

ON THE GAUGE-NATURAL OPERATORS SIMILAR TO THE TWISTED DORFMAN–COURANT BRACKET

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Communicated by P.A. Cojuhari

Abstract. All $\mathcal{VB}_{m,n}$ -gauge-natural operators C sending linear 3-forms $H \in \Gamma_E^l(\wedge^3 T^*E)$ on a smooth (C^∞) vector bundle E into \mathbf{R} -bilinear operators

$$C_H : \Gamma_E^l(T E \oplus T^* E) \times \Gamma_E^l(T E \oplus T^* E) \rightarrow \Gamma_E^l(T E \oplus T^* E)$$

transforming pairs of linear sections of $T E \oplus T^* E \rightarrow E$ into linear sections of $T E \oplus T^* E \rightarrow E$ are completely described. The complete descriptions is given of all generalized twisted Dorfman–Courant brackets C (i.e. C as above such that C_0 is the Dorfman–Courant bracket) satisfying the Jacobi identity for closed linear 3-forms H . An interesting natural characterization of the (usual) twisted Dorfman–Courant bracket is presented.

Keywords: natural operator, linear vector field, linear form, twisted Dorfman–Courant bracket, the Jacobi identity in Leibniz form.

Mathematics Subject Classification: 53A55, 53A45, 53A99.

1. INTRODUCTION

All manifolds considered in the paper are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class C^∞). Maps between manifolds are assumed to be C^∞ .

In [3], the authors completely described bilinear operators on sections of the Whitney sum $TN \oplus T^*N \rightarrow N$ of the tangent and cotangent bundles (for N a smooth manifold), which are $\mathcal{M}f_m$ -natural, i.e. invariant under the morphisms in the category $\mathcal{M}f_m$ of m -dimensional manifolds and their submersions. The Courant bracket, defined in [2], is an example of such operators and it is of particular interest, because it involves in the concepts of Dirac and generalized complex structures on N , see [2, 4, 5]

A simple (but very important) modification of the Courant bracket is the so called twisted (or H -twisted) Courant bracket $[-, -]_H$ on sections of $TN \oplus T^*N \rightarrow N$

for any 3-form H on a smooth manifold N . The properties of $[-, -]_H$ (for closed H) were used in [8, 12] to define the concept of exact Courant algebroid. In [9], we completely described all $\mathcal{M}f_m$ -natural operators which send 3-forms H on N into bilinear operators on sections of $TN \oplus T^*N \rightarrow N$ (for N a smooth manifold).

The restriction of the Courant bracket to linear sections of $TE \oplus T^*E \rightarrow E$ (for $E \rightarrow M$ a smooth vector bundle) is called the Dorfman–Courant bracket, see [6]. It is of particular interest, because $(TE \oplus T^*E; E, TM \oplus E^*; M)$ is the *standard VB-Courant algebroid* and the Dorfman–Courant bracket is the part of this structure. (The Dorfman–Courant bracket can be also interpreted as the bracket of the Omni-Lie algebroid $Der(E^*) \oplus J^1(E^*)$, studied in [1].)

In [10], we completely described all bilinear operators on linear sections of $TE \oplus T^*E \rightarrow E$ (for $E \rightarrow M$ a smooth vector bundle), which are $\mathcal{VB}_{m,n}$ -gauge-natural, i.e. invariant under the morphisms in the category $\mathcal{VB}_{m,n}$ of rank- n vector bundles over m -dimensional bases and their vector bundle isomorphisms onto images. The Dorfman–Courant bracket is an example of such a $\mathcal{VB}_{m,n}$ -gauge natural bilinear operator

$$A : \Gamma_E^l(TE \oplus T^*E) \times \Gamma_E^l(TE \oplus T^*E) \rightarrow \Gamma_E^l(TE \oplus T^*E),$$

where $\Gamma_E^l(TE \oplus T^*E)$ is the space of linear sections of $TE \oplus T^*E \rightarrow E$.

In [11], we completely described all $\mathcal{VB}_{m,n}$ -gauge-natural (i.e. invariant under the morphisms in the category $\mathcal{VB}_{m,n}$) operators

$$C : \Gamma^{l-\text{clos}}(\bigwedge^3 T^*) \rightsquigarrow \text{Lin}_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

which, like the twisted Dorfman–Courant bracket, transform closed linear 3-forms

$$H \in \Gamma_E^{l-\text{clos}}(\bigwedge^3 T^*E)$$

on E into bilinear operators

$$C_H : \Gamma_E^l(TE \oplus T^*E) \times \Gamma_E^l(TE \oplus T^*E) \rightarrow \Gamma_E^l(TE \oplus T^*E)$$

(for E a $\mathcal{VB}_{m,n}$ -object).

In the present paper, we completely describe all $\mathcal{VB}_{m,n}$ -gauge-natural operators of the same type as in [11], but with bigger domain. Namely, we classify all $\mathcal{VB}_{m,n}$ -gauge-natural operators

$$C : \Gamma^l(\bigwedge^3 T^*) \rightsquigarrow \text{Lin}_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

transforming linear 3-forms $H \in \Gamma_E^l(\bigwedge^3 T^*E)$ on E into bilinear operators

$$C_H : \Gamma_E^l(TE \oplus T^*E) \times \Gamma_E^l(TE \oplus T^*E) \rightarrow \Gamma_E^l(TE \oplus T^*E)$$

(for E a $\mathcal{VB}_{m,n}$ -object). Thus, the main result of the paper is the following theorem.

Theorem 1.1. *Let $m \geq 3$ and $n \geq 1$ be fixed integers. Any $\mathcal{VB}_{m,n}$ -gauge-natural operator $C : \Gamma^l(\bigwedge^3 T^*) \rightsquigarrow \text{Lin}_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$ is of the form*

$$C_H(\rho^1, \rho^2) = a[X^1, X^2] \oplus \{b_1 \mathcal{L}_{X^1} \omega^2 + b_2 \mathcal{L}_{X^2} \omega^1 + b_3 di_{X^1} \omega^2 + b_4 di_{X^2} \omega^1 + b_5 \mathcal{L}_{X^1} di_L \omega^2 + b_6 \mathcal{L}_{X^2} di_L \omega^1 + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH + c_3 i_L i_{X^2} di_{X^1} H + c_4 i_L i_{X^1} di_{X^2} H + c_5 i_L di_{X^2} i_{X^1} H\} \quad (1.1)$$

for arbitrary (uniquely determined by C) reals $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5$, where $\rho^i = X^i \oplus \omega^i \in \Gamma_E^l(TE \oplus T^*E)$, $H \in \Gamma_E^l(\bigwedge^3 T^*E)$, and where $[-, -]$ is the usual bracket on vector fields, \mathcal{L} is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.

We have non-trivial operator $0 \oplus i_L i_{X^1} i_{X^2} dH$, which is 0 on closed linear 3-forms H . So, the present paper is an essential extension of [11].

The second result of the paper is the following.

Theorem 1.2. *Let $m \geq 4$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C (i.e. operator C as above such that C_0 is the usual Dorfman–Courant bracket) satisfying the Jacobi identity in Leibniz form for closed linear 3-forms (i.e.*

$$C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3))$$

for all closed linear 3-forms $H \in \Gamma_E^l(\bigwedge^3 T^*E)$ and all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma_E^l(TE \oplus T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$ -objects E) is of the form

$$C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [X^1, X^2] \oplus \{\mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1 + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH\} \quad (1.2)$$

for any (not necessarily closed) linear 3-form $H \in \Gamma_E^l(\bigwedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$ -object E , where c_1, c_2 are arbitrary (uniquely determined by C) real numbers.

From Theorem 1.2, we have the following interesting natural characterization of the (usual) twisted Dorfman–Courant bracket.

Corollary 1.3. *Let $m \geq 4$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C_H satisfying the Jacobi identity in Leibniz form for closed linear 3-forms satisfies*

$$C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_{cH} \quad (1.3)$$

for any closed linear 3-form $H \in \Gamma_E^l(\bigwedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus T^*E)$, where $[[-, -]]_H$ is the usual twisted (H -twisted) Dorfman–Courant bracket and c is an arbitrary (uniquely determined by C) real number.

Roughly speaking, the above corollary means that (for $m \geq 4$) the (usual) twisted Dorfman–Courant bracket $[[-, -]]_H$ (for closed linear 3-forms H) is the unique (up to multiplication of H by a real number c) $\mathcal{VB}_{m,n}$ -gauge-natural extension of the (usual) Dorfman–Courant bracket $[[-, -]]_0$ (by means of closed linear 3-forms H) satisfying the Jacobi identity in Leibniz form.

From now on, let $\mathbf{R}^{m,n}$ be the trivial vector bundle over \mathbf{R}^m with the standard fibre \mathbf{R}^n and let $x^1, \dots, x^m, y^1, \dots, y^n$ be the usual coordinates on $\mathbf{R}^{m,n}$.

2. THE DORFMAN-COURANT LIKE BRACKETS

Let $E = (E \rightarrow M)$ be a vector bundle.

Applying the tangent and the cotangent functors to $E \rightarrow M$, we obtain double vector bundles $(TE; E, TM; M)$ and $(T^*E; E, E^*; M)$.

A vector field X on E is called linear if it is a vector bundle map $X : E \rightarrow TE$ between $E \rightarrow M$ and $TE \rightarrow TM$. Equivalently, a vector field X on E is linear iff it has expression

$$X = \sum_{i=1}^m a^i(x^1, \dots, x^m) \frac{\partial}{\partial x^i} + \sum_{j,k=1}^n b_j^k(x^1, \dots, x^m) y^j \frac{\partial}{\partial y^k}$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E . The Euler vector field L on E is an example of a linear vector field on E . (We recall that the coordinate expression of L is $L = \sum_{j=1}^n y^j \frac{\partial}{\partial y^j}$.) Equivalently, a vector field X on E is linear iff $\mathcal{L}_L X = 0$, where \mathcal{L} denotes the Lie derivative.

A 1-form ω on E is called linear if it is a vector bundle map $\omega : E \rightarrow T^*E$ between $E \rightarrow M$ and $T^*E \rightarrow E^*$. Equivalently, a 1-form ω on E is linear iff it has expression

$$\omega = \sum_{i=1}^m \sum_{j=1}^n a_{ij}(x^1, \dots, x^m) y^j dx^i + \sum_{j=1}^n b_j(x^1, \dots, x^m) dy^j$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E . Equivalently, a 1-form ω on E is linear iff $\mathcal{L}_L \omega = \omega$, where L is the Euler vector field on E .

We have the following definition being respective modification of the general one from the fundamental monograph [7].

Definition 2.1. A $\mathcal{VB}_{m,n}$ -gauge-natural bilinear operator

$$A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$$

is a $\mathcal{VB}_{m,n}$ -invariant family of \mathbf{R} -bilinear operators

$$A : \Gamma_E^l(TE \oplus T^*E) \times \Gamma_E^l(TE \oplus T^*E) \rightarrow \Gamma_E^l(TE \oplus T^*E)$$

for all $\mathcal{VB}_{m,n}$ -objects E , where $\Gamma_E^l(TE \oplus T^*E)$ is the vector space of linear sections of $TE \oplus T^*E$ (couples $X \oplus \omega$ of linear vector fields X and linear 1-forms ω on E).

Remark 2.2. The $\mathcal{VB}_{m,n}$ -invariance of A means that if

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma_E^l(TE \oplus T^*E) \times \Gamma_E^l(TE \oplus T^*E)$$

and

$$(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in \Gamma_{\tilde{E}}^l(T\tilde{E} \oplus T^*\tilde{E}) \times \Gamma_{\tilde{E}}^l(T\tilde{E} \oplus T^*\tilde{E})$$

are φ -related by an $\mathcal{VB}_{m,n}$ -map $\varphi : E \rightarrow \tilde{E}$ (i.e. $\tilde{X}^i \circ \varphi = T\varphi \circ X^i$ and $\tilde{\omega}^i \circ \varphi = T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$.

Remark 2.3. The Dorfman–Courant bracket

$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_0 := [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - i_{X^2}d\omega^1)$$

is an example of a $\mathcal{VB}_{m,n}$ -gauge-natural bilinear operator

$$\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*).$$

Theorem 2.4 ([10]). *Let $m \geq 2$ and $n \geq 1$. Any $\mathcal{VB}_{m,n}$ -gauge-natural bilinear operator*

$$A : \Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*) \rightsquigarrow \Gamma^l(T \oplus T^*)$$

is of the form

$$\begin{aligned} A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = & a[X^1, X^2] \oplus \{b_1\mathcal{L}_{X^1}\omega^2 \\ & + b_2\mathcal{L}_{X^2}\omega^1 + b_3di_{X^1}\omega^2 + b_4di_{X^2}\omega^1 \\ & + b_5\mathcal{L}_{X^1}di_L\omega^2 + b_6\mathcal{L}_{X^2}di_L\omega^1\} \end{aligned} \quad (2.1)$$

for arbitrary (uniquely determined by A) reals $a, b_1, b_2, b_3, b_4, b_5, b_6$, where $[-, -]$ is the usual bracket on vector fields, \mathcal{L} is the Lie derivative, d is the exterior derivative, i is the insertion derivative and L is the Euler vector field.

3. THE TWISTED DORFMAN–COURANT LIKE BRACKETS

A p -form ω on E is called linear if $\mathcal{L}_L\omega = \omega$, where L is the Euler vector field on E . Equivalently, a p -form ω on E is linear iff it has expression

$$\omega = \sum a_{i_1, \dots, i_p, j}(x)y^j dx^{i_1} \wedge \dots \wedge dx^{i_p} + \sum b_{i_1, \dots, i_{p-1}, j}(x)dy^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{p-1}}$$

in any local vector bundle trivialization $x^1, \dots, x^m, y^1, \dots, y^n$ on E .

Definition 3.1. A $\mathcal{VB}_{m,n}$ -gauge-natural operator

$$C : \Gamma^l(\bigwedge^3 T^*) \rightsquigarrow Lin_2(\Gamma^l(T \oplus T^*) \times \Gamma^l(T \oplus T^*), \Gamma^l(T \oplus T^*))$$

sending linear 3-forms $H \in \Gamma^l_E(\bigwedge^3 T^*E)$ on $\mathcal{VB}_{m,n}$ -objects E into \mathbf{R} -bilinear operators

$$C_H : \Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E) \rightarrow \Gamma^l_E(TE \oplus T^*E)$$

is a $\mathcal{VB}_{m,n}$ -invariant family of regular operators (functions)

$$C : \Gamma^l_E(\bigwedge^3 T^*E) \rightarrow Lin_2(\Gamma^l_E(TE \oplus T^*E) \times \Gamma^l_E(TE \oplus T^*E), \Gamma^l_E(TE \oplus T^*E))$$

for all $\mathcal{VB}_{m,n}$ -objects E , where $Lin_2(U \times V, W)$ denotes the vector space of all bilinear (over \mathbf{R}) functions $U \times V \rightarrow W$ for any real vector spaces U, V, W .

Remark 3.2. The invariance of C means that if $H \in \Gamma_E^l(\wedge^3 T^*E)$ and $\tilde{H} \in \Gamma_{\tilde{E}}^l(\wedge^3 T^*\tilde{E})$ are φ -related by a $\mathcal{VB}_{m,n}$ -map $\varphi : E \rightarrow \tilde{E}$, and

$$(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \Gamma_E^l(TE \oplus T^*E) \times \Gamma_E^l(TE \oplus T^*E)$$

and

$$(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2) \in \Gamma_{\tilde{E}}^l(T\tilde{E} \oplus T^*\tilde{E}) \times \Gamma_{\tilde{E}}^l(T\tilde{E} \oplus T^*\tilde{E})$$

are also φ -related, then so are $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $C_{\tilde{H}}(\tilde{X}^1 \oplus \tilde{\omega}^1, \tilde{X}^2 \oplus \tilde{\omega}^2)$. The regularity of C means that C transforms smoothly parametrized families $(H_t, X_t^1 \oplus \omega_t^1, X_t^2 \oplus \omega_t^2)$ into smoothly parametrized families $C_{H_t}(X_t^1 \oplus \omega_t^1, X_t^2 \oplus \omega_t^2)$.

Definition 3.3. A $\mathcal{VB}_{m,n}$ -gauge-natural operator C in the sense of Definition 3.1 is of order 1 if the following implication

$$(j_x^1 H = j_x^1 \tilde{H}, j_x^1 \rho^1 = j_x^1 \tilde{\rho}^1, j_x^1 \rho^2 = j_x^1 \tilde{\rho}^2) \Rightarrow C_H(\rho^1, \rho^2)|_{E_x} = C_{\tilde{H}}(\tilde{\rho}^1, \tilde{\rho}^2)|_{E_x}$$

holds for any $H, \tilde{H} \in \Gamma_E^l(\wedge^3 T^*E)$ and any $\rho^1, \rho^2, \tilde{\rho}^1, \tilde{\rho}^2 \in \Gamma_E^l(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$ -object $E \rightarrow M$ and any $x \in M$.

Remark 3.4. The twisted Dorfman–Courant bracket

$$[[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_H := [X^1, X^2] \oplus \{\mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1 + i_{X^1} i_{X^2} H\} \quad (3.1)$$

is a gauge natural operator (of order 1) in the sense of Definition 3.1.

The main result is the following classification theorem.

Theorem 3.5. *Let C be a $\mathcal{VB}_{m,n}$ -gauge-natural operator in the sense of Definition 3.1. Assume that $m \geq 3$ and $n \geq 1$. Then there exist uniquely determined real numbers $a, b_1, b_2, b_3, b_4, b_5, b_6, c_1, c_2, c_3, c_4, c_5$ such that*

$$\begin{aligned} C_H(\rho^1, \rho^2) = & a[X^1, X^2] \oplus \{b_1 \mathcal{L}_{X^1} \omega^2 + b_2 \mathcal{L}_{X^2} \omega^1 \\ & + b_3 di_{X^1} \omega^2 + b_4 di_{X^2} \omega^1 + b_5 \mathcal{L}_{X^1} di_L \omega^2 \\ & + b_6 \mathcal{L}_{X^2} di_L \omega^1 + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH \\ & + c_3 i_L i_{X^2} di_{X^1} H + c_4 i_L i_{X^1} di_{X^2} H + c_5 i_L di_{X^2} i_{X^1} H\} \end{aligned} \quad (3.2)$$

for any $H \in \Gamma_E^l(\wedge^3 T^*E)$ and any $\rho^1, \rho^2 \in \Gamma_E^l(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$ -object E , where $\rho^1 = X^1 \oplus \omega^1$ and $\rho^2 = X^2 \oplus \omega^2$.

Proof. Operator $A := C_0$, where 0 is the zero linear 3-form, can be treated as the $\mathcal{VB}_{m,n}$ -gauge-natural bilinear operator in the sense of Definition 2.1. Then C_0 is described in Theorem 2.4. So, replacing C by $C - C_0$, we can assume

$$C_0 = 0. \quad (3.3)$$

We will keep this assumption in the rest of this section. The proof of our Theorem 3.5 will be continued after proving several lemmas.

By the $\mathcal{VB}_{m,n}$ -invariance of C , such C is determined by the values

$$C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e \in T_e \mathbf{R}^{m,n} \oplus T_e^* \mathbf{R}^{m,n} \quad (3.4)$$

for all $H \in \Gamma_{\mathbf{R}^{m,n}}^1 (\bigwedge^3 T^* \mathbf{R}^{m,n})$ and all $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_{\mathbf{R}^{m,n}}^1 (T\mathbf{R}^{m,n} \oplus T^* \mathbf{R}^{m,n})$ and all $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n = \mathbf{R}_0^{m,n}$.

Given $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n = \mathbf{R}_0^{m,n}$, let $T_e(\mathbf{R}^m \times \mathbf{R}^n) = \mathbf{R}^m \times \mathbf{R}^n$ and $T_e^*(\mathbf{R}^m \times \mathbf{R}^n) = \mathbf{R}^{m*} \times \mathbf{R}^{n*}$ be the usual identifications. Let

$$\begin{aligned} C_H^{1,1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e &= \text{the } \mathbf{R}^m\text{-part of } C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e, \\ C_H^{1,2}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e &= \text{the } \mathbf{R}^n\text{-part of } C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e, \\ C_H^{2,1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e &= \text{the } \mathbf{R}^{m*}\text{-part of } C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e, \\ C_H^{2,2}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e &= \text{the } \mathbf{R}^{n*}\text{-part of } C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e. \end{aligned} \quad (3.5)$$

We will keep this notion (3.5) in the rest of this section.

Lemma 3.6. *C is of order 1 and $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ is linear in H . Moreover, $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ is independent of both ω^1 and ω^2 .*

Proof. By the invariance of C with respect to $h_t = (\frac{1}{t}x^1, \dots, \frac{1}{t}x^m, y^1, \dots, y^n)$, we have the homogeneity conditions

$$\begin{aligned} t^{k(\mu,\nu)} C_H^{\mu,\nu}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e \\ = C_{(h_t)_* H}^{\mu,\nu}(t(h_t)_* X^1 \oplus t(h_t)_* \omega^1, t(h_t)_* X^2 \oplus t(h_t)_* \omega^2)_e \end{aligned} \quad (3.6)$$

for $\mu, \nu = 1, 2$, where $k(1, 1) = 1$, $k(1, 2) = 2$, $k(2, 1) = 3$, $k(2, 2) = 2$. By Corollary 19.9 of the non-linear Petree theorem in [7], we may assume $H, X^1, X^2, \omega^1, \omega^2$ are polynomial of degree not more than r , where r is an arbitrary finite number. We can write

$$\begin{aligned} (h_t)_* H &= a_2(H)t^2 + \dots + a_{r+3}(H)t^{r+3}, \\ t(h_t)_* X^1 &= b_0(X^1) + \dots + b_{r+3}(X^1)t^{r+3}, \\ t(h_t)_* \omega^1 &= c_1(\omega^1)t + \dots + c_{r+3}(\omega^1)t^{r+3}, \\ t(h_t)_* X^2 &= b_0(X^2) + \dots + b_{r+3}(X^2)t^{r+3}, \\ t(h_t)_* \omega^2 &= c_1(\omega^2)t + \dots + c_{r+3}(\omega^2)t^{r+3}. \end{aligned} \quad (3.7)$$

(The first above expression is because of H is a linear 3-form.) Then the homogeneous function theorem and the homogeneity condition (3.6) and the assumption $C_0 = 0$ complete the first sentence of the lemma. Moreover, they imply that $C_H^{1,1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ and $C_H^{1,2}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ and $C_H^{2,2}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ are independent of ω^1 and ω^2 for any e in question.

It remains to prove that $C_F^{2,1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ is independent of both ω^1 and ω^2 , too. For, it is sufficient to show that $C_H^{2,1}(0 \oplus \omega^1, 0 \oplus \omega^2)_e = 0$ and $C_H^{2,1}(X^1 \oplus 0, 0 \oplus \omega^2)_e = 0$, and $C_H^{2,1}(0 \oplus \omega^1, X^2 \oplus 0)_e = 0$.

For any $\tau \in \mathbf{R}$, we can write

$$C_H^{2,1}(0 \oplus \omega^1, X^2 \oplus 0)_{\tau e} = \sum_{i=1}^m a_i \tau d_{\tau e} x^i,$$

where a_i are the real numbers (depending on ω^1 and X^2 and e and independent of τ). Using the invariance of C with respect to $(x^1, \dots, x^m, \frac{1}{t}y^1, \dots, \frac{1}{t}y^n)$ (preserving X^2 (as X^2 is linear) and sending H into tH (as H is linear) and ω^1 into $t\omega^1$ (as ω^1 is linear) and τe into $\frac{1}{t}\tau e$) and that $C_H(-, -)$ is linear in H , we get

$$t^2 \sum_{i=1}^m a_i \frac{1}{t} \tau d_{\frac{1}{t}\tau e} x^i = \sum_{i=1}^m a_i \tau d_{\frac{1}{t}\tau e} x^i.$$

Then $ta_i = a_i$, and then $a_i = 0$ for $i = 1, \dots, m$. Then $C_H^{2,1}(0 \oplus \omega^1, X^2 \oplus 0)_e = 0$, as well. The proofs of the two other equalities are quite similar.

The lemma is complete. \square

Lemma 3.7. *The vector field part of $C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ is zero.*

Proof. Let $H \in \Gamma_{\mathbf{R}^{m,n}}^l(\wedge^3 T^* \mathbf{R}^{m,n})$ and $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_{\mathbf{R}^{m,n}}^l(T \mathbf{R}^{m,n} \oplus T^* \mathbf{R}^{m,n})$ and $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n = \mathbf{R}_0^{m,n}$. By the homogeneity condition from the proof of Lemma 3.6 and the homogeneous function theorem, we derive that $C_H^{1,1}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e$ is independent of H , and then it is zero because of the assumption $C_0 = 0$. Further, for any $\tau \in \mathbf{R}$, we can write

$$C_H^{1,2}(X^1 \oplus 0, X^2 \oplus 0)_{\tau e} = \sum_{k=1}^n a^k \tau \frac{\partial}{\partial y^k} \Big|_{\tau e},$$

where a^k are real numbers (depending on X^1 and X^2 and e and independent of τ). Then, using the invariance of C with respect to $\tilde{h}_t = (x^1, \dots, x^m, \frac{1}{t}y^1, \dots, \frac{1}{t}y^n)$ (preserving X^1 and X^2 as they are linear, sending H into tH as it is linear, and sending $\frac{\partial}{\partial y^k} \Big|_{\tau e}$ into $\frac{1}{t} \frac{\partial}{\partial y^k} \Big|_{\frac{1}{t}\tau e}$, and sending τe into $\frac{1}{t}\tau e$), since C_H is linear in H , we get

$$t \sum_{k=1}^n a^k \frac{1}{t} \tau \frac{\partial}{\partial y^k} \Big|_{\frac{1}{t}\tau e} = \sum_{k=1}^n a^k \tau \frac{1}{t} \frac{\partial}{\partial y^k} \Big|_{\frac{1}{t}\tau e}.$$

Then $a^k = 0$ for $k = 1, \dots, n$. Then, applying Lemma 3.6, we get

$$C_H^{1,2}(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_e = C_H^{1,2}(X^1 \oplus 0, X^2 \oplus 0)_e = 0.$$

The lemma is complete. \square

Lemma 3.8. *Under the assumption $m \geq 3$, C is determined by the collection*

$$\begin{aligned}
 & C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1} \left(\frac{\partial}{\partial x^i} \oplus 0, \frac{\partial}{\partial x^{i_1}} \oplus 0 \right)_{e_1}, \\
 & C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^i} \oplus 0, y^k \frac{\partial}{\partial y^{k_1}} \oplus 0 \right)_{e_1}, \\
 & C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(y^k \frac{\partial}{\partial y^{k_1}} \oplus 0, \frac{\partial}{\partial x^i} \oplus 0 \right)_{e_1}, \\
 & C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^i} \oplus 0, x^3 \frac{\partial}{\partial x^{i_1}} \oplus 0 \right)_{e_1}, \\
 & C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(x^3 \frac{\partial}{\partial x^{i_1}} \oplus 0, \frac{\partial}{\partial x^i} \oplus 0 \right)_{e_1}, \\
 & C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2} \left(\frac{\partial}{\partial x^i} \oplus 0, \frac{\partial}{\partial x^{i_1}} \oplus 0 \right)_{e_1}
 \end{aligned} \tag{3.8}$$

for all $i, i_1 = 1, \dots, m$ and $k, k_1 = 1, \dots, n$, where $e_1 = (1, 0, \dots, 0) \in \mathbf{R}^n = \mathbf{R}_0^{m,n}$.

Proof. By Lemmas 3.6 and 3.7 (and their proofs), C is determined by the collection

$$\begin{aligned}
 & C_{\varphi(y)df^1(x) \wedge df^2(x) \wedge \overline{df^3(x)}}(X^1 \oplus 0, X^2 \oplus 0)_e, \\
 & C_{f^3(x)d\varphi(y) \wedge df^1(x) \wedge df^2(x)}(X^1 \oplus 0, X^2 \oplus 0)_e, \\
 & C_{d\varphi(y) \wedge dg^1(x) \wedge dg^2(x)}(X^1 \oplus 0, X^2 \oplus 0)_e
 \end{aligned} \tag{3.9}$$

for all $X^1, X^2 \in \Gamma_{\mathbf{R}^{m,n}}^l(T\mathbf{R}^{m,n})$ and all $e \in \mathbf{R}^n = \{0\} \times \mathbf{R}^n \subset \mathbf{R}^{m,n}$ and all maps

$$f^1, f^2, f^3, g^1, g^2 : \mathbf{R}^m \rightarrow \mathbf{R}$$

with

$$f^1(0) = f^2(0) = f^3(0) = g^1(0) = g^2(0) = 0$$

and all linear maps $\varphi : \mathbf{R}^m \rightarrow \mathbf{R}$. Of course, we can assume $\varphi(e) = 1$ and the rank of $(d_0 f^1, d_0 f^2, d_0 f^3)$ is maximal and the rank of $(d_0 g^1, d_0 g^2)$ is maximal. Then, using the $\mathcal{VB}_{m,n}$ -invariance of C , we can assume $e = e_1$, $\varphi = y^1$, $f^1 = x^1$, $f^2 = x^2$, $f^3 = x^3$ (we use $m \geq 3$) and $g^1 = x^1$ and $g^2 = x^2$. Further, using the invariance of C with respect to $(x^1, \dots, x^m, y^1 + x^3 y^1, y^1, \dots, y^n)^{-1}$, we can see that the values $C_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$ determine the values $C_{d(y^1 + x^3 y^1) \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. Then the values $C_{dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$ together with the values $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$ determine the values $C_{x^3 dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. So, the values $C_{x^3 dy^1 \wedge dx^1 \wedge dx^2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$ may be omitted. So, C is determined by the collection of values

$$\begin{aligned}
 & C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}, \\
 & C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}, \\
 & C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}, \\
 & C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}
 \end{aligned} \tag{3.10}$$

for all $\alpha, \beta, \gamma, \delta \in (\mathbf{N} \cup \{0\})^m$ and $i, i_1 = 1, \dots, m$ and $j, k, j_1, k_1 = 1, \dots, n$, where $(X^1 = x^\alpha \frac{\partial}{\partial x^i}$ or $X^1 = x^\beta y^j \frac{\partial}{\partial y^k}$) and $(X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$ or $X^2 = x^\delta y^{j_1} \frac{\partial}{\partial y^{k_1}}$), where (of course) $x^\alpha := (x^1)^{\alpha_1} \dots (x^m)^{\alpha_m}$. We are going to study this collection (3.10).

(i) We start with $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. If $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$ then by invariance of C with respect to $h_t = (\frac{1}{t}x^1, \dots, \frac{1}{t}x^m, y^1, \dots, y^n)$, we get

$$t^{3+|\alpha|+|\gamma|-2} C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = t C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$$

and then

$$C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\alpha| + |\gamma| \neq 0$. Quite similarly,

$$C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

in the rest three sub-cases.

(ii) Now, we pass to $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. If $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$ then by invariance of C with respect to h_t (as above), we get

$$t^{3+|\alpha|+|\gamma|-2} C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1},$$

and then we have

$$C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0.$$

Quite similarly, we get

$$C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

in the rest three sub-cases.

(iii) Now, we study $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. If $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$ then by the invariance of C with respect to h_t , we get

$$t^{2+|\alpha|+|\gamma|-2} C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = t C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1},$$

and then

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\alpha| + |\gamma| \neq 1$. Similarly, if $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\delta y^{j_1} \frac{\partial}{\partial y^{k_1}}$ then

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\alpha| + |\delta| \neq 0$. Similarly, if $X^1 = x^\beta y^j \frac{\partial}{\partial y^k}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i_1}}$, then

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\beta| + |\gamma| \neq 0$. Similarly,

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

in the rest sub-case. Further, we can see that the values $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^i} \oplus 0, x^{i2} \frac{\partial}{\partial x^{i1}} \oplus 0)_{e_1}$ are determined by the values $C_{dy^1 \wedge df \wedge dg}^{2,1}(X^1 \oplus 0, hX^2 \oplus 0)_{e_1}$ for all “constant” vector fields X^1 and X^2 on \mathbf{R}^m (treated as linear vector fields on $\mathbf{R}^{m,n}$) and all linear maps $f, g, h : \mathbf{R}^m \rightarrow \mathbf{R}$. Then (of course) we can assume that f, g, h are linearly independent (we use $m \geq 3$). Then, using the invariance of C with respect to $(\varphi(x^1, \dots, x^m), y^1, \dots, y^n)$ for a linear isomorphism $\varphi : \mathbf{R}^m \rightarrow \mathbf{R}^m$, we can assume $f = x^1, g = x^2$ and $h = x^3$. Because of the bi-linearity of C_H , we can also assume that $X^1 = \frac{\partial}{\partial x^i}$ and $X^2 = \frac{\partial}{\partial x^{i1}}$. Quite similarly, one can proceed with $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(x^{i2} \frac{\partial}{\partial x^{i1}} \oplus 0, \frac{\partial}{\partial x^i} \oplus 0)_{e_1}$ instead of $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^i} \oplus 0, x^{i2} \frac{\partial}{\partial x^{i1}} \oplus 0)_{e_1}$.

(iv) Finally, we study $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1}$. If $X^1 = x^\alpha \frac{\partial}{\partial x^i}$ and $X^2 = x^\gamma \frac{\partial}{\partial x^{i1}}$ then by the invariance of C with respect to h_t we get

$$t^{2+|\alpha|+|\gamma|-2} C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1},$$

and then

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$$

if $|\alpha| + |\gamma| \neq 0$. Quite similarly, $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(X^1 \oplus 0, X^2 \oplus 0)_{e_1} = 0$ in the rest three sub-cases.

The lemma is complete. □

Lemma 3.9. *All values $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(\frac{\partial}{\partial x^i} \oplus 0, \frac{\partial}{\partial x^{i1}} \oplus 0)_{e_1}$ are zero except (eventually) of $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned} C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}\left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} &= \tilde{a} d_{e_1} y^1, \\ C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}\left(\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0\right)_{e_1} &= -\tilde{a} d_{e_1} y^1, \end{aligned} \tag{3.11}$$

where \tilde{a} is the real number (determined by the operator C).

Proof. We can write

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}\left(\frac{\partial}{\partial x^i} \oplus 0, \frac{\partial}{\partial x^{i1}} \oplus 0\right)_{e_1} = \sum_{k=1}^n a_{ii_1 k} d_{e_1} y^k,$$

where $a_{ii_1 k} \in \mathbf{R}$. Then by invariance of C with respect to $(\frac{1}{\tau^1} x^1, \dots, \frac{1}{\tau^m} x^m, y^1, \dots, y^n)$ we get $\tau^1 \tau^2 \frac{1}{\tau^i} \frac{1}{\tau^{i1}} a_{ii_1 k} = a_{ii_1 k}$. Then $a_{ii_1 k} = 0$ if $\{i, i_1\} \neq \{1, 2\}$. Further, by the invariance of C with respect to $(x^1, \dots, x^m, y^1, \frac{1}{t} y^2, \dots, \frac{1}{t} y^n)$ we get $a_{12k} = t a_{12k}$ for $k = 2, \dots, n$. Then $a_{12k} = 0$ for $k = 2, \dots, n$. (If $n = 1$, this is trivial.) Further, by the invariance of C with respect to $(x^2, x^1, x^3, \dots, x^m, y^1, \dots, y^n)$ (replacing x^1 by x^2 and vice-versa) we get $a_{12k} = -a_{21k}$ for $k = 1, \dots, n$. Summing up,

all values $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(\frac{\partial}{\partial x^i} \oplus 0, \frac{\partial}{\partial x^{i_1}} \oplus 0)_{e_1}$ are zero except (eventually) of $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}(\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$. Moreover,

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}\left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} = -C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,2}\left(\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0\right)_{e_1} = a_{121} d_{e_1} y^1.$$

The lemma is complete. \square

Lemma 3.10. *Let $m \geq 3$. All values $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(\frac{\partial}{\partial x^i} \oplus 0, \frac{\partial}{\partial x^{i_1}} \oplus 0)_{e_1}$ are equal to zero except (eventually) of $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ and $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(\frac{\partial}{\partial x^3} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ and $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0)_{e_1}$ and $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0)_{e_1}$ and $C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}(\frac{\partial}{\partial x^3} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned} C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}\left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} &= \tilde{b} d_{e_1} x^3, \\ C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}\left(\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0\right)_{e_1} &= -\tilde{b} d_{e_1} x^3, \\ C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}\left(\frac{\partial}{\partial x^3} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0\right)_{e_1} &= \tilde{b} d_{e_1} x^2, \\ C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}\left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0\right)_{e_1} &= -\tilde{b} d_{e_1} x^2, \\ C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}\left(\frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0\right)_{e_1} &= \tilde{b} d_{e_1} x^1, \\ C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}\left(\frac{\partial}{\partial x^3} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} &= -\tilde{b} d_{e_1} x^1, \end{aligned} \tag{3.12}$$

where \tilde{b} is the real number (determined by the operator C).

Proof. We can write

$$C_{y^1 dx^1 \wedge dx^2 \wedge dx^3}^{2,1}\left(\frac{\partial}{\partial x^i} \oplus 0, \frac{\partial}{\partial x^{i_1}} \oplus 0\right)_{e_1} = \sum_{j=1}^m b_{ii_1 j} d_{e_1} x^j,$$

where $b_{ii_1 j} \in \mathbf{R}$. Then by the invariance of C with respect to base homotheties $(\frac{1}{\tau} x^1, \dots, \frac{1}{\tau^m} x^m, y^1, \dots, y^n)$, we get $\tau^1 \tau^2 \tau^3 \frac{1}{\tau^i} \frac{1}{\tau^{i_1}} b_{ii_1 j} = \tau^j b_{ii_1 j}$. Then $b_{ii_1 j} = 0$ if $\{i, i_1, j\} \neq \{1, 2, 3\}$. Further, applying the invariance of C with respect to the permutations of x^1, x^2, x^3 we easily see that $b_{123} = -b_{213} = b_{312} = -b_{132} = -b_{321} = b_{231}$. The lemma is complete. \square

Lemma 3.11. *All values $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^i} \oplus 0, y^k \frac{\partial}{\partial y^{k_1}} \oplus 0)_{e_1}$ are equal to zero except (eventually) of $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^1} \oplus 0, y^1 \frac{\partial}{\partial y^1} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^2} \oplus 0, y^1 \frac{\partial}{\partial y^1} \oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned} C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}\left(\frac{\partial}{\partial x^1} \oplus 0, y^1 \frac{\partial}{\partial y^1} \oplus 0\right)_{e_1} &= \tilde{c} d_{e_1} x^2 \\ C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}\left(\frac{\partial}{\partial x^2} \oplus 0, y^1 \frac{\partial}{\partial y^1} \oplus 0\right)_{e_1} &= -\tilde{c} d_{e_1} x^1, \end{aligned} \tag{3.13}$$

where \tilde{c} is the real number (determined by the operator C).

Proof. We can write

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^i} \oplus 0, y^k \frac{\partial}{\partial y^{k_1}} \oplus 0 \right)_{e_1} = \sum_{j=1}^m c_{ikk_1j} d_{e_1} x^j,$$

where c_{ikk_1j} are the real numbers. Then by the invariance of C with respect to the base homotheties $(\frac{1}{\tau^1}x^1, \dots, \frac{1}{\tau^m}x^m, y^1, \dots, y^n)$ we get $\tau^1\tau^2\frac{1}{\tau^i}c_{ikk_1j} = \tau^j c_{ikk_1j}$. Then $c_{ikk_1j} = 0$ if $\{i, j\} \neq \{1, 2\}$. Further, by the invariance of C with respect to replacing x^1 by x^2 (and vice-versa) we get $c_{1kk_12} = -c_{2kk_11}$. Further, by invariance of C with respect to $(x^1, \dots, x^m, \frac{1}{\tau^1}y^1, \frac{1}{\tau^2}y^2, \dots, \frac{1}{\tau^n}y^n)$ with $\tau^1 = 1$, we get $\tau^k \frac{1}{\tau^{k_1}} c_{1kk_12} = c_{1kk_12}$. Then $c_{1kk_12} = 0$ if $k \neq k_1$. Further, if $k \in \{2, \dots, n\}$, there exists a \mathcal{VB}_m -map $\psi = (x^1, \dots, x^m, y^1, \tilde{\psi}(x^2, \dots, x^m, y^2, \dots, y^n))$ sending $\frac{\partial}{\partial x^2}$ into $\frac{\partial}{\partial x^2} + y^k \frac{\partial}{\partial y^k}$. Then, using the invariance of C with respect to ψ , from

$$C_{dy^1 \wedge dx^1 \wedge dx^2} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = 0 \oplus \tilde{a} d_{e_1} y^1,$$

where \tilde{a} is from Lemma 3.9, we get

$$C_{dy^1 \wedge dx^1 \wedge dx^2} \left(\frac{\partial}{\partial x^1} \oplus 0, \left(\frac{\partial}{\partial x^2} + y^k \frac{\partial}{\partial y^k} \right) \oplus 0 \right)_{e_1} = 0 \oplus \tilde{a} d_{e_1} y^1.$$

(That $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1} = 0$, see the proof of Lemma 3.8.) Then

$$C_{dy^1 \wedge dx^1 \wedge dx^2} \left(\frac{\partial}{\partial x^1} \oplus 0, y^k \frac{\partial}{\partial y^k} \oplus 0 \right)_{e_1} = 0 \oplus 0$$

for $k = 2, \dots, n$. The lemma is complete. □

Lemma 3.12. *All values $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(y^k \frac{\partial}{\partial y^{k_1}} \oplus 0, \frac{\partial}{\partial x^i} \oplus 0)_{e_1}$ are zero except (eventually) of $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(y^1 \frac{\partial}{\partial y^1} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(y^1 \frac{\partial}{\partial y^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned} C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(y^1 \frac{\partial}{\partial y^1} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0 \right)_{e_1} &= \tilde{e} d_{e_1} x^2 \\ C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(y^1 \frac{\partial}{\partial y^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} &= -\tilde{e} d_{e_1} x^1, \end{aligned} \tag{3.14}$$

where \tilde{e} is the real number (determined by C).

Proof. The proof is quite the same as the one of Lemma 3.11. In fact, this lemma is Lemma 3.11 for C^{op} instead of C , where

$$C_H^{\text{op}}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) := C_H(X^2 \oplus \omega^2, X^1 \oplus \omega^1). \tag{□}$$

Lemma 3.13. *Let $m \geq 3$. All values $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^i} \oplus 0, x^3 \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ are equal to zero except (eventually) of $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^1} \oplus 0, x^3 \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^2} \oplus 0, x^3 \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^3} \oplus 0, x^3 \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(\frac{\partial}{\partial x^3} \oplus 0, x^3 \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned} C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^1} \oplus 0, x^3 \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} &= \tilde{f} d_{e_1} x^3, \\ C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^2} \oplus 0, x^3 \frac{\partial}{\partial x^1} \oplus 0 \right)_{e_1} &= -\tilde{f} d_{e_1} x^3, \\ C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^3} \oplus 0, x^3 \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} &= \tilde{g} d_{e_1} x^1, \\ C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^3} \oplus 0, x^3 \frac{\partial}{\partial x^1} \oplus 0 \right)_{e_1} &= -\tilde{g} d_{e_1} x^2, \end{aligned} \tag{3.15}$$

where \tilde{f} and \tilde{g} are the real numbers (determined by the operator C).

Proof. We can write

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^i} \oplus 0, x^3 \frac{\partial}{\partial x^{i_1}} \oplus 0 \right)_{e_1} = \sum_{j=1}^m q_{ii_1j} d_{e_1} x^j,$$

where $q_{ii_1j} \in \mathbf{R}$ are the numbers. Then by the invariance of C with respect to $(\frac{1}{\tau^1} x^1, \dots, \frac{1}{\tau^m} x^m, y^1, \dots, y^n)$ we get $\tau^1 \tau^2 \tau^3 \frac{1}{\tau^i} \frac{1}{\tau^{i_1}} q_{ii_1j} = \tau^j q_{ii_1j}$. Then $q_{ii_1j} = 0$ if $\{i, i_1, j\} \neq \{1, 2, 3\}$. Further, there exists a 0-preserving embedding $\varphi : \mathbf{R} \rightarrow \mathbf{R}$ sending (the germ at 0 of) $\frac{\partial}{\partial x}$ into $\frac{\partial}{\partial x} + x \frac{\partial}{\partial x}$. Then, by the invariance of C with respect to $(x^1, x^2, \varphi(x^3), \dots, x^m, y^1, \dots, y^n)$, from

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0 \right)_{e_1} = 0$$

we get

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^1} \oplus 0, \left(\frac{\partial}{\partial x^3} + x^3 \frac{\partial}{\partial x^3} \right) \oplus 0 \right)_{e_1} = 0,$$

and then

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^1} \oplus 0, x^3 \frac{\partial}{\partial x^3} \oplus 0 \right)_{e_1} = 0,$$

i.e. $q_{132} = 0$. Then using the invariance of C with respect to changing x^1 by x^2 (and vice-versa) we get that $q_{231} = -q_{132} = 0$ and $q_{321} = -q_{312}$ and $q_{123} = -q_{213}$. We put $\tilde{f} := q_{123}$ and $\tilde{g} := q_{321}$. The lemma is complete. \square

Lemma 3.14. *Let $m \geq 3$. All values $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(x^3 \frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ are equal to zero except (eventually) of $B_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(x^3 \frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(x^3 \frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(x^3 \frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0)_{e_1}$ and $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}(x^3 \frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0)_{e_1}$. Moreover, we have*

$$\begin{aligned}
 C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}\left(x^3 \frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0\right)_{e_1} &= \tilde{h}d_{e_1}x^3, \\
 C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}\left(x^3 \frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} &= -\tilde{h}d_{e_1}x^3, \\
 C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}\left(x^3 \frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0\right)_{e_1} &= \tilde{k}d_{e_1}x^1, \\
 C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}\left(x^3 \frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^3} \oplus 0\right)_{e_1} &= -\tilde{k}d_{e_1}x^2,
 \end{aligned}
 \tag{3.16}$$

where \tilde{h} and \tilde{k} are the real numbers (determined by the operator C).

Proof. The proof is almost the same as the one of Lemma 3.13. In fact, this lemma is Lemma 3.13 for C^{op} instead of C . □

Lemma 3.15. *Let $m \geq 3$. We have*

$$\tilde{f} = \tilde{a} + \tilde{c}
 \tag{3.17}$$

where \tilde{a} is the real number from Lemma 3.9 and \tilde{c} is the real number from Lemma 3.11 and \tilde{f} is the number from Lemma 3.13.

Proof. Given a positive number τ , $\psi_\tau := (x^1, \frac{x^2}{1+\tau x^3}, x^3, \dots, x^m, y^1, \dots, y^n)$ preserves e_1 and $\frac{\partial}{\partial x^1}$ and sends $dy^1 \wedge dx^1 \wedge dx^2$ into $dy^1 \wedge dx^1 \wedge d(x^2 + \tau x^2 x^3)$ and $\frac{\partial}{\partial x^2}$ into $\frac{1}{1+\tau x^3} \frac{\partial}{\partial x^2}$. Moreover, by the invariance of C with respect to the base homotheties $(\frac{1}{t}x^1, \dots, \frac{1}{t}x^m, y^1, \dots, y^n)$, we can easily see that

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}\left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} = 0.$$

Then, by the invariance of C with respect to ψ_τ , we get

$$C_{dy^1 \wedge dx^1 \wedge d(x^2 + \tau x^2 x^3)}^{2,1}\left(\frac{\partial}{\partial x^1} \oplus 0, \frac{1}{1 + \tau x^3} \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} = 0.$$

Then by the order argument, we get

$$C_{dy^1 \wedge dx^1 \wedge d(x^2 + \tau x^2 x^3)}^{2,1}\left(\frac{\partial}{\partial x^1} \oplus 0, (1 - \tau x^3) \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} = 0.$$

Then, comparing the coefficients on τ of both sides of this equality, we easily get

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1}\left(\frac{\partial}{\partial x^1} \oplus 0, x^3 \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1} = C_{dy^1 \wedge dx^1 \wedge d(x^2 x^3)}^{2,1}\left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0\right)_{e_1}.
 \tag{3.18}$$

Further, by the invariance of C with respect to $(x^1, \dots, x^m, \frac{1}{1+x^3}y^1, y^2, \dots, y^n)$, from

$$C_{dy^1 \wedge dx^1 \wedge dx^2} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = 0 \oplus \tilde{a} d_{e_1} y^1,$$

we get

$$C_{d(y^1+x^3y^1) \wedge dx^1 \wedge dx^2} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = 0 \oplus (\tilde{a} d_{e_1} y^1 + \tilde{a} d_{e_1} x^3),$$

and then

$$C_{d(x^3y^1) \wedge dx^1 \wedge dx^2} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = \tilde{a} d_{e_1} x^3, \quad (3.19)$$

where \tilde{a} is the number from Lemma 3.9.

Further, by the invariance of C with respect to $(x^1, \frac{1}{t}x^2, x^3, \dots, x^m, y^1, \dots, y^n)$, we can easily see that

$$C_{dy^1 \wedge dx^1 \wedge dx^3} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = 0 \oplus 0.$$

Then, by the invariance of C with respect to $(x^1, \dots, x^m, \frac{1}{1+\tau x^2}y^1, y^2, \dots, y^n)$, we get

$$C_{d(y^1+\tau x^2y^1) \wedge dx^1 \wedge dx^3} \left(\frac{\partial}{\partial x^1} \oplus 0, \left(\frac{\partial}{\partial x^2} - \frac{\tau}{1+\tau x^2} y^1 \frac{\partial}{\partial y^1} \right) \oplus 0 \right)_{e_1} = 0 \oplus 0,$$

and then (by the order argument and comparing the coefficients on τ) we get

$$C_{d(x^2y^1) \wedge dx^1 \wedge dx^3} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = C_{dy^1 \wedge dx^1 \wedge dx^3} \left(\frac{\partial}{\partial x^1} \oplus 0, y^1 \frac{\partial}{\partial y^1} \oplus 0 \right)_{e_1}.$$

Further, by the invariance of C with respect to $(x^1, x^2 + x^3, x^3, \dots, x^m, y^1, \dots, y^n)$, from the first equality of (3.13), we get

$$C_{dy^1 \wedge dx^1 \wedge (dx^2 - dx^3)} \left(\frac{\partial}{\partial x^1} \oplus 0, y^1 \frac{\partial}{\partial y^1} \oplus 0 \right)_{e_1} = \tilde{c} d_{e_1} (x^2 - x^3),$$

and then

$$C_{dy^1 \wedge dx^1 \wedge dx^3} \left(\frac{\partial}{\partial x^1} \oplus 0, y^1 \frac{\partial}{\partial y^1} \oplus 0 \right)_{e_1} = \tilde{c} d_{e_1} x^3,$$

where \tilde{c} is the number from Lemma 3.11. Then

$$C_{d(x^2y^1) \wedge dx^1 \wedge dx^3} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = \tilde{c} d_{e_1} x^3. \quad (3.20)$$

Now, from (3.15), (3.18), (3.19) and (3.20), since

$$d(x^3y^1) \wedge dx^1 \wedge dx^2 + d(x^2y^1) \wedge dx^1 \wedge dx^3 = dy^1 \wedge dx^1 \wedge d(x^2x^3),$$

we get

$$\tilde{f} d_{e_1} x^3 = C_{dy^1 \wedge dx^1 \wedge dx^2} \left(\frac{\partial}{\partial x^1} \oplus 0, x^3 \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = (\tilde{a} + \tilde{c}) d_{e_1} x^3,$$

as well. The lemma is complete. \square

Lemma 3.16. *Let $m \geq 3$. We have*

$$\tilde{h} = -\tilde{a} + \tilde{e}, \tag{3.21}$$

where \tilde{a} is the real number from Lemma 3.9 and \tilde{e} is the real number from Lemma 3.12 and \tilde{h} is the number from Lemma 3.14.

Proof. In fact, this lemma is Lemma 3.15 for C^{op} instead of C . So, the proof is almost the same as the one of Lemma 3.15. \square

Lemma 3.17. *Let $m \geq 3$. We have*

$$\tilde{f} + \tilde{g} + \tilde{k} + \tilde{h} = 0, \tag{3.22}$$

where \tilde{f} and \tilde{g} are the numbers from Lemma 3.13 and \tilde{h} and \tilde{k} are the numbers from Lemma 3.14.

Proof. By the invariance of C with respect to $(x^1 + \tau x^3, x^2, \dots, x^m, y^1, \dots, y^n)$, from the third equality of (3.15) we get

$$C_{dy^1 \wedge d(x^1 - \tau x^3) \wedge dx^2}^{2,1} \left(\left(\frac{\partial}{\partial x^3} + \tau \frac{\partial}{\partial x^1} \right) \oplus 0, x^3 \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = \tilde{g} d_{e_1}(x^1 - \tau x^3),$$

and then considering the coefficients on τ and using the first equation of (3.15) we obtain

$$-C_{dy^1 \wedge dx^3 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^3} \oplus 0, x^3 \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} + \tilde{f} d_{e_1} x^3 = -\tilde{g} d_{e_1} x^3.$$

Then using (in particular) the invariance of C with replacing x^3 by x^1 (and vice-versa) we get

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^1} \oplus 0, x^1 \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} = (\tilde{g} + \tilde{f}) d_{e_1} x^1. \tag{3.23}$$

Quite similarly, using (3.16) instead of (3.15) we get

$$C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(x^1 \frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0 \right)_{e_1} = (\tilde{k} + \tilde{h}) d_{e_1} x^1. \tag{3.24}$$

(In fact, the equality (3.24) is the equality (3.23) for C^{op} instead of C .)

Next, by invariance of C with respect to $(x^1, x^2 + \tau(x^1)^2, x^3, \dots, x^m, y^1, \dots, y^n)$, from $C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^1} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0 \right)_{e_1} = 0$, we get

$$C_{dy^1 \wedge dx^1 \wedge d(x^2 - \tau(x^1)^2)}^{2,1} \left(\left(\frac{\partial}{\partial x^1} + 2\tau x^1 \frac{\partial}{\partial x^2} \right) \oplus 0, \left(\frac{\partial}{\partial x^1} + 2\tau x^1 \frac{\partial}{\partial x^2} \right) \oplus 0 \right)_{e_1} = 0,$$

and then considering the coefficients on τ we get

$$\begin{aligned} & C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(\frac{\partial}{\partial x^1} \oplus 0, x^1 \frac{\partial}{\partial x^2} \oplus 0 \right)_{e_1} \\ & + C_{dy^1 \wedge dx^1 \wedge dx^2}^{2,1} \left(x^1 \frac{\partial}{\partial x^2} \oplus 0, \frac{\partial}{\partial x^1} \oplus 0 \right)_{e_1} = 0. \end{aligned} \tag{3.25}$$

From (3.25), (3.24) and (3.23) we obtain (3.22), as well. The lemma is complete. \square

We continue the proof of Theorem 3.5. By Lemmas 3.6–3.17, any $\mathcal{VB}_{m,n}$ -gauge-natural operator C with $C_0 = 0$ is uniquely determined by the corresponding 5-tuple $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{g}, \tilde{k})$. Further, one can easily directly compute the corresponding 5-tuples of $\mathcal{VB}_{m,n}$ -gauge natural operators $0 \oplus i_{X^1} i_{X^2} H$ and $0 \oplus i_L i_{X^1} i_{X^2} dH$ and $0 \oplus i_L i_{X^2} di_{X^1} H$ and $0 \oplus i_L i_{X^1} di_{X^2} H$ and $0 \oplus i_L di_{X^2} i_{X^1} H$. They are equal to $(-1, -1, 1, 0, 0)$ and $(0, -1, 0, 0, 0)$ and $(0, -1, 0, 0, 1)$ and $(0, 1, -1, 1, 0)$ and $(0, 1, -1, 0, 0)$, respectively. The determinant of the matrix of the above vectors is 1. So, the dimension argument complete Theorem 3.5. \square

4. THE GENERALIZED TWISTED DORFMAN–COURANT BRACKETS WITH THE JACOBI IDENTITY IN LEIBNIZ FORM

Definition 4.1. Let C be a $\mathcal{VB}_{m,n}$ -gauge-natural operator in the sense of Definition 3.1. We say that C is a generalized twisted Dorfman–Courant bracket if C_0 is the (usual) Dorfman–Courant bracket.

Corollary 4.2. Let $m \geq 3$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C is of the form

$$\begin{aligned}
 C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = & [X^1, X^2] \oplus \{ \mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1 \\
 & + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH + c_3 i_L i_{X^2} di_{X^1} H \quad (4.1) \\
 & + c_4 i_L i_{X^1} di_{X^2} H + c_5 i_L di_{X^2} i_{X^1} H \}
 \end{aligned}$$

for any $H \in \Gamma_E^l(\wedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$ -object E , where c_1, c_2, c_3, c_4, c_5 are (uniquely determined by C) real numbers.

Proof. It is a immediate consequence of Theorem 3.5. \square

Definition 4.3. We say that a generalized twisted Dorfman–Courant bracket C satisfies the Jacobi identity in Leibniz form for closed linear 3-forms if

$$C_H(\rho^1, C_H(\rho^2, \rho^3)) = C_H(C_H(\rho^1, \rho^2), \rho^3) + C_H(\rho^2, C_H(\rho^1, \rho^3)) \quad (4.2)$$

for all closed linear 3-forms $H \in \Gamma_E^l(\wedge^3 T^*E)$ and all linear sections $\rho^i = X^i \oplus \omega^i \in \Gamma_E^l(TE \oplus T^*E)$ for $i = 1, 2, 3$ and all $\mathcal{VB}_{m,n}$ -objects E .

Remark 4.4. It is well-known that the twisted Dorfman–Courant bracket (i.e. the generalized one satisfying (4.1) with $(c_1, c_2, c_3, c_4, c_5) = (1, 0, 0, 0, 0)$) satisfies the Jacobi identity in Leibniz form for closed linear 3-forms.

Lemma 4.5. *Let C be a generalized twisted Dorfman–Courant bracket of the form (4.1). If C satisfies the Jacobi identity in Leibniz form for closed linear 3-forms, then*

$$\begin{aligned}
& c_1 \mathcal{L}_{X^1} i_{X^2} i_{X^3} H + c_3 \mathcal{L}_{X^1} i_L i_{X^3} di_{X^2} H \\
& + c_4 \mathcal{L}_{X^1} i_L i_{X^2} di_{X^3} H + c_5 \mathcal{L}_{X^1} i_L di_{X^3} i_{X^2} H \\
& + c_1 i_{X^1} i_{[X^2, X^3]} H + c_3 i_L i_{[X^2, X^3]} di_{X^1} H \\
& + c_4 i_L i_{X^1} di_{[X^2, X^3]} H + c_5 i_L di_{[X^2, X^3]} i_{X^1} H \\
& = -c_1 i_{X^3} di_{X^1} i_{X^2} H - c_3 i_{X^3} di_L i_{X^2} di_{X^1} H \\
& \quad - c_4 i_{X^3} di_L i_{X^1} di_{X^2} H - c_5 i_{X^3} di_L di_{X^2} i_{X^1} H \\
& + c_1 i_{[X^1, X^2]} i_{X^3} H + c_3 i_L i_{X^3} di_{[X^1, X^2]} H \\
& + c_4 i_L i_{[X^1, X^2]} di_{X^3} H + c_5 i_L di_{X^3} di_{[X^1, X^2]} H \\
& + c_1 \mathcal{L}_{X^2} i_{X^1} i_{X^3} H + c_3 \mathcal{L}_{X^2} i_L i_{X^3} di_{X^1} H \\
& + c_4 \mathcal{L}_{X^2} i_L i_{X^1} di_{X^3} H + c_5 \mathcal{L}_{X^2} i_L di_{X^3} i_{X^1} H \\
& + c_1 i_{X^2} i_{[X^1, X^3]} H + c_3 i_L i_{[X^1, X^3]} di_{X^2} H \\
& + c_4 i_L i_{X^2} di_{[X^1, X^3]} H + c_5 i_L di_{[X^1, X^3]} i_{X^2} H
\end{aligned} \tag{4.3}$$

for any linear vector fields X^1, X^2, X^3 and any closed linear 3-form H on E .

Proof. For any linear vector fields X^1, X^2, X^3 on E and any closed linear 3-form H on E , we can write

$$\begin{aligned}
C_H(X^1 \oplus 0, C_H(X^2 \oplus 0, X^3 \oplus 0)) &= [X^1, [X^2, X^3]] \oplus \Omega, \\
C_H(C_H(X^1 \oplus 0, X^2 \oplus 0), X^3 \oplus 0) &= [[X^1, X^2], X^3] \oplus \Theta, \\
C_H(X^2 \oplus 0, C_H(X^1 \oplus 0, X^3 \oplus 0)) &= [X^2, [X^1, X^3]] \oplus \mathcal{T},
\end{aligned}$$

where

$$\begin{aligned}
\Omega &= c_1 \mathcal{L}_{X^1} i_{X^2} i_{X^3} H + c_3 \mathcal{L}_{X^1} i_L i_{X^3} di_{X^2} H \\
& + c_4 \mathcal{L}_{X^1} i_L i_{X^2} di_{X^3} H + c_5 \mathcal{L}_{X^1} i_L di_{X^3} i_{X^2} H \\
& + c_1 i_{X^1} i_{[X^2, X^3]} H + c_3 i_L i_{[X^2, X^3]} di_{X^1} H \\
& + c_4 i_L i_{X^1} di_{[X^2, X^3]} H + c_5 i_L di_{[X^2, X^3]} i_{X^1} H, \\
\Theta &= -c_1 i_{X^3} di_{X^1} i_{X^2} H - c_3 i_{X^3} di_L i_{X^2} di_{X^1} H \\
& - c_4 i_{X^3} di_L i_{X^1} di_{X^2} H - c_5 i_{X^3} di_L di_{X^2} i_{X^1} H \\
& + c_1 i_{[X^1, X^2]} i_{X^3} H + c_3 i_L i_{X^3} di_{[X^1, X^2]} H \\
& + c_4 i_L i_{[X^1, X^2]} di_{X^3} H + c_5 i_L di_{X^3} di_{[X^1, X^2]} H, \\
\mathcal{T} &= c_1 \mathcal{L}_{X^2} i_{X^1} i_{X^3} H + c_3 \mathcal{L}_{X^2} i_L i_{X^3} di_{X^1} H \\
& + c_4 \mathcal{L}_{X^2} i_L i_{X^1} di_{X^3} H + c_5 \mathcal{L}_{X^2} i_L di_{X^3} i_{X^1} H \\
& + c_1 i_{X^2} i_{[X^1, X^3]} H + c_3 i_L i_{[X^1, X^3]} di_{X^2} H \\
& + c_4 i_L i_{X^2} di_{[X^1, X^3]} H + c_5 i_L di_{[X^1, X^3]} i_{X^2} H.
\end{aligned}$$

If C satisfies the Jacobi identity in Leibniz form of C for closed linear 3-forms, then $\Omega = \Theta + \mathcal{T}$, i.e. (4.3). The lemma is complete. \square

Lemma 4.6. *Let C be a generalized twisted Dorfman–Courant bracket of the form (4.1). If C satisfies the Jacobi identity in Leibniz form for closed linear 3-forms, then*

$$\begin{aligned}
& c_3 \mathcal{L}_{X^1} i_L i_{X^3} di_{X^2} H \\
& + c_4 \mathcal{L}_{X^1} i_L i_{X^2} di_{X^3} H + c_5 \mathcal{L}_{X^1} i_L di_{X^3} i_{X^2} H \\
& + c_3 i_L i_{[X^2, X^3]} di_{X^1} H \\
& + c_4 i_L i_{X^1} di_{[X^2, X^3]} H + c_5 i_L di_{[X^2, X^3]} i_{X^1} H \\
& = -c_3 i_{X^3} di_L i_{X^2} di_{X^1} H + \\
& \quad - c_4 i_{X^3} di_L i_{X^1} di_{X^2} H - c_5 i_{X^3} di_L di_{X^2} i_{X^1} H \\
& \quad + c_3 i_L i_{X^3} di_{[X^1, X^2]} H \\
& \quad + c_4 i_L i_{[X^1, X^2]} di_{X^3} H + c_5 i_L di_{X^3} di_{[X^1, X^2]} H \\
& \quad + c_3 \mathcal{L}_{X^2} i_L i_{X^3} di_{X^1} H \\
& \quad + c_4 \mathcal{L}_{X^2} i_L i_{X^1} di_{X^3} H + c_5 \mathcal{L}_{X^2} i_L di_{X^3} i_{X^1} H \\
& \quad + c_3 i_L i_{[X^1, X^3]} di_{X^2} H \\
& \quad + c_4 i_L i_{X^2} di_{[X^1, X^3]} H + c_5 i_L di_{[X^1, X^3]} i_{X^2} H
\end{aligned} \tag{4.4}$$

for any linear vector fields X^1, X^2, X^3 and any closed linear 3-form H on $\mathbf{R}^{m,n}$.

Proof. It is well-known that the (usual) twisted Dorfman–Courant bracket satisfies the Jacobi identity in Leibniz form for closed linear 3-forms. So, we have (4.3) in the case $c_3 = c_4 = c_5 = 0$. So, formula (4.3) is equivalent to (4.4) for all linear vector fields X^1, X^2, X^3, X^4 and all closed linear 3-forms H on E . \square

Lemma 4.7. *Let C be a generalized twisted Dorfman–Courant bracket of the form (4.1). Assume $m \geq 4$ and $n \geq 1$. If C satisfies the Jacobi identity in Leibniz form for closed linear 3-forms, then $c_3 = c_4 = c_5 = 0$.*

Proof. Putting linear vector fields $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = \frac{\partial}{\partial x^2}$ and $X^3 = L$ and closed linear 3-form $H = x^1 dx^1 \wedge dx^2 \wedge dy^1$ into (4.4), we get

$$\begin{aligned}
& c_3 \cdot 0 + c_4 \cdot (y^1 dx^1) + c_5 \cdot (y^1 dx^1) + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
& = -c_3 \cdot y^1 dx^1 - c_4 \cdot 0 - c_5 \cdot (-y^1 dx^1) + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
& \quad + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0.
\end{aligned}$$

Hence $c_3 = -c_4$.

Similarly, putting linear vector fields $X^1 = x^2 \frac{\partial}{\partial x^1}$ and $X^2 = \frac{\partial}{\partial x^2}$ and $X^3 = L$ and closed linear 3-form $H = dx^1 \wedge dx^2 \wedge dy^1$ into (4.4), we get

$$\begin{aligned}
& c_3 \cdot 0 + c_4 \cdot y^1 dx^2 + c_5 \cdot y^1 dx^2 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\
& = -c_3 \cdot 0 - c_4 \cdot 0 - c_5 \cdot (-y^1 dx^2) + c_3 \cdot 0 + c_4 \cdot y^1 dx^2 + c_5 \cdot 0 \\
& \quad + c_3 \cdot 0 + c_4 \cdot (-y^1 dx^2) + c_5 \cdot (-y^1 dx^2) + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0.
\end{aligned}$$

Hence $c_4 = -c_5$.

Similarly, putting linear vector fields $X^1 = \frac{\partial}{\partial x^1}$ and $X^2 = x^1 \frac{\partial}{\partial x^2}$ and $X^3 = \frac{\partial}{\partial x^3}$ and closed linear 3-form $H = d(x^2x^4) \wedge dx^3 \wedge dy^1$ into (4.4), we get

$$\begin{aligned} & c_3 \cdot y^1 dx^4 + c_4 \cdot 0 + c_5 \cdot (-y^1 dx^4) + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 \\ & = -c_3 \cdot 0 - c_4 \cdot (y^1 dx^4 + x^4 dy^1) - c_5 \cdot 0 + c_3 \cdot y^1 dx^4 + c_4 \cdot 0 + c_5 \cdot 0 \\ & \quad + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0 + c_3 \cdot 0 + c_4 \cdot 0 + c_5 \cdot 0. \end{aligned}$$

Hence $c_4 = 0$.

Consequently, $c_3 = c_4 = c_5 = 0$, as well. □

Thus we have the following result.

Theorem 4.8. *Let $m \geq 4$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C satisfying the Jacobi identity in Leibniz form for closed linear 3-forms is of the form*

$$\begin{aligned} C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) &= [X^1, X^2] \oplus \{ \mathcal{L}_{X^1} \omega^2 - i_{X^2} d\omega^1 \\ & \quad + c_1 i_{X^1} i_{X^2} H + c_2 i_L i_{X^1} i_{X^2} dH \} \end{aligned} \tag{4.5}$$

for any $H \in \Gamma_E^l(\wedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus T^*E)$ and any $\mathcal{VB}_{m,n}$ -object E , where c_1, c_2 are (uniquely determined by C) real numbers.

Moreover, given $c_1, c_2 \in \mathbf{R}$, the generalized twisted Dorfman–Courant bracket C of the form (4.5) satisfies the Jacobi identity in Leibniz form for closed linear 3-forms.

Proof. The last sentence is an easy consequence of the fact that the usual twisted Dorfman–Courant bracket satisfies the Jacobi identity in Leibniz form for closed linear 3-forms. □

From Theorem 4.8, we have the following interesting natural characterization of the (usual) twisted Dorfman–Courant bracket.

Corollary 4.9. *Let $m \geq 4$ and $n \geq 1$. Any generalized twisted Dorfman–Courant bracket C_H satisfying the Jacobi identity in Leibniz form for closed linear 3-forms satisfies*


$$C_H(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [[X^1 \oplus \omega^1, X^2 \oplus \omega^2]]_{cH} \tag{4.6}$$

for any closed linear 3-form $H \in \Gamma_E^l(\wedge^3 T^*E)$ and any $X^1 \oplus \omega^1, X^2 \oplus \omega^2 \in \Gamma_E^l(TE \oplus T^*E)$, where $[[-, -]]_H$ is the usual twisted (H -twisted) Dorfman–Courant bracket and c is an arbitrary (uniquely determined by C) real number.

Remark 4.10. Roughly speaking, the above corollary means that (for $m \geq 4$) the (usual) twisted Dorfman–Courant bracket $[[-, -]]_H$ (for closed linear 3-forms H) is the unique (up to multiplication of H by a real number c) $\mathcal{VB}_{m,n}$ -gauge-natural extension of the (usual) Dorfman–Courant bracket $[[-, -]]_0$ (by means of closed linear 3-forms H) satisfying the Jacobi identity in Leibniz form.

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Received: November 12, 2020.

Accepted: January 27, 2021.