Opuscula Math. 41, no. 1 (2021), 113–143
https://doi.org/10.7494/OpMath.2021.41.1.113 Opuscula Mathematica

REITERATED PERIODIC HOMOGENIZATION OF INTEGRAL FUNCTIONALS WITH CONVEX AND NONSTANDARD GROWTH INTEGRANDS

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Communicated by P.A. Cojuhari

Abstract. Multiscale periodic homogenization is extended to an Orlicz–Sobolev setting. It is shown by the reiteraded periodic two-scale convergence method that the sequence of minimizers of a class of highly oscillatory minimizations problems involving convex functionals, converges to the minimizers of a homogenized problem with a suitable convex function.

Keywords: convex function, reiterated two-scale convergence, relaxation, Orlicz–Sobolev spaces.

Mathematics Subject Classification: 35B27, 35B40, 35J25, 46J10, 49J45.

1. INTRODUCTION

The method of two-scale convergence introduced by Nguetseng [35] and later developed by Allaire [2] have been widely adopted in homogenization of PDEs in classical Sobolev spaces neglecting materials where microstructure cannot be conveniently captured by modeling exclusively by means of those spaces. Recently in [21] some of the above methods were extended to Orlicz–Sobolev setting. On the other hand, an increasing number of works in homogenization and dimension reduction (see [26–32, 38]), among the others) are devoted to deal with this more general setting. We also refer to [42–44] for two scale homogenization in variable exponent spaces, which also evidence Lavrentieff phenomena.

In order to model multiscale phenomena, i.e., to provide homogenization results closer to reality, more than two-scales should be considered. Indeed the aim of this work is to show that the two-scale convergence method can be extended and generalized to tackle reiterated homogenization problems in the Orlicz–Sobolev setting.

In details, we intend to study the asymptotic behaviour as $\varepsilon \to 0^+$ of the sequence of solutions of the problem

$$\min\left\{F_\varepsilon(v) : v \in W_0^1 L^B(\Omega)\right\} \tag{1.1}$$

where, for each $\varepsilon > 0$, the functional F_ε is defined on $W_0^1L^B(\Omega)$ by

$$F_\varepsilon(v) = \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Dv(x)\right) dx, \ v \in W_0^1L^B(\Omega), \tag{1.2}$$

Ω being a bounded open set in $\mathbb{R}^N_x$, $n, N \in \mathbb{N}$, D denoting the gradient operator in Ω with respect to x and the function $f : \mathbb{R}^N_y \times \mathbb{R}^N_z \times \mathbb{R}^{nN} \to [0, +\infty)$ being an integrand, that satisfies the following hypotheses:

(H_1) for all $\lambda \in \mathbb{R}^N$, $f(\cdot, z, \lambda)$ is measurable for all $z \in \mathbb{R}^N$ and $f(y, \cdot, \lambda)$ is continuous for almost all $y \in \mathbb{R}^N$;

(H_2) $f(y, z, \cdot)$ is strictly convex for a.e. $y \in \mathbb{R}^N_y$ and all $z \in \mathbb{R}^N_z$;

(H_3) for each $(k, k') \in \mathbb{Z}^{2N}$ we have $f(y + k, z + k', \lambda) = f(y, z, \lambda)$ for all $(z, \lambda) \in \mathbb{R}^N_z \times \mathbb{R}^N$ and a.e. $y \in \mathbb{R}^N_y$;

(H_4) there exist two constants $c_1, c_2 > 0$ such that:

$$c_1 B(|\lambda|) \le f(y, z, \lambda) \le c_2 (1 + B(|\lambda|))$$

for all $\lambda \in \mathbb{R}^{nN}$and for a.e. $y \in \mathbb{R}^N_y$ and all $z \in \mathbb{R}^N_z$.

We observe that problems of the type (1.1) have been studied by many authors in many contexts (see, among the others, [2–8, 10, 11, 17, 18, 20, 22, 34, 40]. But in all the above papers the two-scale approach or other methods (see in particular unfolding) have been always considered in classical Sobolev setting. The novelty here is the multiscale approach beyond classical Sobolev spaces. For the sake of exposition we consider the scales ε and ε^2, but more general choices are possible, as in [3]. We also refer to [24] for extensions of the present results to higher order Orlicz–Sobolev spaces.

In particular we introduce the following setting.

Let B an N-function and $\widetilde{B}$ its conjugate both verifying the $\triangle_2$ (in words: delta-2) condition (see (2.1) below), let Ω be a bounded open set in $\mathbb{R}^N_x$, $Y = Z = (-\frac{1}{2}, \frac{1}{2})^N$, $N \in \mathbb{N}$ and ε any sequence of positive numbers converging to 0. Assume that $(u_\varepsilon)_\varepsilon$ is bounded in $W^1L^B(\Omega)$. Then, there exist not relabelled subsequences $\varepsilon, (u_\varepsilon)_\varepsilon, u_0 \in W^1L^B(\Omega)$,

$$(u_1, u_2) \in L^1\left(\Omega; W^1_\# L^B(Y)\right) \times L^1\left(\Omega; L^1_{per}\left(Y; W^1_\# L^B(Z)\right)\right)$$

such that: $u_\varepsilon \rightharpoonup u_0$ in $W^1L^B(\Omega)$ weakly, and

$$\int_\Omega D_{x_i} u_\varepsilon \varphi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx \to \iiint_{\Omega\times Y\times Z} (D_{x_i}u_0 + D_{y_i}u_1 + D_{z_i}u_2)\, \varphi(x, y, z)\, dxdydz \tag{1.3}$$

as $\varepsilon \to 0$,

$1 \leq i \leq N$, and for all $\varphi \in L^{\widetilde{B}}(\Omega; \mathcal{C}_{per}(Y \times Z))$, where D_{x_i}, D_{y_i} and D_{z_i} denote the distributional derivatives with respect to the variables $x_i.y_i, z_i$, (also denoted by $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial z_i}$, respectively). (See Section 2 for detailed notations and Definition 2.4 and Proposition 2.12 for rigorous results.)

Next, we define, following the same type of notation adopted in [21], (and referring to subsection 2.1 for notation, norms and properties of functions spaces below) the space

$$\mathbb{F}_0^1 L^B = W_0^1 L^B(\Omega) \times L^B_{D_y}(\Omega; W^1_{\#} L^B(Y)) \times L^B_{D_z}(\Omega; L^1_{per}(Y; W^1_{\#} L^B(Z))), \tag{1.4}$$

where

$$\begin{aligned} &L^B_{D_y}(\Omega; W^1_{\#} L^B(Y)) = \left\{ u \in L^1(\Omega; W^1_{\#} L^B(Y)) : D_y u \in L^B_{per}(\Omega \times Y)^N \right\}, \\ &L^B_{D_z}(\Omega; L^1_{per}(Y; W^1_{\#} L^B(Z))) \\ &= \left\{ u \in L^1(\Omega; L^1_{per}(Y; W^1_{\#} L^B(Z))) : D_z u \in L^B_{per}(\Omega \times Y \times Z)^N \right\}. \end{aligned} \tag{1.5}$$

Observe that D_x, D_y and D_z denote the vector of distributional derivatives with respect to $x \equiv (x_1, \ldots, x_N)$, $y \equiv (y_!, \ldots, y_N)$ and $z \equiv (z_1, \ldots, z_N)$, respectively.

We equip $\mathbb{F}_0^1 L^B$ with the norm

$$\|u\|_{\mathbb{F}_0^1 L^B} = \|Du_0\|_{B,\Omega} + \|D_y u_1\|_{B,\Omega\times Y} + \|D_z u_2\|_{B,\Omega\times Y\times Z}, \quad u = (u_0, u_1, u_2) \in \mathbb{F}_0^1 L^B,$$

which makes it a Banach space.

Finally, for $v = (v_0, v_1, v_2) \in \mathbb{F}_0^1 L^B$, denote by $\mathbb{D}v$ the sum $Dv_0 + D_y v_1 + D_z v_2$ and define the functional $F : \mathbb{F}_0^1 L^B \to \mathbb{R}^+$ by

$$F(v) = \iiint\limits_{\Omega\times Y\times Z} f(\cdot, \mathbb{D}v)\,dxdydz. \tag{1.6}$$

With the tool of multiscale convergence at hand in the Orlicz–Sobolev setting, we prove the following result.

Theorem 1.1. *Let Ω be a bounded open set in $\mathbb{R}^N_x$ and let $f : \mathbb{R}^N_y \times \mathbb{R}^N_z \times \mathbb{R}^N \to [0, +\infty)$ be an integrand satisfying (H_1)–(H_4). For each $\varepsilon > 0$, let u_ε be the unique solution of* (1.1)*, then as $\varepsilon \to 0$,*

(a) *$u_\varepsilon \rightharpoonup u_0$ weakly in $W_0^1 L^B(\Omega)$;*
(b) *$Du_\varepsilon \rightharpoonup \mathbb{D}u = Du_0 + D_y u_1 + D_z u_2$ weakly reiteratively two-scale in $L^B(\Omega)^N$ (i.e. in the sense of* (1.3)*), where $u = (u_0, u_1, u_2) \in \mathbb{F}_0^1 L^B$ is the unique solution of the minimization problem*

$$F(u) = \min_{v \in \mathbb{F}_0^1 L^B} F(v), \tag{1.7}$$

where $\mathbb{F}_0^1 L^B$ and F are as in (1.4) *and* (1.6)*, respectively.*

The paper is organized as follows, Section 2 deals with notations, preliminary results on Orlicz–Sobolev spaces, introduction of suitable function spaces to deal with multiple scales homogenization, and compactness result for reiterated two-scale convergence, while Section 3 contains the main results devoted to the proof of Theorem 1.1, together with Corollary 3.6 which allows to recast the main result in the framework of Γ convergence (see also [23] for the single scale case).

2. NOTATION AND PRELIMINARIES

In what follows, X and V denote a locally compact space and a Banach space, respectively, and $C(X;V)$ stands for the space of continuous functions from X into V, and $C_b(X;V)$ stands for those functions in $C(X;F)$ that are bounded. The space $C_b(X;V)$ is endowed with the supremum norm $\|u\|_\infty = \sup_{x\in X}\|u(x)\|$, where $\|\cdot\|$ denotes the norm in V, (in particular, given an open set $A \subset \mathbb{R}^N$ by $\mathcal{C}_b(A)$ we denote the space of real valued continuous and bounded functions defined in A). Likewise the spaces $L^p(X;V)$ and $L^p_{\text{loc}}(X;V)$ (X provided with a positive Radon measure) are denoted by $L^p(X)$ and $L^p_{\text{loc}}(X)$, respectively, when $V=\mathbb{R}$ (we refer to [12, 13, 16] for integration theory).

In the sequel we denote by Y and Z two identical copies of the cube $(-\frac{1}{2},\frac{1}{2})^N$.

In order to enlighten the space variable under consideration we will adopt the notation $\mathbb{R}^N_x, \mathbb{R}^N_y$, or $\mathbb{R}^N_z$ to indicate, where x, y or z belong to.

The family of open subsets in $\mathbb{R}^N_x$ will be denoted by $\mathcal{A}(\mathbb{R}^N_x)$.

For any subset E of $\mathbb{R}^m$, $m\in\mathbb{N}$, by $\overline{E}$, we denote its closure in the relative topology.

For every $x\in\mathbb{R}^N$ we denote by $[x]$ its integer part, namely the vector in $\mathbb{Z}^N$, which has as a component the integer parts of the components of x.

By $\mathcal{L}^N$ we denote the Lebesgue measure in $\mathbb{R}^N$.

2.1. ORLICZ–SOBOLEV SPACES

Let $B : [0,+\infty[\to[0,+\infty[$ be an N-function [1], i.e., B is continuous, convex, with $B(t)>0$ for $t>0$, $\frac{B(t)}{t}\to 0$ as $t\to 0$, and $\frac{B(t)}{t}\to\infty$ as $t\to\infty$. Equivalently, B is of the form $B(t)=\int_0^t b(\tau)d\tau$, where $b:[0,+\infty[\to[0,+\infty[$ is non decreasing, right continuous, with $b(0)=0, b(t)>0$ if $t>0$ and $b(t)\to+\infty$ if $t\to+\infty$.

We denote by $\widetilde{B}$ the complementary N-function of B defined by

$$\widetilde{B}(t)=\sup_{s\geq 0}\{st-B(s), t\geq 0\}.$$

It follows that

$$\frac{tb(t)}{B(t)}\geq 1 \ \ (\text{or } > \text{if } b \text{ is strictly increasing}),$$

$$\widetilde{B}(b(t))\leq tb(t)\leq B(2t) \text{ for all } t>0.$$

An N-function B is of class $\triangle_2$ at ∞ (denoted $B\in\triangle_2$) if there are $\alpha>0$ and $t_0\geq 0$ such that

$$B(2t)\leq \alpha B(t) \tag{2.1}$$

for all $t\geq t_0$.

In what follows, B and $\widetilde{B}$ are conjugates N-functions satisfying the $\triangle_2$ condition and c refers to a constant. Let Ω be a bounded open set in $\mathbb{R}^N (N\in\mathbb{N})$. The Orlicz space

$$L^B(\Omega)=\left\{u:\Omega\to\mathbb{R}; u \text{ is measurable}, \lim_{\delta\to 0^+}\int_\Omega B(\delta\,|u(x)|)dx=0\right\}$$

is a Banach space with respect to the Luxemburg norm:

$$\|u\|_{B,\Omega} = \inf\left\{k > 0 : \int_{\Omega} B\left(\frac{|u(x)|}{k}\right)dx \leq 1\right\} < +\infty.$$

It follows that: $\mathcal{D}(\Omega)$ is dense in $L^B(\Omega)$, $L^B(\Omega)$ is separable and reflexive, the dual of $L^B(\Omega)$ is identified with $L^{\widetilde{B}}(\Omega)$, and the norm on $L^{\widetilde{B}}(\Omega)$ is equivalent to $\|\cdot\|_{\widetilde{B},\Omega}$. We will denote the norm of elements in $L^B(\Omega)$, both by $\|\cdot\|_{L^B(\Omega)}$ and with $\|\cdot\|_{B,\Omega}$, the latter symbol being useful when we want emphasize the domain Ω.

Furthermore, it is also convenient to recall that:

(i) $\left|\int_{\Omega} u(x)v(x)dx\right| \leq 2\|u\|_{B,\Omega}\|v\|_{\widetilde{B},\Omega}$ for $u \in L^B(\Omega)$ and $v \in L^{\widetilde{B}}(\Omega)$,

(ii) given $v \in L^{\widetilde{B}}(\Omega)$ the linear functional L_v on $L^B(\Omega)$ defined by

$$L_v(u) = \int_{\Omega} u(x)v(x)dx, \quad (u \in L^B(\Omega))$$

belongs to the dual $\left[L^B(\Omega)\right]' = L^{\widetilde{B}}(\Omega)$ with $\|v\|_{\widetilde{B},\Omega} \leq \|L_v\|_{[L^B(\Omega)]'} \leq 2\|v\|_{\widetilde{B},\Omega}$,

(iii) the property $\lim_{t\to+\infty} \frac{B(t)}{t} = +\infty$ implies $L^B(\Omega) \subset L^1(\Omega) \subset L^1_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$, each embedding being continuous.

For the sake of notations, given any $d \in \mathbb{N}$, when $u : \Omega \to \mathbb{R}^d$, such that each component (u^i), of u, lies in $L^B(\Omega)$ we will denote the norm of u with the symbol

$$\|u\|_{L^B(\Omega)^d} := \sum_{i=1}^{d} \|u^i\|_{B,\Omega}.$$

Analogously one can define the Orlicz–Sobolev functional space as follows:

$$W^1L^B(\Omega) = \left\{u \in L^B(\Omega) : \frac{\partial u}{\partial x_i} \in L^B(\Omega), 1 \leq i \leq d\right\},$$

where derivatives are taken in the distributional sense on Ω. Endowed with the norm

$$\|u\|_{W^1L^B(\Omega)} = \|u\|_{B,\Omega} + \sum_{i=1}^{d}\left\|\frac{\partial u}{\partial x_i}\right\|_{B,\Omega}, \quad u \in W^1L^B(\Omega),$$

$W^1L^B(\Omega)$ is a reflexive Banach space. We denote by $W^1_0L^B(\Omega)$, the closure of $\mathcal{D}(\Omega)$ in $W^1L^B(\Omega)$ and the semi-norm

$$u \to \|u\|_{W^1_0L^B(\Omega)} = \|Du\|_{B,\Omega} = \sum_{i=1}^{d}\left\|\frac{\partial u}{\partial x_i}\right\|_{B,\Omega}$$

is a norm on $W^1_0L^B(\Omega)$ equivalent to $\|\cdot\|_{W^1L^B(\Omega)}$.

By $W^1_\# L^B(Y)$, we denote the space of functions $u \in W^1L^B(Y)$ such that $\int_Y u(y)dy = 0$. It is endowed with the gradient norm. Given a function space S defined in Y, Z or $Y \times Z$, the subscript S_{per} means that the functions are periodic in Y, Z or $Y \times Z$, as it will be clear from the context. In particular $C_{per}(Y \times Z)$ denotes the space of periodic functions in $C(\mathbb{R}^N_y \times \mathbb{R}^N_z)$, i.e. that verify $w(y+k, z+h) = w(y,z)$ for $(y,z) \in \mathbb{R}^N \times \mathbb{R}^N$ and $(k,h) \in \mathbb{Z}^N \times \mathbb{Z}^N$. $C^\infty_{per}(Y \times Z) = C_{per}(Y \times Z) \cap C^\infty(\mathbb{R}^N_y \times \mathbb{R}^N)$, and $L^p_{per}(Y \times Z)$ is the space of $Y \times Z$-periodic functions in $L^p_{loc}(\mathbb{R}^N_y \times \mathbb{R}^N_z)$.

2.2. FUNDAMENTALS OF REITERATED HOMOGENIZATION IN ORLICZ SPACES

This subsection is devoted to show some results which are useful for an explicit construction of reiterated multiscale convergence in the Orlicz setting. Indeed all the definitions are given starting from spaces of regular functions, then several norms are introduced together with proofs of functions spaces' properties. On the other hand we will not present neither arguments which are very similar to the ones used to deal with standard two scale convergence in the Orlicz setting, nor those related to reiterated two-scale convergence in the standard Sobolev setting (for the latter we refer to [25, Sections 2 and 4]).

We start by defining rigorously the traces of the form $u\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)$, $x \in \Omega$, $\varepsilon > 0$. We will consider several cases, according to the regularity of u.

Case 1. $u \in \mathcal{C}\left(\Omega \times \mathbb{R}^N_y \times \mathbb{R}^N_z\right)$

We define

$$u^\varepsilon(x) := u\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right).$$

Obviously $u^\varepsilon \in \mathcal{C}(\Omega)$. We define the trace operator of order $\varepsilon > 0, (t_\varepsilon)$ by

$$t_\varepsilon : u \in \mathcal{C}\left(\Omega \times \mathbb{R}^N_y \times \mathbb{R}^N_z\right) \longrightarrow u^\varepsilon \in \mathcal{C}(\Omega). \tag{2.2}$$

It results that the operator t^ε in (2.2) is linear and continuous.

Case 2. $u \in \mathcal{C}\left(\overline{\Omega}; \mathcal{C}_b\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)\right)$.

$$\mathcal{C}\left(\overline{\Omega}; \mathcal{C}_b\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)\right) \subset \mathcal{C}\left(\overline{\Omega}; \mathcal{C}\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)\right) \cong \mathcal{C}\left(\overline{\Omega} \times \mathbb{R}^N_y \times \mathbb{R}^N_z\right).$$

Then, we can consider $\mathcal{C}\left(\overline{\Omega}; \mathcal{C}_b\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)\right)$ as a subspace of $\mathcal{C}\left(\overline{\Omega} \times \mathbb{R}^N_y \times \mathbb{R}^N_z\right)$. Since $\overline{\Omega}$ is compact in $\mathbb{R}^N_x$, then $u^\varepsilon \in \mathcal{C}_b(\Omega)$ and the above operator can be considered from $\mathcal{C}\left(\overline{\Omega}; \mathcal{C}_b\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)\right)$ to $\mathcal{C}_b(\Omega)$, as linear and continuous.

Case 3. $u \in L^B(\Omega; V)$ where V is a closed vector subspace of $\mathcal{C}_b\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)$.

Recall that $u \in L^B(\Omega; V)$ means the function $x \to \|u(x)\|_\infty$, from Ω into $\mathbb{R}$, belongs to $L^B(\Omega)$ and

$$\|u\|_{L^B(\Omega; \mathcal{C}_b(\mathbb{R}^N_y \times \mathbb{R}^N_z))} = \inf\left\{k > 0 : \int_\Omega B\left(\frac{\|u(x)\|_\infty}{k}\right) dx \leq 1\right\} < +\infty.$$

Let $u \in \mathcal{C}\left(\overline{\Omega}; C_b\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)\right)$, then

$$|u^\varepsilon(x)| = \left|u\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)\right| \le \|u(x)\|_\infty .$$

As N-functions are non decreasing we deduce that

$$B\left(\frac{|u^\varepsilon(x)|}{k}\right) \le B\left(\frac{\|u(x)\|_\infty}{k}\right), \text{ for all } k > 0 \text{ and all } x \in \overline{\Omega}.$$

Hence we get

$$\int_\Omega B\left(\frac{|u^\varepsilon(x)|}{k}\right) dx \le \int_\Omega B\Big(\frac{\|u(x)\|_\infty}{k}\Big) dx,$$

thus $\int_\Omega B\left(\frac{\|u(x)\|_\infty}{k}\right) dx \le 1$ implies $\int_\Omega B\left(\frac{|u^\varepsilon(x)|}{k}\right) dx \le 1$, that is,

$$\|u^\varepsilon\|_{L^B(\Omega)} \le \|u\|_{L^B\left(\Omega; \mathcal{C}_b\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)\right)} .$$

Therefore the trace operator $u \to u^\varepsilon$ from $\mathcal{C}\left(\overline{\Omega}; V\right)$ into $L^B(\Omega)$, extends by density and continuity to a unique operator from $L^B(\Omega; \mathcal{C}_b(V))$.

It will be still denoted by

$$t^\varepsilon : u \to u^\varepsilon$$

and it verifies

$$\|u^\varepsilon\|_{L^B(\Omega)} \le \|u\|_{L^B\left(\Omega; \mathcal{C}_b\left(\mathbb{R}^N_y \times \mathbb{R}^N_z\right)\right)} \text{ for all } u \in L^B\left(\Omega; (V)\right). \tag{2.3}$$

In order to deal with reiterated multiscale convergence we need to have good definition for the measurability of test functions, so we should ensure measurability for the trace of elements $u \in L^\infty\left(\mathbb{R}^N_y; \mathcal{C}_b\left(\mathbb{R}^N_z\right)\right)$ and $u \in \mathcal{C}\left(\overline{\Omega}; L^\infty\left(\mathbb{R}^N_y; \mathcal{C}_b\left(\mathbb{R}^N_z\right)\right)\right)$, but we omit these proofs, referring to [25, Section 2].

Let $M : \mathcal{C}_{per}\left(Y \times Z\right) \to \mathbb{R}$ be the mean value functional (or equivalently "averaging operator") defined as

$$u \to M(u) := \iint_{Y \times Z} u\left(x, y\right) dxdy. \tag{2.4}$$

It results that

(i) M is nonnegative, i.e. $M(u) \ge 0$ for all $u \in \mathcal{C}_{per}(Y \times Z), u \ge 0$;
(ii) M is continuous on $\mathcal{C}_{per}\left(Y \times Z\right)$ (for the sup norm);
(iii) $M\left(1\right) = 1$;
(iv) M is translation invariant.

In the same spirit of [25], for the given N-function B, we define $\Xi^B\left(\mathbb{R}^N_y; \mathcal{C}_b\left(\mathbb{R}^N_z\right)\right)$ (or simply $\Xi^B\left(\mathbb{R}^N_y; \mathcal{C}_b\right)$) the following space

$$\Xi^B\left(\mathbb{R}^N_y; \mathcal{C}_b\right) := \Bigg\{ u \in L^B_{loc}(\mathbb{R}^N_x; C_b(\mathbb{R}^N_z)) : \text{for every } U \in \mathcal{A}(\mathbb{R}^N_x) : \sup_{0<\varepsilon\le 1} \inf\left\{k > 0\colon \int_U B\Big(\frac{\left\|u\left(\frac{x}{\varepsilon}, \cdot\right)\right\|_{L^\infty}}{k}\Big) dx \le 1\right\} < \infty \Bigg\}. \tag{2.5}$$

Hence putting

$$\|u\|_{\Xi^B\left(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z)\right)} = \sup_{0<\varepsilon\leq 1} \inf\left\{k>0\colon \int_{B_N(0,1)} B\left(\frac{\|u\left(\frac{x}{\varepsilon},\cdot\right)\|_{L^\infty}}{k}\right)dx \leq 1\right\}, \qquad (2.6)$$

with $B_N(0,1)$ being the unit ball of $\mathbb{R}^N_x$ centered at the origin, we have a norm on $\Xi^B\left(\mathbb{R}^N_y;\mathcal{C}_b\left(\mathbb{R}^N_z\right)\right)$ which makes it a Banach space.

We also denote by $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y;\mathcal{C}_b\right)$ the closure of $\mathcal{C}_{per}\left(Y\times Z\right)$ in $\Xi^B\left(\mathbb{R}^N_y;\mathcal{C}_b\right)$.

Recall that $L^B_{per}\left(Y\times Z\right)$ denotes the space of functions in $L^B_{\text{loc}}(\mathbb{R}^N_y\times\mathbb{R}^N_z)$ which are $Y\times Z$-periodic.

Clearly $\|\cdot\|_{B,Y\times Z}$ is a norm on $L^B_{per}\left(Y\times Z\right)$, namely it suffices to consider the L^B norm just on the unit period.

Let $u\in\mathcal{C}_{per}\left(Y\times Z\right)$, we have

$$\left|\int_{B_N(0,1)} u\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)dx\right| \leq \int_{B_N(0,1)} \left\|u\left(\frac{x}{\varepsilon},\cdot\right)\right\|_\infty dx \leq 2\left\|1\right\|_{\widetilde{B},B_N(0,1)}\|u\|_{\Xi^B\left(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z)\right)}.$$

The following result, useful to prove estimates which involve test functions on oscillating arguments (see for instance Proposition 2.7), is a preliminary instrument which aims at comparing the L^B norm in $Y\times Z$ with the one in (2.6).

Lemma 2.1. *There exists $C\in\mathbb{R}^+$ such that $\|u^\varepsilon\|_{B,B_N(0,1)}\leq C\|u\|_{B,Y\times Z}$ for every $0<\varepsilon\leq 1$ and $u\in\mathfrak{X}^B_{per}(\mathbb{R}^N_y;\mathcal{C}_b)$.*

Proof. Let $\varepsilon>0$. We start observing that we can always find a compact set $H\subset\mathbb{R}^N$ (independent on ε) such that

$$B_N(0,1)\subseteq\bigcup_{k\in Z_{\varepsilon^2}}\varepsilon^2(k+Z)\subseteq H,$$

where

$$Z_{\varepsilon^2}=\left\{k\in\mathbb{Z}^N : \varepsilon^2(k+Z)\cap\overline{B_N(0,1)}\neq\emptyset\right\}.$$

Define also

$$B_{N,\varepsilon^2}:=\text{int}\left(\bigcup_{k\in Z_{\varepsilon^2}}\varepsilon^2(k+\overline{Z})\right).$$

Then $B_N(0,1) \subset B_{N,\varepsilon^2}$. Thus

$$\begin{aligned}\int\limits_{B_N(0,1)} B\left(\left|u\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|\right)dx &\le \int\limits_{\bigcup_{k\in Z_{\varepsilon^2}}\varepsilon^2(k+\overline{Z})} B\left(\left|u\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|\right)dx\\ &= \sum_{i=1}^{n(\varepsilon^2)}\varepsilon^{2N}\int\limits_Z B\left(\left|u\left(\frac{\varepsilon^2k_i+\varepsilon^2 z}{\varepsilon},\frac{\varepsilon^2k_i+\varepsilon^2 z}{\varepsilon^2}\right)\right|\right)dz\\ &= \sum_{i=1}^{n(\varepsilon^2)}\varepsilon^{2N}\int\limits_Z B\left(\left|u\left(\varepsilon k_i+\varepsilon z, z\right)\right|\right)dz,\end{aligned}$$

where we have used the change of variables $x = \varepsilon^2(k_i + z)$, in each cube $\varepsilon^2(k_i + Z)$, the periodicity of u in the second variable, the fact that we can cover $B_N(0,1)$ with a finite number of cubes $\varepsilon^2(k_i + Z)$, depending on ε^2 and denoted by $n(\varepsilon^2)$.

Since $\left[\frac{x}{\varepsilon^2}\right] = k_i$ and $[z] = 0$ for every $x \in \varepsilon^2(k_i + Z)$ and $z \in Z$ and $\mathcal{L}^N(\varepsilon^2(k_i + Z)) = \varepsilon^{2N}$, we can write

$$\begin{aligned}\int\limits_{B_N(0,1)} B\left(\left|u\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|\right)dx &\le \sum_{i=1}^{n(\varepsilon^2)}\varepsilon^{2N}\int\limits_Z B\left(\left|u\left(\varepsilon\left[\frac{x}{\varepsilon^2}\right]+\varepsilon z, z\right)\right|\right)dz\\ &\le \sum_{i=1}^{n(\varepsilon^2)}\int\limits_{\varepsilon^2(k_i+Z)}\int\limits_Z B\left(\left|u\left(\varepsilon\left[\frac{x}{\varepsilon^2}\right]+\varepsilon z, z\right)\right|\right)dzdx\\ &\le \iint\limits_{B_{N,\varepsilon^2}\times Z} B\left(\left|u\left(\varepsilon\left[\frac{x}{\varepsilon^2}\right]+\varepsilon z, z\right)\right|\right)dzdx\\ &= \iint\limits_{B_{N,\varepsilon^2}\times Z} B\left(\left|u\left(\frac{x}{\varepsilon}, z\right)\right|\right)dxdz,\end{aligned}$$

where in the third line above we have used the fact that $\frac{x}{\varepsilon} = \varepsilon\left[\frac{x}{\varepsilon^2}\right] + \varepsilon z$.

Now, making again another change of variable of the same type, i.e. $y + h_i = x/\varepsilon$, after a covering of B_{N,ε^2} made by $\bigcup_{h_i\in Z_\varepsilon}\varepsilon(h_i + Y)$, where

$$Z_\varepsilon = \left\{h \in \mathbb{Z}^N : \varepsilon(h+Y) \cap \overline{B_{N,\varepsilon^2}} \neq \emptyset\right\},$$

we have

$$\begin{aligned}\iint\limits_{B_{N,\varepsilon^2}\times Z} B\left(\left|u\left(\frac{x}{\varepsilon}, z\right)\right|\right)dxdz &\le \sum_{i=1}^{n(\varepsilon)}\varepsilon^N\iint\limits_{h_i+Y\times Z} B\left(\left|u\left(\frac{\varepsilon h_i+\varepsilon y}{\varepsilon}, z\right)\right|\right)dydz\\ &\le \sum_{i=1}^{n(\varepsilon)}\varepsilon^N\iint\limits_{Y\times Z} B\left(|u(y,z)|\right)dydz.\end{aligned}$$

Up to another choice of $0 < \varepsilon_0 \leq 1$, we can observe that, given $\varepsilon < \varepsilon_0$, $B_N(0,1) \subset B_{N,\varepsilon^2}$ and also $B_N(0,1) \subset \bigcup_{i=1}^{n(\varepsilon)} \varepsilon(h_i + Y)$. On the other hand there is a compact H, which contains $\bigcup_{i=1}^{n(\varepsilon)} \varepsilon(h_i + Y)$ and whose measure satisfies the following inequality $\mathcal{L}^N(H) \geq \sum_{i=1}^{n(\varepsilon)} \varepsilon^N$.

Essentially repeating the same above computations, we have for every $k \in \mathbb{R}^+$, and $0 < \varepsilon \leq \varepsilon_0$ and $u \in L^B_{\mathrm{per}}(Y \times Z)$:

$$\int\limits_{B_N(0,1)} B\left(\left|\frac{u\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)}{k}\right|\right) dx \leq \varepsilon^N \sum_{i=1}^{n(\varepsilon)} \iint\limits_{Y\times Z} B\left(\left|\frac{u(y,z)}{k}\right|\right) dydz.$$

For $k = \|u\|_{B,Y\times Z}$ using the convexity of B, and the fact that $B(0) = 0$, we get

$$\begin{aligned}
&\int\limits_{B_N(0,1)} B\left(\left|\frac{u\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)}{\left(1+\mathcal{L}^N(H)\right)\|u\|_{B,Y\times Z}}\right|\right) dx \\
&\leq \frac{1}{\left(1+\mathcal{L}^N(H)\right)} \int\limits_{B_N(0,1)} B\left(\left|\frac{u\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)}{\|u\|_{B,Y\times Z}}\right|\right) dx \\
&\leq \varepsilon^N \sum_{i=1}^{n(\varepsilon)} \iint\limits_{Y\times Z} B\left(\left|\frac{u(y,z)}{\|u\|_{B,Y\times Z}}\right|\right) dydz \times \frac{1}{\left(1+\mathcal{L}^N(H)\right)} \\
&\leq \frac{n(\varepsilon)\varepsilon^N}{\left(1+\mathcal{L}^N(H)\right)} \iint\limits_{Y\times Z} B\left(\left|\frac{u(y,z)}{\|u\|_{B,Y\times Z}}\right|\right) dydz \\
&\leq \frac{\mathcal{L}^N(H)}{\left(1+\mathcal{L}^N(H)\right)} \iint\limits_{Y\times Z} B\left(\left|\frac{u(y,z)}{\|u\|_{B,Y\times Z}}\right|\right) dydz < 1,
\end{aligned}$$

where the non decreasing behaviour of B has been exploited. Therefore, by the definition of norm in $B_N(0,1)$,

$$\|u^\varepsilon\|_{B,B_N(0,1)} \leq \left(1+\mathcal{L}^N(H)\right)\|u\|_{B,Y\times Z}. \qquad \square$$

Lemma 2.2. *The mean value operator M defined on $\mathcal{C}_{per}(Y \times Z)$ by (2.4) can be extended by continuity to a unique linear and continuous functional denoted in the same way from $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right)$ to $\mathbb{R}$ such that:*

- *M is non negative, i.e. for all $u \in \mathfrak{X}^B_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right), u \geq 0 \Longrightarrow M(u) \geq 0$,*
- *M is translation invariant.*

Proof. It is a consequence of (2.5) and the definition of $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right)$, of the density of $\mathcal{C}_{per}(Y \times Z)$ in $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right)$, of the continuity of M on $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right)$ and of the continuity of $v \to v^\varepsilon$ from $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right)$ to $L^B(\Omega)$ (see (2.3)). $\square$

Now we endow $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y;\mathcal{C}_b\right)$ with another norm. Indeed we define $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y\times\mathbb{R}^N_z\right)$ the closure of $\mathcal{C}_{per}\left(Y\times Z\right)$ in $L^B_{loc}\left(\mathbb{R}^N_y\times\mathbb{R}^N_z\right)$ with the norm

$$\|u\|_{\Xi^B} := \sup_{0<\varepsilon\leq 1}\left\|u\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right\|_{B,2B_N}.$$

Via Riemann–Lebesgue lemma the above norm is equivalent to $\|u\|_{L^B(Y\times Z)}$, thus in the sequel we will consider this one.

For the sake of completeness, we state the following result which proves that the latter norm is controlled by the one defined in (2.6), thus together with Lemma 2.1, it provides the eqivalence among the introduced norms in $\mathfrak{X}^B_{per}(\mathbb{R}^N_y;\mathcal{C}_b)$. The proof is postponed to the Appendix.

Proposition 2.3. *It results that*

$$\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y;\mathcal{C}_b\right)\subset L^B_{per}\left(Y\times Z\right)=\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y\times\mathbb{R}^N_z\right)$$

and $\|u\|_{B,Y\times Z}\leq c\,\|u\|_{\Xi^B\left(\mathbb{R}^N_y;\mathcal{C}_b\left(\mathbb{R}^N_z\right)\right)}$ *for all* $u\in\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y;\mathcal{C}_b\right)$.

2.3. REITERATED TWO-SCALE CONVERGENCE IN ORLICZ SPACES

Generalizing definitions in [21, 25, 39] we introduce

$$L^B_{per}(\Omega\times Y\times Z)=\Big\{u\in L^B_{loc}(\Omega\times\mathbb{R}^N_y\times\mathbb{R}^N_z):\text{for } a.e\ x\in\Omega, u\left(x,\cdot,\cdot\right)\in L^B_{per}\left(Y\times Z\right)$$
$$\text{and}\iiint\limits_{\Omega\times Z}B\left(\left|u\left(x,y,z\right)\right|\right)dxdydz<\infty\Big\}.$$

We are in position to define reiterated two-scale convergence:

Definition 2.4. A sequence of functions $\left(u_\varepsilon\right)_\varepsilon\subseteq L^B(\Omega)$ is said to be:

- weakly reiteratively two-scale convergent in $L^B(\Omega)$ to a function $u_0\in L^B_{per}(\Omega\times Y\times Z)$ if

$$\int\limits_{\Omega}u_\varepsilon f^\varepsilon dx\to\iiint\limits_{\Omega\times Y\times Z}u_0 f\,dxdydz,\ \text{for all } f\in L^{\widetilde{B}}\left(\Omega;\mathcal{C}_{per}\left(Y\times Z\right)\right),\tag{2.7}$$

  as $\varepsilon\to 0$,
- strongly reiteratively two-scale convergent in $L^B(\Omega)$ to $u_0\in L^B_{per}\left(\Omega\times Y\times Z\right)$ if for $\eta>0$ and $f\in L^B\left(\Omega;\mathcal{C}_{per}\left(Y\times Z\right)\right)$ verifying $\|u_0-f\|_{B,\Omega\times Y\times Z}\leq\frac{\eta}{2}$ there exists $\rho>0$ such that $\|u_\varepsilon-f^\varepsilon\|_{B,\Omega}\leq\eta$ for all $0<\varepsilon\leq\rho$.

When (2.7) happens we denote it by "$u_\varepsilon\rightharpoonup u_0$ in $L^B\left(\Omega\right)-$ weakly reiteratively two-scale" and we will say that u_0 is the weak reiterated two-scale limit in $L^B\left(\Omega\right)$ of the sequence $\left(u_\varepsilon\right)_\varepsilon$.

Remark 2.5. The above definition extends in a canonical way, arguing in components, to vector valued functions.

Lemma 2.6. *If* $u \in L^{B}\left(\Omega; \mathcal{C}_{per}\left(Y \times Z\right)\right)$*, then* $u^{\varepsilon} \rightharpoonup u$ *in* $L^{B}\left(\Omega\right)$ *weakly reiteratively two-scale, and we have* $\lim_{\varepsilon \to 0} \|u^{\varepsilon}\|_{B,\Omega} = \|u\|_{B,\Omega \times Y \times Z}$.

Proof. Let $u \in L^{B}(\Omega; \mathcal{C}_{per}(Y \times Z))$ and $f \in L^{\widetilde{B}}(\Omega; \mathcal{C}_{per}(Y \times Z))$, then $uf \in L^{1}\left(\Omega; \mathcal{C}_{per}\left(Y \times Z\right)\right)$ and

$$\lim_{\varepsilon \to 0} \int_{\Omega} u^{\varepsilon} f^{\varepsilon} dx = \iiint_{\Omega \times Y \times Z} uf dxdydz.$$

Similarly, for all $\delta > 0$, $B\left(\left|\frac{u}{\delta}\right|\right) \in L^{1}\left(\Omega; \mathcal{C}_{per}\left(Y \times Z\right)\right)$ and the result follows. □

We are in position of proving a first sequential compactness result.

Proposition 2.7. *Given a bounded sequence* $\left(u_{\varepsilon}\right)_{\varepsilon} \subset L^{B}\left(\Omega\right)$*, one can extract a not relabelled subsequence such that* $\left(u_{\varepsilon}\right)_{\varepsilon}$ *is weakly reiteratively two-scale convergent in* $L^{B}(\Omega)$*.*

Proof. For $\varepsilon > 0$, set

$$L_{\varepsilon}\left(\psi\right) = \int_{\Omega} u_{\varepsilon}(x) \psi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^{2}}\right) dx, \quad \psi \in L^{\widetilde{B}}\left(\Omega; \mathcal{C}_{per}\left(Y \times Z\right)\right).$$

Clearly L_{ε} is a linear form and we have

$$\left|L_{\varepsilon}\left(\psi\right)\right| \leq 2 \left\|u_{\varepsilon}\right\|_{B,\Omega} \left\|\psi^{\varepsilon}\right\|_{\widetilde{B},\Omega} \leq c \left\|\psi\right\|_{L^{\widetilde{B}}(\Omega; \mathcal{C}_{per}(Y \times Z))}, \tag{2.8}$$

for a constant c independent on ε and ψ. Thus $\left(L_{\varepsilon}\right)_{\varepsilon}$ is bounded in

$$\left[L^{\widetilde{B}}(\Omega; \mathcal{C}_{per}(Y \times Z))\right]'.$$

Since $L^{\widetilde{B}}(\Omega; \mathcal{C}_{per}(Y \times Z))$ is a separable Banach space, we can extract a not relabelled subsequence, such that, as $\varepsilon \to 0$,

$$L_{\varepsilon} \to L_{0} \text{ in } \left[L^{\widetilde{B}}\left(\Omega; \mathcal{C}_{per}\left(Y \times Z\right)\right)\right]' \text{ weakly} * .$$

In order to characterize L_0 note that (2.8) ensures

$$\left|L_{0}\left(\psi\right)\right| \leq c \left\|\psi\right\|_{\widetilde{B},\Omega \times Y \times Z} \text{ for every } \psi \in L^{\widetilde{B}}\left(\Omega; \mathcal{C}_{per}\left(Y \times Z\right)\right).$$

Recalling that $L^{\widetilde{B}}(\Omega; \mathcal{C}_{per}(Y \times Z))$ is dense in $L^{\widetilde{B}}_{per}\left(\Omega \times Y \times Z\right)$, L_0 can be extended by continuity to an element of

$$\left[L^{\widetilde{B}}_{per}(\Omega \times Y \times Z)\right]' \equiv L^{B}_{per}\left(\Omega \times Y \times Z\right).$$

Thus there exists $u_0 \in L^B_{per}(\Omega \times Y \times Z)$ such that

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx = \iiint_{\Omega \times Y \times Z} u_0(x,y,z)\,\psi(x,y,z)\,dxdydz,$$

for all $\psi \in L^{\widetilde{B}}(\Omega; \mathcal{C}_{per}(Y \times Z))$. □

The proof of the following results are omitted, since they are consequence of "standard" density results and are very similar to the (non reiterated) two-scale case (see for instance [21]).

Proposition 2.8. *If a sequence $(u_\varepsilon)_\varepsilon$ is weakly reiteratively two-scale convergent in $L^B(\Omega)$ to $u_0 \in L^B_{per}(\Omega \times Y \times Z)$, then*

(i) *$u_\varepsilon \rightharpoonup \int_Z u_0(\cdot,\cdot,z)\,dz$ in $L^B(\Omega)$ weakly two-scale, and*
(ii) *$u_\varepsilon \rightharpoonup \widetilde{u_0}$ in $L^B(\Omega)$-weakly as $\varepsilon \to 0$ where $\widetilde{u_0}(x) = \iint_{Y \times Z} u_0(x,\cdot,\cdot)dydz$.*

Proposition 2.9. *Let*

$$\mathfrak{X}^{B,\infty}_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right) := \mathfrak{X}^B_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right) \cap L^\infty(\mathbb{R}^N_y \times \mathbb{R}^N_z).$$

If a sequence $(u_\varepsilon)_\varepsilon$ is weakly reiteratively two-scale convergent in $L^B(\Omega)$ to $u_0 \in L^B_{per}(\Omega \times Y \times Z)$ we also have

$$\int_\Omega u_\varepsilon f^\varepsilon dx \to \iiint_{\Omega \times Y \times Z} u_0 f\,dxdydz,$$

for all $f \in \mathcal{C}\left(\overline{\Omega}\right) \otimes \mathfrak{X}^{B,\infty}_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right)$.

Corollary 2.10. *Let $v \in \mathcal{C}\left(\overline{\Omega}; \mathfrak{X}^{B,\infty}_{per}(\mathbb{R}^N_y; \mathcal{C}_b)\right)$. Then $v^\varepsilon \rightharpoonup v$ in $L^B(\Omega)$-weakly reiteratively two-scale as $\varepsilon \to 0$.*

Remark 2.11.

(1) If $v \in L^B(\Omega; \mathcal{C}_{per}(Y \times Z))$, then $v^\varepsilon \to v$ in $L^B(\Omega)$-strongly reiteratively two-scale as $\varepsilon \to 0$.
(2) If $(u_\varepsilon)_\varepsilon \subset L^B(\Omega)$ is strongly reiteratively two-scale convergent in $L^B(\Omega)$ to $u_0 \in L^B_{per}(\Omega \times Y \times Z)$, then:
   (i) $u_\varepsilon \rightharpoonup u_0$ in $L^B(\Omega)$ weakly reiteratively two-scale as $\varepsilon \to 0$,
   (ii) $\|u_\varepsilon\|_{B,\Omega} \to \|u_0\|_{B,\Omega \times Y \times Z}$ as $\varepsilon \to 0$.

The following result is crucial to provide a notion of weakly reiterated two-scale convergence in Orlicz–Sobolev spaces and for the sequential compactness result on $W^1L^B(\Omega)$. It extends and presents an alternative proof of [21, Theorem 4.1].

To this end, recall first that $L^1_{per}\left(Y; W^1_\# L^B(Z)\right)$ denotes the space of functions $u \in L^1_{per}(Y \times Z)$, such that $u(y,\cdot) \in W^1_\# L^B(Z)$, for a.e. $y \in Y$.

Proposition 2.12. *Let Ω be a bounded open set in $\mathbb{R}^N_x$, and $(u_\varepsilon)_\varepsilon$ bounded in $W^1L^B(\Omega)$. There exist a not relabelled subsequence, $u_0 \in W^1L^B(\Omega)$, $(u_1, u_2) \in L^1\left(\Omega; W^1_\# L^B(Y)\right) \times L^1\left(\Omega; L^1_{per}\left(Y; W^1_\# L^B(Z)\right)\right)$ such that:*

(i) $u^\varepsilon \rightharpoonup u_0$ *weakly reiteratively two-scale in* $L^B(\Omega)$,
(ii) $D_{x_i}u^\varepsilon \rightharpoonup D_{x_i}u_0 + D_{y_i}u_1 + D_{z_i}u_2$ *weakly reiteratively two-scale in* $L^B(\Omega)$, $1 \le i \le N$,

as $\varepsilon \to 0$.

Corollary 2.13. *If $(u_\varepsilon)_\varepsilon$ is such that $u_\varepsilon \rightharpoonup v_0$ weakly reiteratively two-scale in $W^1L^B(\Omega)$, we have:*

(i) $u_\varepsilon \rightharpoonup \int_Z v_0(\cdot,\cdot,z)\,dz$ *weakly two-scale in* $W^1L^B(\Omega)$,
(ii) $u_\varepsilon \rightharpoonup \widetilde{v_0}$ *in* $W^1L^B(\Omega)$*-weakly, where* $\widetilde{v_0}(x) = \iint_{Y\times Z} v_0(x,\cdot,\cdot)dydz$.

Proof of Proposition 2.12. We recall that

$$L^B(\Omega_1 \times \Omega_2) \subset L^1\left(\Omega_1; L^B(\Omega_2)\right).$$

Moreover since B satisfies $\triangle_2$, there exist $q > p > 1$ such that

$$L^q(\Omega) \hookrightarrow L^B(\Omega) \hookrightarrow L^p(\Omega),$$

(relying on [15, Proposition 2.4] (see also [9, Proposition 3.5]) and a standard argument based on decreasing rearrangements), where the arrows $\hookrightarrow$ stand for continuous embedding.

Let $(u_\varepsilon)_\varepsilon$ be bounded in $L^B(\Omega)$. Then it is bounded in $L^p(\Omega)$ and we have:

(i) $u_\varepsilon \rightharpoonup U_0$ weakly reiteratively two-scale in $L^B(\Omega)$,
(ii) $u_\varepsilon \rightharpoonup u_0$ in $W^1L^B(\Omega)$,
(i)' $u_\varepsilon \rightharpoonup U_0'$ weakly reiteratively two-scale in $L^p(\Omega)$,
(ii)' $u_\varepsilon \rightharpoonup u_0'$ in $W^{1,p}(\Omega)$.

By classical results (see for instance [3] and [20]), we know that

$$u_0' = U_0'.$$

On the other hand, using the embeddings $W^{1,p}(\Omega)$-weak $\hookrightarrow \mathcal{D}'(\Omega)$-weak and $W^1L^B(\Omega)$-weak $\hookrightarrow \mathcal{D}'(\Omega)$-weak, we deduce that $u_0' = u_0 \in W^1L^B(\Omega)$. Moreover, since $L^{p'}(\Omega) \hookrightarrow L^{\tilde{B}}(\Omega)$, it results then $L^{p'}(\Omega; \mathcal{C}_{per}(Y \times Z)) \subset L^{\widetilde{B}}(\Omega; \mathcal{C}_{per}(Y \times Z))$, thus

$$U_0 = U_0',$$

and

$$U_0 = U_0' = u_0 = u_0'.$$

We also have

(iii) $D_{x_i}u_\varepsilon \rightharpoonup \tilde{w}$ weakly reiteratively two-scale in $L^B(\Omega)$, $1 \le i \le N$,
(iii)' $D_{x_i}u_\varepsilon \rightharpoonup D_{x_i}u_0 + D_{y_i}u_1 + D_{z_i}u_2$ weakly reiteratively two-scale in $L^p(\Omega)$, $1 \le i \le N$, with $(u_1, u_2) \in L^p_{per}\left(\Omega; W^{1,p}_\#(Y)\right) \times L^p\left(\Omega; L^p_{per}\left(Y; W^{1,p}_\#(Z)\right)\right)$ (see [3] and [20]).

Arguing in components, as done above, we are lead to conclude that

$$\tilde{w} = D_{x_i} u_0 + D_{y_i} u_1 + D_{z_i} u_2 \in L^B_{per}\left(\Omega \times Y \times Z\right)$$

and $D_{x_i} u_0 \in L^B(\Omega) \subset L^B_{per}\left(\Omega \times Y \times Z\right)$, as $u_0 \in W^1 L^B(\Omega)$. Therefore

$$\tilde{w} - D_{x_i} u_0 = D_{y_i} u_1 + D_{z_i} u_2 \in L^B_{per}(\Omega \times Y \times Z).$$

By Jensen's inequality,

$$B\Bigg(\int_Z |\tilde{w}| dz\Bigg) \leq \Bigg(\int_Z B(|\tilde{w}|) dz\Bigg),$$

then

$$\iint_{\Omega\times Y} B\left(\int_Z |\tilde{w}|\, dz\right) dxdy \leq \iint_{\Omega\times Y} \int_Z B\left(|\tilde{w}|\right) dzdxdy < \infty.$$

Since B satisfies $\triangle_2$,

$$\int_Z \tilde{w} dz = D_{x_i} u_0 + D_{y_i} u_1 \in L^B_{per}\left(\Omega \times Y\right)$$

with $D_{x_i} u_0 \in L^B(\Omega) \subset L^B_{per}(\Omega \times Y)$. Therefore

$$\int_Z \tilde{w} dz - D_{x_i} u_0 = D_{y_i} u_1 \in L^B_{per}\left(\Omega \times Y\right) \subset L^1\left(\Omega; L^B_{per}(Y)\right).$$

On the other hand, $u_1 \in L^p_{per}\left(\Omega; W^{1,p}_{\#}(Y)\right)$, i.e. for almost all x,

$$u_1\left(x,\cdot\right) \in W^{1,p}_{\#}(Y) = \left\{ v \in W^{1,p}_{per}(Y) : \int_Y v dy = 0 \right\}$$

and $D_{y_i} u_1\left(x,\cdot\right) \in L^B_{per}(Y)$. In particular $u_1\left(x,\cdot\right) \in L^p_{per}(Y) \subset L^1_{per}(Y)$.

To complete the proof it remains to show that every $v \in L^p(Y)$ with $D_{y_i} v \in L^B_{per}(Y)$ is in $L^B_{per}(Y)$.

Set $u = u - M(u) + M(u)$, where M is the averaging operator in (2.4). Then, by Poincaré's inequality, it results

$$\begin{aligned} \|u\|_{B,Y} &\leq \|u - M(u)\|_{B,Y} + \|M(u)\|_{B,Y} \leq c\,\|Du\|_{B,Y} + \|M(u)\|_{B,Y} \\ &\leq c\,\|Du\|_{B,Y} + c_1\left(1 + \|u\|_{L^1(Y)}\right) < \infty, \end{aligned}$$

the last inequality being consequence of the fact that $\lim_{t\to 0} B(t) = 0$, and there exists $c_1 > 0$ such that $B\left(\frac{1}{c_1}\right) < 1$. Hence,

$$\int_Y B\left(\frac{|M(u)|}{(1+|M(u)|)\,c_1}\right) dy \le \int_Y B\left(\frac{1}{c_1}\right) dy \le 1,$$

that is,

$$\|M(u)\|_{B,Y} \le (1+|M(u)|)\,c_1 = \left(1+\left|\int_Y u dy\right|\right) c_1 \le c_1\left(1+\|u\|_{L^1(Y)}\right).$$

Thus we can conclude that $u_1 \in L^1_{per}\left(\Omega; W^1_\# L^B(Y)\right)$.

For what concerns u_2 we can argue in a similar way. Recall that

$$\begin{aligned}\tilde{w} = D_{x_i}u_0 + D_{y_i}u_1 + D_{z_i}u_2 \in L^B_{per}(\Omega\times Y\times Z), \quad D_{x_i}u_0 \in L^B(\Omega),\\ u_1 \in L^1\left(\Omega; W^1_\# L^B(Y)\right), u_2 \in L^p\left(\Omega; L^p_{per}\left(Y; W^{1,p}_\#(Z)\right)\right).\end{aligned}$$

So

$$D_{z_i}u_2 = \tilde{w} - (D_{x_i}u_0 + D_{y_i}u_1) \in L^B_{per}(\Omega\times Y\times Z) \subset L^1\left(\Omega; L^1_{per}\left(Y; L^B(Z)\right)\right),$$

thus $D_{z_i}u_2(x,y,\cdot) \in L^B_{per}(Z)$ for almost all $(x,y) \in \Omega\times\mathbb{R}^N_y$.

$$\int_Z u_2(x,y,\cdot)\,dz = 0$$

as $u_2(x,y,\cdot) \in W^{1,p}_\#(Z)$. Consequently, since

$$u_2(x,y,\cdot) \in L^p_{per}(Z) \subset L^1_{per}(Z), \quad D_{z_i}u_2(x,y,\cdot) \in L^B_{per}(Z),$$

exploiting Poincare's inequality with the averaging operator M, as done above, it results that $u_2(x,y,\cdot) \in W^1_\# L^B(Z)$. Since

$$\begin{aligned}L^p(\Omega; L^p_{per}(Y; W^{1,p}_\#(Z))) &= L^p_{per}(\Omega\times Y; W^{1,p}_\#(Z)) \subset L^1_{per}(\Omega\times Y; W^{1,p}_\#(Z))\\ &= L^1(\Omega; L^1_{per}(Y; W^{1,p}_\#(Z))),\end{aligned}$$

we deduce that $u_2 \in L^1_{per}\left(\Omega; L^1\left(Y; W^1_\# L^B(Z)\right)\right)$. □

In view of the next applications, we underline that, under the assumptions of the above proposition, the canonical injection $W^1L^B(\Omega) \hookrightarrow L^B(\Omega)$ is compact.

3. HOMOGENIZATION OF INTEGRAL ENERGIES WITH CONVEX AND NON STANDARD GROWTH

In this section we study the asymptotic behaviour of (1.1) under the assumptions (H_1)–(H_4), stated above. We start by recalling the properties satisfied by F_ε in (1.2).

Since the function f in (1.2) is convex in the last argument and satisfies (H_4), it results that (cf. [21]) there exists a constant $c>0$ such that

$$|f(y,z,\lambda)-f(y,z,\mu)|\leq c\frac{1+B(2(1+|\lambda|+|\mu|))}{1+|\lambda|+|\mu|}\,|\lambda-\mu| \tag{3.1}$$

for all $\lambda,\mu\in\mathbb{R}^{nN}$ and for a.e. $y\in\mathbb{R}^N_y$ and for all $z\in\mathbb{R}^N_z$. Hence for fixed $\varepsilon>0$ and for $v\in W^1_0L^B(\Omega;\mathbb{R}^{nN})$, the function $x\mapsto f(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v(x))$ from Ω into $\mathbb{R}_+$ denoted by $f^\varepsilon(\cdot,\cdot,v)$, is well defined as an element of $L^1(\Omega)$ and it results (arguing as in [21, Proposition 3.1])

$$\begin{aligned}&\|f^\varepsilon(\cdot,\cdot,v)-f^\varepsilon(\cdot,\cdot,w)\|_{L^1(\Omega)}\\&\leq c(\|1\|_{\widetilde{B},\Omega}+\|b(1+|v|+|w|)\|_{\widetilde{B},\Omega})\,\|v-w\|_{(L^B(\Omega))^{nN}}\,.\end{aligned} \tag{3.2}$$

Moreover, (H_4) ensures that for $v\in W^1_0L^B(\Omega;\mathbb{R}^n)$ such that $\|Dv\|_{(L^B(\Omega))^{nN}}\geq 1$, we have

$$c_1\,\|Dv\|_{(L^B(\Omega))^{nN}}\leq\|f^\varepsilon(\cdot,\cdot,Dv)\|_{L^1(\Omega)}\leq c_2(1+\|Dv\|_{(L^B(\Omega))^{nN}}).$$

Consequently it results that F_ε is continuous, strictly convex and coercive thus there exists a unique $u_\varepsilon\in W^1_0L^B(\Omega)$ solution of the minimization problem $\min\limits_{v\in W^1_0L^B(\Omega)}F_\varepsilon(v)$, i.e.

$$F_\varepsilon(u_\varepsilon)=\min_{v\in W^1_0L^B(\Omega)}F_\varepsilon(v).$$

Let $\psi\in\mathcal{C}(\overline{\Omega};\mathcal{C}_{per}(Y\times Z))^N$. For fixed $x\in\overline{\Omega}$ the function

$$(y,z)\in\mathbb{R}^N_y\times\mathbb{R}^N_z\mapsto f(y,z,\psi(x,y,z))\in\mathbb{R}_+,$$

denoted by $f(\cdot,\cdot,\psi(x,\cdot,\cdot))$, lies in $L^\infty(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z))$. Hence one can define the function $x\in\overline{\Omega}\mapsto f(\cdot,\cdot,\psi(x,\cdot,\cdot))$ and denote it by $f(\cdot,\cdot,\psi))$ as an element of $\mathcal{C}(\overline{\Omega};L^\infty(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z)))$.

Therefore, for fixed $\varepsilon>0$, the function

$$x\mapsto f\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\psi\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\Big)$$

denoted by $f^\varepsilon(\cdot,\cdot,\psi^\varepsilon)$ is an element of $L^\infty(\Omega)$. Moreover, in view of the periodicity of $f(\cdot,\cdot,\psi)$, which is in $\mathcal{C}(\overline{\Omega};L^\infty_{per}(Y;\mathcal{C}^\infty_{per}(Z)))$ for all $\psi\in\mathcal{C}(\overline{\Omega};\mathcal{C}_{per}(Y\times Z))^N$, the following result holds.

Proposition 3.1. *For every $v \in \mathcal{C}\left(\overline{\Omega};\mathcal{C}_{per}\left(Y\times Z\right)\right)^N$ one has*

$$\lim_{\varepsilon\to 0}\int_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right)dx = \iiint_{\Omega\times Y\times Z} f\left(y,z,v\left(x,y,z\right)\right)dxdydz.$$

Furthermore, the mapping

$$v\in\mathcal{C}\left(\overline{\Omega};\mathcal{C}_{per}\left(Y\times Z\right)\right)^N \mapsto f\left(\cdot,\cdot,v\right)\in L^1_{per}\left(\Omega\times Y\times Z\right)$$

extends by continuity to a mapping still denoted by $v \mapsto f(\cdot,\cdot,v)$ from $(L^B_{per}(\Omega\times Y\times Z))^N$ into $L^1_{per}(\Omega\times Y\times Z)$ such that

$$\begin{aligned}&\left\|f\left(\cdot,\cdot,v\right)-f\left(\cdot,\cdot,w\right)\right\|_{L^1(\Omega\times Y\times Z)}\\ &\leq c\left(\|1\|_{\widetilde{B},\Omega}+\left\|b\left(1+|v|+|w|\right)\right\|_{\widetilde{B},\Omega\times Y\times Z}\right)\|v-w\|_{\left(L^B_{per}(\Omega\times Y\times Z)\right)^N}\end{aligned}\tag{3.3}$$

for all $v,w\in\left(L^B_{per}\left(\Omega\times Y\times Z\right)\right)^N$.

Proof. It is a simple adaptations of the proof of [21, Proposition 5.1], relying in turn on Corollary 2.10. Moreover, (3.3) follows by (3.1) and by arguments identical to those used to deduce (3.2), and omitted here since already presented in [21, Proposition 3.1], which in turn require the application of Lemma 2.1 □

Corollary 3.2. *Let*

$$\phi_\varepsilon(x) := \psi_0+\varepsilon\psi_1\left(x,\frac{x}{\varepsilon}\right)+\varepsilon^2\psi_1\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)$$

for $x\in\Omega$, where

$$\psi_0\in\mathcal{C}^\infty_0(\Omega),\quad \psi_1\in\left[\mathcal{C}^\infty_0(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)\right]\quad \textit{and}\quad \psi_2\in\left[\mathcal{C}^\infty_0(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)\otimes\mathcal{C}^\infty_{per}(Z)\right],$$

then, as $\varepsilon\to 0$,

$$\lim_{\varepsilon\to 0}\int_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},D\phi_\varepsilon\right)dx = \iiint_{\Omega\times Y\times Z} f\left(y,z,D\psi_0+D_y\psi_1+D_z\psi_2\right)dxdydz.$$

Proof. It is a simple adaptation of [21, Corollary 5.1], relying on (3.1) and (3.2), observing that

$$f^\varepsilon(\cdot,\cdot,(D\psi_0+D_y\psi_1+D_z\psi_2)^\varepsilon)\in C(\overline{\Omega};\mathfrak{X}^{B,\infty}_{per}(\mathbb{R}^N_y;\mathcal{C}_b))$$

and Corollary 2.10 applies. □

Now, we observe that, thanks to the density of $\mathcal{D}(\Omega)$ in $W^1_0L^B(\Omega)$, of $\mathcal{C}^\infty_{per}(Y)/\mathbb{R}$ in $W^1_\# L^B_{per}(Y)$ and that of $\mathcal{C}^\infty_{per}(Y)\otimes\mathcal{C}^\infty_{per}(Z)/\mathbb{R}$ in $L^1_{per}(Y;W^1_\# L^B(Z))$, the space

$$F^\infty_0 := \mathcal{D}(\Omega)\times\left[\mathcal{D}(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)/\mathbb{R}\right]\times\left[\mathcal{D}(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)\otimes\mathcal{C}^\infty_{per}(Z)/\mathbb{R}\right]\tag{3.4}$$

is dense in $\mathbb{F}^1_0L^B$.

By hypotheses (H_1)–(H_4), it is easily seen that the following result holds.

Lemma 3.3. *There exists a unique $u=(u_0,u_1,u_2)\in\mathbb{F}^1_0L^B$ which solves* (1.7)*.*

3.1. PROOF OF THEOREM 1.1

This subsection is devoted to provide an application of reiterated two-scale convergence to the study of minimum problems involving integral functionals, i.e. to prove Theorem 1.1. The proof will be achieved by means of several steps. First, following the same strategy in [37], (see also [33]) we regularize the integrands in order to get an approximating family of differentiable integrands with some extra properties which will be detailed in the sequel.

Let $f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ be such that (H_1)–(H_4) hold. Set

$$f_m : (y, z, \lambda) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{nN} \mapsto \int_{\mathbb{R}^{nN}} \theta_m(\eta) f(y, z, \lambda - \eta) d\eta, \tag{3.5}$$

where θ_m is a symmetric mollifier, namely $\theta_m \in \mathcal{D}(\mathbb{R}^{nN})$ (integer $m \geq 1$) with $0 \leq \theta_m$, $\mathrm{supp}(\theta_m) \subset \frac{1}{m}\overline{B}_{nN}(0,1)$, ($B_{nN}(0,1)$ being the open unit ball in $\mathbb{R}^{nN}$), and

$$\int_{\overline{B_{nN}(0,1)}} \theta_m(\eta) d\eta = 1.$$

It is easily to verify the following conditions.

$(H_1)_m$ $f_m(\cdot, z, \lambda)$ is measurable for every $(z, \lambda) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ and $f_m(y, \cdot, \lambda)$ is continuous for almost all $y \in \mathbb{R}^N_y$.

$(H_2)_m$ $f_m(y, z, \cdot)$ is strictly convex for almost all $(y, z) \in \mathbb{R}^N_y \times \mathbb{R}^N_z$.

$(H_3)_m$ There exists a constant $c > 0$ such that

$$f_m(y, z, \lambda) \leq c(1 + b(|\lambda|)),$$

for every $(z, \lambda) \in \mathbb{R} \times \mathbb{R}^{nN}$, and for almost all $y \in \mathbb{R}^N$

$(H_4)_m$ $f_m(\cdot, \cdot, \lambda)$ is periodic for all $\lambda \in \mathbb{R}^{nN}$

$(H_5)_m$ $\frac{\partial f_m}{\partial \lambda}(y, z, \lambda)$ exists for all $\lambda \in \mathbb{R}^{nN}$ and for almost all (y, z) and there exists a constant $c = c(m) > 0$ such that

$$\left| \frac{\partial f_m}{\partial \lambda}(y, z, \lambda) \right| \leq c(m)(1 + b(|\lambda|))$$

for all $\lambda \in \mathbb{R}^{nN}$ and for almost all $(y, z) \in \mathbb{R}^N \times \mathbb{R}^N$.

All the convergence results established in Proposition 3.1 and Corollary 3.2 for f, remain valid with f_m. Moreover for every $v \in L^B_{per}(\Omega \times Y \times Z)^{nN}$, one has $f_m(\cdot, \cdot, v) \to f(\cdot, \cdot, v)$ in $L^1(\Omega; L^1_{per}(Y \times Z))$, as $m \to +\infty$.

The next result extends to the Orlicz setting an argument presented in [37] to prove Corollary 2.10 therein.

Proposition 3.4. *Let* (v_ε) *be a sequence in* $L^B(\Omega)^{nN}$ *which reiteratively two-scale converges (in each component) to* $v \in L^B_{per}(\Omega \times Y \times Z)^{nN}$*, then, for any integer* $m \geq 1$*, we have that there exists a constant* C' *such that*

$$\iiint_{\Omega \times Y \times Z} f_m(y, z, v) dx dy dz - \frac{C'}{m} \leq \liminf_{\varepsilon \to 0} \int_\Omega f\Big(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, v_\varepsilon(x)\Big) dx.$$

Proof. Let $(v_l)_{l\geq 1}$ be a sequence in $\mathcal{D}(\Omega;\mathbb{R})\otimes\mathcal{C}^\infty_{per}(Y;\mathbb{R})\otimes\mathcal{C}^\infty_{per}(Z;\mathbb{R})$ such that $v_l \to v$ in $L^B_{per}(\Omega\times Y\times Z)^{nN}$ as $l\to\infty$. The convexity and differentiability of $f_m(y,z,\cdot)$ imply (for any integer $l\geq 1$),

$$\int_\Omega f_m\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon(x)\Big)dx \geq \int_\Omega f_m\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_l\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\Big)dx + \int_\Omega \frac{\partial f_m}{\partial\lambda}\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_l\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\Big)\cdot\Big(v_\varepsilon(x)-v_l\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\Big)dx.$$

$(H_1)_m,(H_2)_m$ and $(H_5)_m$ guarantee that

$$x\longmapsto \frac{\partial f_m}{\partial\lambda}(\cdot,\cdot,v_l)\in\mathcal{C}(\overline{\Omega};L^\infty_{per}(Y;\mathcal{C}^\infty_{per}(Z))).$$

Hence, by Proposition 3.1, it results

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{\partial f_m}{\partial\lambda}\Big(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_l\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\Big)\cdot\Big(v_\varepsilon(x)-v_l\Big(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\Big)\Big)dx = \iiint_{\Omega\times Y\times Z}\frac{\partial f_m}{\partial\lambda}(y,z,v_l(x,y,z))\cdot(v(x,y,z)-v_l(x,y,z))dxdydz.$$

Next, we observe that for a.e. y and every z,λ and a suitable positive constant c, one has

$$f_m(y,z,\lambda)\leq f(y,z,\lambda)+\frac{1}{m}c(1+b(2(1+|\lambda|))). \tag{3.6}$$

Indeed, for a.e. y, every z,λ,μ, by (3.1),

$$\begin{aligned} f(y,z,\lambda) &\leq f(y,z,\mu)+c\frac{B(2(1+|\lambda|+|\mu|))}{1+|\lambda|+|\mu|}\left|\lambda-\mu\right| \\ &\leq f(y,z,\mu)+c(1+b(1+|\lambda|+|\mu|))\left|\lambda-\mu\right|. \end{aligned}$$

Replacing λ by $\lambda-\eta$ and μ by λ respectively, we obtain

$$\begin{aligned} f(y,z,\lambda-\eta) &\leq f(y,z,\lambda)+c(1+b(1+|\lambda-\eta|+|\lambda|))\left|\eta\right| \\ &\leq f(y,z,\lambda)+c(1+b(1+|\eta|+2\left|\lambda\right|))\left|\eta\right|. \end{aligned}$$

Let $m>0$, and assume $|\eta|\leq\frac{1}{m}\leq 1$. Hence,

$$f\left(y,z,\lambda-\eta\right)\leq f\left(x,y,\lambda\right)+c\left(1+b\left(2\left(1+|\lambda|\right)\right)\right)\frac{1}{m}.$$

Multiplying both side of the inequality by θ_m, we get

$$f\left(y,z,\lambda-\eta\right)\theta_m\left(\eta\right)\leq f\left(y,z,\lambda\right)\theta_m\left(\eta\right)+\frac{1}{m}c\left(1+b\left(2\left(1+|\lambda|\right)\right)\right)\theta_m\left(\eta\right).$$

Integration leads to (3.6). Hence, given v_ε, we have

$$f_m\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon\right)\leq f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon\ \right)+\frac{1}{m}c\left(1+b\left(2\left(1+|v_\varepsilon|\right)\right)\right)$$

thus

$$\int\limits_\Omega f_m\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon\right)dx\leq\int\limits_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon\ \right)dx+\frac{1}{m}C|\Omega|+\frac{c}{m}\int\limits_\Omega\alpha\frac{b\left(2\left(1+|v_\varepsilon|\right)\right)}{\alpha}dx,\qquad 0<\alpha\leq 1.$$

But

$$\alpha\frac{b\left(2\left(1+|v_\varepsilon|\right)\right)}{\alpha}\leq\widetilde{B}\left(\alpha b\left(2\left(1+|v_\varepsilon|\right)\right)\right)+B\left(\frac{1}{\alpha}\right)\leq\alpha\widetilde{B}\left(b\left(2\left(1+|v_\varepsilon|\right)\right)\right)+B\left(\frac{1}{\alpha}\right).$$

Set

$$\Omega_1=\left\{x\in\Omega:2(1+|v_\varepsilon(x)|)>t_0\right\},\quad \Omega_2=\Omega\backslash\Omega_1.$$

Hence, we get

$$\begin{aligned}\int\limits_\Omega\alpha\frac{b\left(2\left(1+|v_\varepsilon|\right)\right)}{\alpha}dx&\leq\int\limits_\Omega\alpha\widetilde{B}\left(b\left(2\left(1+|v_\varepsilon|\right)\right)\right)dx+B\left(\frac{1}{\alpha}\right)|\Omega|\\&\leq\int\limits_{\Omega_1}\alpha\widetilde{B}\left(b\left(2\left(1+|v_\varepsilon|\right)\right)\right)dx\\&\quad+\int\limits_{\Omega_2}\alpha\widetilde{B}\left(b\left(2\left(1+|v_\varepsilon|\right)\right)\right)dx+B\left(\frac{1}{\alpha}\right)|\Omega|\\&\leq|\Omega_2|\alpha\widetilde{B}\left(b\left(t_0\right)\right)+B\left(\frac{1}{\alpha}\right)|\Omega|+\alpha\int\limits_{\Omega_1}B\left(4\left(1+|v_\varepsilon|\right)\right)dx.\end{aligned}$$

Let $C>1+\|4\left(1+|v_\varepsilon|\right)\|_{B,\Omega}$, then $\int_\Omega B\left(\frac{4(1+|v_\varepsilon|)}{C}\right)dx\leq 1$.
Indeed,

$$B\left(4\left(1+|v_\varepsilon|\right)\right)=B\left(C\frac{4\left(1+|v_\varepsilon|\right)}{C}\right)\leq K\left(C\right)B\left(\frac{4\left(1+|v_\varepsilon|\right)}{C}\right)$$

whenever $\frac{4(1+|v_\varepsilon|)}{C}\geq t_0$.
Set

$$\Omega_3=\left\{x\in\Omega_1:\frac{4\left(1+|v_\varepsilon|\right)}{C}\geq t_0\right\},\quad \Omega_4=\Omega_1\backslash\Omega_3.$$

Hence,

$$\begin{aligned}
&\int_{\Omega_1} B\left(4\left(1+|v_\varepsilon|\right)\right)dx \\
&= \int_{\Omega_4} B\left(4\left(1+|v_\varepsilon|\right)\right)dx + \int_{\Omega_3} B\left(4\left(1+|v_\varepsilon|\right)\right)dx \\
&\leq |\Omega_4| B\left(Ct_0\right) + \int_{\Omega_3} B\left(4\left(1+|v_\varepsilon|\right)\right)dx \\
&\leq |\Omega_4| B\left(Ct_0\right) + \int_{\Omega_3} B\left(C\frac{4\left(1+|v_\varepsilon|\right)}{C}\right)dx \\
&\leq |\Omega_4| B\left(Ct_0\right) + K\left(C\right)\int_{\Omega_3} B\left(\frac{4\left(1+|v_\varepsilon|\right)}{C}\right)dx \\
&\leq |\Omega_4| B\left(Ct_0\right) + K\left(C\right)\int_{\Omega} B\left(\frac{4\left(1+|v_\varepsilon|\right)}{C}\right)dx \\
&\leq |\Omega_4| B\left(Ct_0\right) + K\left(C\right)\int_{\Omega} B\left(\frac{4\left(1+|v_\varepsilon|\right)}{C}\right)dx.
\end{aligned}$$

Since $B \in \triangle_2$, and $(v_\varepsilon)_\varepsilon$ is bounded in $L^B(\Omega)$ it results that $\int_\Omega B\left(4\left(1+|v_\varepsilon|\right)\right)dx$ is also bounded.

Then, we have

$$\begin{aligned}
\int_{\Omega} f_m\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon\right)dx &\leq \int_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon\right)dx + \frac{1}{m}C|\Omega| \\
&\quad + \frac{c}{m}\Bigg(\alpha|\Omega|\widetilde{B}\left(b\left(t_0\right)\right) + B\left(\frac{1}{\alpha}\right)|\Omega| + \alpha(|\Omega_4|B(Ct_0) \\
&\qquad + K(C))\int_{\Omega} B\left(\frac{4\left(1+|v_\varepsilon|\right)}{C}\right)dx\Bigg) \\
&\leq \int_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon\right)dx + \frac{1}{m}C'
\end{aligned}$$

for a suitably big constant C'.

Thus,

$$\liminf_{\varepsilon\to 0}\int_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon(x)\right)dx$$
$$\geq \iiint_{\Omega\times Y\times Z} f_m\left(y,z,v_l(x,y,z)\right)dxdydz$$
$$-\frac{C'}{m}+\iiint_{\Omega\times Y\times Z}\frac{\partial f_m}{\partial\lambda}\left(y,z,v_l(x,y,z)\right)\cdot\left(v(x,y,z)-v_l(x,y,z)\right)dxdydz.$$

Using $(H_5)_m$ we get

$$\left|\iiint_{\Omega\times Y\times Z}\frac{\partial f_m}{\partial\lambda}\left(y,z,v_l(x,y,z)\right)\cdot\left(v(x,y,z)-v_l(x,y,z)\right)dxdydz\right|$$
$$\leq c\left\|v-v_l\right\|_{B,\Omega\times Y\times Z}\cdot\left\|1+b(v_l)\right\|_{\widetilde{B},\Omega\times Y\times Z}.$$

Since $v_l\to v$ in $L^B_{per}(\Omega\times Y\times Z)^{nN}$ as $l\to\infty$, it follows that, for $\delta>0$ arbitrarily fixed, there exists $l_0\in\mathbb{N}$ such that

$$\left|\iiint_{\Omega\times Y\times Z}\frac{\partial f_m}{\partial\lambda}\left(y,z,v_l(x,y,z)\right)\cdot\left(v(x,y,z)-v_l(x,y,z)\right)dxdydz\right|\leq\delta$$

for every $l\geq l_0$. Hence, for every $l\geq l_0$,

$$\liminf_{\varepsilon\to 0}\int_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon(x)\right)dx\geq\iiint_{\Omega\times Y\times Z} f_m\left(y,z,v_l(x,y,z)\right)dxdydz-\delta-\frac{C'}{m}.$$

Now sending $l\to\infty$ we have

$$\liminf_{\varepsilon\to 0}\int_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},v_\varepsilon(x)\right)dx\geq\iiint_{\Omega\times Y\times Z} f_m\left(y,z,v(x,y,z)\right)dxdydz-\delta-\frac{C'}{m}.$$

The arbitrariness of δ concludes the proof. □

Letting $m\to+\infty$, and replacing v_ε by Du_ε, with u_ε reiteratively two-scale convergent to

$$u(x,y,z):=u_0(x)+u_1(x,y)+u_2(x,y,z)$$

in $W^1L^B(\Omega;\mathbb{R}^n)$, one obtains the following result:

Corollary 3.5. *Let $(u_\varepsilon)_\varepsilon$ be a sequence in $W^1_0L^B(\Omega;\mathbb{R}^n)$ reiteratively two-scale convergent to $u=(u_0,u_1,u_2)\in\mathbb{F}^1_0L^B$. Then*

$$\iiint_{\Omega\times Y\times Z} f\left(y,z,\mathbb{D}u(x,y,z)\right)dxdydz\leq\liminf_{\varepsilon\to 0}\int_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du_\varepsilon(x)\right)dx,$$

where $\mathbb{D}u=Du_0+D_yu_1+D_zu_2$.

Now we are in position to put together all the previous results in order to prove our main result.

Proof of Theorem 1.1. For every ε, let u_ε be a minimizer of F_ε. Hypothesis (H_4) guarantees that $(u_\varepsilon)_\varepsilon$ is bounded in $W_0^1 L^B(\Omega;\mathbb{R})^n$. On the other hand, since the real sequence $(F_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is bounded, we can extract a not relabelled subsequence, such that we have $(a)-(b)$, in the statement, and $\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)$ exists.

It remains to verify that $u=(u_0,u_1,u_2)$ is the solution of the minimization problem (3.3). Let $\phi=(\psi_0,\psi_1,\psi_2)\in F_0^\infty$ with $\psi_0\in\mathcal{D}(\Omega)^n, \psi_1\in\left[\mathcal{D}(\Omega)\otimes\mathcal{C}_{per}^\infty(Y)/\mathbb{R}\right]^n$, $\psi_2\in\left[\mathcal{C}_0^\infty(\Omega)\otimes\mathcal{C}_{per}^\infty(Y)\otimes\mathcal{C}_{per}^\infty(Z)/\mathbb{R}\right]^n$. Define

$$\phi_\varepsilon := \psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2.$$

Then $\phi_\varepsilon\in W_0^1L^B(\Omega;\mathbb{R})^n$ so that we have

$$\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du_\varepsilon(x)\right)dx \le \int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},D\phi_\varepsilon(x)\right)dx.$$

Therefore, taking the limit as $\varepsilon\to 0$, using the arbitrariness of ϕ, the density of F_0^∞ in $\mathbb{F}_0^1L^B$, the above inequality leads us to

$$\lim_{\varepsilon\to 0}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du_\varepsilon(x)\right)dx \le \inf_{v\in\mathbb{F}_0^1L^B}\iiint_{\Omega\times Y\times Z} f(y,z,\mathbb{D}v(x,y,z))\,dxdydz.$$

This inequality, together with Corollary 3.5, leads to the equality

$$\iiint_{\Omega\times Y\times Z} f(y,z,\mathbb{D}u(x,y,z))\,dxdydz = \inf_{v\in\mathbb{F}_0^1L^B}\iiint_{\Omega\times Y\times Z} f(y,z,\mathbb{D}v(x,y,z))\,dxdydz.$$

Since (1.7) has a unique solution, we can conclude that the whole sequence $(u_\varepsilon)_\varepsilon$ verifies (a)–(b) and the proof is completed. □

The following corollary recasts the above results in terms of Γ-convergence with respect to reiterated two-scale convergence, thus extending the result proven in the single scale case in [23] (see [14] for details about Γ-convergence).

Corollary 3.6. *Let Ω and f be as in Theorem 1.1. Then, for every $u=(u_0,u_1,u_2)\in\mathbb{F}_0^1L^B$, it results*

$$\begin{aligned}&\inf\left\{\liminf_{\varepsilon\to 0}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du_\varepsilon\right)dx : u_\varepsilon \rightharpoonup u \text{ weakly reiteratively two-scale}\right\}\\ &=\inf\left\{\limsup_{\varepsilon\to 0}\int_\Omega f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du_\varepsilon\right)dx : u_\varepsilon \rightharpoonup u \text{ weakly reiteratively two-scale}\right\}\\ &=\iiint_{\Omega\times Y\times Z} f(y,x,\mathbb{D}u(x,y,z))dxdydz,\end{aligned}\tag{3.7}$$

where $\mathbb{D}u = Du_0 + D_yu_1 + D_zu_2$.

Proof. The statement will be proven if we show that

$$\iiint\limits_{\Omega\times Y\times Z} f(y,x,\mathbb{D}u(x,y,z))dxdydz \leq \liminf_{\varepsilon\to 0}\int\limits_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},Du_\varepsilon\right)dx,$$

for any sequence $u_\varepsilon \rightharpoonup u \in \mathbb{F}^1_0 L^B$ reiteratively two-scale, and we exhibit a sequence $\overline{u}_\varepsilon$ such that $\overline{u}_\varepsilon \rightharpoonup u \in \mathbb{F}^1_0 L^B$ reiteratively two-scale, and

$$\limsup_{\varepsilon\to 0}\int\limits_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},D\overline{u}_\varepsilon\right)dx \leq \iiint\limits_{\Omega\times Y\times Z} f(y,x,\mathbb{D}u(x,y,z))dxdydz.$$

The first inequality is consequence of Corollary 3.5. For what concerns the upper bound we preliminarily observe that a standard argument in the Orlicz setting allows us to consider, for any given N-function B, a generating continuous function b such that B verifies the $\triangle_2$ condition near 0.

Now let

$$\phi_\varepsilon(x) := \psi_0 + \varepsilon\psi_1\left(x,\frac{x}{\varepsilon}\right) + \varepsilon^2\psi_1\left(x,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)$$

for $x \in \Omega$, where

$$\psi_0 \in \mathcal{C}^\infty_0(\Omega), \psi_1 \in \left[\mathcal{C}^\infty_0(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)\right]$$

and $\psi_2 \in \left[\mathcal{C}^\infty_0(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)\otimes\mathcal{C}^\infty_{per}(Z)\right]$. Then

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega} f\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},D\phi_\varepsilon\right)dx = \iiint\limits_{\Omega\times Y\times Z} f\left(y,z,D\psi_0 + D_y\psi_1 + D_z\psi_2\right)dxdydz.$$

Let

$$\mathbb{F}^1 L^B := W^1L^B(\Omega)\times L^B_{D_y}\left(\Omega; W^1_\# L^B(Y)\right)\times L^B_{D_z}\left(\Omega; L^1_{per}\left(Y; W^1_\# L^B(Z)\right)\right)$$

where $L^B_{D_y}\left(\Omega; W^1_\# L^B(Y)\right)$, $L^B_{D_z}\left(\Omega; L^1_{per}\left(Y; W^1_\# L^B(Z)\right)\right)$ have been defined in (1.5). Recalling also that $\mathbb{F}^1 L^B$, equipped with the norm

$$\|u_0\|_{\mathbb{F}^1L^B} = \|Du\|_{B,\Omega} + \|D_y u_1\|_{B,\Omega\times Y} + \|D_z u_2\|_{B,\Omega\times Y\times Z}, \quad u_0 = (u,u_1,u_2)\in\mathbb{F}^1_0 L^B,$$

is Banach space, thanks to the density of $\mathcal{C}^\infty(\overline{\Omega})$ in $W^1L^B(\Omega)$, of $\mathcal{C}^\infty_{per}(Y)/\mathbb{R}$ in $W^1_\# L^B_{per}(Y)$ and of $\mathcal{C}^\infty_{per}(Y)\otimes\mathcal{C}^\infty_{per}(Z)/\mathbb{R}$ in $L^1_{per}\left(Y; W^1_\# L^B(Z)\right)$, the space

$$F^\infty := \mathcal{C}^\infty(\overline{\Omega})\times\left[\mathcal{D}(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)/\mathbb{R}\right]\times\left[\mathcal{D}(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)\otimes\mathcal{C}^\infty_{per}(Z)/\mathbb{R}\right]$$

is dense in $\mathbb{F}^1 L^B$.

As above, for $v_0 = (v,v_1,v_2)\in\mathbb{F}^1 L^B$ we denote by $\mathbb{D}v_0$ the sum $Dv + D_y v_1 + D_z v_2$. In view of the stated density, given $\delta > 0$, there exist

$$u_\delta \in \mathcal{C}^\infty(\overline{\Omega}), v_\delta \in \left[\mathcal{D}(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)/\mathbb{R}\right], \quad w_\delta \in \left[\mathcal{D}(\Omega)\otimes\mathcal{C}^\infty_{per}(Y)\otimes\mathcal{C}^\infty_{per}(Z)/\mathbb{R}\right]$$

such that

$$\|v - u_\delta\|_{W^1L^B(\Omega)} + \|v_1 - v_\delta\|_{L^1\left(\Omega;W^1_\# L^B(Y)\right)} + \|v_2 - w_\delta\|_{L^1\left(\Omega;L^B_{per}\left(Y;W^1_\# L^B(Z)\right)\right)} < \delta.$$

For every $\delta, \varepsilon > 0$ and for every $x \in \Omega$, define

$$u_{\delta,\varepsilon}(x) =: u_\delta(x) + \varepsilon v_\delta\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 w_\delta\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right).$$

It results that

$$\begin{aligned} D_x u_{\delta,\varepsilon}(x) = D_x u_\delta(x) + \varepsilon D_x v_\delta\left(x, \frac{x}{\varepsilon}\right) \\ + \varepsilon^2 D_x w_\delta\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) + D_y v_\delta\left(x, \frac{x}{\varepsilon}\right) \\ + \varepsilon D_y w_\delta\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) + D_z w_\delta\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right). \end{aligned}$$

As immediate consequence, for δ fixed,

$$\begin{aligned} &u_{\delta,\varepsilon} \to u_\delta \text{ in } L^B(\Omega), \\ &D_x u_{\delta,\varepsilon} \to D_x u_\delta + D_y v_\delta + D_z w_\delta \text{ strongly reiteratively two-scale in } L^B_{per}(\Omega \times Y \times Z), \end{aligned}$$

as $\varepsilon \to 0$.

Next, setting

$$c_{\delta,\varepsilon} =: \|u_{\delta,\varepsilon} - v\|_{W^1L^B(\Omega)} + \left| \|Du_{\delta,\varepsilon}\|_{L^B(\Omega)} - \|Dv + D_y v_1 + D_y v_2\|_{L^B(\Omega\times Y\times Z)} \right|,$$

using the above density results:

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} c_{\delta,\varepsilon} = 0.$$

Then, via diagonalization, we can construct a sequence $\delta(\varepsilon) \to 0$, as $\varepsilon \to 0$ and such that:

(i) $\lim_{\delta(\varepsilon)\to 0} c_{\delta(\varepsilon),\varepsilon} = 0$,
(ii) $u_{\delta(\varepsilon),\varepsilon} \to v$ in $L^B(\Omega)$,
(iii) $Du_{\delta(\varepsilon),\varepsilon} \rightharpoonup D_x v + D_y v_1 + D_z v_2$ strongly reiteratively in $L^B_{per}(\Omega \times Y \times Z)$.

In particular, it follows that $Du_{\delta(\varepsilon),\varepsilon} \rightharpoonup D_x v$ weakly in $L^B(\Omega)$, and

$$\lim_{\varepsilon\to 0} \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du_{\delta(\varepsilon),\varepsilon}(x)\right) dx = \iiint_{\Omega\times Y\times Z} f(y, z, D_x v + D_y v_1 + D_z v_2)\, dxdydz.$$

Since the above construction can be performed for every triple $(v, v_1, v_2) \in \mathbb{F}^1 L^B$, it is enough to repeat the construction for $u_0 = (u, u_1, u_2) \in \mathbb{F}^1_0 L^B$ as claimed. □

Remark 3.7. It is worth to observe that the result in Corollary 3.6 holds, with the exact same proof under weaker assumptions than those in Theorem 1.1: namely (H_2) can be replaced by convexity, and in (H_4) it is not crucial to have f non-negative, it is enough to have a bound from below. Moreover the same proof can be performed if u_ε and u are vector valued and not just scalar valued functions.

4. APPENDIX

Here we present the proof Proposition 2.3 which establishes the equivalence between the norms $\|\cdot\|_{B,Y\times Z}$ and $\|\cdot\|_{\Xi^B\left(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z)\right)}$ in $\mathfrak{X}^B_{per}\left(\mathbb{R}^N_y;\mathcal{C}_b\right)$.

Proof of Proposition 2.3. The inclusion is a direct consequence of the definition, and clearly every element in $L^B_{per}(Y\times Z)$, can be obtained as limit in $\|\cdot\|_{B,Y\times Z}$ norm of sequences in $\mathcal{C}_{per}(Y\times Z)$.

On the other hand, by the very definition of $\mathfrak{X}^B_{per}(\mathbb{R}^N_y;\mathcal{C}_b)$, $v\in\mathfrak{X}^B_{per}(\mathbb{R}^N_y;\mathcal{C}_b)$ if and only if there exists $(v_n)_{n\in\mathbb{N}}\in\mathcal{C}_{per}(Y\times Z)$ such that $(v_n)_{n\in\mathbb{N}}$ converge to v for the norm $\|\cdot\|_{\Xi^B(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z))}$.

Thus for every $w\in\mathfrak{X}^B_{per}(\mathbb{R}^N_y;\mathcal{C}_b)$ there exist $(w_n)_{n\in\mathbb{N}}\subset\mathcal{C}_{per}(Y\times Z)$ such that as $n\to\infty, w_n\to w$ in $\Xi^B(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z))$.

We claim that for every $u\in C_{per}(Y\times Z)$, it results

$$\|u\|_{B,Y\times Z}\le\|u\|_{\Xi^B(\mathbb{R}^N_y;\mathcal{C}_b)}.$$

From the claim it follows that

$$\|w_n-w_m\|_{B,Y\times Z}\le\|w_n-w_m\|_{\Xi^B(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z))},$$

for all $m,n\in\mathbb{N}$. Therefore $(w_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{X}^B_{per}(\mathbb{R}^N_y\times\mathbb{R}^N_z)$ and in $\mathfrak{X}^B_{per}(\mathbb{R}^N_y;\mathcal{C}_b)$. Hence there exist $w^1\in\mathfrak{X}^B_{per}(\mathbb{R}^N_y\times\mathbb{R}^N_z), w^2\in\mathfrak{X}^B_{per}(\mathbb{R}^N_y;\mathcal{C}_b)$ such that

$$\lim_{n\to\infty}\left\|w_n-w^1\right\|_{B,Y\times Z}=\lim_{n\to\infty}\left\|w_n-w^2\right\|_{\Xi^B(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z))}=0.$$

Moreover the passage to the limit guarantees that

$$\left\|w^1\right\|_{B,Y\times Z}\le\left\|w^2\right\|_{\Xi^B(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z))}.$$

It is also clear, considering the convergence in the sense of distributions, that $w^1=w^2$.

It remains to prove the claim. To this end, let $u,v\in\mathcal{C}_{per}\left(Y\times Z\right)$; we have

$$\begin{aligned}\left|\int_{B_N(0,1)}u\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)v\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)dx\right|&\le\int_{B_N(0,1)}\left\|u\left(\frac{x}{\varepsilon},\cdot\right)\right\|_\infty\left|v\left(\frac{x}{\varepsilon},\frac{x}{\varepsilon^2}\right)\right|dx\\&\le 2\left\|v^\varepsilon\right\|_{\widetilde{B},B_N(0,1)}\|u\|_{\Xi^B\left(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z)\right)}.\end{aligned}$$

Passing to limit, as $\varepsilon\to 0$, we obtain:

$$\left|\int_{Y\times Z}u\left(y,z\right)v\left(y,z\right)dydz\right|\le 2\left\|v\right\|_{\widetilde{B},Y\times Z}\|u\|_{\Xi^B\left(\mathbb{R}^N_y;\mathcal{C}_b(\mathbb{R}^N_z)\right)}.$$

Using the density of $\mathcal{C}_{per}(Y \times Z)$ in $L^{\widetilde{B}}_{per}(Y \times Z)$ we obtain (with the topology of the norm)

$$\left| \int_{Y \times Z} u(y,z)\, v(y,z)\, dydz \right| \leq 2 \|v\|_{\widetilde{B}, Y \times Z} \|u\|_{\Xi^B\left(\mathbb{R}^N_y; \mathcal{C}_b(\mathbb{R}^N_z)\right)},$$

for all $v \in L^{\widetilde{B}}_{per}(Y \times Z)$. Thus

$$\|u\|_{B, Y \times Z} \leq 2 \|u\|_{\Xi^B\left(\mathbb{R}^N_y; \mathcal{C}_b(\mathbb{R}^N_z)\right)},$$

for all $u \in \mathcal{C}_{per}(Y \times Z)$, and we get the result for all $u \in \mathfrak{X}^B_{per}\left(\mathbb{R}^N_y; \mathcal{C}_b\right)$, via standard density arguments. □

Acknowledgements
This paper has been written during the visit of Joel Fotso Tachago at Dipartimento di Ingegneria Industriale (INdAM unit) at University of Salerno. The authors gratefully acknowledge the support of the INdAM-ICTP Research in pairs programme. Elvira Zappale is a member of INdAM-GNAMPA, and gratefully acknowledges the support of the project GNAMPA 2019 "Analisi ed ottimizzazione di strutture sottili".

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Received: February 20, 2020.
Revised: January 8, 2021.
Accepted: January 8, 2021.