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SOME EXISTENCE RESULTS FOR A NONLOCAL NON-ISOTROPIC PROBLEM

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Abstract. In this paper we deal with the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N}\left[\left(a+b\int\limits_{\Omega}|\partial_i u|^{p_i}\,dx\right)\partial_i\left(|\partial_i u|^{p_i-2}\,\partial_i u\right)\right]=\frac{f(x)}{u^{\gamma}}\pm g(x)u^{q-1} & in\ \Omega,\\ u\geq 0 & in\ \Omega,\\ u=0 & on\ \partial\Omega, \end{cases}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$. We will assume without loss of generality that $1\leq p_1\leq p_2\leq\ldots\leq p_N$ and that f and g are non-negative functions belonging to a suitable Lebesgue space $L^m(\Omega)$, $1<q<\overline{p}^*$, $a>0, b>0$ and $0<\gamma<1$.

Keywords: anisotropic operator, integro-differential problem, variational methods.

Mathematics Subject Classification: 35A15, 35B09, 35E15, 35J20.

1. INTRODUCTION

We consider in this paper the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N}\left[\left(a+b\int\limits_{\Omega}|\partial_i u|^{p_i}\,dx\right)\partial_i\left(|\partial_i u|^{p_i-2}\,\partial_i u\right)\right]=\frac{f(x)}{u^{\gamma}}\pm g(x)u^{q-1} & in\ \Omega,\\ u\geq 0 & in\ \Omega,\\ u=0 & on\ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$ $(N\geq 2)$ and, without loss of generality, $1\leq p_1\leq p_2\leq\ldots\leq p_N$. We will assume that f and g are non-negative functions belonging to suitable Lebesgue spaces and $1<q<\overline{p}^*$, $a>0$, $b>0$, $0<\gamma<1$. In the whole paper we will denote $\frac{\partial u}{\partial x_i}=\partial_i u$.

When $b = 0$, problems involving the anisotropic operator

$$Lu = \sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i - 2} \, \partial_i u \right]$$

are widely studied in the literature. We cite, for example, [10–12] and the references therein. There is also a huge literature about elliptic problems with singular nonlinearities, see for instance [6, 18], when the considered differential operator is the Laplacian operator, and we refer to [9], where the author deals with the p-Laplacian. In the works [1, 12, 25] and [26], in addition to the Laplacian, an anisotropic operator and the p-Laplacian, respectively, a first order term is considered. We also mention some very recent works dealing with anisotropic operators of different kinds in [3, 16, 29, 30, 34].

Let us also turn the reader attention to the leading works [4, 15, 17] and the very recent one [8], where the anisotropic operator is associated to a nonlinearity, and existence, uniqueness, multiplicity and non-existence results are obtained by various ways.

When a singular term is associated to the anisotropic operator L there are some few recent results. For example in [20, 27], the existence and regularity of solutions to the equation

$$Lu = \sum_{i=1}^{N} \partial_i \left[|\partial_i u|^{p_i - 2} \, \partial_i u \right] = \frac{f(x)}{u^{\gamma(x)}} + \lambda g(x, u)$$

where g presents a regular nonlinearity in u, are obtained for the case $\lambda = 0$ by approximation methods, while in [13], the authors obtained existence results for $\lambda \neq 0$ by monotonicity methods.

Systems involving operator L are less studied. We cite in this direction the works [13] and [7]. To the best of our knowledge, there are only two works in the literature studying problems, where the anisotropic operator L is associated to nonlocal terms. These are [14] and [5]. We also mention [21] and the reference therein for results about variational techniques applied to the study of problems involving L.

There is a huge literature about the study of different types of Kirchhoff problems, that are problems involving the nonlocal operator

$$\mathcal{L}_p u = -\left(a + b \int_{\Omega} |\nabla u|^p \, dx \right) \Delta_p u,$$

especially when $p = 2$. We limit our presentation to problems involving the operator $\mathcal{L}_p$ associated to singular nonlinearities. In the work [24], the authors obtained the existence and uniqueness result for the solution to equation

$$\mathcal{L}_2 u = \frac{f(x)}{u^\gamma} - \lambda u^{q-1}$$

associated to the Dirichlet boundary condition. In [22], the authors succeed in proving a multiplicity result for the problem

$$\mathcal{L}_2 u = \frac{\lambda}{u^\gamma} + u^{q-1}.$$

The reader is referred, for instance, to [2, 23, 31, 33] and the references therein, where such kind of problems were considered.

Inspired by all these previous works, we propose to study a problem involving a differential operator that combines in a particular way both operators L and $\mathcal{L}_p$ associated to singular terms, that is, (1.1).

The paper is organized as follows. After this brief introduction, we present some preliminaries dealing with the functional setting associated to our problem. In the third section we study the problem (1.1), where the regular nonlinearity is placed as a reaction term. We also obtain a multiplicity result for some approximating problems. In the last section we prove the existence of solution to the problem (1.1), but with the regular nonlinearity placed as an absorption term. We also obtain a partial uniqueness result for a particular range of parameters.

2. PRELIMINARIES

The natural function spaces associated to the problem (3.1) are the anisotropic Sobolev spaces

$$W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,1}(\Omega) : \partial_i v \in L^{p_i}(\Omega) \right\}$$

and

$$W_0^{1,(p_i)}(\Omega) = W^{1,(p_i)}(\Omega) \cap W_0^{1,1}(\Omega)$$

endowed by the usual norm

$$\|v\|_{W_0^{1,(p_i)}(\Omega)} = \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}.$$

We will say that $u \in W_0^{1,(p_i)}(\Omega)$ is solution to (3.1) if and only if for all $\varphi \in W_0^{1,(p_i)}(\Omega)$

$$\sum_{i=1}^{N} \left[\left(a + b \int_\Omega |\partial_i u|^{p_i} \right) \int_\Omega |\partial_i u|^{p_i - 1} \partial_i \varphi \right] = \int_\Omega \frac{f\varphi}{u^\gamma} + \int_\Omega g u^{q-1} \varphi.$$

We shall also often use the following indices

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}$$

and

$$\overline{p}^* = \frac{N\overline{p}}{N - \overline{p}}, \quad p_\infty = \max\left\{ p_N, \overline{p}^* \right\}.$$

It will be assumed throughout this paper that $\overline{p} < N$, so in this case we will have

$$W_0^{1,(p_i)}(\Omega) \subset L^r(\Omega) \quad \text{for all } r \in [1, \overline{p}^*].$$

This imbedding is compact whenever $r < \overline{p}^*$. Let us recall the following Sobolev type inequalities. We refer to the early works [19, 28] and [32].

Theorem 2.1.

$$\|v\|^{p_N}_{L^{\overline{p}^*}(\Omega)} \leq C \sum_{i=1}^{N} \|\partial_i v\|^{p_i}_{L^{p_i}(\Omega)}, \tag{2.1}$$

$$\|v\|_{L^r(\Omega)} \leq C \prod_{i=1}^{N} \|\partial_i v\|^{\frac{1}{N}}_{L^{p_i}(\Omega)} \quad \text{for all } r \in [1, \overline{p}^*] \tag{2.2}$$

and for all $v \in W_0^{1,(p_i)}(\Omega) \cap L^\infty(\Omega)$, $\overline{p} < N$,

$$\left(\int_\Omega |v|^r \right)^{\frac{N}{p}-1} \leq C \prod_{i=1}^{N} \left(\int_\Omega |\partial_i v|^{p_i} |v|^{t_i p_i} \right)^{\frac{1}{p_i}}, \tag{2.3}$$

for every r *and* t_j *chosen such a way to have*

$$\begin{cases} \frac{1}{r} = \frac{\gamma_i (N-1) - 1 + \frac{1}{p_i}}{t_i + 1}, \\ \sum\limits_{i=1}^{N} \gamma_i = 1. \end{cases}$$

We need as well to recall the following truncating functions:

$$T_n(s) = \begin{cases} n\frac{s}{|s|} & \text{if } |s| > n, \\ s & \text{if } |s| \leq n. \end{cases}$$

3. THE REACTION CASE

We consider the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \left[\left(a + b \int\limits_\Omega |\partial_i u|^{p_i} dx \right) \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u \right) \right] = \frac{f(x)}{u^\gamma} + g(x) u^{q-1} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $f \in L^{\frac{\overline{p}^*}{\overline{p}^* + \gamma - 1}}(\Omega)$ and $g \in L^{\overline{p}^{*\prime}}(\Omega)$. For $u \in W_0^{1,(p_i)}(\Omega)$, we define the energy functional associated to (3.1) by

$$J(u) = \sum_{i=1}^{N} \left[\frac{a}{p_i} \int_\Omega |\partial_i u|^{p_i} + \frac{b}{2p_i} \left(\int_\Omega |\partial_i u|^{p_i} \right)^2 \right] - \frac{1}{1-\gamma} \int_\Omega f \left(u^+\right)^{1-\gamma} - \frac{1}{q} \int_\Omega g \left(u^+\right)^q,$$

where $u^+ = \max\{u, 0\}$. We will use u instead of u^+ to simplify the notation.

Lemma 3.1. *The functional J is coercive.*

Proof. Observe that

$$\begin{aligned}
J(u) &= \sum_{i=1}^{N}\left[\frac{a}{p_i}\int_\Omega |\partial_i u|^{p_i} + \frac{b}{2p_i}\left(\int_\Omega |\partial_i u|^{p_i}\right)^2\right] - \frac{1}{1-\gamma}\int_\Omega f u^{1-\gamma} - \frac{1}{q}\int_\Omega g u^q \\
&\geq \sum_{i=1}^{N}\frac{b}{2p_i}\left(\int_\Omega |\partial_i u|^{p_i}\right)^2 - \frac{1}{1-\gamma}\int_\Omega f u^{1-\gamma} - \frac{1}{q}\int_\Omega g u^q \\
&\geq \frac{b}{2p_N}\sum_{i=1}^{N}\left(\int_\Omega |\partial_i u|^{p_i}\right)^2 - \frac{1}{1-\gamma}\|f\|_{L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}}\|u\|_{L^{\overline{p}^*}}^{1-\gamma} - \|u\|_{L^{p_N}}^q \|g\|_{L^{p'_N}}^{\frac{p_N-q}{p}} \\
&\geq \frac{b}{2p_N}\sum_{i=1}^{N}\left(\int_\Omega |\partial_j u|^{p_j}\right)^2 - \frac{1}{1-\gamma}\|f\|_{L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}}\|u\|_{L^{\overline{p}^*}}^{1-\gamma} - \|u\|_{L^{p_N}}^q \|g\|_{L^{p'_N}}^{\frac{p_N-q}{p}},
\end{aligned}$$

where

$$\int_\Omega |\partial_j u|^{p_j} = \max\left\{\int_\Omega |\partial_i u|^{p_i},\ i=1,\dots,N\right\}.$$

Now putting

$$\left(\int_\Omega |\partial_k u|^{p_k}\right)^{\frac{1}{p_k}} = \max\left\{\left(\int_\Omega |\partial_i u|^{p_i}\right)^{\frac{1}{p_i}},\ i=1,\dots,N\right\}$$

we have

$$\left(\int_\Omega |\partial_j u|^{p_j}\right)^{\frac{1}{p_j}} \leq \left(\int_\Omega |\partial_k u|^{p_k}\right)^{\frac{1}{p_k}} \leq \|u\| \leq N\left(\int_\Omega |\partial_k u|^{p_k}\right)^{\frac{1}{p_k}} \leq N\left(\int_\Omega |\partial_j u|^{p_j}\right)^{\frac{p_j}{p_k}}, \tag{3.2}$$

so we obtain

$$\frac{1}{N^{2p_k}}\|u\|^{2p_k} \leq \left(\int_\Omega |\partial_k u|^{pk}\right)^2.$$

Returning to the energy functional we get

$$\begin{aligned}
J(u) &\geq \frac{b}{2p_N}\frac{1}{N^{2p_k}}\|u\|^{2p_k} - \frac{1}{1-\gamma}\|f\|_{L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}}\|u\|_{L^{\overline{p}^*}}^{1-\gamma} - \|u\|_{L^{\overline{p}^*}}^q \|g\|_{L^{\overline{p}^{*\prime}}}^{\frac{\overline{p}^*-q}{\overline{p}^*}} \\
&\geq \frac{b}{2p_N}\frac{1}{N^{2p_k}}\|u\|^{2p_k} - C_1\|u\|^{1-\gamma} - C_2\|u\|^q.
\end{aligned}$$

Since $p_k > 1$, the lemma is proved. $\square$

Lemma 3.2. *There exist $r > 0$ and $\rho > 0$ such that $J(u) \geq r$ for $\|u\| = \rho$.*

Proof. Let us note that

$$J(u) = \sum_{i=1}^{N} \left[\frac{a}{p_i} \int_\Omega |\partial_i u|^{p_i} + \frac{b}{2p_i} \left(\int_\Omega |\partial_i u|^{p_i} \right)^2 \right] - \frac{1}{1-\gamma} \int_\Omega f u^{1-\gamma} - \frac{1}{q} \int_\Omega g u^q.$$

As $\|u\| = \rho$ with $\rho > 0$, one can always assume without loss of generality that

$$\|\partial_i u\|_{L^{p_i}} \leq 1 \quad \text{for every } i = 1, \dots, N.$$

Using the Hölder and Sobolev inequalities we obtain

$$\begin{aligned} J(u) &\geq \frac{a}{p_N} \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}}^{p_N} + \frac{b}{2p_N} \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}}^{2p_N} \\ &\quad - \frac{1}{1-\gamma} \|f\|_{L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}} \|u\|_{L^{\overline{p}^*}}^{1-\gamma} - \|u\|_{L^{\overline{p}^*}}^{q} \|g\|_{L^{\overline{p}^{*\prime}}}^{\frac{\overline{p}^*-q}{\overline{p}^*}} \\ &\geq \frac{a}{p_N} \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}}^{p_N} + \frac{b}{2p_N} \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}}^{2p_N} - C_1 \|u\|^{1-\gamma} - C_2 \|u\|^q \\ &\geq k_1 \|u\|^{p_N} + k_2 \|u\|^{2p_N} - C_1 \|u\|^{1-\gamma} - C_2 \|u\|^q \end{aligned}$$

by the fact that $1 < q < p_N$ and $0 < \gamma < 1$. The lemma is proved. □

Lemma 3.3. *There exists $\varphi \in W_0^{1,(p_i)}$, $\varphi \geq 0$, φ not identically equal to zero, such that $J(t\varphi) < 0$ for $t > 0$ small enough.*

Proof. We have

$$\begin{aligned} J(t\varphi) &= \sum_{i=1}^{N} \left[\frac{at^{p_i}}{p_i} \int_\Omega |\partial_i \varphi|^{p_i} + \frac{bt^{2p_i}}{2p_i} \left(\int_\Omega |\partial_i \varphi|^{p_i} \right)^2 \right] - \frac{t^{1-\gamma}}{1-\gamma} \int_\Omega f \varphi^{1-\gamma} - \frac{t^q}{q} \int_\Omega g \varphi^q \\ &\leq \frac{at^{p_1}}{p_1} \sum_{i=1}^{N} \int_\Omega |\partial_i \varphi|^{p_i} + \frac{bt^{2p_1}}{2p_1} \sum_{i=1}^{N} \left(\int_\Omega |\partial_i \varphi|^{p_i} \right)^2 - \frac{t^{1-\gamma}}{1-\gamma} \int_\Omega f \varphi^{1-\gamma} \\ &\leq \frac{t^{p_1}}{p_1} \left[a \sum_{i=1}^{N} \int_\Omega |\partial_i \varphi|^{p_i} + \frac{b}{2} \sum_{i=1}^{N} \left(\int_\Omega |\partial_i \varphi|^{p_i} \right)^2 \right] - \frac{t^{1-\gamma}}{1-\gamma} \int_\Omega f \varphi^{1-\gamma}. \end{aligned}$$

By choosing

$$t < T^{\frac{1}{p_1 - (1-\gamma)}}$$

with

$$0 < T < \min \left\{ 1, \frac{\frac{1}{1-\gamma} \int_\Omega f \varphi^{1-\gamma}}{a \sum_{i=1}^{N} \int_\Omega |\partial_i \varphi|^{p_i} + \frac{b}{2} \sum_{i=1}^{N} \left(\int_\Omega |\partial_i \varphi|^{p_i} \right)^2} \right\},$$

we obtain

$$J(t\varphi) < 0$$

which completes the proof. □

Theorem 3.4. *Under the hypotheses that* $1 < q < p_N$ *and* $0 < \gamma < 1$, *with positive functions* $f \in L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}(\Omega)$ *and* $g \in L^{p'_N}(\Omega)$, *and real positive numbers* a, b, *the energy functional* J *reaches its global minimizer in* $W_0^{1,(p_i)}(\Omega)$.

Proof. Let

$$m = \min_{u \in W_0^{1,(p_i)}(\Omega)} J(u).$$

Observe that m is well-defined by the previous computations, and consider a minimizing sequence $\{u_n\}_n$ such that

$$J(u_n) \longrightarrow m \quad \text{as} \quad n \longrightarrow +\infty.$$

Obviously $\{u_n\}_n$ is bounded in $W_0^{1,(p_i)}(\Omega)$. If not, this will be a contradiction to the coercivity of J. As a consequence, (up to a subsequence) we have that

$$\begin{aligned} u_n &\rightharpoonup u \ \text{ weakly in } W_0^{1,(p_i)}(\Omega), \\ u_n &\longrightarrow u \ \text{ strongly in } L^r(\Omega) \ \text{ for all } r \in [1, \overline{p}^*], \\ u_n &\longrightarrow u \ \text{ almost everywhere in } \Omega. \end{aligned}$$

The weak convergence of $\{u_n\}_n$ associated with (3.2) implies that

$$\|\partial_i u\|_{L^{p_i}} \le \liminf \|\partial_i u_n\|_{L^{p_i}} \ \text{ for every } i = 1, \dots, N.$$

Now if we note that

$$M(t) = a + bt \quad \text{for } t > 0$$

and

$$\widehat{M}(t) = \int_0^t M(s)ds = at + b\frac{t^2}{2},$$

then $\widehat{M}$ is an increasing function, and thus

$$\widehat{M}\left(\|\partial_i u\|_{L^{p_i}}^{p_i}\right) \le \widehat{M}\left((\liminf \|\partial_i u_n\|_{L^{p_i}})^{p_i}\right) \ \text{ for every } i = 1, \dots, N.$$

By a continuity argument of $\widehat{M}$, we have

$$\begin{aligned} \widehat{M}\left((\liminf \|\partial_i u_n\|_{L^{p_i}})^{p_i}\right) &= \liminf \widehat{M}(\|\partial_i u_n\|_{L^{p_i}}^{p_i}) \\ &= \liminf \left[\frac{a}{p_i}\int_\Omega |\partial_i u_n|^{p_i} + \frac{b}{2p_i}\left(\int_\Omega |\partial_i u_n|^{p_i}\right)^2\right], \end{aligned}$$

hence

$$\widehat{M}\left(\|\partial_i u\|_{L^{p_i}}^{p_i}\right) \leq \liminf \left[\frac{a}{p_i}\int\limits_\Omega |\partial_i u_n|^{p_i} + \frac{b}{2p_i}\left(\int\limits_\Omega |\partial_i u_n|^{p_i}\right)^2\right].$$

Now we consider the term

$$\int\limits_\Omega f u_n^{1-\gamma}.$$

From one hand, by the Sobolev embedding we know that there exits a constant C such that

$$\|u_n\|_{L^{\overline{p}^*}} < C.$$

On the other hand, we know that (by the absolute continuity) there exits δ such that

$$\int\limits_E f^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}} \leq \varepsilon^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}} \text{ for every } E \subset \Omega \text{ with } \operatorname{meas}(E) < \delta.$$

The Hölder inequality gives

$$\int\limits_E f u_n^{1-\gamma} \leq \|u_n\|_{L^{\overline{p}^*}}^{1-\gamma}\left(\int\limits_E f^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}\right)^{\frac{\overline{p}^*+\gamma-1}{\overline{p}^*}} \leq \varepsilon\, C^{1-\gamma}.$$

By Vitali's theorem, we conclude that

$$\int\limits_\Omega f u_n^{1-\gamma} \longrightarrow \int\limits_\Omega f u^{1-\gamma} \text{ as } n \longrightarrow +\infty.$$

For the last term in the energy we use the Lebesgue dominated convergence theorem to deduce that

$$\int\limits_\Omega g u_n^q \longrightarrow \int\limits_\Omega g u^q \;\text{ as } n \longrightarrow +\infty.$$

As a conclusion, J is weakly lower semi-continuous, thus

$$m \leq J(u) \leq \liminf J(u_n) = m,$$

so $J(u) = m$. $\square$

In what follows, u is a solution to (3.1).

3.1. A MULTIPLICITY RESULT

Let us consider the approximating problems

$$\begin{cases} -\sum\limits_{i=1}^N \left[\left(a + b\int\limits_\Omega |\partial_i u_n|^{p_i}\,dx\right)\partial_i\left(|\partial_i u_n|^{p_i-2}\,\partial_i u\right)\right] = \frac{f(x)}{\left(u+\frac{1}{n}\right)^\gamma} + g(x)u_n^{q-1} & \text{in } \Omega, \\ u_n \geq 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Once we have avoided the singularity, the associated energy functional is now regular (that is, differentiable):

$$
\begin{aligned}
J(u_n) = \sum_{i=1}^{N} &\left[\frac{a}{p_i} \int_\Omega |\partial_i u_n|^{p_i} + \frac{b}{2p_i} \left(\int_\Omega |\partial_i u_n|^{p_i} \right)^2 \right] \\
&- \frac{1}{1-\gamma} \int_\Omega f \left(\left(u_n^+ + \frac{1}{n} \right)^{1-\gamma} - \left(\frac{1}{n} \right)^{1-\gamma} \right) - \frac{1}{q} \int_\Omega g \left(u_n^+ \right)^q ,
\end{aligned}
$$

where $u^+ = \max\{u, 0\}$. We will use u instead of u^+ to simplify the notation.

Lemma 3.5. *There exist $r > 0$ and $\rho > 0$ such that $J(u) \geq r$ for $\|u\| = \rho$.*

Proof. We have

$$
\begin{aligned}
J(u_n) = \sum_{i=1}^{N} &\left[\frac{a}{p_i} \int_\Omega |\partial_i u_n|^{p_i} + \frac{b}{2p_i} \left(\int_\Omega |\partial_i u_n|^{p_i} \right)^2 \right] \\
&- \frac{1}{1-\gamma} \int_\Omega f \left(\left(u_n + \frac{1}{n} \right)^{1-\gamma} - \left(\frac{1}{n} \right)^{1-\gamma} \right) - \frac{1}{q} \int_\Omega g u_n^q .
\end{aligned}
$$

As $\|u\| = \rho$ with $\rho > 0$, one can always assume without loss of generality that

$$
\|\partial_i u\|_{L^{p_i}} \leq 1 \quad \text{for every } i = 1, \ldots, N.
$$

Using the Hölder and Sobolev inequalities we obtain

$$
\begin{aligned}
J(u_n) \geq\ & \frac{a}{p_N} \sum_{i=1}^{N} \|\partial_i u_n\|_{L^{p_i}}^{p_N} + \frac{b}{2p_N} \sum_{i=1}^{N} \|\partial_i u_n\|_{L^{p_i}}^{2p_N} \\
& - \frac{1}{1-\gamma} \|f\|_{L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}} \|u_n\|_{L^{\overline{p}^*}}^{1-\gamma} + \frac{1}{1-\gamma} \|f\|_{L^1} - \|u_n\|_{L^{\overline{p}^*}}^q \|g\|_{L^{\overline{p}^{*\prime}}}^{\frac{\overline{p}^*-q}{\overline{p}^*}} \\
\geq\ & \frac{a}{p_N} \sum_{i=1}^{N} \|\partial_i u_n\|_{L^{p_i}}^{p_N} + \frac{b}{2p_N} \sum_{i=1}^{N} \|\partial_i u_n\|_{L^{p_i}}^{2p_N} \\
& - C_1 \|u_n\|^{1-\gamma} - C_2 \|u_n\|^q + \frac{1}{1-\gamma} \|f\|_{L^1} \\
\geq\ & k_1 \|u_n\|^{p_N} + k_2 \|u_n\|^{2p_N} - C_1 \|u_n\|^{1-\gamma} - C_2 \|u_n\|^q + \frac{1}{1-\gamma} \|f\|_{L^1} . \qquad \square
\end{aligned}
$$

Lemma 3.6. *There exists $\varphi \in W_0^{1,(p_i)}$, $\varphi \geq 0$, φ not identically equal to zero, such that $J(t\varphi) < 0$ for $t > 0$ small enough.*

Proof. Observe that

$$\begin{aligned} J(t\varphi) &= \sum_{i=1}^{N}\left[\frac{at^{p_i}}{p_i}\int_\Omega |\partial_i\varphi|^{p_i} + \frac{bt^{2p_i}}{2p_i}\left(\int_\Omega |\partial_i\varphi|^{p_i}\right)^2\right] \\ &\quad - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega f\varphi^{1-\gamma} - \frac{t^q}{q}\int_\Omega g\varphi^q + \frac{t}{1-\gamma}\|f\|_{L^1}\|\varphi\|_{L^\infty} \\ &\leq \frac{at^{p_1}}{p_1}\sum_{i=1}^{N}\int_\Omega |\partial_i\varphi|^{p_i} + \frac{bt^{2p_1}}{2p_1}\sum_{i=1}^{N}\left(\int_\Omega |\partial_i\varphi|^{p_i}\right)^2 \\ &\quad - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega f\varphi^{1-\gamma} + \frac{t}{1-\gamma}\|f\|_{L^1}\|\varphi\|_{L^\infty} \\ &\leq \frac{t^{p_1}}{p_1}\left[a\sum_{i=1}^{N}\int_\Omega |\partial_i\varphi|^{p_i} + \frac{b}{2}\sum_{i=1}^{N}\left(\int_\Omega |\partial_i\varphi|^{p_i}\right)^2\right] \\ &\quad - \frac{t^{1-\gamma}}{1-\gamma}\int_\Omega f\varphi^{1-\gamma} + \frac{t}{1-\gamma}\|f\|_{L^1}\|\varphi\|_{L^\infty}. \end{aligned}$$

By choosing

$$t < T^{\frac{1}{p_1-(1-\gamma)}}$$

with

$$0 < T < \min\left\{1, \frac{\frac{1}{1-\gamma}\int_\Omega f\varphi^{1-\gamma} - + \frac{t}{1-\gamma}\|f\|_{L^1}\|\varphi\|_{L^\infty}}{a\sum_{i=1}^{N}\int_\Omega |\partial_i\varphi|^{p_i} + \frac{b}{2}\sum_{i=1}^{N}\left(\int_\Omega |\partial_i\varphi|^{p_i}\right)^2}\right\},$$

we obtain

$$J(t\varphi) < 0.$$

□

Lemma 3.7. *J satisfies the Palais–Smale condition under the additional assumption $q < p_1$.*

Proof. Fix n and let $\{u_k\}_k$ be a Palais–Smale sequence of J, that is,

$$J(u_k) \leq C$$

and

$$J'(u_k) \longrightarrow 0 \text{ in } \left(W_0^{1,(p_i)}(\Omega)\right)'.$$

We claim that $\{u_k\}_k$ is bounded. Indeed, recall that

$$J(u_k) = \sum_{i=1}^{N}\left[\frac{a}{p_i}\int_\Omega |\partial_i u_k|^{p_i} + \frac{b}{2p_i}\left(\int_\Omega |\partial_i u_k|^{p_i}\right)^2\right] - \frac{1}{1-\gamma}\int_\Omega f\left(\left(u_k+\frac{1}{n}\right)^{1-\gamma} - \left(\frac{1}{n}\right)^{1-\gamma}\right) - \frac{1}{q}\int_\Omega g u_k^q.$$

By the fact that

$$\langle J'(u_k), u_k\rangle = \left\{\sum_{i=1}^{N}\left[\left(a+b\int_\Omega |\partial_i u_k|^{p_i}\right)\int_\Omega |\partial_i u_k|^{p_i}\right] - \int_\Omega f\left(u_k+\frac{1}{n}\right)^{1-\gamma} - \int_\Omega g u_k^q\right\},$$

we get

$$\langle J'(u_k), u_k\rangle \longrightarrow 0.$$

Using the Hölder and Sobolev inequalities we have

$$\begin{aligned}
\sum_{i=1}^{N}\left[\left(a+b\int_\Omega |\partial_i u_k|^{p_i}\right)\int_\Omega |\partial_i u_k|^{p_i}\right] &= \int_\Omega f\left(u_k+\frac{1}{n}\right)^{1-\gamma} - \int_\Omega g u_k^q + o(1)\\
&\le \int_\Omega f {u_k}^{1-\gamma} - \int_\Omega g u_k^q + o(1)\\
&\le \frac{1}{1-\gamma}\|f\|_{L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}}\|u_k\|_{L^{\overline{p}^*}}^{1-\gamma} + \|u_k\|_{L^{\overline{p}^*}}^{q}\|g\|_{L^{\overline{p}^{*\prime}}}^{\frac{\overline{p}^*-q}{\overline{p}^*}}\\
&\le C_1\|u_k\|^{1-\gamma} + C_2\|u_k\|^q.
\end{aligned}$$

Assume by contradiction that

$$\|u_k\| \longrightarrow +\infty \quad \text{as} \quad k \longrightarrow +\infty.$$

Necessarily there exists l such that

$$\int_\Omega |\partial_l u_k| \longrightarrow +\infty \quad \text{as} \quad k \longrightarrow +\infty,$$

so we asymptotically get

$$b\left(\int_\Omega |\partial_l u_k|^{p_l}\right)^2 \le \|u_k\|^q.$$

As $q < p_1 \le p_l$, we obtain a contradiction, and thus $\{u_k\}_k$ is bounded. Thus up to a subsequence we get

$$\begin{aligned}
u_k &\rightharpoonup u_n \ \text{ weakly in } W_0^{1,(p_i)}(\Omega),\\
u_k &\longrightarrow u_n \ \text{ strongly in } L^r(\Omega) \text{ for all } r\in[1,\overline{p}^*],\\
u_k &\longrightarrow u_n \ \text{ almost everywhere in } \Omega.
\end{aligned}$$

We conclude the proof by using the same arguments as before. $\square$

As a direct consequence of the three previous lemmas, we have shown that problem (3.3) possesses a solution u_n such that $J(u_n) < 0$.

Lemma 3.8. *Problem* (3.3) *possesses a solution* v_n *such that* $J(v_n) > 0$.

Proof. From previous lemmas and applying the mountain-pass lemma, we deduce that there exists a sequence $\{v_k\}_k \subset W_0^{1,(p_i)}$ such that

$$J(v_k) \longrightarrow C > r \quad \text{and} \quad J'(v_k) \longrightarrow 0,$$

where

$$C = \inf_{\beta\in\Gamma} \max_{t\in[0,1]} J(\beta(t))$$

and

$$\Gamma = \left\{\beta \in C\left([0,1], W_0^{1,(p_i)}\right) : \beta(0) = 0, \beta(1) = \rho\right\}.$$

By similar arguments as before and up to a subsequence, we deduce that $\{v_k\}_k$ converges to v_n in $W_0^{1,(p_i)}$ as $k \longrightarrow +\infty$, a solution to (3.3) such that

$$J(v_n) > r > 0. \qquad \square$$

In conclusion, problem (3.3) possesses at least two solutions.

4. THE ABSORPTION CASE

We consider the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \left[\left(a + b\int\limits_{\Omega} |\partial_i u|^{p_i}\, dx\right) \partial_i \left(|\partial_i u|^{p_i-2}\, \partial_i u\right)\right] + g(x)u^{q-1} = \frac{f(x)}{u^{\gamma}} & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

with $f \in L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}(\Omega)$ and $g \in L^{\overline{p}^{*\prime}}(\Omega)$. For $u \in W_0^{1,(p_i)}(\Omega)$, we define the energy functional associated to (4.1) by

$$J(u) = \sum_{i=1}^{N} \left[\frac{a}{p_i} \int\limits_{\Omega} |\partial_i u|^{p_i} + \frac{b}{2p_i} \left(\int\limits_{\Omega} |\partial_i u|^{p_i}\right)^2\right] + \frac{1}{q} \int\limits_{\Omega} g\left(u^+\right)^q - \frac{1}{1-\gamma} \int\limits_{\Omega} f\left(u^+\right)^{1-\gamma}$$

where $u^+ = \max\{u, 0\}$. We will use u instead of u^+ to simplify the notation.

Lemma 4.1. *The functional J is coercive.*

Proof. We use the same arguments as previously, that is,

$$\begin{aligned} J(u) &= \sum_{i=1}^{N}\left[\frac{a}{p_i}\int_\Omega |\partial_i u|^{p_i} + \frac{b}{2p_i}\left(\int_\Omega |\partial_i u|^{p_i}\right)^2\right] + \frac{1}{q}\int_\Omega g u^q - \frac{1}{1-\gamma}\int_\Omega f u^{1-\gamma} \\ &\geq \sum_{i=1}^{N}\frac{b}{2p_i}\left(\int_\Omega |\partial_i u|^{p_i}\right)^2 - \frac{1}{1-\gamma}\int_\Omega f u^{1-\gamma} \\ &\geq \frac{b}{2p_N}\sum_{i=1}^{N}\left(\int_\Omega |\partial_j u|^{p_j}\right)^2 - \frac{1}{1-\gamma}\|f\|_{L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}}\|u\|_{L^{\overline{p}^*}}^{1-\gamma}, \end{aligned}$$

where

$$\int_\Omega |\partial_j u|^{p_j} = \max\left\{\int_\Omega |\partial_i u|^{p_i},\ i=1,\ldots,N\right\}.$$

Now putting

$$\left(\int_\Omega |\partial_k u|^{p_k}\right)^{\frac{1}{p_k}} = \max\left\{\left(\int_\Omega |\partial_i u|^{p_i}\right)^{\frac{1}{p_i}},\ i=1,\ldots,N\right\}$$

we have

$$\left(\int_\Omega |\partial_j u|^{p_j}\right)^{\frac{1}{p_j}} \leq \left(\int_\Omega |\partial_k u|^{p_k}\right)^{\frac{1}{p_k}} \leq \|u\| \leq N\left(\int_\Omega |\partial_k u|^{p_k}\right)^{\frac{1}{p_k}} \leq N\left(\int_\Omega |\partial_j u|^{p_j}\right)^{\frac{p_j}{p_k}},$$

so we obtain

$$\frac{1}{N^{2p_k}}\|u\|^{2p_k} \leq \left(\int_\Omega |\partial_k u|^{pk}\right)^2.$$

Returning to the energy functional we get

$$\begin{aligned} J(u) &\geq \frac{b}{2p_N}\frac{1}{N^{2p_k}}\|u\|^{2p_k} - \frac{1}{1-\gamma}\|f\|_{L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}}\|u\|_{L^{\overline{p}^*}}^{1-\gamma} \\ &\geq \frac{b}{2p_N}\frac{1}{N^{2p_k}}\|u\|^{2p_k} - C_1\|u\|^{1-\gamma}. \end{aligned}$$

As $p_k > 1$, the lemma is proved. □

Theorem 4.2. *Under the hypotheses $1 < q < p_N$ and $0 < \gamma < 1$, with positive functions $f \in L^{\frac{\overline{p}^*}{\overline{p}^*+\gamma-1}}(\Omega)$ and $g \in L^{p'_N}(\Omega)$, and real positive numbers a, b, the energy functional J reaches its global minimizer in $W_0^{1,(p_i)}(\Omega)$.*

Proof. Let
$$m=\min_{u\in W_0^{1,(p_i)}(\Omega)} J(u).$$
Observe that m is well-defined by the previous computations, and consider a minimizing sequence $\{u_n\}_n$ such that
$$J(u_n)\longrightarrow m \ \text{ as } \ n\longrightarrow+\infty.$$
Obviously $\{u_n\}_n$ is bounded in $W_0^{1,(p_i)}(\Omega)$. If not, this will be a contradiction to the coercivity of J. As a consequence, (up to a subsequence) we get
$$\begin{aligned} u_n &\rightharpoonup u \ \text{ weakly in } W_0^{1,(p_i)}(\Omega),\\ u_n &\longrightarrow u \ \text{ strongly in } L^r(\Omega) \ \text{ for all } r\in[1,\overline{p}^*],\\ u_n &\longrightarrow u \ \text{ almost everywhere in } \Omega. \end{aligned}$$
As a direct consequence of Vitali's theorem, as done previously, we have
$$\int_\Omega f u_n^{1-\gamma}\longrightarrow\int_\Omega f u^{1-\gamma} \ \text{ as } \ n\longrightarrow+\infty.$$
Using the Lebesgue dominated convergence theorem we obtain
$$\int_\Omega g u_n^{q}\longrightarrow\int_\Omega g u^{q} \quad \text{as} \ \ n\longrightarrow+\infty.$$
Now, by Brezis-Lieb's theorem and (3.2), we have
$$\begin{aligned} \|\partial_i u_n\|_{L^{p_i}}^{p_i} &= \|\partial_i(u_n-u)\|_{L^{p_i}}^{p_i}+\|\partial_i u\|_{L^{p_i}}^{p_i}+o(1),\\ \|\partial_i u_n\|_{L^{p_i}}^{2p_i} &= \|\partial_i(u_n-u)\|_{L^{p_i}}^{2p_i}+\|\partial_i u\|_{L^{p_i}}^{2p_i}+2\|\partial_i(u_n-u)\|_{L^{p_i}}^{p_i}\|\partial_i u\|_{L^{p_i}}^{2p_i}+o(1), \end{aligned}$$
so we obtain
$$\begin{aligned} m &= \lim_{n\longrightarrow+\infty} J(u_n)\\ &= \lim_{n\longrightarrow+\infty}\left(\sum_{i=1}^N\left[\frac{a}{p_i}\int_\Omega|\partial_i u_n|^{p_i}+\frac{b}{2p_i}\left(\int_\Omega|\partial_i u_n|^{p_i}\right)^2\right]+\frac{1}{q}\int_\Omega g u_n^q-\frac{1}{1-\gamma}\int_\Omega f u_n^{1-\gamma}\right)\\ &= \lim_{n\longrightarrow+\infty}\sum_{i=1}^N\left[\frac{a}{p_i}\left(\|\partial_i(u_n-u)\|_{L^{p_i}}^{p_i}+\|\partial_i u\|_{L^{p_i}}^{p_i}\right)\right.\\ &\quad\left.+\frac{b}{2p_i}\int_\Omega\left(\|\partial_i(u_n-u)\|_{L^{p_i}}^{2p_i}+\|\partial_i u\|_{L^{p_i}}^{2p_i}+2\|\partial_i(u_n-u)\|_{L^{p_i}}^{p_i}\|\partial_i u\|_{L^{p_i}}^{2p_i}\right)\right]\\ &\quad+\frac{1}{q}\int_\Omega g u^q-\frac{1}{1-\gamma}\int_\Omega f u^{1-\gamma}\\ &\ge J(u). \end{aligned}$$
By the definition of m, the lemma is proved. □

In what follows, u is a solution to (4.1).

4.1. A UNIQUENESS RESULT

Under the additional assumption $p_1 = p_2 = \ldots = p_N = 2$ we have the following result.

Theorem 4.3. *Problem* (4.1) *possesses a unique solution.*

Proof. Assume that u_1 and u_2 are two solutions to (3.1). Then

$$\sum_{i=1}^{N}\left[\left(a+b\int_{\Omega}|\partial_i u_1|^2\right)\int_{\Omega}|\partial_i u_1|\,\partial_i\varphi\right]+\int_{\Omega}gu_1^q\varphi\ =\int_{\Omega}\frac{f\varphi}{u_1^\gamma},\quad \varphi\in W_0^{1,(p_i)}(\Omega),$$

and

$$\sum_{i=1}^{N}\left[\left(a+b\int_{\Omega}|\partial_i u_2|^2\right)\int_{\Omega}|\partial_i u_2|\,\partial_i\varphi\right]+\int_{\Omega}gu_2^q\varphi=\int_{\Omega}\frac{f\varphi}{u_2^\gamma},\quad \varphi\in W_0^{1,(p_i)}(\Omega).$$

Choosing $\varphi = u_1 - u_2$ we get

$$\sum_{i=1}^{N}\left[\left(a+b\int_{\Omega}|\partial_i u_1|^2\right)\int_{\Omega}|\partial_i u_1|\,\partial_i\left(u_1-u_2\ \right)\right]+\int_{\Omega}gu_1^q\left(u_1-u_2\ \right)=\int_{\Omega}\frac{f\left(u_1-u_2\ \right)}{u_1^\gamma}$$

and

$$\sum_{i=1}^{N}\left[\left(a+b\int_{\Omega}|\partial_i u_2|^2\right)\int_{\Omega}|\partial_i u_2|\,\partial_i\left(u_1-u_2\ \right)\right]+\int_{\Omega}gu_2^q\left(u_1-u_2\ \right)=\int_{\Omega}\frac{f\left(u_1-u_2\ \right)}{u_2^\gamma}.$$

The difference between the last two equations gives

$$\begin{aligned}&\sum_{i=1}^{N}a\int_{\Omega}\left(|\partial_i u_1|\,\partial_i\left(u_1-u_2\ \right)-|\partial_i u_2|\,\partial_i\left(u_1-u_2\right)\right)\\&+\sum_{i=1}^{N}b\left\{\int_{\Omega}|\partial_i u_1|^2\int_{\Omega}|\partial_i u_1|\,\partial_i\left(u_1-u_2\ \right)-\int_{\Omega}|\partial_i u_2|^2\int_{\Omega}|\partial_i u_2|\,\partial_i\left(u_1-u_2\ \right)\right\}\\&=\int_{\Omega}f\left(u_1-u_2\ \right)\left(u_1^{-\gamma}-u_2^{-\gamma}\right)-\int_{\Omega}g\left(u_1-u_2\ \right)\left(u_1^q-u_2^q\right),\end{aligned}$$

so it follows that

$$\begin{aligned}
&\sum_{i=1}^{N} a \int_{\Omega} \left(|\partial_i u_1|^2 + |\partial_i u_2|^2 - 2\partial_i u_1 \partial_i u_2 \right) \\
&+ \sum_{i=1}^{N} \left\{ b \left(\left(\int_{\Omega} |\partial_i u_1|^2 \right)^2 + \left(\int_{\Omega} |\partial_i u_2|^2 \right)^2 \right) \right. \\
&\qquad \left. - \int_{\Omega} |\partial_i u_1|^2 \int_{\Omega} |\partial_i u_1| \, \partial_i u_2 \; - \int_{\Omega} |\partial_i u_2|^2 \int_{\Omega} |\partial_i u_2| \, \partial_i u_1 \right\} \\
&= \int_{\Omega} f \left(u_1 - u_2 \right) \left(u_1^{-\gamma} - u_2^{-\gamma} \right) - \int_{\Omega} g \left(u_1 - u_2 \right) \left(u_1^q - u_2^q \right).
\end{aligned}$$

Noting

$$\left(\int_{\Omega} |\partial_i u|^2 \right)^{\frac{1}{2}} = |||u|||,$$

from one hand, using the Hölder inequality in the general term in the second sum, we have

$$\begin{aligned}
&|||u_1|||^4 + |||u_2|||^4 - |||u_1|||^2 \int_{\Omega} |\partial_i u_1| \, \partial_i u_2 \; - |||u_2|||^2 \int_{\Omega} |\partial_i u_2| \, \partial_i u_1 \\
&\geq |||u_1|||^4 + |||u_2|||^4 - |||u_1|||^3 |||u_2||| - |||u_1|||.|||u_2|||^3 \\
&= \left(|||u_1||| - |||u_2||| \right)^2 \left(|||u_1|||^2 + |||u_1|||.|||u_2||| + |||u_1|||^2 \right) \\
&\geq 0.
\end{aligned}$$

On the other hand, as $0 < \gamma < 1$ and $q > 0$, by the algebraic inequalities

$$\left(X^{-\gamma} - Y^{-\gamma} \right) (X - Y) \leq 0 \quad \text{and} \quad \left(X^q - Y^q \right) (X - Y) \geq 0,$$

we have

$$\int_{\Omega} f \left(u_1 - u_2 \right) \left(u_1^{-\gamma} - u_2^{-\gamma} \right) - \int_{\Omega} g \left(u_1 - u_2 \right) \left(u_1^q - u_2^q \right) \leq 0.$$

As $a > 0$ and $b > 0$, we must have $u_1 = u_2$. □

5. GENERAL REMARKS

We give here some remarks and observations.

1. Observe that the energy functionals are such that $J(X) = J(|X|)$, so one can always assume that the considered minimizing sequences are positive, which leads to a solution $u \geq 0$.

2. To prove that the considered solutions are strictly positive $u > 0$ we usually use a strong maximum principle. This can be done only for the case $p_1 \geq 2$. Indeed, to the best of our knowledge such a maximum principle exits for the anisotropic operator L only in this case.
3. Even if we did not succeed to prove the multiplicity result for problem (3.1), but only for the approximating problems (3.3), we still believe that this result holds also for (3.1), and this is left to some future works.
4. Even if we have proved the uniqueness result for problem (4.1) only in a particular case and for particular values of p_i, we still believe that this result still holds for more general values and this is left to some future works.

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