Opuscula Math. 41, no. 1 (2021), 71–94
https://doi.org/10.7494/OpMath.2021.41.1.71

REMARKS ON THE EXISTENCE OF NONOSCILLATORY SOLUTIONS OF HALF-LINEAR ORDINARY DIFFERENTIAL EQUATIONS, I

Manabu Naito

Communicated by Petr Stehlík

Abstract. We consider the half-linear differential equation of the form

$$(p(t)|x'|^\alpha \operatorname{sgn} x')' + q(t)|x|^\alpha \operatorname{sgn} x = 0, \quad t \geq t_0,$$

under the assumption $\int_{t_0}^\infty p(s)^{-1/\alpha} ds = \infty$. It is shown that if a certain condition is satisfied, then the above equation has a pair of nonoscillatory solutions with specific asymptotic behavior as $t \to \infty$.

Keywords: asymptotic behavior, nonoscillatory solution, half-linear differential equation, Hardy-type inequality.

Mathematics Subject Classification: 34C11, 34C10, 26D10.

1. INTRODUCTION

In this paper we consider the half-linear ordinary differential equation

$$(p(t)|x'|^\alpha \operatorname{sgn} x')' + q(t)|x|^\alpha \operatorname{sgn} x = 0, \quad t \geq t_0, \tag{1.1}$$

where α is a positive constant, and $p(t)$ and $q(t)$ are real-valued continuous functions on $[t_0, \infty)$ and $p(t) > 0$ for $t \geq t_0$.

If $\alpha = 1$, then (1.1) becomes the linear equation

$$(p(t)x')' + q(t)x = 0, \quad t \geq t_0. \tag{1.2}$$

It is known that basic results and qualitative results for the linear equation (1.2) can be generalized to the half-linear equation (1.1). The important works for (1.1) are summarized in the book of Došlý and Řehák [2].

Recently the problem of asymptotic behavior of nonoscillatory solutions of (1.2) or (1.1) is investigated in the framework of regular variation. We refer the reader, for instance, to [5–10, 12, 13] and the references therein.

The classical theory on regularly varying functions is suited for the case $p(t) \equiv 1$ in (1.1). However the classical theory is not sufficient to properly describe the possible asymptotic behavior of nonoscillatory solutions of (1.1) for the case $p(t) \not\equiv 1$ in (1.1). In fact, the asymptotic behavior of a nonoscillatory solution of (1.1) is strongly affected by the condition on $p(t)$, more precisely, by the condition

$$\int_{t_0}^{\infty} \frac{1}{p(s)^{1/\alpha}} ds = \infty \quad \text{or} \quad \int_{t_0}^{\infty} \frac{1}{p(s)^{1/\alpha}} ds < \infty.$$

Then the notion of regular variation in the classical sense has been generalized by Jaroš and Kusano [5].

Let $R(t)$ be a continuously differentiable function on some neighborhood $[T_0, \infty)$ of infinity and satisfy

$$R'(t) > 0 \quad \text{for} \quad t \geq T_0 \quad \text{and} \quad \lim_{t\to\infty} R(t) = \infty.$$

For simplicity, we suppose that a function $f(t)$ is positive and continuously differentiable on $[T_0, \infty)$. Then, $f(t)$ is said to be a regularly varying function with respect to $R(t)$ if and only if $f(t)$ can be written in the form

$$f(t) = c(t) \exp\left\{ \int_T^t \frac{R'(s)}{R(s)} \lambda(s) ds \right\}, \quad t \geq T, \tag{1.3}$$

for some $T > T_0$ and some continuous functions $c(t)$ and $\lambda(t)$ such that

$$\lim_{t\to\infty} c(t) = c \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \lambda(t) = \lambda \in \mathbb{R}.$$

The real number λ is called the index of $f(t)$. If $c(t) \equiv c$ (positive constant) in (1.3), then $f(t)$ is said to be a normalized regularly varying function (of index λ) with respect to $R(t)$. We omit the description of the original definition of regular variation with respect to $R(t)$. For details, see the paper [5]. It is worth noting that $f(t)$ is a normalized regularly varying function of index λ with respect to $R(t)$ if and only if

$$\lim_{t\to\infty} \frac{R(t)}{R'(t)} \frac{f'(t)}{f(t)} = \lambda.$$

The set of normalized regularly varying functions of index λ with respect to $R(t)$ is denoted by n-$\mathrm{RV}_R(\lambda)$.

In this paper the half-linear equation (1.1) is considered under the condition

$$\int_{t_0}^{\infty} \frac{1}{p(s)^{1/\alpha}} ds = \infty. \tag{1.4}$$

Then we define the function $P(t)$ by

$$P(t) = \int_{t_0}^{t} \frac{1}{p(s)^{1/\alpha}} ds, \quad t \geq t_0. \tag{1.5}$$

For a while, we assume that $q(t)$ is conditionally integrable in the sense that

$$\int_{t_0}^{\infty} q(s)ds = \lim_{t\to\infty} \int_{t_0}^{t} q(s)ds \quad \text{exists and is finite,} \tag{1.6}$$

and define the function $Q(t)$ by

$$Q(t) = \int_{t}^{\infty} q(s)ds, \quad t \geq t_0. \tag{1.7}$$

Now, put

$$E(\alpha) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}, \tag{1.8}$$

and let c be an arbitrary and fixed real number such that $c < E(\alpha)$. Then the equation

$$|\rho|^{(\alpha+1)/\alpha} - \rho + c = 0 \tag{1.9}$$

has two real roots ρ_1, ρ_2 ($\rho_1 < \rho_2$). They satisfy

$$\rho_1 < (\alpha+1)E(\alpha) < \rho_2.$$

The condition

$$\lim_{t\to\infty} P(t)^\alpha Q(t) = c \tag{1.10}$$

plays an important role in the asymptotic analysis of nonoscillatory solutions of (1.1). The following theorem has been proved by Jaroš, Kusano and Tanigawa [7, Theorem 2.1]. As usual, the asterisk notation

$$\xi^{\gamma*} = |\xi|^\gamma \operatorname{sgn} \xi, \quad \xi \in \mathbb{R}, \gamma > 0,$$

is used.

Theorem 1.1. *Consider the equation* (1.1) *under the condition* (1.4). *Assume that* (1.6) *holds. Let* $c \in (-\infty, E(\alpha))$ *be fixed and let* ρ_1, ρ_2 ($\rho_1 < \rho_2$) *be the real roots of the equation* (1.9). *If* (1.1) *has a solution* $x_i(t) \in \text{n-RV}_P(\rho_i^{(1/\alpha)*})$ ($i = 1$ *or* 2), *then* (1.10) *is satisfied. Conversely, if* (1.10) *is satisfied, then* (1.1) *has a pair of solutions* $x_i(t) \in \text{n-RV}_P(\rho_i^{(1/\alpha)*})$ ($i = 1$ *and* 2).

It should be remarked that $x_i(t)$ belongs to n-RV$_P(\rho_i^{(1/\alpha)*})$ if and only if

$$\lim_{t\to\infty} p(t)^{1/\alpha} P(t) \frac{x_i'(t)}{x_i(t)} = \rho_i^{(1/\alpha)*}. \tag{1.11}$$

Throughout the paper the following fact plays an essential part. Let $x(t)$ be a nonoscillatory solution of (1.1). We suppose that $x(t) > 0$ for $t \geq T$ $(> t_0)$. Put

$$y(t) = p(t) \left(\frac{x'(t)}{x(t)} \right)^{\alpha*}, \quad t \geq T. \tag{1.12}$$

Then, $y(t)$ satisfies the generalized Riccati differential equation

$$y'(t) = -q(t) - \alpha p(t)^{-1/\alpha} |y(t)|^{(\alpha+1)/\alpha}, \quad t \geq T. \tag{1.13}$$

Conversely, if $y(t)$ is a solution of (1.13) on $[T, \infty)$, then

$$x(t) = \exp \left(\int_T^t p(s)^{-1/\alpha} y(s)^{(1/\alpha)*} ds \right), \quad t \geq T, \tag{1.14}$$

is a positive solution of (1.1) on $[T, \infty)$. The proof is immediate.

In Theorem 1.1, the condition (1.6) is previously assumed. We first note that (1.6) is necessary for the existence of a solution $x(t)$ which belongs to the class n-RV$_P(\lambda)$ for some $\lambda \in \mathbb{R}$. Then a natural integral form of (1.13) is

$$y(t) = Q(t) + \alpha \int_t^\infty p(s)^{-1/\alpha} |y(s)|^{(\alpha+1)/\alpha} ds, \quad t \geq T, \tag{1.15}$$

where $Q(t)$ is defined by (1.7). Actually, we have the following proposition.

Proposition 1.2. Consider the equation (1.1) under the condition (1.4). Define the function $P(t)$ by (1.5). If the equation (1.1) has a nonoscillatory solution $x(t)$, $x(t) > 0$ $(t \geq T)$, such that

$$\lim_{t\to\infty} p(t)^{1/\alpha} P(t) \frac{x'(t)}{x(t)} = \lambda \tag{1.16}$$

for some $\lambda \in \mathbb{R}$, then (1.6) holds, and moreover, the function $y(t)$ defined by (1.12) satisfies (1.15).

The main purpose of this paper is to show that the last statement of Theorem 1.1 can be refined as follows:

Theorem 1.3. *Consider the equation* (1.1) *under the condition* (1.4). *Define* $P(t)$ *by* (1.5). *Assume that* (1.6) *holds and define* $Q(t)$ *by* (1.7). *Let* $c \in (-\infty, E(\alpha))$ *be fixed and let* ρ_1, ρ_2 *be the real roots of* (1.9) *such that* $\rho_1 < \rho_2$ *and* $\rho_1 \neq 0$. *Suppose that* (1.10) *holds and put*

$$\varepsilon(t) = P(t)^{\alpha} Q(t) - c, \quad t \geq t_0. \tag{1.17}$$

If

$$\int^{\infty} \frac{|\varepsilon(t)|}{p(t)^{1/\alpha} P(t)} dt < \infty, \tag{1.18}$$

then (1.1) *has a pair of solutions* $x_i(t)$ ($i = 1$ *and* 2) *such that*

$$\begin{cases} x_i(t) \sim P(t)^{\lambda_i} \quad \textit{as} \;\; t \to \infty, \\ x_i'(t) \sim \lambda_i p(t)^{-1/\alpha} P(t)^{\lambda_i - 1} \quad \textit{as} \;\; t \to \infty, \end{cases} \tag{1.19}$$

where $\lambda_i = \rho_i^{(1/\alpha)*}$ $(i = 1, 2)$.

Here, the notation $f(t) \sim g(t)$ as $t \to \infty$ means that $\lim_{t\to\infty}[f(t)/g(t)] = 1$.

Note that the solution $x_i(t)$ obtained in Theorem 1.3 belongs to the class n-$\mathrm{RV}_P(\lambda_i)$, $i = 1, 2$.

The results in [8,9,12,13] are related to Theorem 1.3. In [8] Kusano and Manojlović considered the equation (1.1) of the case $p(t) \equiv 1$, and in the recent paper [9] they have established the accurate asymptotic formulas for the generalized regularly varying solutions $x_i(t) \in$ n-$\mathrm{RV}_P(\lambda_i)$ ($i = 1$ and 2) of (1.1). However, in either paper, no attention is paid to the asymptotic behavior of the derivatives of solutions. Řehák [12] and Řehák and Taddei [13] considered the equation (1.1) of the case $q(t) < 0$ and obtained the precise asymptotic behavior of nonoscillatory (monotone) solutions of (1.1) with the aid of the Karamata theory of regular variation and the de Haan theory. It should be remarked that, in [12,13], the conditions in order that all (nonoscillatory) solutions of (1.1) are regularly varying are established.

The results in this paper are new even for the case $p(t) \equiv 1$. The results are also new for the linear equation (1.2), however, an analogous result to Theorem 1.3 is derived from Theorem 9.1 in [4]. As an auxiliary equation, consider the Euler equation

$$(p(t)y')' + \frac{c}{p(t)P(t)^2} y = 0,$$

which has a principal solution $y_1(t) = P(t)^{\lambda_1}$ and a nonprincipal solution $y_2(t) = P(t)^{\lambda_2}$. Applying Theorem 9.1 in [4], we have the following result. If

$$\int^{\infty} \frac{|p(t)P(t)^2 q(t) - c|}{p(t)P(t)} dt < \infty, \tag{1.20}$$

then (1.2) has a pair of solutions $x_i(t)$ ($i = 1$ and 2) such that (1.19) with $\alpha = 1$ holds. Moreover (see Exercise 9.4 in [4]), if $p(t)P(t)^2q(t) - c$ does not change signs, and (1.2) has a solution $x_i(t)$ satisfying (1.19) with $\alpha = 1$ for either $i = 1$ or $i = 2$, then (1.20) holds.

The proof of Theorem 1.3 is similar to that used in [7]. The key idea of the proof of Theorem 1.3 is to use Hardy-type integral inequalities. For the classical Hardy inequality, see Hardy et al. [3, Theorem 330]. In [1], Beesack gave a systematic treatment of some analogues and extensions of the classical Hardy inequality. Then Beesack employed half-linear differential equations of the special form and corresponding generalized Riccati differential equations. We adapt the method of Beesack [1], and give two kinds of Hardy-type integral inequalities which are necessary for the proof of Theorem 1.3. These are stated and proved in Section 2. The proofs of Proposition 1.2 and Theorem 1.3 are given in Section 3.

In the papers [7, 9], the case

$$\int_{t_0}^{\infty} \frac{1}{p(s)^{1/\alpha}} ds < \infty$$

is also discussed. For this case, certain results which correspond to Proposition 1.2 and Theorem 1.3 can be proved. However, the results will appear elsewhere as another paper because the statements and the proofs are rather long.

2. HARDY-TYPE INEQUALITIES

Let $g(t)$ be a continuous and positive function on an interval $[a, \infty)$. Further, we suppose that

$$\int_a^{\infty} g(s) ds = \infty. \tag{2.1}$$

Then we define the function $G(t)$ by

$$G(t) = \int_a^t g(s) ds, \quad t \geq a. \tag{2.2}$$

As a general inequality we have, for $p > 1$,

$$\xi^p + (p-1)\eta^p - p\xi\eta^{p-1} \geq 0 \quad (\xi \geq 0,\ \eta \geq 0). \tag{2.3}$$

In this section the inequality (2.3) will play an important part.

Theorem 2.1. *Let p and r be constants such that $p > 1$ and $r < 1$, respectively. Suppose that $g(t)$ is continuous on $[a, \infty)$, $g(t) > 0$ $(t \geq a)$, and (2.1) holds. Define the function $G(t)$ by (2.2). Let $b > a$. Suppose furthermore that $f(t)$ is continuous on $[b, \infty)$ and satisfies*

$$\int_b^\infty |f(s)|ds < \infty \tag{2.4}$$

and

$$\int_b^\infty g(s)^{-p+1} G(s)^{p-r} |f(s)|^p ds < \infty. \tag{2.5}$$

Then we have

$$\int_b^\infty g(s)G(s)^{-r} \left[\int_s^\infty |f(\sigma)|d\sigma \right]^p ds \leq \left(\frac{p}{1-r} \right)^p \int_b^\infty g(s)^{-p+1} G(s)^{p-r} |f(s)|^p ds. \tag{2.6}$$

Equality holds if and only if $f(t) \equiv 0$ on $[b, \infty)$.

Proof. We adapt the method of Beesack [1]. Put

$$\varphi(t) = \left(\frac{p}{1-r} \right)^p G(t)^{p-r} \quad \text{and} \quad \psi(t) = G(t)^{-r}$$

for $t \geq b$. It is clear that $\varphi(t) > 0$ and $\psi(t) > 0$ $(t \geq b)$. We can easily check that the function

$$x(t) = G(t)^{-(1-r)/p}, \quad t \geq b,$$

satisfies $x'(t) < 0$ $(t \geq b)$, and $x = x(t)$ is a positive solution of the half-linear differential equation

$$(g(t)^{-p+1} \varphi(t) |x'|^{p-1} \operatorname{sgn} x')' + g(t) \psi(t) |x|^{p-1} \operatorname{sgn} x = 0, \quad t \geq b.$$

Therefore the function

$$y(t) = -\left(\frac{x'(t)}{x(t)} \right)^{(p-1)*} \quad (> 0)$$

satisfies the generalized Riccati differential equation

$$(g(t)^{-p+1} \varphi(t) y(t))' = g(t) \psi(t) + (p-1) g(t)^{-p+1} \varphi(t) y(t)^{p/(p-1)} \tag{2.7}$$

for $t \geq b$.

Applying the inequality (2.3) to the case

$$\xi = |f(t)|, \quad \eta = y(t)^{1/(p-1)} \int_t^\infty |f(\sigma)|d\sigma,$$

we get

$$|f(t)|^p + (p-1)y(t)^{p/(p-1)} \left[\int_t^\infty |f(\sigma)|d\sigma\right]^p - p|f(t)|y(t) \left[\int_t^\infty |f(\sigma)|d\sigma\right]^{p-1} \geq 0, \quad t \geq b.$$

Multiply the above inequality by $g(t)^{-p+1}\varphi(t)$ and integrate from b to t $(t \geq b)$. It is seen that the function

$$\begin{aligned} I(t) \equiv & \int_b^t g(s)^{-p+1}\varphi(s)|f(s)|^p ds \\ & + (p-1) \int_b^t g(s)^{-p+1}\varphi(s)y(s)^{p/(p-1)} \left[\int_s^\infty |f(\sigma)|d\sigma\right]^p ds \\ & - p \int_b^t g(s)^{-p+1}\varphi(s)|f(s)|y(s) \left[\int_s^\infty |f(\sigma)|d\sigma\right]^{p-1} ds \end{aligned}$$

is nonnegative for $t \geq b$. Denote the last term of $I(t)$ by $I_3(t)$. By an integration by parts and use of (2.7), we find that

$$\begin{aligned} I_3(t) = & \, g(t)^{-p+1}\varphi(t)y(t) \left[\int_t^\infty |f(\sigma)|d\sigma\right]^p \\ & - g(b)^{-p+1}\varphi(b)y(b) \left[\int_b^\infty |f(\sigma)|d\sigma\right]^p \\ & - \int_b^t \left\{g(s)\psi(s) + (p-1)g(s)^{-p+1}\varphi(s)y(s)^{p/(p-1)}\right\} \left[\int_s^\infty |f(\sigma)|d\sigma\right]^p ds. \end{aligned}$$

Therefore, since $I(t) \geq 0$ $(t \geq b)$, we obtain

$$\begin{aligned} & \int_b^t g(s)\psi(s) \left[\int_s^\infty |f(\sigma)|d\sigma\right]^p ds \\ & \leq \int_b^t g(s)^{-p+1}\varphi(s)|f(s)|^p ds + g(t)^{-p+1}\varphi(t)y(t) \left[\int_t^\infty |f(\sigma)|d\sigma\right]^p \qquad (2.8) \\ & \quad - g(b)^{-p+1}\varphi(b)y(b) \left[\int_b^\infty |f(\sigma)|d\sigma\right]^p, \quad t \geq b. \end{aligned}$$

We claim that the second term of the right-hand side of (2.8) tends to 0 as $t \to \infty$:

$$\lim_{t\to\infty} g(t)^{-p+1}\varphi(t)y(t)\left[\int_t^\infty |f(\sigma)|d\sigma\right]^p = 0. \tag{2.9}$$

To prove (2.9), we note that $y(t)$ is explicitly given by

$$y(t) = \left(\frac{p}{1-r}\right)^{-p+1} g(t)^{p-1}G(t)^{-p+1},$$

and so

$$g(t)^{-p+1}\varphi(t)y(t) = \frac{p}{1-r}G(t)^{1-r}.$$

Using the Hölder inequality, we have

$$\begin{aligned}\int_t^\infty |f(\sigma)|d\sigma &\leq \left[\int_t^\infty g(\sigma)G(\sigma)^{-(p-r)/(p-1)}d\sigma\right]^{(p-1)/p}\left[\int_t^\infty g(\sigma)^{-p+1}G(\sigma)^{p-r}|f(\sigma)|^p d\sigma\right]^{1/p}\\ &= AG(t)^{-(1-r)/p}\left[\int_t^\infty g(\sigma)^{-p+1}G(\sigma)^{p-r}|f(\sigma)|^p d\sigma\right]^{1/p},\end{aligned}$$

where

$$A = \left(\frac{p-1}{1-r}\right)^{(p-1)/p} > 0.$$

Therefore it is seen that

$$\begin{aligned}0 &\leq g(t)^{-p+1}\varphi(t)y(t)\left[\int_t^\infty |f(\sigma)|d\sigma\right]^p\\ &\leq \frac{p}{1-r}A^p\int_t^\infty g(\sigma)^{-p+1}G(\sigma)^{p-r}|f(\sigma)|^p d\sigma.\end{aligned}$$

Then, by the condition (2.5), we get (2.9) as claimed.

Let $t \to \infty$ in (2.8). Then, by (2.9), we see that

$$\begin{aligned}&\int_b^\infty g(s)\psi(s)\left[\int_s^\infty |f(\sigma)|d\sigma\right]^p ds\\ &\leq \int_b^\infty g(s)^{-p+1}\varphi(s)|f(s)|^p ds - g(b)^{-p+1}\varphi(b)y(b)\left[\int_b^\infty |f(\sigma)|d\sigma\right]^p,\end{aligned}$$

and, in consequence,

$$\int_b^\infty g(s)\psi(s)\left[\int_s^\infty |f(\sigma)|d\sigma\right]^p ds \le \int_b^\infty g(s)^{-p+1}\varphi(s)|f(s)|^p ds. \tag{2.10}$$

The inequality (2.10) is identical with the inequality (2.6). The equality holds in (2.10) if and only if

$$g(b)^{-p+1}\varphi(b)y(b)\left[\int_b^\infty |f(\sigma)|d\sigma\right]^p = 0, \quad \text{i.e.} \quad f(t) \equiv 0 \text{ on } [b,\infty).$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let p and r be constants such that $p > 1$ and $r > 1$, respectively. Suppose that $g(t)$ is continuous on $[a,\infty)$, $g(t) > 0$ $(t \ge a)$, and (2.1) holds. Define the function $G(t)$ by (2.2). Let $a < b < c$. Suppose that $f(t)$ is continuous on $[b,c]$. Then we have*

$$\int_b^c g(s)G(s)^{-r}\left[\int_b^s |f(\sigma)|d\sigma\right]^p ds \le \left(\frac{p}{r-1}\right)^p \int_b^c g(s)^{-p+1}G(s)^{p-r}|f(s)|^p ds. \tag{2.11}$$

Equality holds if and only if $f(t) \equiv 0$ on $[b,c]$.

Proof. The proof of Theorem 2.2 is quite similar to that of Theorem 2.1. Put

$$\varphi(t) = \left(\frac{p}{r-1}\right)^p G(t)^{p-r} \quad \text{and} \quad \psi(t) = G(t)^{-r}$$

for $t \in [b,c]$. It is clear that $\varphi(t) > 0$ and $\psi(t) > 0$ $(b \le t \le c)$. The function

$$x(t) = G(t)^{(r-1)/p}, \quad b \le t \le c,$$

satisfies $x'(t) > 0$ $(b \le t \le c)$, and that $x = x(t)$ is a positive solution of the half-linear differential equation

$$(g(t)^{-p+1}\varphi(t)|x'|^{p-1}\operatorname{sgn} x')' + g(t)\psi(t)|x|^{p-1}\operatorname{sgn} x = 0, \quad b \le t \le c.$$

Therefore the function

$$y(t) = \left(\frac{x'(t)}{x(t)}\right)^{(p-1)*} (> 0)$$

satisfies the generalized Riccati differential equation

$$(g(t)^{-p+1}\varphi(t)y(t))' = -g(t)\psi(t) - (p-1)g(t)^{-p+1}\varphi(t)y(t)^{p/(p-1)} \tag{2.12}$$

for $t \in [b,c]$.

Applying the general inequality (2.3) to the case

$$\xi = |f(t)|, \quad \eta = y(t)^{1/(p-1)} \int_b^t |f(\sigma)| d\sigma,$$

we see that the number

$$\begin{aligned} J \equiv & \int_b^c g(s)^{-p+1} \varphi(s) |f(s)|^p ds \\ & + (p-1) \int_b^c g(s)^{-p+1} \varphi(s) y(s)^{p/(p-1)} \left[\int_b^s |f(\sigma)| d\sigma \right]^p ds \\ & - p \int_b^c g(s)^{-p+1} \varphi(s) |f(s)| y(s) \left[\int_b^s |f(\sigma)| d\sigma \right]^{p-1} ds \end{aligned}$$

is nonnegative. Integrate the last term of J by parts and use (2.12). We find that

$$\begin{aligned} & \int_b^c g(s) \psi(s) \left[\int_b^s |f(\sigma)| d\sigma \right]^p ds \\ & \leq \int_b^c g(s)^{-p+1} \varphi(s) |f(s)|^p ds - g(c)^{-p+1} \varphi(c) y(c) \left[\int_b^c |f(\sigma)| d\sigma \right]^p, \end{aligned}$$

which yields

$$\int_b^c g(s) \psi(s) \left[\int_b^s |f(\sigma)| d\sigma \right]^p ds \leq \int_b^c g(s)^{-p+1} \varphi(s) |f(s)|^p ds. \tag{2.13}$$

The inequality (2.13) is identical with (2.11). The equality holds in (2.13) if and only if

$$g(c)^{-p+1} \varphi(c) y(c) \left[\int_b^c |f(\sigma)| d\sigma \right]^p = 0, \quad \text{i.e.} \quad f(t) \equiv 0 \text{ on } [b, c].$$

The proof of Theorem 2.2 is complete. $\square$

3. PROOFS OF PROPOSITION 1.2 AND THEOREM 1.3

Proof of Proposition 1.2. Suppose that (1.1) has a nonoscillatory solution $x(t)$ which satisfies (1.16) for some $\lambda \in \mathbb{R}$. We suppose that $x(t) > 0$ for $t \geq T$. Define $y(t)$ by (1.12). As stated above, $y(t)$ satisfies the generalized Riccati equation (1.13). Further, by (1.16), the function $y(t)$ satisfies $P(t)^{\alpha}|y(t)| \to |\lambda|^{\alpha}$ as $t \to \infty$, and so

$$|y(t)| \leq \frac{|\lambda|^{\alpha}+1}{P(t)^{\alpha}} \quad \text{for all large } t.$$

Therefore we see that

$$\lim_{t\to\infty} |y(t)| = 0 \quad \text{and} \quad \int_T^{\infty} p(s)^{-1/\alpha}|y(s)|^{(\alpha+1)/\alpha} ds < \infty. \tag{3.1}$$

Integrating (1.13) from t to τ ($T \leq t \leq \tau$), we obtain

$$y(\tau) - y(t) = -\int_t^{\tau} q(s)ds - \alpha \int_t^{\tau} p(s)^{-1/\alpha}|y(s)|^{(\alpha+1)/\alpha} ds$$

for $t \geq T$. Let $\tau \to \infty$ in the above equality. Then, since (3.1) is satisfied, we conclude that (1.6) and (1.15) hold. The proof of Proposition 1.2 is complete. $\square$

To prove Theorem 1.3, we use the following lemmas.

Lemma 3.1. *Let $\lambda \neq 0$ be fixed. Let w and ε be real numbers with $|\varepsilon| \leq |\lambda|^{\alpha}/4$. The function*

$$F(w,\varepsilon) = |w + \lambda^{\alpha*} + \varepsilon|^{(\alpha+1)/\alpha} - |\lambda^{\alpha*} + \varepsilon|^{(\alpha+1)/\alpha} - \frac{\alpha+1}{\alpha}(\lambda^{\alpha*}+\varepsilon)^{(1/\alpha)*} w \tag{3.2}$$

satisfies

$$0 \leq F(w,\varepsilon) \leq K(\alpha)|\lambda|^{-\alpha+1}w^2 \quad \left(|w| \leq \frac{|\lambda|^{\alpha}}{4},\ |\varepsilon| \leq \frac{|\lambda|^{\alpha}}{4}\right),$$

where

$$K(\alpha) = \begin{cases} \dfrac{\alpha+1}{2\alpha^2}\left(\dfrac{3}{2}\right)^{(-\alpha+1)/\alpha}, & 0 < \alpha \leq 1, \\ \dfrac{\alpha+1}{2\alpha^2}\left(\dfrac{1}{2}\right)^{(-\alpha+1)/\alpha}, & \alpha > 1. \end{cases}$$

Note that the function $F(w,\varepsilon)$ defined by (3.2) arises naturally in [6,7]. For a brief proof of Lemma 3.1, see Naito [11, Lemma 2.4].

Lemma 3.2. *Let* $\lambda \neq 0$. *Then*

$$\left| |\lambda^{\alpha *} + \varepsilon|^{(\alpha+1)/\alpha} - |\lambda|^{\alpha+1} \right| \leq 2\frac{\alpha+1}{\alpha}|\lambda||\varepsilon| \tag{3.3}$$

and

$$|(\lambda^{\alpha *} + \varepsilon)^{(1/\alpha)*} - \lambda| \leq \frac{2}{\alpha}|\lambda|^{-\alpha+1}|\varepsilon| \tag{3.4}$$

for all sufficiently small $|\varepsilon|$.

Since

$$\lim_{\varepsilon \to 0} \frac{|\lambda^{\alpha *} + \varepsilon|^{(\alpha+1)/\alpha} - |\lambda|^{\alpha+1}}{\varepsilon} = \frac{\alpha+1}{\alpha}\lambda$$

and

$$\lim_{\varepsilon \to 0} \frac{(\lambda^{\alpha *} + \varepsilon)^{(1/\alpha)*} - \lambda}{\varepsilon} = \frac{1}{\alpha}|\lambda|^{-\alpha+1} \quad (\lambda \neq 0),$$

Lemma 3.2 is obvious.

Now, let c be a real constant such that $c < E(\alpha)$, where $E(\alpha)$ is given by (1.8). We suppose that (1.6) and (1.10) hold. Here, $P(t)$ and $Q(t)$ are defined by (1.5) and (1.7), respectively. For the roots ρ_1 and ρ_2 ($\rho_1 < \rho_2$) of (1.9), set $\lambda_i = \rho_i^{(1/\alpha)*}$ ($i = 1, 2$). Then, λ_1 and λ_2 are solutions of the equation

$$|\lambda|^{\alpha+1} - \lambda^{\alpha *} + c = 0,$$

and satisfy $\lambda_1 < \alpha/(\alpha+1) < \lambda_2$. Let $x(t)$ be a nonoscillatory solution of (1.1) and satisfy the asymptotic condition of the form (1.19). We suppose that $x(t) > 0$ for $t \geq T$ ($> t_0$), and define the function $y(t)$ by (1.12). According to Proposition 1.2, the function $y(t)$ satisfies (1.15).

Let $\lambda = \lambda_1 \neq 0$ or $\lambda = \lambda_2$. We define the function $w(t)$ by

$$w(t) = P(t)^\alpha y(t) - \lambda^{\alpha *} - \varepsilon(t), \quad t \geq T,$$

where $\varepsilon(t)$ is given by (1.17). Since $\varepsilon(t) \to 0$ ($t \to \infty$), we can suppose that $|\varepsilon(t)| \leq |\lambda|^\alpha/4$ for $t \geq T$. Noting that $\lambda^{\alpha *} = |\lambda|^{\alpha+1} + c$ and using the formula (1.15), we have

$$w(t) = -|\lambda|^{\alpha+1} + \alpha P(t)^\alpha \int_t^\infty p(s)^{-1/\alpha} \frac{|w(s) + \lambda^{\alpha *} + \varepsilon(s)|^{(\alpha+1)/\alpha}}{P(s)^{\alpha+1}} ds \tag{3.5}$$

for $t \geq T$. Then it is easy to see that

$$p(t)^{1/\alpha} w'(t) = \frac{\alpha}{P(t)} w(t) + \frac{\alpha|\lambda|^{\alpha+1}}{P(t)} - \frac{\alpha}{P(t)}|w(t) + \lambda^{\alpha *} + \varepsilon(t)|^{(\alpha+1)/\alpha}$$

for $t \geq T$.

This equality can be rewritten as

$$\begin{aligned} p(t)^{1/\alpha}w'(t) = &-\frac{1}{P(t)}\{(\alpha+1)\lambda-\alpha\}\,w(t) \\ &-\frac{\alpha}{P(t)}\left\{|\lambda^{\alpha *}+\varepsilon(t)|^{(\alpha+1)/\alpha}-|\lambda|^{\alpha+1}\right\} \\ &-\frac{\alpha+1}{P(t)}\left\{(\lambda^{\alpha *}+\varepsilon(t))^{(1/\alpha)*}-\lambda\right\}w(t) \\ &-\frac{\alpha}{P(t)}F(w(t),\varepsilon(t)), \quad t \ge T, \end{aligned}$$

where $F(w,\varepsilon)$ is defined by (3.2).

For simplicity of notation, we put

$$f_1(t) = |\lambda^{\alpha *}+\varepsilon(t)|^{(\alpha+1)/\alpha}-|\lambda|^{\alpha+1}, \quad f_2(t) = (\lambda^{\alpha *}+\varepsilon(t))^{(1/\alpha)*}-\lambda, \tag{3.6}$$

and so

$$\begin{aligned} p(t)^{1/\alpha}w'(t) = &-\frac{1}{P(t)}\{(\alpha+1)\lambda-\alpha\}\,w(t) \\ &-\frac{\alpha}{P(t)}f_1(t)-\frac{\alpha+1}{P(t)}f_2(t)w(t) \\ &-\frac{\alpha}{P(t)}F(w(t),\varepsilon(t)), \quad t \ge T. \end{aligned} \tag{3.7}$$

Since $\varepsilon(t) \to 0$ $(t \to \infty)$, it follows from Lemma 3.2 that

$$|f_1(t)| \le 2\frac{\alpha+1}{\alpha}|\lambda||\varepsilon(t)|, \quad |f_2(t)| \le \frac{2}{\alpha}|\lambda|^{-\alpha+1}|\varepsilon(t)| \tag{3.8}$$

for all large t. Without loss of generality we assume that (3.8) holds for $t \ge T$.

In the following, we distinguish the cases

$$(\alpha+1)\lambda-\alpha<0 \quad \text{and} \quad (\alpha+1)\lambda-\alpha>0.$$

Let us consider the first case $(\alpha+1)\lambda-\alpha<0$, i.e., $\lambda=\lambda_1$ $(\neq 0)$. In this case we set

$$\beta = -(\alpha+1)\lambda_1+\alpha \ (>0).$$

Then, (3.7) yields

$$\begin{aligned} (P(t)^{-\beta}w(t))' = &-\alpha p(t)^{-1/\alpha}P(t)^{-\beta-1}f_1(t) \\ &-(\alpha+1)p(t)^{-1/\alpha}P(t)^{-\beta-1}f_2(t)w(t) \\ &-\alpha p(t)^{-1/\alpha}P(t)^{-\beta-1}F(w(t),\varepsilon(t)) \end{aligned} \tag{3.9}$$

for $t \ge T$. Note that $x(t)$ satisfies the asymptotic condition of the form (1.11) with $i=1$. Therefore we have

$$P(t)^{\alpha}y(t) = p(t)P(t)^{\alpha}(x'(t)/x(t))^{\alpha *} \to \rho_1 = \lambda_1^{\alpha *} \quad \text{as} \ \ t \to \infty,$$

and so $w(t)$ with $\lambda = \lambda_1$ tends to 0 as $t \to \infty$. In particular, $P(t)^{-\beta} w(t)$ tends to 0 as $t \to \infty$. Therefore, it follows from (3.9) that

$$\begin{aligned} w(t) = \alpha P(t)^{\beta} \int_t^{\infty} p(s)^{-1/\alpha} P(s)^{-\beta-1} f_1(s) ds \\ + (\alpha+1) P(t)^{\beta} \int_t^{\infty} p(s)^{-1/\alpha} P(s)^{-\beta-1} f_2(s) w(s) ds \\ + \alpha P(t)^{\beta} \int_t^{\infty} p(s)^{-1/\alpha} P(s)^{-\beta-1} F(w(s), \varepsilon(s)) ds \end{aligned} \tag{3.10}$$

for $t \geq T$. We are now ready for the proof of Theorem 1.3 of the case $i = 1$.

Proof of Theorem 1.3 of the case $i = 1$. Since $\varepsilon(t) \to 0$ $(t \to \infty)$, we can take $T > t_0$ sufficiently large so that $|\varepsilon(t)| \leq |\lambda_1|^{\alpha}/4$ for $t \geq T$. Define the functions $f_1(t)$ and $f_2(t)$ by (3.6) with $\lambda = \lambda_1$. We may suppose that (3.8) $(\lambda = \lambda_1)$ holds for $t \geq T$. Put

$$\eta(t) = P(t)^{\beta} \int_t^{\infty} p(s)^{-1/\alpha} P(s)^{-\beta-1} |\varepsilon(s)| ds, \quad t \geq T. \tag{3.11}$$

Since $\varepsilon(t) \to 0$ $(t \to \infty)$, the function $\eta(t)$ is well defined and $\eta(t) \to 0$ $(t \to \infty)$, and so we can suppose that

$$4(\alpha+1)|\lambda_1|\eta(t) \leq |\lambda_1|^{\alpha}/4, \quad t \geq T.$$

Denote by W the set of all functions $w \in C[T, \infty)$ such that

$$|w(t)| \leq 4(\alpha+1)|\lambda_1|\eta(t), \quad t \geq T. \tag{3.12}$$

Moreover, keeping (3.10) in mind, we define the operator $\mathcal{F} : W \to C[T, \infty)$ by

$$\begin{aligned} (\mathcal{F}w)(t) = \alpha P(t)^{\beta} \int_t^{\infty} p(s)^{-1/\alpha} P(s)^{-\beta-1} f_1(s) ds \\ + (\alpha+1) P(t)^{\beta} \int_t^{\infty} p(s)^{-1/\alpha} P(s)^{-\beta-1} f_2(s) w(s) ds \\ + \alpha P(t)^{\beta} \int_t^{\infty} p(s)^{-1/\alpha} P(s)^{-\beta-1} F(w(s), \varepsilon(s)) ds \end{aligned}$$

for $t \geq T$. Here, $F(w, \varepsilon)$ is given by (3.2) with $\lambda = \lambda_1$. As is easily verified, the set W is a nonempty closed convex subset of the Fréchet space $C[T, \infty)$ of all continuous functions on $[T,\infty)$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$. Note that if $w \in W$, then $|w(t)| \le |\lambda_1|^\alpha/4$ for $t \ge T$, and so, by Lemma 3.1,

$$0 \le F(w(t),\varepsilon(t)) \le K(\alpha)|\lambda_1|^{-\alpha+1}w(t)^2, \quad t \ge T. \tag{3.13}$$

Then it is easily checked that $\mathcal{F}w$ is well defined for $w \in W$.

Let $w \in W$. Then, by (3.8) with $\lambda = \lambda_1$ and (3.12) and (3.13), we have

$$\begin{aligned}|(\mathcal{F}w)(t)| \le{}& 2(\alpha+1)|\lambda_1|P(t)^\beta \int_t^\infty p(s)^{-1/\alpha}P(s)^{-\beta-1}|\varepsilon(s)|ds \\ &+ \frac{8(\alpha+1)^2}{\alpha}|\lambda_1|^{-\alpha+2}P(t)^\beta \int_t^\infty p(s)^{-1/\alpha}P(s)^{-\beta-1}|\varepsilon(s)|\eta(s)ds \\ &+ 16\alpha(\alpha+1)^2K(\alpha)|\lambda_1|^{-\alpha+3}P(t)^\beta \int_t^\infty p(s)^{-1/\alpha}P(s)^{-\beta-1}\eta(s)^2ds\end{aligned}$$

for $t \ge T$. For the integral in the last term of the right-hand side of the above, apply the Hardy-type inequality (2.6) in Theorem 2.1 to the case $a = t_0$, $b = t$, $p = 2$, $r = -\beta+1$, and

$$g(t) = p(t)^{-1/\alpha} \quad \text{and} \quad f(t) = p(t)^{-1/\alpha}P(t)^{-\beta-1}|\varepsilon(t)|.$$

Then it can be deduced that

$$\begin{aligned}&\int_t^\infty p(s)^{-1/\alpha}P(s)^{-\beta-1}\eta(s)^2ds \\ &= \int_t^\infty p(s)^{-1/\alpha}P(s)^{\beta-1}\left[\int_s^\infty p(\sigma)^{-1/\alpha}P(\sigma)^{-\beta-1}|\varepsilon(\sigma)|d\sigma\right]^2 ds \\ &\le \left(\frac{2}{\beta}\right)^2 \int_t^\infty p(s)^{-1/\alpha}P(s)^{-\beta-1}|\varepsilon(s)|^2ds, \quad t \ge T.\end{aligned}$$

Therefore, $|(\mathcal{F}w)(t)|$ is estimated as follows:

$$\begin{aligned}|(\mathcal{F}w)(t)| \le{}& 2(\alpha+1)|\lambda_1|\eta(t) + \frac{8(\alpha+1)^2}{\alpha}|\lambda_1|^{-\alpha+2}\left[\sup_{s\ge t}\eta(s)\right]\eta(t) \\ &+ 16\alpha(\alpha+1)^2K(\alpha)|\lambda_1|^{-\alpha+3}\left(\frac{2}{\beta}\right)^2\left[\sup_{s\ge t}|\varepsilon(s)|\right]\eta(t)\end{aligned}$$

for $t \ge T$. Since $\sup_{s\ge t}\eta(s) \to 0$ and $\sup_{s\ge t}|\varepsilon(s)| \to 0$ as $t \to \infty$, we can suppose that

$$\frac{8(\alpha+1)^2}{\alpha}|\lambda_1|^{-\alpha+2}\left[\sup_{s\ge t}\eta(s)\right] \le (\alpha+1)|\lambda_1|, \quad t \ge T,$$

and

$$16\alpha(\alpha+1)^2K(\alpha)|\lambda_1|^{-\alpha+3}\left(\frac{2}{\beta}\right)^2\left[\sup_{s\geq t}|\varepsilon(s)|\right]\leq(\alpha+1)|\lambda_1|,\quad t\geq T.$$

Then we get

$$|(\mathcal{F}w)(t)|\leq 4(\alpha+1)|\lambda_1|\eta(t),\quad t\geq T.$$

This means that

(i) $\mathcal{F}$ maps W into W.

Moreover, it can be checked that

(ii) $\mathcal{F}$ is continuous on W,
(iii) $\mathcal{F}W$ is uniformly bounded and equicontinuous at every point of $[T,\infty)$.

Therefore, by the Schauder–Tychonoff fixed point theorem, we conclude that $\mathcal{F}$ has a fixed element $w\in W$: $w(t)=(\mathcal{F}w)(t)$, $t\geq T$.

It is clear that the above fixed element $w(t)$ satisfies (3.10) and (3.12). Since $\eta(t)\to 0$ $(t\to\infty)$, it follows from (3.12) that $w(t)\to 0$ as $t\to\infty$. We can verify without difficulty that $w(t)$ satisfies (3.5) $(\lambda=\lambda_1)$ for $t\geq T$. Put

$$y(t)=\frac{w(t)+\lambda_1^{\alpha*}+\varepsilon(t)}{P(t)^{\alpha}},\quad t\geq T. \tag{3.14}$$

It is shown that $y(t)$ satisfies (1.15), and hence it satisfies (1.13). Therefore, the function $x(t)$ which is defined by (1.14) is a positive solution of (1.1) on $[T,\infty)$. Furthermore, we have

$$p(t)P(t)^{\alpha}\left(\frac{x'(t)}{x(t)}\right)^{\alpha*}=P(t)^{\alpha}y(t)=w(t)+\lambda_1^{\alpha*}+\varepsilon(t)\to\lambda_1^{\alpha*}$$

as $t\to\infty$, and so

$$\lim_{t\to\infty}p(t)^{1/\alpha}P(t)\frac{x'(t)}{x(t)}=\lambda_1. \tag{3.15}$$

Since $\lambda_1=\rho_1^{(1/\alpha)*}$, this implies that $x(t)$ belongs to the class n-$\mathrm{RV}_P(\rho_1^{(1/\alpha)*})$.

Note that the arguments up to now give a proof of the last statement $(i=1)$ of Theorem 1.1. The advantage of the arguments here is the bound (3.12). In fact, it follows from (1.12) and (3.14) that

$$\begin{aligned}\frac{x'(t)}{x(t)}&=\frac{(w(t)+\lambda_1^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}}{p(t)^{1/\alpha}P(t)}\\&=\frac{\lambda_1}{p(t)^{1/\alpha}P(t)}+\frac{(w(t)+\lambda_1^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}-\lambda_1}{p(t)^{1/\alpha}P(t)}\end{aligned} \tag{3.16}$$

for $t\geq T$. Noting that $\varepsilon(t)\to 0$ and $w(t)\to 0$ $(t\to\infty)$ and using (3.4) in Lemma 3.2, we see that

$$|(w(t)+\lambda_1^{\alpha*}+\varepsilon(t))^{(1/\alpha)*}-\lambda_1|\leq\frac{2}{\alpha}|\lambda_1|^{-\alpha+1}\{|w(t)|+|\varepsilon(t)|\} \tag{3.17}$$

for all large t.

By (3.16), we have

$$x(t) = \frac{x(T_1)}{P(T_1)^{\lambda_1}} \exp\left(\int\limits_{T_1}^{t} \frac{(w(s) + \lambda_1^{\alpha *} + \varepsilon(s))^{(1/\alpha)*} - \lambda_1}{p(s)^{1/\alpha} P(s)} ds \right) P(t)^{\lambda_1} \tag{3.18}$$

for $t \geq T_1$, where T_1 is a constant and is taken sufficiently large. It should be noticed that the assumption (1.18), together with the condition $\varepsilon(t) \to 0$ $(t \to \infty)$, implies

$$\int\limits_{T_1}^{\infty} \frac{\eta(t)}{p(t)^{1/\alpha} P(t)} dt < \infty, \tag{3.19}$$

where $\eta(t)$ is given by (3.11). The verification of this fact is left to the reader. Then, by (3.12) and (3.19), we have

$$\int\limits_{T_1}^{\infty} \frac{|w(t)|}{p(t)^{1/\alpha} P(t)} dt < \infty. \tag{3.20}$$

By (1.18), (3.17), (3.18) and (3.20), it is found that $x(t)$ is written in the form

$$x(t) = c_0(t) P(t)^{\lambda_1} \quad \text{with} \quad c_0(t) \to c_0 \in (0, \infty) \ \text{ as } \ t \to \infty. \tag{3.21}$$

Then we have

$$x'(t) = c_0(t) p(t)^{-1/\alpha} P(t)^{\lambda_1 - 1} p(t)^{1/\alpha} P(t) \frac{x'(t)}{x(t)},$$

and so (3.15) implies that $x'(t)$ is written in the form

$$x'(t) = c_1(t) p(t)^{-1/\alpha} P(t)^{\lambda_1 - 1} \quad \text{with} \quad c_1(t) \to c_0 \lambda_1 \ \text{ as } \ t \to \infty. \tag{3.22}$$

In general, if $x(t)$ is a solution of the half-linear equation (1.1) and if c is a constant, then $cx(t)$ is also a solution of (1.1). Therefore, without loss of generality, we may suppose that $c_0 = 1$ in (3.21) and (3.22). This shows (1.19) with $i = 1$. The proof of Theorem 1.3 of the case $i = 1$ is complete. □

In order to discuss the second case $(\alpha + 1)\lambda - \alpha > 0$, let us return to (3.7). Note that $\lambda = \lambda_2 > 0$ for this case. We set

$$\beta = (\alpha + 1)\lambda_2 - \alpha \ (> 0).$$

Then, (3.7) yields

$$\begin{aligned} (P(t)^{\beta} w(t))' = & -\alpha p(t)^{-1/\alpha} P(t)^{\beta - 1} f_1(t) \\ & - (\alpha + 1) p(t)^{-1/\alpha} P(t)^{\beta - 1} f_2(t) w(t) \\ & - \alpha p(t)^{-1/\alpha} P(t)^{\beta - 1} F(w(t), \varepsilon(t)), \quad t \geq T. \end{aligned}$$

Therefore we have

$$\begin{aligned}w(t) = P(T)^\beta w(T)P(t)^{-\beta} - \alpha P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} f_1(s)ds \\ - (\alpha+1)P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} f_2(s)w(s)ds \\ - \alpha P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} F(w(s), \varepsilon(s))ds\end{aligned} \tag{3.23}$$

for $t \geq T$. We can now give the proof of Theorem 1.3 of the case $i = 2$.

Proof of Theorem 1.3 of the case $i = 2$. Since $\varepsilon(t) \to 0$ $(t \to \infty)$, we can choose $T > t_0$ sufficiently large so that $|\varepsilon(t)| \leq |\lambda_2|^\alpha/4$ for $t \geq T$. Define the functions $f_1(t)$ and $f_2(t)$ by (3.6) with $\lambda = \lambda_2$. We may suppose that (3.8) $(\lambda = \lambda_2)$ holds for $t \geq T$. Put

$$\eta(t;T) = P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} |\varepsilon(s)|ds, \quad t \geq T. \tag{3.24}$$

Since $\varepsilon(t) \to 0$ $(t \to \infty)$, we have $\eta(t;T) \to 0$ $(t \to \infty)$, and further

$$\eta(t;T) \leq P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} ds \left[\sup_{s \geq T} |\varepsilon(s)|\right] \leq \frac{1}{\beta} \left[\sup_{s \geq T} |\varepsilon(s)|\right] \tag{3.25}$$

for $t \geq T$. By virtue of the fact that $\sup_{s \geq T} |\varepsilon(s)| \to 0$ as $T \to \infty$, we may suppose that T is sufficiently large so that

$$\sup_{s \geq T} |\varepsilon(s)| \leq \frac{\beta \lambda_2^{\alpha-1}}{16(\alpha+1)}.$$

Denote by W the set of all functions $w \in C[T, \infty)$ such that

$$|w(t)| \leq 4(\alpha+1)\lambda_2 \eta(t;T), \quad t \geq T. \tag{3.26}$$

For $w \in W$, we have $w(T) = 0$. Then, keeping (3.23) in mind, we define the operator $\mathcal{F} : W \to C[T, \infty)$ by

$$\begin{aligned}(\mathcal{F}w)(t) = -\alpha P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} f_1(s) ds \\ - (\alpha+1) P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} f_2(s) w(s) ds \\ - \alpha P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} F(w(s), \varepsilon(s)) ds\end{aligned}$$

for $t \geq T$. Here, $F(w, \varepsilon)$ is given by (3.2) with $\lambda = \lambda_2$.

Let $w \in W$. Note that

$$|w(t)| \leq 4(\alpha+1)\lambda_2 \frac{1}{\beta} \left[\sup_{s \geq T} |\varepsilon(s)| \right] \leq \lambda_2^{\alpha}/4, \quad t \geq T,$$

and so, by Lemma 3.1,

$$0 \leq F(w(t), \varepsilon(t)) \leq K(\alpha) \lambda_2^{-\alpha+1} w(t)^2, \quad t \geq T. \tag{3.27}$$

Therefore, by (3.8) with $\lambda = \lambda_2$ and (3.26) and (3.27), we have

$$\begin{aligned}|(\mathcal{F}w)(t)| \leq 2(\alpha+1)\lambda_2 P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} |\varepsilon(s)| ds \\ + \frac{8(\alpha+1)^2}{\alpha} \lambda_2^{-\alpha+2} P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} |\varepsilon(s)| \eta(s;T) ds \\ + 16\alpha(\alpha+1)^2 K(\alpha) \lambda_2^{-\alpha+3} P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} \eta(s;T)^2 ds\end{aligned}$$

for $t \geq T$. For the second term of the right-hand side of the above, we use (3.25). For the last term of the right-hand side, we apply the Hardy-type inequality (2.11) in Theorem 2.2 to the case $a = t_0$, $b = T$, $c = t$, $p = 2$, $r = \beta + 1$ and

$$g(t) = p(t)^{-1/\alpha} \quad \text{and} \quad f(t) = p(t)^{-1/\alpha} P(t)^{\beta-1} |\varepsilon(t)|.$$

Then we find that

$$\begin{aligned}
&\int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} \eta(s;T)^2 ds \\
&= \int_T^t p(s)^{-1/\alpha} P(s)^{-\beta-1} \left[\int_T^s p(\sigma)^{-1/\alpha} P(\sigma)^{\beta-1} |\varepsilon(\sigma)| d\sigma \right]^2 ds \\
&\leq \left(\frac{2}{\beta}\right)^2 \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} |\varepsilon(s)|^2 ds, \quad t \geq T.
\end{aligned}$$

Therefore, $|(\mathcal{F}w)(t)|$ is estimated in the following way:

$$\begin{aligned}
|(\mathcal{F}w)(t)| \leq 2(\alpha+1)\lambda_2 \eta(t;T) &+ \frac{8(\alpha+1)^2}{\alpha} \lambda_2^{-\alpha+2} \frac{1}{\beta} \left[\sup_{s \geq T} |\varepsilon(s)| \right] \eta(t;T) \\
&+ 16\alpha(\alpha+1)^2 K(\alpha) \lambda_2^{-\alpha+3} \left(\frac{2}{\beta}\right)^2 \left[\sup_{s \geq T} |\varepsilon(s)| \right] \eta(t;T)
\end{aligned}$$

for $t \geq T$. Since $\sup_{s \geq T} |\varepsilon(s)| \to 0$ as $T \to \infty$, we can suppose that

$$\frac{8(\alpha+1)^2}{\alpha} \lambda_2^{-\alpha+2} \frac{1}{\beta} \left[\sup_{s \geq T} |\varepsilon(s)| \right] \leq (\alpha+1)\lambda_2$$

and

$$16\alpha(\alpha+1)^2 K(\alpha) \lambda_2^{-\alpha+3} \left(\frac{2}{\beta}\right)^2 \left[\sup_{s \geq T} |\varepsilon(s)| \right] \leq (\alpha+1)\lambda_2.$$

Then we deduce that

$$|(\mathcal{F}w)(t)| \leq 4(\alpha+1)\lambda_2 \eta(t;T), \quad t \geq T.$$

This means that

(i) $\mathcal{F}$ maps W into W.

Moreover it can be checked without difficulty that

(ii) $\mathcal{F}$ is continuous on W,
(iii) $\mathcal{F}W$ is uniformly bounded and equicontinuous at every point of $[T, \infty)$.

The Schauder–Tychonoff fixed point theorem implies that $\mathcal{F}$ has a fixed element w in W. This fixed element $w(t)$ satisfies

$$\begin{aligned} w(t) = -\alpha P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} f_1(s) ds \\ - (\alpha+1) P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} f_2(s) w(s) ds \\ - \alpha P(t)^{-\beta} \int_T^t p(s)^{-1/\alpha} P(s)^{\beta-1} F(w(s), \varepsilon(s)) ds, \quad t \geq T, \end{aligned}$$

and has the bound (3.26). Since $\eta(t;T) \to 0$ $(t \to \infty)$, it follows from (3.26) that $w(t) \to 0$ $(t \to \infty)$. Further, $w(t)$ satisfies (3.5) $(\lambda = \lambda_2)$ for $t \geq T$. Put

$$y(t) = \frac{w(t) + \lambda_2^\alpha + \varepsilon(t)}{P(t)^\alpha}, \quad t \geq T. \tag{3.28}$$

Then it is found that $y(t)$ satisfies (1.15), and hence it satisfies (1.13). Then, as in the first case, the function $x(t)$ which is defined by (1.14) is a positive solution of (1.1) on $[T, \infty)$, and satisfies

$$p(t) P(t)^\alpha \left(\frac{x'(t)}{x(t)} \right)^{\alpha *} = P(t)^\alpha y(t) = w(t) + \lambda_2^\alpha + \varepsilon(t) \to \lambda_2^\alpha$$

as $t \to \infty$. Consequently,

$$\lim_{t \to \infty} p(t)^{1/\alpha} P(t) \frac{x'(t)}{x(t)} = \lambda_2. \tag{3.29}$$

Since $\lambda_2 = \rho_2^{(1/\alpha)*}$, this implies that $x(t)$ belongs to the class n-RV$_P(\rho_2^{(1/\alpha)*})$.

As in the first case, it follows from (1.12) and (3.28) that

$$\begin{aligned} \frac{x'(t)}{x(t)} &= \frac{(w(t) + \lambda_2^\alpha + \varepsilon(t))^{(1/\alpha)*}}{p(t)^{1/\alpha} P(t)} \\ &= \frac{\lambda_2}{p(t)^{1/\alpha} P(t)} + \frac{(w(t) + \lambda_2^\alpha + \varepsilon(t))^{(1/\alpha)*} - \lambda_2}{p(t)^{1/\alpha} P(t)}, \quad t \geq T. \end{aligned} \tag{3.30}$$

Using (3.4) in Lemma 3.2, we see that

$$|(w(t) + \lambda_2^\alpha + \varepsilon(t))^{(1/\alpha)*} - \lambda_2| \leq \frac{2}{\alpha} \lambda_2^{-\alpha+1} \{|w(t)| + |\varepsilon(t)|\}$$

for all large t. By (3.30), the solution $x(t)$ is expressed as

$$x(t) = \frac{x(T_1)}{P(T_1)^{\lambda_2}} \exp \left(\int_{T_1}^t \frac{(w(s) + \lambda_2^\alpha + \varepsilon(s))^{(1/\alpha)*} - \lambda_2}{p(s)^{1/\alpha} P(s)} ds \right) P(t)^{\lambda_2}$$

for $t \geq T_1$, where T_1 is taken sufficiently large.

Note that the assumption (1.18) implies

$$\int_{T_1}^{\infty} \frac{\eta(t;T)}{p(t)^{1/\alpha}P(t)}dt < \infty.$$

Then, by (3.26), we have

$$\int_{T_1}^{\infty} \frac{|w(t)|}{p(t)^{1/\alpha}P(t)}dt < \infty.$$

Therefore, as in the first case, it is seen that $x(t)$ is written in the form

$$x(t) = c_0(t)P(t)^{\lambda_2} \quad \text{with} \quad c_0(t) \to c_0 \in (0,\infty) \;\; \text{as} \;\; t \to \infty. \tag{3.31}$$

Then we have

$$x'(t) = c_1(t)p(t)^{-1/\alpha}P(t)^{\lambda_2 - 1} \quad \text{with} \quad c_1(t) \to c_0\lambda_2 \;\; \text{as} \;\; t \to \infty. \tag{3.32}$$

Without loss of generality, we can suppose that $c_0 = 1$ in (3.31) and (3.32). This shows (1.19) with $i = 2$. The proof of Theorem 1.3 of the case $i = 2$ is complete. □

Acknowledgements

The author would like to express his sincere thanks to the referees for many valuable comments and suggestions.

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Manabu Naito
jpywm078@yahoo.co.jp

Ehime University
Faculty of Science
Department of Mathematics
Matsuyama 790–8577, Japan

Received: July 29, 2020.
Revised: November 13, 2020.
Accepted: December 11, 2020.