore, by Corollary 2.4, equation (2.22) is oscillatory.

Remark 2.8. The results of this paper can be easily extended to the second-order nonlinear differential equation with a superlinear neutral term and a damping term

$$(a(t)z'(t))' + d(t)z'(t) + q(t)x^\beta(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \exp\left(- \int_{t_0}^{t} d(s)/a(s) ds \right) dt = \infty,$$

where $a \in C([t_0, \infty), (0, \infty))$, $z(t) = x(t) + p(t)x^\alpha(\tau(t))$, and the other functions and constants α and β in the equation are defined as in this paper.

Remark 2.9. The results of this paper can be extended to the second-order nonlinear differential equation with a superlinear neutral term and a damping term

$$(a(t)(z'(t))^\alpha)' + d(t)(z'(t))^\alpha + q(t)f(t, x(\sigma(t))) = 0, \quad t \geq t_0 > 0,$$

under the condition

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left[\exp\left(- \int_{t_0}^{t} \frac{d(s)}{a(s)} ds \right) \right]^{1/\alpha} dt = \infty,$$

where $a \in C([t_0,\infty),(0,\infty))$, $z(t) = x(t) + p(t)x^{\alpha}(\tau(t))$, $f(t,u) : [t_0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $uf(t,u) > 0$ for all $u \neq 0$ and there exists a positive constant M such that

$$f(t,u)/u^{\beta} \geq M \quad \text{for} \quad u \neq 0,$$

and the other functions and constants α and β in the equation are defined as in this paper.

Remark 2.10. It would also be of interest to study equation (1.1) for the case where $p(t) \to -\infty$ as $t \to -\infty$.

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