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UNIQUENESS OF SERIES IN THE FRANKLIN SYSTEM AND THE GEVORKYAN PROBLEMS

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Abstract. In 1870 G. Cantor proved that if $\lim_{N\to\infty}\sum_{n=-N}^{N} c_n e^{inx} = 0$, $\bar{c}_n = c_n$, then $c_n = 0$ for $n \in \mathbb{Z}$. In 2004 G. Gevorkyan raised the issue that if Cantor's result extends to the Franklin system. He solved this conjecture in 2015. In 2014 Z. Wronicz proved that there exists a Franklin series for which a subsequence of its partial sums converges to zero, where not all coefficients of the series are zero. In the present paper we show that to the uniqueness of the Franklin system $\lim_{n\to\infty}\sum_{n=0}^{\infty} a_n f_n$ it suffices to prove the convergence its subsequence s_{2^n} to zero by the condition $a_n = o(\sqrt{n})$. It is a solution of the Gevorkyan problem formulated in 2016.

Keywords: Franklin system, orthonormal spline system, uniqueness of series.

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1. INTRODUCTION

In 1870 G. Cantor ([2]) proved the following result.

Theorem 1.1. *If* $\lim_{N\to\infty}\sum_{n=-N}^{N} c_n e^{inx} = 0$ *for every real number* x, *where* $\bar{c}_n = c_n$, *then* $c_n = 0$ *for* $n \in \mathbb{Z}$.

By the Gram–Schmidt process to the Schauder basis Ph. Franklin constructed an orthonormal system of continuous piecewise linear functions with dyadic knots ([4]). It is an orthonormal Schauder basis in the space $\mathcal{C}[0,1]$, and also in the space $L^2[0,1]$. In 1963 Z. Ciesielski ([3]) proved exponential type estimates for the Franklin functions. Since then, it has been studied by many authors from different points of view. In 2004 G. Gevorkyan ([5]) raised the issue if Cantor's result extends to the Franklin system. He solved this problem in 2015 (see [6, 7]). In 2014 the author proved the following theorem.

Theorem 1.2 ([9]). *There exists a nontrivial series in the Franklin system*

$$\sum_{n=0}^{\infty} a_n f_n(x) \tag{1.1}$$

for which

$$\lim_{\nu\to\infty} \sum_{\nu=0}^{2^\nu} a_n f_n(x) = 0, \quad x \in [0,1]. \tag{1.2}$$

The purpose of the paper is to prove the ensuing result.

Theorem 1.3. *Let the coefficients of the series* (1.1) *satisfy the condition*

$$a_n = o(\sqrt{n})$$

and let (1.2) *hold. Then all the coefficients of this series vanish.*

It is a solution of a problem of G. Gevorkyan given in [7].

2. PRELIMINARIES

In this section we present some properties of the Franklin system and the Egorov theorem which play the fundamental role in the proof of Theorem 1.3.

Consider the following sequence $\{\Delta_n\}_{n=1}^{\infty}$ of dyadic partitions of the interval $I = [0,1]$: $\Delta_n = \{s_{n,i}\}_{i=0}^{n}$, $s_{1,0} = 0$, $s_{1,1} = 1$,

$$s_{n,i} = \begin{cases} \frac{i}{2^{\mu+1}}, & \text{for } i = 0,1,\ldots,2\nu, \\ \frac{i-\nu}{2^{\mu}}, & \text{for } i = 2\nu+1,\ldots,n \end{cases} \tag{2.1}$$

for $n = 2^\mu + \nu$, $\mu = 0,1,\ldots$, $\nu = 1,2,\ldots,2^\mu$.

We can obtain the Franklin system by means of cubic splines. We put

$$f_0 = 1, \quad f_1 = \sqrt{3}(2x-1).$$

Let g_n be a cubic spline with respect to the partition Δ_n, i.e. $g_n \in C^2(I)$ and it is a polynomial of degree at most 3 in each interval $[s_{n,i-1}, s_{n,i}]$. We assume that $g_n(s_{n-1,j}) = 0$ for $j = 0,1,\ldots,n-1$ and $g_n(s_{n,k}) = 1$ for $s_{n,k} = \Delta_n \setminus \Delta_{n-1}$ with $g_n'(0) = g_n'(1) = 0$. The spline g_n is unique. For the proof we refer to [1]. Integrating by parts, we check that the system $\{f_n\}_{n=0}^{\infty}$, where

$$f_n = \frac{g_n''}{\| g_n'' \|}, \quad \| g_n'' \|^2 = \int_0^1 [g_n''(x)]^2 dx, \quad n = 2,3,\ldots$$

is orthonormal in the interval I (see [1,10,11]).

In the sequel we shall need the Ciesielski inequality and the Egorov theorem.

Theorem 2.1 ([3]). *Let $\{f_n\}_{n=0}^{\infty}$ be the Franklin system defined by means (2.1), $\Delta_n = \{x_0, x_1, \ldots, x_n\}$, $x_n = \Delta_n \setminus \Delta_{n-1}$. There exist constants $M > 0$ and $0 < r < 1$ such that*

$$|f_n(x)| \le M\sqrt{n} r^{n|x-x_n|}$$

for every $x \in I$ and $n = 0, 1, \ldots$

Theorem 2.2 ([8]). *Let $\{f_n\}$ be a sequence of measurable functions w.r.t. the Lebesgue measure on the interval I. Assume that $f_n \to f$ pointwise. Then for any $\delta > 0$, there exists a measurable set E_δ of I such that $m(E_\delta) > |I| - \delta$ and $f_n \to f$ uniformly on $I \setminus E_\delta$.*

3. PROOF OF THEOREM 1.3

Let $F_n(x) = \frac{g_n(x)}{\|g_n''\|}$. Then $F_n''(x) = f_n(x)$. We define

$$s_n(x) = \sum_{i=0}^{n} a_i f_n(x), \quad S_n(x) = \sum_{i=0}^{n} a_i F_i(x).$$

Then $S_n''(x) = s_n(x)$.

We assume that (1.2) holds for the series (1.1). We apply the Egorov theorem to the sequence $\{s_{2^n}\}_{n=0}^{\infty}$. Let E be a set of points $x \in I$ such that $x \in I \setminus E_\delta$ for all $\delta > 0$. By the Egorov theorem, for all $x_1, x_2 \in E$ there exists $x \in I \setminus E$ such that $x_1 < x < x_2$. By the continuity of the functions s_n, we prove that the points of the set E are isolated or they are accumulation points.

Let α and β be two consecutive points of E. Then the sequence s_n is convergent uniformly on every closed interval $F \subset (\alpha, \beta)$. S_n is a cubic spline with respect to the partition

$$\Delta_n = \{x_k\}_{k=0}^{n} = \{0 = t_0 < t_1 < \ldots < t_n = 1\},$$

$S_n(x_j) = S_k(x_j)$ for $x_j \in \Delta_k, k < n$. Let $t_i, t_j \in \Delta_k$ for some $k, [t_i, t_j] \subset (\alpha, \beta)$. Then

$$\forall \varepsilon > 0 \, \exists n_0 \, \forall n > n_0 \, \forall x \in [t_i, t_j] \, : \, |s_n(x)| < \varepsilon.$$

Further, for any $n > n_0$,

$$\frac{S_n(t_j) - S_n(t_i)}{t_j - t_i} = S_n'(\zeta_n) = C = \text{const}$$

for some $\zeta_n \in (t_i, t_j)$, $n > k$. This follows from the fact that $S_n(t_l) = S_k(t_l)$ for $n > k$ and $t_l \in \Delta_k$. Hence

$$S_n'(x) = S_n'(\zeta_n) + \int_{\zeta}^{x} s_n(t)dt \tag{3.1}$$

and the sequence $\{s_n\}$ is uniformly convergent to the constant $C = S_n'(\zeta_n)$ in the interval $[t_i, t_j]$. Repeating this reasoning, we prove that the sequence $\{S_n\}$ is uniformly convergent to the function $S_n(t_i)+C(x-t_i)$ in the interval $[t_i,t_j]$. Applying the Rolle theorem to the functions F_n, and (3.1) with Theorem 2.1 we obtain the following inequalities:

$$|F_n'(x)| \le \frac{M_1}{\sqrt{n}}\, r^{n|x-x_n|} \quad \text{and} \quad |F_n(x)| \le \frac{M_2}{n\sqrt{n}}\, r^{n|x-x_n|},$$

where M_1 and M_2 are constants.

Let

$$c_n = \max_{i\ge n} \frac{|a_i|}{\sqrt{i}}, \quad n \ge 0. \tag{3.2}$$

Hence

$$\left|\sum_{k=m}^{m+l} a_k F_k(x)\right| \le \left|\sum_{k=m}^{m+l} c_k\sqrt{k}F_k(x)\right| < M_2 \sum_{n=2^i}^{\infty} \frac{c_n}{n} r^{n|x-x_n|},$$

where $2^i < m < m+l \le 2^{i+j}$.

Let

$$\Delta_{2^{j+1}} = \{0 = t_0 < t_1 < \ldots < t_{2^{j+1}} = 1\}.$$

Then

$$t_{k+1} - t_k = \frac{1}{2^{j+1}}, \quad k = 0,1,\ldots,2^{j+1}-1.$$

Hence for $x \in [t_k, t_{k+1}]$

$$\sum_{n=2^j+1}^{2^{j+1}} \frac{1}{n} r^{n|x-x_n|} \le \sum_{n=2^j+1}^{2^{j+1}} \frac{1}{n} r^{2^j k_n 2^{-j-1}} \le \frac{1}{2^{j-1}} \sum_{i=0}^{\infty} (\sqrt{r})^i = \frac{1}{2^{j-1}} \frac{1}{1-\sqrt{r}},$$

where $0 \le k_n \le 2^{j+1}$, $k_n \ne k_m$ for $n \ne m$.

By summation over j, we obtain

$$\forall m,l \in \mathbb{N} : \sum_{k=m}^{m+l} \sqrt{k}|F_k(x)| < \frac{2M_2}{1-\sqrt{r}}.$$

Proceeding as in the proof of the Dirichlet criterion and applying the fact that the sequence (3.2) is diminishing, we prove that the sequence $\{S_n\}$ is uniformly convergent to the continuous function S. Hence the function S is a broken line with knots in the set E. We shall prove that the set E is empty.

We assume that α, β and γ are consecutive points of the set E. Then the function $S(x) = \lim_{n\to\infty} S_n(x)$ is linear in the intervals $[\alpha,\beta]$ and $[\beta,\gamma]$.

Let $t_\alpha = t_{j_n} \in (\alpha,\beta)$, $t_\gamma = t_{l_n} \in (\beta,\gamma)$ and $\beta \in [t_{k_n}, t_{k_n+1})$. Later we shall written j instead of j_n and l instead of l_n. Since an addition a linear function to the function S does not change its second derivative, we may assume that $S(x) = 0$ for $x \in [\alpha,\beta]$ and $S(x) = ax+b$ for $x \in [\beta,\gamma]$, where a and b are some constants.

Let $S'(t_\alpha) = \alpha_n$ and $S'(t_\gamma) = \gamma_n$. The cubic spline S_n is defined in the interval $[t_\alpha, t_\gamma]$ by the following conditions:

$$\begin{aligned} S_n(t_i) &= S(t_i) = 0 \ for \ t_i \in \Delta_n \cap [t_\alpha, \beta], \\ S_n(t_j) &= S(t_j) = 0 \ for \ t_j \in \Delta_n \cap [\beta, t_\gamma], \\ S_n'(t_\alpha) &= \alpha_n, \ S_n'(t_\gamma) = \gamma_n. \end{aligned}$$

We define the cubic spline S_n by the following system of equations (see [1]):

$$\begin{aligned} 4M_j + M_{j+1} &= 2d_j, \\ M_{i-1} + 4M_i + M_{i+1} &= 2d_i, \quad i = j+1, \dots, l-1, \\ M_{l-1} + 4M_l &= 2d_l, \end{aligned} \tag{3.3}$$

where $M_i = S''(t_i)$, $i = j, j+1, \dots, l$,

$$\begin{aligned} d_j &= \frac{6}{t_{j+1} - t_j} \cdot \left(\frac{S_n(t_{j+1}) - S_n(t_j)}{t_{j+1} - t_j} - \alpha_n \right) = S_n''(\xi_{n,j}), \quad \xi_{n,j} \in (t_j, t_{j+1}), \\ d_l &= \frac{6}{t_l - t_{l-1}} \cdot \left(\gamma_n - \frac{S_n(t_l) - S_n(t_{l-1})}{t_l - t_{l-1}} \right) = S_n''(\xi_{n,l}), \quad \xi_{n,l} \in (t_{l-1}, t_l), \\ d_i &= 6 \frac{\frac{S_n(t_{i+1}) - S_n(t_i)}{t_{i+1} - t_i} - \frac{S_n(t_i) - S_n(t_{i-1})}{t_i - t_{i-1}}}{t_{i+1} - t_{i-1}} = S_n''(\xi_{n,i}), \quad \xi_{n,i} \in (t_{i-1}, t_{i+1}), j < i < l. \end{aligned}$$

The function S_n interpolate the function S at the points t_i, $j \le i \le l$. Since the sequences $\{S_n'\}$ and $\{S_n''\}$ are uniformly convergent in the intervals $[z_1, z_2] \subset (\alpha, \beta)$ and $[z_3, z_4] \subset (\beta, \gamma)$, $t_j \in [z_1, z_2]$, $t_l \in [z_3, z_4]$, we conclude that

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} S_n''(\xi_{n,j}) = 0, \quad \xi_{n,j} \in (t_j, t_{j+1})$$

and

$$\lim_{n\to\infty} (\gamma_n - a) = \lim_{n\to\infty} S_n'(\xi_{n,l}) = 0, \quad \xi_{n,l} \in (t_{l-1}, t_l).$$

We may write the system (3.3) as follows:

$$A_n M_n = D_n. \tag{3.4}$$

Further,

$$S_n = F_n + H_n,$$

where the cubic splines F_n and H_n are defined by the following conditions:

$$\begin{aligned} &F_n(t_i) = S(t_i) = 0 && \text{for } t_i \in \Delta_n \cap [t_\alpha, \beta], \\ &F_n(t_j) = S(t_j) && \text{for } t_j \in \Delta_n \cap [\beta, t_\gamma], \\ &F_n'(t_\alpha) = 0, \quad F_n'(t_\gamma) = a \end{aligned}$$

and

$$\begin{aligned} H_n(t_i) &= 0 \text{ for } t_i \in \Delta_n \cap [t_\alpha, t_\gamma], \\ H_n'(t_\alpha) &= \alpha_n, \quad H_n'(t_\gamma) = \gamma_n. \end{aligned}$$

Note that

$$d_{H_n,t_\alpha} = H_n''(\zeta_{n,j}), \quad d_{H_n,t_\gamma} = H_n''(\zeta_{n,l}).$$

Since

$$\lim_{n\to\infty} H_n''(\zeta_{n,j}) = \lim_{n\to\infty} H_n''(\zeta_{n,l}) = 0,$$

then the function H_n is convergent uniformly to 0 on the interval $[t_\alpha, t_\gamma]$. Hence, it suffices to prove that

$$\lim_{n\to\infty} |F_n''(\beta)| = \infty.$$

The function F_n is determined by the system (3.4) with

$$D_n = [0, \ldots, 2d_k, 2d_{k+1}, 0, \ldots, 0]^T.$$

Let

$$\beta = t_k + th, \quad h = t_{i+1} - t_i = \frac{1}{n}, \quad 0 \le t \le 1,\ i = 0, 1, \ldots, n-1.$$

Then

$$2d_k = 6\frac{a(1-t)}{h}, \quad 2d_{k+1} = 6\frac{at}{h}, \quad 2d_i = 0 \quad \text{for } j \le i < l, i \ne k, k+1. \tag{3.5}$$

We write the system (3.3) for the function F_n as follows:

$$A_n M_n = D_{F,n}, \tag{3.6}$$

where

$$D_{F,n} = [0, \ldots, 0, 2d_k, 2d_{k+1}, 0, \ldots, 0]^T.$$

We may write the determinant $\det A_n$ in the form

$$\det A_n = \begin{vmatrix}
4 & 1 & 0 & 0 \ldots & 0 & 0 & 0 & 0 & 0 \ldots 0 & 0 & 0 & 0 \\
0 & \alpha_1 & 1 & 0 \ldots & 0 & 0 & 0 & 0 & 0 \ldots 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_2 & 1 \ldots & 0 & 0 & 0 & 0 & 0 \ldots 0 & 0 & 0 & 0 \\
- & - & - & - \, - & - & - & - & - & - \, - \, - & - & - & - \\
0 & 0 & 0 & 0 \ldots & \alpha_{k-1} & 1 & 0 & 0 & 0 \ldots 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \ldots & 0 & \alpha_k & 1 & 0 & 0 \ldots 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \ldots & 0 & 1 & \alpha_{l-k-1} & 0 & 0 \ldots 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \ldots & 0 & 0 & 1 & \alpha_{l-k-2} & 0 \ldots 0 & 0 & 0 & 0 \\
- & - & - & - \, - & - & - & - & - & - \, - \, - & - & - & - \\
0 & 0 & 0 & 0 \ldots & 0 & 0 & 0 & 0 & 0 \ldots 0 & \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 \ldots & 0 & 0 & 0 & 0 & 0 \ldots 0 & 1 & \alpha_1 & 0 \\
0 & 0 & 0 & 0 \ldots & 0 & 0 & 0 & 0 & 0 \ldots 0 & 0 & 1 & 4
\end{vmatrix}$$

$$= 4\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_{k-1}(\alpha_k\alpha_{l-k-1})\alpha_{l-k-2} \cdot \ldots \cdot \alpha_2\alpha_1 \cdot 4,$$

where $\alpha_1 = \frac{15}{4}, \alpha_{i+1} = 4 - \frac{1}{\alpha_i}, i = 1, 2, \ldots$

We have

$$3 < \alpha_{i+1} < \alpha_i < 4, \quad i = 1, 2, \ldots,$$

and

$$\lim_{m\to\infty} \alpha_m = 2 + \sqrt{3} > \frac{5}{2}.$$

By the Cramer formula for the system (3.6) and (3.5), we obtain

$$M_k = \frac{12an[\alpha_{l-k-1} - (\alpha_{l-k-1} + 1)t]}{\alpha_k \alpha_{l-k-1} - 1}$$

and

$$M_{k+1} = \frac{12an[(\alpha_k + 1)t - 1]}{\alpha_k \alpha_{l-k-1} - 1}.$$

Further, we have

$$S_n''(t) = M_k(1-t) + M_{k+1}t, \quad t \in [0,1], \quad \beta = t_k + \frac{t}{n}$$

and

$$\begin{aligned}
S_n''(t) &= \frac{12an}{\alpha_k \alpha_{l-k-1} - 1}\{[\alpha_{l-k-1} - (\alpha_{l-k-1} + 1)t](1-t) + [(\alpha_k + 1)t - 1]t\} \\
&= \frac{12an}{\alpha_k \alpha_{l-k-1}} g_n(t), \\
g_n(t) &= (\alpha_k + \alpha_{l-k-1} + 2)t^2 - 2(\alpha_{l-k-1} + 1)t + \alpha_{l-k-1}, \\
\Delta &= 4(1 - \alpha_k \alpha_{l-k-1}) < 0, \\
g_n(0) &= \alpha_{l-k-1}.
\end{aligned}$$

Hence

$$\lim_{n\to\infty} |s_n(\beta)| = \lim_{n\to\infty} |S_n''(\beta)| = \infty$$

and it is a contradiction to the assumption that $\lim_{n\to\infty} |s_n(\beta)| = 0$. Hence $\beta \notin E$. In the same way we prove that each knot of the broken line S does not belong to the set E and we have proved that S is a linear function.

Thus $S_n'' = S'' = 0$. Because of the fact that the Franklin system is the Schauder basis in the space $\mathcal{C}[0,1]$, we conclude that all the coefficients of the series (1.1) vanish and we have proved the theorem.

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