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MULTI-VARIABLE QUATERNIONIC SPECTRAL ANALYSIS

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Abstract. In this paper, we consider finite dimensional vector spaces $\mathbb{H}^n$ over the ring $\mathbb{H}$ of all quaternions. In particular, we are interested in certain functions acting on $\mathbb{H}^n$, and corresponding functional equations. Our main results show that (i) all quaternions of $\mathbb{H}$ are classified by the spectra of their realizations under representation, (ii) all vectors of $\mathbb{H}^n$ are classified by a canonical extended setting of (i), and (iii) the usual spectral analysis on the matricial ring $M_n(\mathbb{C})$ of all $(n \times n)$-matrices over the complex numbers $\mathbb{C}$ has close connections with certain "non-linear" functional equations on $\mathbb{H}^n$ up to the classification of (ii).

Keywords: the quaternions $\mathbb{H}$, vector spaces $\mathbb{H}^n$ over $\mathbb{H}$, q-spectral forms, q-spectral functions.

Mathematics Subject Classification: 20G20, 46S10, 47S10.

1. INTRODUCTION

Motivated by applications, we consider systems of functional equations over the quaternions, as well we their associated spectral theory. A main tool in our analysis is the use of a new noncommutative harmonic analysis. It takes the form of particular representations, q-spectral forms, and an induced quaternionic spectral calculus. A further tool in our harmonic analysis is a new quaternion-spectral relation. Then we extend such spectral-theoretic tools on the quaternions to those on the multi-dimensional vector spaces over the quaternions.

In this paper, we study finite-dimensional vector spaces $\mathbb{H}^n$ over $\mathbb{H}$ of the *quaternions*, for $n \in \mathbb{N}$. In particular, we are interested in certain functions acting on $\mathbb{H}^n$. Let

$$\mathbb{C} = \{x + yi : x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

be the set of all *complex numbers*, where $\mathbb{R}$ denotes the *real numbers*. Then the set

$$\mathbb{H} = \{x + yi + uj + vk : x, y, u, v \in \mathbb{R}, i^2 = j^2 = k^2 = -1, \text{and } ijk = -1\}$$

of all *quaternions* (or *quaternion numbers*) is defined.

From a representation of [19], every quaternion $q \in \mathbb{H}$ is realized to be a matrix $[q] \in M_2(\mathbb{C})$ on the 2-dimensional complex vector space $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. For instance, the matrices $[i]$, $[j]$, and $[k]$ become the Pauli matrices,

$$[i] = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ [j] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ and } [k] = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

in $M_2(\mathbb{C})$. The *spectral properties of* $[q] \in M_2(\mathbb{C})$ is considered in [1]. And, by using the main results of [1], we solved monomial equations, and some quadratic equations on $\mathbb{H}$, in [2]. For the self-containedness of this paper, we introduce the concepts and main results of these preprints [1] and [2] in Sections 2, 3 and 4.

1.1. MOTIVATION

The quaternions $\mathbb{H}$ is an interesting object not only in pure mathematics (e.g., [9, 10] and [16]), but also in applied mathematics (e.g., [3] and [18]). Algebra on $\mathbb{H}$ is considered in e.g., [20]; analysis on $\mathbb{H}$ is studied in e.g., [11] and [17]; and physics on $\mathbb{H}$ is investigated in e.g., [6]. Also, the matrices over the quaternions $\mathbb{H}$ have been studied (e.g., [4, 5, 14] and [17]); and the eigenvalue problems on such matrices form a branch of linear, or multilinear algebra (e.g., [12, 13] and [15]).

At this moment, we emphasize that, even though our works are motivated by the recent studies of $\mathbb{H}$, they are approached differently from the earlier works. Starting from the representation of $\mathbb{H}$ (in the sense of [20]), the quaternions $q \in \mathbb{H}$ are considered as (2×2)-matrices of the matricial ring $M_2(\mathbb{C})$ over $\mathbb{C}$ (e.g., see Section 2). And then the realizations $[q] \in M_2(\mathbb{C})$ of $q \in \mathbb{H}$ are classified by their spectra (equivalently, by the similarity) (see Sections 3 and 4).

We here study finite-dimensional vector spaces $\mathbb{H}^n$ over the quaternions $\mathbb{H}$, for $n \in \mathbb{N}$, and certain types of functions on $\mathbb{H}^n$. As in [1] and [2], we focus on certain representatives of the images of such functions acting on a complex vector space over $\mathbb{C}$.

1.2. OVERVIEW

In Sections 2, 3 and 4, the spectral analysis of the realizations of quaternions is reconsidered (also, see [1] and [2]). Even though the representation of the quaternions, introduced in Section 2, is well-known (e.g., [20]), the classification of quaternions obtained from the eigenvalues of their realizations under representation is newly introduced here in Sections 3 and 4.

In Section 5, we study finite-dimensional vector space $\mathbb{H}^n$ over $\mathbb{H}$, for $n \in \mathbb{N}$, and consider natural homomorphic vector spaces of $\mathbb{H}^n$ over $\mathbb{C}$. And certain functions on $\mathbb{H}^n$, whose ranges are contained in $\mathbb{C}^n$ (embedded in $\mathbb{H}^n$). As application, we consider a set $\sum_n(\mathbb{H})$ of such functions which forms a noncommutative unital ring over $\mathbb{R}$.

In Sections 6 and 7, we study fundamental functional equations induced by $\sum_n(\mathbb{H})$, and characterize the solvability of those equations.

By using the main results of previous sections, in Section 8, the relations between the usual spectral theory on $M_n(\mathbb{C})$ and a certain type of functional equations induced by $\sum_n(\mathbb{H})$ are studied. Remark that our functional equations and their solutions are not only interesting pure-algebraically, but also applicable to quaternionic geometry, or $\mathbb{C}$-manifold theory, etc. And, our tools and solutions may/can provide techniques how to handle $\mathbb{H}$-manifolds as $\mathbb{C}$-manifolds under our classification, so-called the q-spectral relation.

2. A REPRESENTATION $(\mathbb{C}^2, \pi)$ OF $\mathbb{H}$

In this section, we review a representation of the quaternions $\mathbb{H}$. In particular, we understand each quaternion $q \in \mathbb{H}$ as a (2×2)-matrix $[q] \in M_2(\mathbb{C})$ acting on the 2-dimensional space $\mathbb{C}^2$ (e.g., see [1, 15] and [20]).

2.1. QUATERNIONS $\mathbb{H}$

Let a and b be complex numbers,

$$a = x + yi \quad \text{and} \quad b = u + vi \quad \text{in } \mathbb{C},$$

where $x, y, u, v \in \mathbb{R}$, and $i = \sqrt{-1}$ in $\mathbb{C}$.

From the complex numbers a, $b \in \mathbb{C}$, the corresponding quaternion $q \in \mathbb{H}$ is canonically constructed by

$$q = a + bj = (x + yi) + (u + vi)j = x + yi + uj + vij = x + yi + uj + vk,$$

in $\mathbb{H}$, satisfying

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2.1}$$

The set $\mathbb{H}$ has a well-defined *addition* $(+)$, and *multiplication* $(\cdot)$; for any

$$q_l = a_l + b_l j \in \mathbb{H}, \quad \text{with } a_l, b_l \in \mathbb{C},$$

(in the sense of (2.1)) for $l = 1, 2$, one has

$$\begin{aligned} q_1 + q_2 &= (a_1 + a_2) + (b_1 + b_2)j, \\ q_1 q_2 &= (a_1 a_2 - b_1 \overline{b_2}) + (a_1 b_2 + \overline{a_2} b_1)j \end{aligned} \tag{2.2}$$

in $\mathbb{H}$, where $\overline{z}$ are the *conjugates of* $z \in \mathbb{C}$.

By (2.2),

$$q_1 q_2 \neq q_2 q_1 \quad \text{in } \mathbb{H}, \text{in general}. \tag{2.3}$$

Under the operations of (2.2), the quaternions $\mathbb{H}$ forms a ring algebraically, moreover it is a "noncommutative field" (in the sense of [20]). A *noncommutative field* $(F, +, \cdot)$ is an algebraic structure satisfying that: the algebraic pair $(F, +)$ forms an abelian group; and the pair $(F^{\times}, \cdot)$ forms a "noncommutative" group, where $F^{\times} = F \setminus \{0_F\}$, where 0_F is the $(+)$-identity of $(F, +)$; and $(+)$ and $(\cdot)$ are left-and-right distributive.

If $q \in \mathbb{H}$ is a quaternion (2.1), then one can define the *quaternion-conjugate* $\overline{q} \in \mathbb{H}$ by

$$\overline{q} = x - yi - ui - vi. \tag{2.4}$$

So, one has that

$$\overline{q}q = q\overline{q} = |a|^2 + |b|^2 = x^2 + y^2 + u^2 + v^2,$$

by (2.4). Thus, by (2.1),

$$\overline{q}q = q\overline{q} \geq 0 \quad \text{in } \mathbb{R} \subset \mathbb{H}, \text{for all } q \in \mathbb{H}. \tag{2.5}$$

By (2.5), one can define the *quaternion-modulus* $\|\cdot\|$ on $\mathbb{H}$ by

$$\|q\| = \sqrt{q\overline{q}}, \quad \text{for all } q \in \mathbb{H}. \tag{2.6}$$

This quaternion-modulus $\|\cdot\|$ of (2.6) is a well-defined norm on $\mathbb{H}$.

If $q \neq 0$ in $\mathbb{H}$, then the quaternion-reciprocal q^{-1} of q,

$$q^{-1} = \left(\frac{\overline{a}}{|a|^2 + |b|^2}\right) + \left(\frac{-b}{|a|^2 + |b|^2}\right) j \tag{2.7}$$

is well-defined in $\mathbb{H}$, by (2.1) and (2.6).

2.2. A REPRESENTATION $(\mathbb{C}^2, \pi)$ OF $\mathbb{H}$

In this section, we consider a representation of the quaternions $\mathbb{H}$, introduced in [20], realized on the 2-dimensional space $\mathbb{C}^2$ over the complex numbers $\mathbb{C}$. As in (2.1), let us understand each quaternion $q \in \mathbb{H}$ as

$$q = a + bj \quad \text{in } \mathbb{H}, \text{ with } a, b \in \mathbb{C},$$

where

$$a = x + yi, \quad \text{and} \quad b = u + vi \quad \text{in } \mathbb{C}.$$

Define an injective representation,

$$\pi : \mathbb{H} \to M_2(\mathbb{C}),$$

by

$$\pi(q) = \pi(a + bj) = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} x + yi & -u - vi \\ u - vi & x - yi \end{pmatrix}, \tag{2.8}$$

where $\overline{z}$ are the complex-conjugates of $z \in \mathbb{C}$, respectively, and $M_2(\mathbb{C})$ is the *matricial ring* of all (2×2)-matrices over $\mathbb{C}$.

This morphism π of (2.8) satisfies that

$$\pi\left(q_1+q_2\right)=\pi(q_1)+\pi(q_2),$$

and

$$\pi\left(q_1q_2\right)=\pi(q_1)\pi(q_2), \tag{2.9}$$

for all $q_1, q_2 \in \mathbb{H}$, by (2.2). Then the quaternion-conjugate $\overline{q}$ of $q \in \mathbb{H}$ satisfies that

$$\pi(\overline{q})=\pi(\overline{a}-bj)=\begin{pmatrix}\overline{a} & b\\ -\overline{b} & a\end{pmatrix}=\begin{pmatrix}a & -b\\ \overline{b} & \overline{a}\end{pmatrix}^{*}=\pi(q)^{*}, \tag{2.10}$$

in $M_2(\mathbb{C})$ by (2.4), where A^* are the *adjoints* (or, the conjugate-transposes) of $A \in M_2(\mathbb{C})$. Furthermore,

$$\det\left(\pi(q)\right)=\det\begin{pmatrix}a & -b\\ \overline{b} & \overline{a}\end{pmatrix}=\left|a\right|^2+\left|b\right|^2,$$

and hence, one can have

$$\|q\|=\sqrt{\det\left(\pi(q)\right)},\quad \text{for all } q\in\mathbb{H}, \tag{2.11}$$

by (2.5) and (2.6).

Lemma 2.1. *Let π be in the sense of* (2.8). *Then*

$$\left(\mathbb{C}^2,\pi\right) \text{ is a topological representation of } \mathbb{H}. \tag{2.12}$$

Proof. The morphism π of (2.8) is a well-defined injective ring-homomorphism from $\mathbb{H}$ into $M_2(\mathbb{C})$, by (2.9) and (2.10). Moreover, the relation (2.11) shows that the usual topology for $\mathbb{H}$, determined by the quaternion-modulus $|\cdot|$ is preserved by the norm $\|\cdot\|$ on $M_2(\mathbb{C})$, and hence, this representation is topological. □

Let $q \in \mathbb{H}$, and $\pi(q)$, the realization of q in $M_2(\mathbb{C})$. For convenience, we denote $\pi(q)$ by $[q]$.

Let us define a subset $\mathcal{H}_2$ of $M_2(\mathbb{C})$ by the set of all realizations of $\mathbb{H}$, i.e.,

$$\mathcal{H}_2 \overset{def}{=} \{[q]\in M_2(\mathbb{C}) : q\in\mathbb{H}\}=\pi\left(\mathbb{H}\right). \tag{2.13}$$

Theorem 2.2. *The quaternions $\mathbb{H}$ and the set $\mathcal{H}_2$ of* (2.13) *are isomorphic noncommutative fields, i.e.,*

$$\mathbb{H} \overset{NF}{=} \mathcal{H}_2, \tag{2.14}$$

where "$\overset{NF}{=}$" means "being noncommutative-field-isomorphic".

Proof. Take the action π of (2.8) acting on $\mathbb{C}^2$. By the injectivity of π, and by the definition (2.13), two sets $\mathbb{H}$ and $\mathcal{H}_2$ are bijective (or equipotent), i.e., $\pi : \mathbb{H} \to \mathcal{H}_2$ is a bijection. Moreover, π is a well-defined topological-ring-homomorphism from $\mathbb{H}$ onto $\mathcal{H}_2$ by (2.12), i.e., π is a continuous ring-isomorphism from $\mathbb{H}$ onto $\mathcal{H}_2$. Thus the relation (2.14) holds. □

3. QUATERNION-SPECTRAL FORMS

Let $\mathcal{H}_2$ be the noncommutative field (2.13), isomorphic to the quaternions $\mathbb{H}$. In this section, we regard each quaternion $q \in \mathbb{H}$ as a (2×2)-matrix $[q] \in \mathcal{H}_2$ in $M_2(\mathbb{C})$ by (2.14), and study spectral analysis on $\mathcal{H}_2$ (and hence, that on $\mathbb{H}$).

3.1. QUATERNION-SPECTRAL FORMS OF $\mathbb{H}$

In this section, by regarding the realizations $[q] \in \mathcal{H}_2$ of quaternions $q \in \mathbb{H}$ as (2×2)-matrices in $M_2(\mathbb{C})$, the spectra *spec* $([q])$ of $[q]$ are studied canonically. Suppose $q = a + bj \in \mathbb{H}$ is a quaternion with

$$a = x + yi,\ b = u + vi \in \mathbb{C},$$

and let

$$[q] = \begin{pmatrix} a & -b \\ b & \overline{a} \end{pmatrix} = \begin{pmatrix} x+yi & -u-vi \\ u-vi & x-yi \end{pmatrix} \in \mathcal{H}_2$$

be the realization of q.

Observe that for $z \in \mathbb{C}$,

$$\begin{aligned} \det\left([q] - zI_2\right) &= \det\left(\begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} - \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}\right) \\ &= z^2 - (x + yi + x - yi)\, z + \left(|a|^2 + |b|^2\right) \\ &= z^2 - 2xz + \left(x^2 + y^2 + u^2 + v^2\right). \end{aligned} \tag{3.1}$$

The above formula (3.1) is nothing but the *characteristic polynomial* of the realization $[q]$ in $z \in \mathbb{C}$. Consider the equation

$$\det\left([q] - zI_2\right) = 0,$$

or equivalently

$$z^2 - 2xz + \left(x^2 + y^2 + u^2 + v^2\right) = 0, \tag{3.2}$$

by (3.1). Then the equation (3.2) has its solutions

$$z = x \pm i\sqrt{y^2 + u^2 + v^2} \quad \text{in } \mathbb{C}. \tag{3.3}$$

Theorem 3.1. *Let $q = a + bj \in \mathbb{H}$ be a quaternion, realized to be $[q] \in \mathcal{H}_2$. Then the spectrum spec $([q])$ of $[q]$ is the subset,*

$$spec\left([q]\right) = \left\{\lambda,\ \overline{\lambda}\right\} \ of\ \mathbb{C},$$

where

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2} \quad in\ \mathbb{C}. \tag{3.4}$$

Proof. Under hypothesis, the characteristic polynomial of the realization $[q]$ is identical to the quadratic function (3.1), providing the equation (3.2). Thus one can get the eigenvalues,

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2},$$

and

$$\overline{\lambda} = x - i\sqrt{y^2 + u^2 + v^2},$$

in $\mathbb{C}$, by (3.3). So, the set-equality (3.4) holds. □

Motivated by (3.4), we define the following concept.

Definition 3.2. Let $q = x + yi + uj + vk \in \mathbb{H}$ be a quaternion, realized to be $[q] \in \mathcal{H}_2$. If $u = 0 = v$ in $\mathbb{R}$, equivalently, if $q = x + yi + 0j + 0k$ in $\mathbb{H}$, equivalently, if $q \in \mathbb{C} \subset \mathbb{H}$, then the matrix,

$$\mathbf{q} \overset{\text{denote}}{=} \begin{pmatrix} x + yi & 0 \\ 0 & x - yi \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & \overline{q} \end{pmatrix} = [q] \in \mathcal{H}_2$$

is called the quaternion-spectral form (in short, the q-spectral form) of q. Meanwhile, if either $u \neq 0$, or $v \neq 0$ in $\mathbb{R}$, equivalently, if $q \in (\mathbb{H} \setminus \mathbb{C}) \subset \mathbb{H}$, then the matrix

$$\mathbf{q} \overset{\text{denote}}{=} \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \in \mathcal{H}_2,$$

with

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C}$$

is called the quaternion-spectral form (in short, the q-spectral form) of q.

By definition, the q-spectral form $\mathbf{q} \in \mathcal{H}_2$ of a quaternion $q \in \mathbb{H}$ is the diagonal matrix of $M_2(\mathbb{C})$ whose diagonal entries are the eigenvalues of the realization $[q] \in \mathcal{H}_2$, which is "contained in $\mathcal{H}_2$", by (3.4). Note that if $q = x + yi + 0j + 0k$ in $\mathbb{H}$, then the realization $[q] \in \mathcal{H}_2$ has its spectrum

$$\{\lambda, \overline{\lambda}\}$$

with

$$\lambda = x + i\sqrt{y^2 + 0^2 + 0^2} = x + |y|\, i.$$

So, Definition 3.2 is meaningful, i.e., the q-spectral form

$$\mathbf{q} = \begin{pmatrix} x + yi & 0 \\ 0 & x - yi \end{pmatrix} = [q]$$

is well-determined in $\mathcal{H}_2$.

3.2. SIMILARITY ON q-SPECTRALFORMS IN $\mathcal{H}_2$

Throughout this section, let

$$a = x + yi,\ b = u + vi \in \mathbb{C}, \quad \text{with } \ x, y, u, v \in \mathbb{R},$$

and

$$q = a + bj = x + yi + uj + vk \in \mathbb{H}. \tag{3.5}$$

In Section 3.1, we showed that every quaternion $q \in \mathbb{H}$ of (3.5), realized to be $[q] \in \mathcal{H}_2$, has its q-spectral form,

$$\mathbf{q} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}, \quad \text{with } \ \lambda = x + i\sqrt{y^2 + u^2 + v^2}, \tag{3.6}$$

if either $u \neq 0$, or $v \neq 0$ in $\mathbb{R}$, and

$$\mathbf{q} = [q] = \begin{pmatrix} x + yi & 0 \\ 0 & x - yi \end{pmatrix}, \tag{3.7}$$

in $\mathcal{H}_2$, if $u = 0 = v$ in $\mathbb{R}$ by Definition 3.2. In this section, we consider the similarity on q-spectral forms "in $\mathcal{H}_2$".

Suppose $b \in \mathbb{C}^{\times}$ in (3.5). For $t \in \mathbb{C}^{\times}$, define a (2×2)-matrix $Q_t(q)$ by

$$Q_t(q) = \begin{pmatrix} t & -\overline{t\left(\frac{a-\lambda}{b}\right)} \\ t\left(\frac{a-\lambda}{b}\right) & \overline{t} \end{pmatrix}, \tag{3.8}$$

in $M_2(\mathbb{C})$, where $q \in \mathbb{H}$ is in the sense of (3.5).

By the assumption that $t, b \in \mathbb{C}^{\times}$, the nonzero matrix $Q_t(q)$ of (3.8) is well-defined in $M_2(\mathbb{C})$. Note that this matrix $Q_t(q)$ is invertible, since

$$\det\left(Q_t(q)\right) = |t|^2 \left(1 + \left|\frac{a-\lambda}{b}\right|^2\right) \neq 0 \quad \text{in } \mathbb{C}, \tag{3.9}$$

by the condition that $t, b \in \mathbb{C}^{\times}$. Observe now that

$$\begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} t & -\overline{\left(\frac{a-\lambda}{b}\right) t} \\ \left(\frac{a-\lambda}{b}\right) t & \overline{t} \end{pmatrix} = \begin{pmatrix} \lambda t & \left(\frac{\overline{\lambda}a - |a|^2}{\overline{b}}\right)\overline{t} - b\overline{t} \\ \overline{\left(\frac{\overline{\lambda}a - |a|^2}{\overline{b}}\right)\overline{t} - b\overline{t}} & \overline{\lambda t} \end{pmatrix}, \tag{3.10}$$

and

$$\begin{pmatrix} t & -\overline{\left(\frac{a-\lambda}{b}\right) t} \\ \left(\frac{a-\lambda}{b}\right) t & \overline{t} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} = \begin{pmatrix} \lambda t & -\overline{\left(\frac{a\lambda - \lambda^2}{b}\right) t} \\ \left(\frac{a\lambda - \lambda^2}{b}\right) t & \overline{\lambda t} \end{pmatrix}. \tag{3.11}$$

In the computations (3.10) and (3.11), let us compare their $(1,2)$-entries:

$$\begin{aligned}
&\left(\frac{\overline{\lambda}a-|a|^2}{\overline{b}}\right)\overline{t}-b\overline{t}\\
&=\overline{t}\left(\frac{(x+yi)\left(x-i\sqrt{y^2+u^2+v^2}\right)-\left(x^2+y^2\right)}{u-vi}-(u+vi)\right)\\
&=\overline{t}\left(\frac{-ix\sqrt{y^2+u^2+v^2}+xyi+y\sqrt{y^2+u^2+v^2}-y^2}{u-vi}-\frac{u^2+v^2}{u-vi}\right)\\
&=\overline{t}\left(\frac{\left(y\sqrt{y^2+u^2+v^2}-y^2-u^2-v^2\right)+i\left(xy-x\sqrt{y^2+u^2+v^2}+xy\right)}{u-vi}\right),
\end{aligned} \tag{3.12}$$

respectively

$$\begin{aligned}
&-\overline{\left(\frac{a\lambda-\lambda^2}{b}\right)t}=\overline{t}\overline{\left(\frac{-a\lambda+\lambda^2}{b}\right)}\\
&=\overline{t}\left(\frac{x^2-2xi\sqrt{y^2+u^2+v^2}-(y^2+u^2+v^2)-\left(x^2-ix\sqrt{y^2+u^2+v^2}-xyi-y\sqrt{y^2+u^2+v^2}\right)}{u-vi}\right)\\
&=\overline{t}\left(\frac{\left(y\sqrt{y^2+u^2+v^2}-y^2-u^2-v^2\right)+i\left(xy-x\sqrt{y^2+u^2+v^2}+xy\right)}{u-vi}\right).
\end{aligned} \tag{3.13}$$

By (3.12) and (3.13), the $(1,2)$-entry of $[q]Q_t(q)$, and that of $Q_t(q)\mathbf{q}$ are same. Therefore,

$$[q]Q_t(q)=Q_t(q)\mathbf{q}, \quad \text{whenever } t,b\in\mathbb{C}^{\times}, \tag{3.14}$$

in $M_2(\mathbb{C})$ by (3.10) and (3.11). Note that the (2×2)-matrix $Q_t(q)$ of (3.8) is contained in the noncommutative field $\mathcal{H}_2$ by (2.13) (which implies the invertibility (3.9) in $M_2(\mathbb{C})$ automatically), whenever $t,b\in\mathbb{C}^{\times}$.

Theorem 3.3. *Let $q=a+bj\in\mathbb{H}$ be a quaternion (3.5) with its realization $[q]\in\mathcal{H}_2$, and let $\mathbf{q}\in\mathcal{H}_2$ be the q-spectral form of q. If $b\neq 0$ in $\mathbb{C}$, then*

$$\mathbf{q}=Q_t(q)^{-1}[q]Q_t(q)\Longleftrightarrow[q]=Q_t(q)\mathbf{q}Q_t(q)^{-1}$$

in $\mathcal{H}_2$, where

$$Q_t(q)=\begin{pmatrix} t & -\overline{\left(\frac{a-\lambda}{b}\right)t} \\ \left(\frac{a-\lambda}{b}\right)t & \overline{t}\end{pmatrix}\in\mathcal{H}_2, \tag{3.15}$$

for all $t\in\mathbb{C}^{\times}$. Meanwhile, if $b=0$ in $\mathbb{C}$, then

$$\mathbf{q}=[w]^{-1}\mathbf{q}[w]=[w]^{-1}[q][w] \quad \textit{in } \mathcal{H}_2, \tag{3.16}$$

where

$$w=w+0j+0k\in\mathbb{C}^{\times} \quad \textit{in } \mathbb{H}. \tag{3.17}$$

Proof. First, suppose that $b = 0$ in $\mathbb{C}$, and hence, $q = a + 0j$ in $\mathbb{H}$. Then, by (3.6), the quaternion q has its q-spectral form

$$\mathbf{q} = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} = [q] \quad \text{in } \mathcal{H}_2,$$

by (3.7). Suppose $w \in \mathbb{C}^{\times}$, and $w = w + 0j + 0k \in \mathbb{H}$, realized to be $[w] \in \mathcal{H}_2$. Then

$$\begin{aligned} \mathbf{q} = [q] = \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} &= \begin{pmatrix} \frac{wa}{w} & 0 \\ 0 & \overline{\left(\frac{wa}{w}\right)} \end{pmatrix} \\ &= \begin{pmatrix} w & 0 \\ 0 & \overline{w} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \begin{pmatrix} w^{-1} & 0 \\ 0 & \overline{w^{-1}} \end{pmatrix} \\ &= [w][q][w^{-1}] = [w]\mathbf{q}[w]^{-1}, \end{aligned}$$

in $\mathcal{H}_2$. Therefore, the relation (3.16) holds true, whenever $w \in \mathbb{C}^{\times} \subset \mathbb{H}$ are in the sense of (3.17).

Assume now that $b \neq 0$ in $\mathbb{C}$. Then, for any $t \in \mathbb{C}^{\times}$, the corresponding matrices $Q_t(q)$ of (3.8) satisfy

$$Q_t(q)\mathbf{q} = [q]Q_t(q),$$

by (3.6) and (3.14). Thus, by the invertibility (3.9) of $Q_t(q)$,

$$Q_t(q)^{-1}\left(Q_t(q)\mathbf{q}\right) = Q_t(q)^{-1}[q]Q_t(q) \quad \text{in } \mathcal{H}_2,$$

if and only if

$$\mathbf{q} = Q_t(q)^{-1}[q]Q_t(q) \quad \text{in } \mathcal{H}_2. \tag{3.18}$$

So, the relation (3.15) holds by (3.18), whenever $b \neq 0$ in $\mathbb{C}$. □

Let $z, a \in \mathbb{C}$ and $b \in \mathbb{C}^{\times}$, and let $q = a + bj \in \mathbb{H}$. Observe that if we regard $z \in \mathbb{C}$ as a quaternion $z + 0j + 0k \in \mathbb{H}$, then

$$[z][q] = \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} = \begin{pmatrix} az & -bz \\ \overline{bz} & \overline{az} \end{pmatrix},$$

and

$$[q][z] = \begin{pmatrix} a & -b \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} = \begin{pmatrix} az & -b\overline{z} \\ \overline{b}z & \overline{az} \end{pmatrix},$$

in $\mathcal{H}_2$, i.e.,

$$[z][q] \neq [q][z] \Longleftrightarrow [z] \neq [q][z][q]^{-1} \quad \text{in } \mathcal{H}_2,$$

in general. Meanwhile, if $z \in \mathbb{R}$ in $\mathbb{H}$, then

$$[z] = [q][z][q]^{-1} \quad \text{in } \mathcal{H}_2.$$

It explains not only that the relation (3.15) is meaningful "in $\mathcal{H}_2$", but also why we put the above condition (3.17) to show the relation (3.16).

The above theorem shows that, for a quaternion $q \in \mathbb{H}$ with its q-spectral form $\mathbf{q} \in \mathcal{H}_2$, there exists at least one nonzero matrix $A \in \mathcal{H}_2$, such that

$$\mathbf{q} = A^{-1}[q]A, \quad \text{or} \quad [q] = A\mathbf{q}A^{-1}, \tag{3.19}$$

in $\mathcal{H}_2$.

Corollary 3.4. *Let $q = a + bj \in \mathbb{H}$ be a quaternion (3.5) with $b \neq 0$ in $\mathbb{C}$, and let*

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C} \quad \text{in } \mathbb{H}.$$

Then there exist

$$y_t = t + \left(-\overline{t\left(\frac{a-\lambda}{b}\right)}\right) j \in \mathbb{H},$$

for any $t \in \mathbb{C}^{\times}$, such that

$$q = y_t \lambda y_t^{-1} \quad \text{in } \mathbb{H}. \tag{3.20}$$

Meanwhile, if $b = 0$ in $\mathbb{C}$, then there exists non-zero $h \in \mathbb{C} \subset \mathbb{H}$, such that

$$q = hqh^{-1} \quad \text{in } \mathbb{H}. \tag{3.21}$$

Proof. Recall first that the noncommutative field $\mathcal{H}_2$ and the quaternions $\mathbb{H}$ are isomorphic by (2.14). So, a matrix $Q_t(q) \in \mathcal{H}_2$ of (3.9) is assigned to be a quaternion,

$$y_t = t + \left(-\overline{t\left(\frac{a-\lambda}{b}\right)}\right) j \in \mathbb{H},$$

equivalently,

$$[y_t] = Q_t(q) \quad \text{in } \mathcal{H}_2, \text{ for all } t \in \mathbb{C}^{\times}.$$

So, the formula (3.15) satisfies that

$$[q] = [y_t][\lambda][y_t]^{-1} \quad \text{in } \mathcal{H}_2, \text{ by (3.21)},$$

if and only if

$$[q] = [y_t \lambda y_t^{-1}] \quad \text{in } \mathcal{H}_2, \text{ by (2.9)},$$

if and only if

$$q = y_t \lambda y_t^{-1} \quad \text{in } \mathbb{H}, \text{ by (2.14)}.$$

Therefore, the relation (3.20) holds true.

Meanwhile, the q-spectral form $\mathbf{q}$ is assigned to be a quaternion,

$$q = \lambda + 0j + 0k \in \mathbb{H}.$$

Then $[\lambda] = \mathbf{q}$ in $\mathcal{H}_2$. Thus, the relation (3.21) holds by (3.16) and (3.17). □

The above corollary shows that the relations (3.15) and (3.16) on the noncommutative field $\mathcal{H}_2$ is equivalent to the relation (3.20) and (3.21), respectively, on the quaternions $\mathbb{H}$. It also shows the relation between a quaternion $q \in \mathbb{H}$ of (3.5) and an eigenvalue $\lambda \in \mathbb{C}$ of the realization $[q]$, satisfying $\mathbf{q} = [\lambda]$ in $\mathcal{H}_2$, i.e., for any $q \in \mathbb{H}$ with its realization $[q] \in \mathcal{H}_2$, there exists at least one nonzero $q_0 \in \mathbb{H}$, such that

$$q = q_0 \lambda q_0^{-1} \quad \text{in } \mathbb{H}, \tag{3.22}$$

where $spec\,([q]) = \{\lambda, \overline{\lambda}\}$ in $\mathbb{C}$, as in (3.19).

Definition 3.5. Let $q \in \mathbb{H}$ be a quaternion with its realization $[q] \in \mathcal{H}_2$, and let

$$\mathbf{q} = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} = [\lambda] \in \mathcal{H}_2$$

be the q-spectral form. Then the (1, 1)-entry $\lambda \in \mathbb{C}$ of $\mathbf{q}$ is called the quaternion-spectral value (in short, q-spectral value) of q.

3.3. QUATERNION-SPECTRAL EQUIVALENCE

In this section, we let an arbitrary fixed quaternion $q \in \mathbb{H}$ be in the sense of (3.5). Define a relation $\mathcal{R}$ on $\mathbb{H}$ by

$$q_1 \mathcal{R} q_2 \overset{def}{\Longleftrightarrow} \lambda_1 = \lambda_2 \text{ in } \mathbb{C}, \tag{3.23}$$

where λ_l are the q-spectral values of q_l, for $l = 1, 2$.

This relation $\mathcal{R}$ of (3.23) is an equivalence relation on $\mathbb{H}$, because

$$q \mathcal{R} q, \quad \text{for all } q \in \mathbb{H},$$

and

$$q_1 \mathcal{R} q_2 \Longleftrightarrow \lambda_1 = \lambda_2 \Longleftrightarrow \lambda_2 = \lambda_1 \Longleftrightarrow q_2 \mathcal{R} q_1,$$

for $q_1, q_2 \in \mathbb{H}$, and

$$q_1 \mathcal{R} q_2, \text{ and } q_2 \mathcal{R} q_3 \Longleftrightarrow \lambda_1 = \lambda_2 = \lambda_3$$

$\Longleftrightarrow$

$$\lambda_1 = \lambda_3 \Longleftrightarrow q_1 \mathcal{R} q_3,$$

for $q_1, q_2, q_3 \in \mathbb{H}$, where λ_l are the q-spectral values of q_l, for all $l = 1, 2, 3$.

Definition 3.6. The equivalence relation $\mathcal{R}$ of (3.23) is called the quaternion-spectral equivalence relation (in short, the q-spectral relation) on $\mathbb{H}$. And two q-spectral equivalent quaternions q_1 and q_2 are said to be q-spectral related.

Let $q_l = a_l + b_l j$ be q-spectral related quaternions in $\mathbb{H}$, with $b_l \neq 0$ in $\mathbb{C}$, and let $\lambda \in \mathbb{C}$ be the identical q-spectral value of q_l, for $l = 1, 2$. Then there exists $y_l \in \mathbb{H}$ such that

$$q_l = y_l \lambda y_l^{-1} \quad \text{in } \mathbb{H}, \text{ for } l = 1, 2, \tag{3.24}$$

by (3.22).

In particular, if $b_l \neq 0$ in $\mathbb{C}$, then

$$y_l = t + \left(- \overline{t\left(\frac{a_l - \lambda}{b_l}\right)} \right) j \in \mathbb{H}, \quad \text{for } l = 1, 2, \tag{3.25}$$

by (3.20). Meanwhile, if $b_l = 0$ in $\mathbb{C}$, then $y_l \in \mathbb{C}^{\times}$ in $\mathbb{H}$, by (3.21). So, one can have that

$$q_2 = y_2 \lambda y_2^{-1} = y_2 \left(y_1^{-1} y_1\right) \lambda \left(y_1^{-1} y_1\right) y_2^{-1}$$

by (3.24)

$$= \left(y_2 y_1^{-1}\right) \left(y_1 \lambda y_1^{-1}\right) \left(y_1 y_2^{-1}\right)$$

by (2.14)

$$= \left(y_2 y_1^{-1}\right) q_1 \left(y_2 y_1^{-1}\right)^{-1}, \tag{3.26}$$

in $\mathbb{H}$.

Recall that two matrices A_1 and A_2 are *similar* in a matricial ring $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, if there exists an invertible matrix $U \in M_n(\mathbb{C})$, such that

$$A_2 = U A_1 U^{-1} \quad \text{in } M_n(\mathbb{C}). \tag{3.27}$$

It is also well-know that if two matrices A_1 and A_2 are similar in the sense of (3.27), then

$$spec(A_1) = spec(A_2) \quad \text{in } \mathbb{C}, \tag{3.28}$$

and vice versa, in $M_n(\mathbb{C})$, i.e., the similarity (3.27) is satisfied, if and only if the set-equality (3.28) holds.

Definition 3.7. Let $q_l \in \mathbb{H}$ be quaternions realized to be $[q_l] \in \mathcal{H}_2$, for $l = 1, 2$. The realizations $[q_1]$ and $[q_2]$ are said to be similar "in $\mathcal{H}_2$" if there exists a nonzero matrix U "in $\mathcal{H}_2$", such that

$$[q_2] = U[q_1]U^{-1} \quad \text{"in } \mathcal{H}_2\text{".} \tag{3.29}$$

By abusing notation, two quaternions q_1 and q_2 are said to be similar in $\mathbb{H}$, if their realizations $[q_1]$ and $[q_2]$ are similar in the sense of (3.29).

Remark that, since $\mathcal{H}_2$ is a noncommutative field (in $M_2(\mathbb{C})$), if $U \in \mathcal{H}_2$ is a nonzero matrix, then it is automatically invertible by (3.9). So, the *similarity* (3.29) *on* $\mathcal{H}_2$ (and hence, the similarity on $\mathbb{H}$) is determined by the similarity (3.27) on $M_2(\mathbb{C})$ under restricted conditions. In this sense, the similarity on $\mathcal{H}_2$ (and hence, that on $\mathbb{H}$) is an equivalence relation, because the similarity (3.27) on $M_2(\mathbb{C})$ is an equivalence relation.

Theorem 3.8. *Two quaternions q_1 and q_2 are q-spectral related, if and only if they are similar in the sense of* (3.29) *in $\mathbb{H}$, i.e., as equivalence relations,*

$$\textit{the q-spectral relation on } \mathbb{H} = \textit{the similarity on } \mathbb{H}. \tag{3.30}$$

Proof. ($\Rightarrow$) Suppose q_1 and q_2 are q-spectral related in $\mathbb{H}$. Then, by (3.26), (3.28) and (3.29), they are similar in $\mathbb{H}$.

($\Leftarrow$) Suppose q_1 and q_2 are similar in $\mathbb{H}$, equivalently, assume that their realizations $[q_1]$ and $[q_2]$ are similar in $\mathcal{H}_2$. If λ_l are the q-spectral values of q_l, then $[q_l]$ and $[\lambda_l]$ are similar in the sense of (3.29) in $\mathcal{H}_2$, too, for all $l = 1, 2$. So, since the similarity on $\mathcal{H}_2$ is an equivalence relation, the q-spectral forms $[\lambda_1]$ and $[\lambda_2]$ are similar in $\mathcal{H}_2$. Since

$$[\lambda_l] = \begin{pmatrix} \lambda_l & 0 \\ 0 & \overline{\lambda_l} \end{pmatrix} \in \mathcal{H}_2, \quad \text{for } l = 1, 2,$$

the similarity of them guarantees that

$$[\lambda_1] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \overline{\lambda_2} \end{pmatrix} = [\lambda_2],$$

by (3.29), and hence,

$$\lambda_1 = \lambda = \lambda_2 \quad \text{in } \mathbb{C}.$$

It means that $\lambda \in \mathbb{C}$ is the q-spectral value of both q_1 and q_2 in $\mathbb{H}$. Therefore, if q_1 and q_2 are similar in $\mathbb{H}$, then they are q-spectral related in $\mathbb{H}$. □

By (3.30), we will use the q-spectral relation $\mathcal{R}$ of (3.23) on $\mathbb{H}$, and the similarity (3.29) on $\mathbb{H}$ alternatively.

3.4. QUATERNION-SPECTRAL MAPPING THEOREM

Throughout this section, we let

$$q = x + yi + uj + vk \in \mathbb{H}$$

be a quaternion with its q-spectral value,

$$\lambda = x + i\sqrt{y^2 + u^2 + v^2},$$

if either $u \neq 0$ or $v \neq 0$ in $\mathbb{R}$, or

$$\lambda = x + yi,$$

if $u = 0 = v$ in $\mathbb{R}$.

Now, let $\mathbb{C}[z]$ be the (pure-algebraic) polynomial ring over a field $\mathbb{C}$ of all complex numbers in a variable z,

$$\mathbb{C}[z] = \{f(z) : f \text{ is a polynomial in } z \text{ over } \mathbb{C}\},$$

i.e., $\mathbb{C}[z]$ is a ring equipped with the polynomial addition, and the polynomial multiplication of all polynomial,

$$\sum_{n=0}^{k} a_n z^n, \quad \text{for } a_n \in \mathbb{C}, \text{for all } n = 1, \ldots, k, \tag{3.31}$$

in a variable z, for all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

It is well-known that if A is a matrix in $M_n(\mathbb{C})$ for $n \in \mathbb{N}$, and if $f \in \mathbb{C}[z]$ is a polynomial (3.31), then

$$spec\,(f(A)) = f\,(spec(A)) \quad \text{in } \mathbb{C}, \tag{3.32}$$

by the *spectral mapping theorem*, where the right-hand side of (3.32) means that

$$f\,(spec(A)) = \{f(t) : t \in spec(A)\}.$$

In the left-hand side of (3.32), a new matrix $f(A) \in M_n(\mathbb{C})$ is

$$a_k A^k + a_{k-1} A^{k-1} + \ldots + a_2 A^2 + a_1 A + a_0 I_n,$$

where I_n is the identity $(n \times n)$-matrix of $M_n(\mathbb{C})$, whenever $f(z)$ is in the sense of (3.31).

By (3.32), one can get that

$$spec\,(f\,([q])) = f\,(spec\,([q]))\,, \quad \text{for all } f \in \mathbb{C}[z], \tag{3.33}$$

"in $M_2\,(\mathbb{C})$", for all $q \in \mathbb{H}$, realized to be $[q] \in \mathcal{H}_2$ in $M_2\,(\mathbb{C})$.

Lemma 3.9. *Let $g(z) \in \mathbb{C}[z]$, and let $q \in \mathbb{H}$ realized to be $[q] \in \mathcal{H}_2$ as a matrix of $M_2(\mathbb{C})$. Then*

$$spec\,(g\,([q])) = g\,(spec\,([q])) \quad \textit{in } \mathbb{C}. \tag{3.34}$$

Proof. The relation (3.34) is proven by (3.32) and (3.33) in $M_2\,(\mathbb{C})$. □

Now, let us define the subset $\mathbb{C}_r[z]$ of $\mathbb{C}[z]$ by

$$\mathbb{C}_r[z] = \bigcup_{N=0}^{\infty} \left\{ \sum_{n=0}^{N} a_n z^n \in \mathbb{C}[z] : a_0, a_1, \ldots, a_N \in \mathbb{R} \right\}, \tag{3.35}$$

i.e., the set $\mathbb{C}_r[z]$ of (3.35) consists of all polynomials in z over $\mathbb{C}$ with $\mathbb{R}$-coefficients.

Theorem 3.10. *Let $q \in \mathbb{H}$ be a quaternion* (3.5) *with its q-spectral value $\lambda \in \mathbb{C}$. If*

$$f(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}_r[z],$$

then

$$f(\lambda) \in \mathbb{C} \textit{ is the } q\textit{-spectral value of } f(q) \in \mathbb{H}, \tag{3.36}$$

where $\mathbb{C}_r[z]$ is the subset (3.35) *of $\mathbb{C}[z]$, and $f(q) = \sum_{n=0}^{N} a_n q^n$ in $\mathbb{H}$.*

Proof. Let $q \in \mathbb{H}$ be a quaternion (3.5) with its q-spectral value $\lambda \in \mathbb{C}$, and let $h(z) \in \mathbb{C}[z]$. If $[q] \in \mathcal{H}_2$ is the realization of q, then

$$spec\,(h\,([q])) = \left\{h\,(\lambda)\,,\; h\left(\overline{\lambda}\right)\right\}, \quad \text{in } \mathbb{C},$$

by (3.34).

Note however that, for $h(z) \in \mathbb{C}[z]$,

$$h\left(\overline{\lambda}\right) \neq \overline{h(\lambda)} \quad \text{in } \mathbb{C}, \text{ in general.}$$

(For instance, if $h(z) = iz$ in $\mathbb{C}[z]$, then $\overline{h(1+i)} = -1 - i \neq 1 + i = h\left(\overline{1+i}\right)$.)

However, if $f(z) = \sum_{n=0}^{N} a_n z^n \in \mathbb{C}_r[z]$ with $a_0, a_1, \ldots, a_N \in \mathbb{R}$, then

$$f\left(\overline{\lambda}\right) = \sum_{n=0}^{N} a_n \left(\overline{\lambda}\right)^n = \sum_{n=0}^{N} a_n \left(\overline{\lambda^n}\right) = \sum_{n=0}^{N} \overline{(a_n \lambda^n)} = \overline{\sum_{n=0}^{N} a_n \lambda^n} = \overline{f(\lambda)},$$

in $\mathbb{C}$. It shows that, if $f(z) \in \mathbb{C}_r[z]$, then

$$spec\left(f\left([q]\right)\right) = \left\{f(\lambda), f\left(\overline{\lambda}\right)\right\} = \left\{f\left(\lambda\right), \overline{f\left(\lambda\right)}\right\},$$

in $\mathbb{C}$, satisfying that

$$\text{the } q\text{-spectral form of } f\left([q]\right) = f\left(\mathbf{q}\right)$$

in $\mathcal{H}_2$, if and only if the q-spectral value of $f(q)$ is identified with $f(\lambda)$ in $\mathbb{C} \subset \mathbb{H}$, where $\mathbf{q}$ is the q-spectral form of $[q]$ in $\mathcal{H}_2$. Therefore, the statement (3.36) holds. □

Remark that the statement (3.36) holds for the polynomials of $\mathbb{C}_r[z]$, not those of $\mathbb{C}[z]$ (in general). Now, let $\mathbb{R}[x]$ be the polynomial ring over $\mathbb{R}$ in a variable x, i.e.,

$$\mathbb{R}[x] = \bigcup_{N=0}^{\infty} \left\{ \sum_{n=0}^{N} a_n x^n : a_0, a_1, \ldots, a_N \in \mathbb{R} \right\}. \tag{3.37}$$

Then, the above theorem can be re-stated as follows.

Corollary 3.11. *Let $f(x) \in \mathbb{R}[x]$, where $\mathbb{R}[x]$ is the polynomial ring (3.37). If $q \in \mathbb{H}$ is a quaternion with its q-spectral value $\lambda \in \mathbb{C}$, realized to be $[q] \in \mathcal{H}_2$, then*

$$spec\left(f\left([q]\right)\right) = \{f(\lambda), \overline{f(\lambda)}\} \quad \textit{in } \mathbb{C}. \tag{3.38}$$

Proof. The quantity $f(\lambda) \in \mathbb{C}$ is the q-spectral value of $f(q) \in \mathbb{H}$ by (3.36) and (3.37). Therefore, the set-equality (3.38) holds. □

We call the relation (3.38), the *quaternion-spectral mapping theorem.*

Theorem 3.12. *Let q_1 and q_2 be q-spectral related in $\mathbb{H}$, with their q-spectral value $\lambda \in \mathbb{C}$. If $f(x) \in \mathbb{R}[x]$, then $f(q_1)$ and $f(q_2)$ are q-spectral related in $\mathbb{H}$, too, with their identical q-spectral value $f(\lambda) \in \mathbb{C}$. Equivalently, if q_1 and q_2 are similar in $\mathbb{H}$, then $f(q_1)$ and $f(q_2)$ are similar in $\mathbb{H}$, for all $f(x) \in \mathbb{R}[x]$.*

Proof. Let q_1 and q_2 be q-spectral related quaternions in $\mathbb{H}$. Assume that $\lambda \in \mathbb{C}$ is the q-spectral value of both q_1 and q_2. Then, for any $f(x) \in \mathbb{R}[x]$, the quantity $f(\lambda) \in \mathbb{C}$ is the q-spectral value of both $f(q_1)$ and $f(q_2)$ by (3.36) and (3.38). Therefore, two quaternions $f(q_1)$ and $f(q_2)$ are q-spectral related in $\mathbb{H}$. By (3.30), the q-spectral relation and the similarity are equivalent on $\mathbb{H}$. So, if q_1 and q_2 are similar, then $f(q_1)$ and $f(q_2)$ are similar in $\mathbb{H}$, for all $f(x) \in \mathbb{R}[x]$. □

3.5. THE QUATERNION-SPECTRALIZATION σ

Motivated by the main results of Sections 3.3 and 3.4, a certain type of functions from $\mathbb{H}$ to $\mathbb{C}$ is considered here. Define a function $\sigma : \mathbb{H} \to \mathbb{H}$ by

$$\sigma(q) \overset{def}{=} \text{ the } q\text{-spectral value of } q, \quad \text{for all } q \in \mathbb{H}. \tag{3.39}$$

For example,

$$\sigma\left(1 + 0i + 2j - 3k\right) = 1 + i\sqrt{0^2 + 2^2 + (-3)^2} = 1 + \sqrt{13}i,$$

and

$$\sigma\left(-2 - i + 0j + 0k\right) = -2 - i,$$

etc.

Definition 3.13. We call the function σ of (3.39), the quaternion-spectralization (in short, the q-spectralization).

Let us consider the range of the q-spectralization σ.

Theorem 3.14. *If σ is the q-spectralization* (3.39), *then*

$$\sigma(\mathbb{H}) = \mathbb{C}. \tag{3.40}$$

Proof. Let $q = x + yi + uj + vk \in \mathbb{H}$ be an arbitrary quaternion. If σ is the q-spectralization (3.39), then

$$\sigma\left(q\right) = x + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C},$$

(if either $u \neq 0$, or $v \neq 0$), or

$$\sigma\left(q\right) = x + yi \in \mathbb{C},$$

(if $u = 0 = v$) in $\mathbb{H}$. So, one has

$$\sigma\left(\mathbb{H}\right) \subset \mathbb{C}.$$

Now, let $t + si \in \mathbb{C}$, with $t, s \in \mathbb{R}$. If $s \geq 0$ in $\mathbb{R}$, then there exists

$$h = t + yi + uj + vk \in \mathbb{H}, \quad \text{with } t, y, u, v \in \mathbb{R},$$

such that

$$\sigma\left(h\right) = t + i\sqrt{y^2 + u^2 + v^2} \in \mathbb{C},$$

satisfying

$$\sqrt{y^2 + u^2 + v^2} = s \quad \text{in } \mathbb{R},$$

by (3.39). While, if $s < 0$ in $\mathbb{R}$, then there exists

$$h = t + si + 0j + 0k \in \mathbb{H},$$

such that

$$\sigma\left(h\right) = t + si \quad \text{in } \mathbb{C},$$

i.e., for any $z \in \mathbb{C}$, there exists at least one quaternion $q \in \mathbb{H}$, such that $\sigma(q) = z$, and hence,

$$\mathbb{C} \subset \sigma(\mathbb{H}).$$

Therefore, set-equality (3.40) holds, i.e.,

$$\sigma(\mathbb{H}) = \{\sigma(q) : q \in \mathbb{H}\} = \mathbb{C}. \qquad \square$$

By (3.40), the q-spectralization σ is onto $\mathbb{C}$, and hence, it is not surjective on $\mathbb{H}$. Also, it is not injective.

4. THE QUATERNION-SPECTRALIZATION

In this section, we consider the q-spectralization (3.39) more in detail.

Corollary 4.1. *Let σ be the q-spectralization* (3.39)*. Then $\sigma(q_1) = \sigma(q_2)$ in $\mathbb{C}$, if and only if q_1 and q_2 are similar in $\mathbb{H}$.*

Proof. The q-spectralization σ satisfies $\sigma(q_1) = \sigma(q_2)$ in $\mathbb{C}$ for $q_1, q_2 \in \mathbb{H}$. Then, by definition, the quaternions q_1 and q_2 are q-spectral related. Conversely, if q_1 and q_2 are q-spectral related, then $\sigma(q_1) = \sigma(q_2)$. So, $\sigma(q_1) = \sigma(q_2)$, if and only if q_1 and q_2 are q-spectral related, if and only if they are similar in $\mathbb{H}$, by (3.30). $\square$

The above corollary confirms that all quaternions are classified by the q-spectralization σ. For a fixed quaternion $q \in \mathbb{H}$, define the subset

$$q^o = \{h \in \mathbb{H} : \sigma(h) = \sigma(q)\}, \tag{4.1}$$

in $\mathbb{H}$. Then, by the above corollary, it forms the equivalence class of $q \in \mathbb{H}$, under the similarity (or, the q-spectral relation). And

$$q^o = (\sigma(q))^o \quad \text{in } \mathbb{H},$$

set-theoretically. Thus,

$$\sigma(q) \in \mathbb{C} \subset \mathbb{H}$$

becomes a representative of all quaternions of q^o in $\mathbb{H}$, by (3.40).

Define now the quotient set $\mathbb{H}^o$ by

$$\mathbb{H}^o \overset{\text{def}}{=} \{q^o : q \in \mathbb{H}\}, \tag{4.2}$$

where q^o are the equivalence classes (4.1).

Theorem 4.2. *Let $\mathbb{H}^o$ be the quotient set* (4.2)*. Then*

$$\mathbb{H}^o = \mathbb{C}, \tag{4.3}$$

set-theoretically.

Proof. Note first that $q^o = (\sigma(q))^o$ in $\mathbb{H}^o$ by (4.1) and Corollary 4.1, for all $q \in \mathbb{H}$. Therefore, by (3.40) and (4.2),

$$\mathbb{H}^o = \{\lambda^o : \lambda \in \mathbb{C}, \text{there exists } q \in \mathbb{H} \text{ such that } \sigma(q) = \lambda\},$$

and hence,

$$\mathbb{H}^o = \{\lambda^o : \lambda \in \mathbb{C}\}. \tag{4.4}$$

Define a function $\varphi : \mathbb{H}^o \to \mathbb{C}$ by

$$\varphi(\lambda^o) = \lambda, \quad \text{for all } \lambda^o \in \mathbb{H}^o,$$

where λ^o is in the sense of (4.4) for $\lambda \in \mathbb{C}$. Then, by Corollary 4.1 and (4.4), the function φ is surjective from $\mathbb{H}^o$ onto $\mathbb{C}$.

Also, for $\lambda_1^o, \lambda_2^o \in \mathbb{H}^o$ (in the sense of (4.4)), if

$$\varphi(\lambda_1^o) = \lambda_1 = \lambda_2 = \varphi(\lambda_2^o) \text{ in } \mathbb{C},$$

then $\lambda_1^o = \lambda_2^o$ in $\mathbb{H}^o$, by Corollary 4.1, (4.2) and (4.4), i.e., φ is injective, too.

Therefore, the function φ is a bijection, and hence, the set-equality (4.3) holds. □

The above theorem shows that the quaternions $\mathbb{H}$ is classified by the q-spectral relation (or, the similarity, or the action of the q-spectralization σ). And the classification is characterized by the set $\mathbb{C}$ of all complex numbers in the sense that: every quantity $\lambda \in \mathbb{C}$ represents all quaternions $q \in \mathbb{H}$ satisfying $\sigma(q) = \lambda$.

5. QUATERNIONIC VECTOR SPACES

In this section, we consider a finite-dimensional vector space $\mathbb{H}^n$ over the quaternions $\mathbb{H}$, for $n \in \mathbb{N}$. Since $\mathbb{H}$ is a noncommutative field (and hence, a ring), vector spaces over $\mathbb{H}$ are well-determined pure algebraically, as a vector space over a ring.

5.1. AN n-DIMENSIONAL VECTOR SPACE $\mathbb{H}^n$ OVER $\mathbb{H}$

Let $\mathbb{H}$ be the quaternions, and let $n \in \mathbb{N}$ be fixed throughout this section. Define naturally a set $\mathbb{H}^n$ by the set

$$\mathbb{H}^n \overset{def}{=} \{(q_1, \ldots, q_n) : q_1, \ldots, q_n \in \mathbb{H}\}, \tag{5.1}$$

of all n-tuples of quaternions, i.e., the set $\mathbb{H}^n$ is the iterated Cartesian product of n-copies of the quaternions $\mathbb{H}$.

Define now a binary operation $(+)$ on $\mathbb{H}^n$ by

$$(q_1, \ldots, q_n) + (h_1, \ldots, h_n) = (q_1 + h_1, \ldots, q_n + h_n), \tag{5.2}$$

for all $(q_1, \ldots, q_n), (h_1, \ldots, h_n) \in \mathbb{H}^n$, where the addition $(+)$ in the right-hand side of (5.2) means the quaternion-addition of (2.2).

Define the left scalar product and the right scalar product on $\mathbb{H}^n$ by

$$q\,(q_1,\ \ldots,\ q_n) = (qq_1, \ldots, qq_n)\,,\qquad (q_1, \ldots, q_n)\,q = (q_1 q, \ldots, q_n q)\,, \tag{5.3}$$

respectively, for all $q \in \mathbb{H}, (q_1,\ \ldots,\ q_n) \in \mathbb{H}^n$, where the multiplication $(\cdot)$ in the right-hand sides of (5.3) is the quaternion-multiplication of (2.2). Then, the scalar products of (5.3) are well-defined on $\mathbb{H}^n$. From below, if there is no confusion, we simply say "$(\cdot)$ is the scalar product (5.3)", meaning the scalar products from both left and right as in (5.3).

Definition 5.1. The mathematical triple $(\mathbb{H}^n,\ +,\ \cdot)$ is called the n-dimensional quaternion-vector space over $\mathbb{H}$ (in short, the n-dimensional $\mathbb{H}$-vector space), where $\mathbb{H}^n$ is the set (5.1), $(+)$ is the addition (5.2), and $(\cdot)$ is the scalar product (5.3). And all elements, the n-tuples, of $(\mathbb{H}^n, +, \cdot)$ are said to be n-dimensional quaternion-vectors (in short, $\mathbb{H}^n$-vectors). For convenience, we denote $(\mathbb{H}^n, +, \cdot)$ simply by $\mathbb{H}^n$ from below.

More generally, one can extend the above definition as follows.

Definition 5.2. Let V be a set containing $\mathbb{H}$ as its subset. Assume that V is equipped with a well-defined addition $(+)$, and a scalar product $(\cdot)$ over $\mathbb{H}$, in the sense that:

$$v_1 + v_2 \in V, \quad \text{for all}\, v_1, v_2 \in V,$$

respectively,

$$qv,\ vq \in V, \quad \text{for all}\, q \in \mathbb{H} \text{ and } v \in V.$$

Then the triple $(V, +, \cdot)$ is called a vector space over the quaternions $\mathbb{H}$ (in short, a $\mathbb{H}$-vector space). All elements of $(V, +, \cdot)$ are said to be $\mathbb{H}$-vectors.

By the above definition, every k-dimensional $\mathbb{H}$-vector space $\mathbb{H}^k$ is a $\mathbb{H}$-vector space, for all $k \in \mathbb{N}$.

Definition 5.3. Let V_1 and V_2 be $\mathbb{H}$-vector spaces. A function $T : V_1 \to V_2$ is said to be a linear transformation over the quaternions $\mathbb{H}$ (or, in short, $\mathbb{H}$-linear transformation) if

$$T\,(v_1 + v_2) = T(v_1) + T(v_2),$$

and

$$T\,(qv) = qT(v), \quad T\,(vq) = T\,(v)\,q, \tag{5.4}$$

for all $q \in \mathbb{H}$, and $v, v_1, v_2 \in V_1$. If a $\mathbb{H}$-linear transformation T of (5.4) is bijective, then it is called a $\mathbb{H}$-vector-space-isomorphism (or, in short, a $\mathbb{H}$-isomorphism). In particular, if T is a $\mathbb{H}$-isomorphism, then two $\mathbb{H}$-vector spaces V_1 and V_2 are said to be $\mathbb{H}$-isomorphic.

Let $\mathcal{H}_2$ be the isomorphic noncommutative field (2.13) of the quaternions $\mathbb{H}$ for the representation $(\mathbb{C}^2,\ \pi)$ by (2.14). Define now a set $\mathcal{H}_2^n$ by the Cartesian product of n-copies of $\mathcal{H}_2$,

$$\mathcal{H}_2^n = \{([q_1],\ \ldots,\ [q_n]) : [q_1],\ \ldots,\ [q_n] \in \mathcal{H}_2\}\,. \tag{5.5}$$

Define a binary operation $(+)$, and a (left-and-right) scalar-product(s) $(\cdot)$ on the set $\mathcal{H}_2^n$ of (5.5) by

$$\begin{aligned}([q_1], \ldots, [q_n]) + ([h_1], \ldots, [h_n]) &\overset{def}{=} ([q_1] + [h_1], \ldots, [q_n] + [n_n]) \\ &= ([q_1 + h_1], \ \ldots, \ [q_n + h_n]),\end{aligned}$$

and

$$\begin{aligned}q\,([q_1], \ldots, [q_n]) &\overset{def}{=} ([q][q_1], \ \ldots, \ [q][q_n]) \\ &= ([qq_1], \ \ldots, \ [qq_n]),\end{aligned} \tag{5.6}$$

and

$$([q_1], \ldots, [q_n])\, q = ([q_1 q], \ldots, [q_n q]),$$

for all $q, q_1, \ldots, q_n, h_1, \ldots, h_n \in \mathbb{H}$. Note that the operations of (5.6) are well-defined on the set $\mathcal{H}_2^n$ of (5.5) by (2.9) and (2.14).

Lemma 5.4. *The triple* $\mathcal{H}_2^n := (\mathcal{H}_2^n, \ +, \ \cdot)$ *of the set* $\mathcal{H}_2^n$ *of* (5.5) *and the operations* $(+)$ *and* $(\cdot)$ *of* (5.6) *is a* $\mathbb{H}$*-vector space.*

Proof. As we discussed in the very above paragraph, the operations of (5.6) are well-defined on the set $\mathcal{H}_2^n$. Equivalently, the triple $\mathcal{H}_2^n$ forms a $\mathbb{H}$-vector space. □

By the isomorphism theorem (2.14), one can verify that two $\mathbb{H}$-vector spaces $\mathbb{H}^n$ and $\mathcal{H}_2^n$ are related as follow.

Theorem 5.5. *The n-dimensional* $\mathbb{H}$*-vector space* $\mathbb{H}^n$ *and the* $\mathbb{H}$*-vector space* $\mathcal{H}_2^n$ *are* $\mathbb{H}$*-isomorphic, i.e.,*

$$\mathbb{H}^n \overset{q\text{-}iso}{=} \mathcal{H}_2^n, \tag{5.7}$$

where "$\overset{q\text{-}iso}{=}$*" means "being* $\mathbb{H}$*-isomorphic".*

Proof. To show the two $\mathbb{H}$-vector spaces $\mathbb{H}^n$ and $\mathcal{H}_2^n$ are $\mathbb{H}$-isomorphic, we construct a $\mathbb{H}$-isomorphism Π from $\mathbb{H}^n$ to $\mathcal{H}_2^n$. Let $(\mathbb{C}^2, \pi)$ be the representation of the quaternions $\mathbb{H}$, inducing the isomorphic noncommutative field $\mathcal{H}_2$ by (2.14). Define a function

$$\Pi : \mathbb{H}^n \to \mathcal{H}_2^n$$

by

$$\Pi\left((q_1, \ \ldots, \ q_n)\right) \overset{def}{=} (\pi(q_1), \ \ldots, \ \pi(q_n)) = ([q_1], \ \ldots, \ [q_n]), \tag{5.8}$$

for all $q_1, \ldots, q_n \in \mathbb{H}$.

Since the action $\pi : \mathbb{H} \to \mathcal{H}_2$ is a bijection, the above function Π of (5.8) is bijective from $\mathbb{H}^n$ onto $\mathcal{H}_2^n$. Also, it satisfies that

$$\Pi\left((q_1, \ldots, q_n) + (h_1, \ldots, h_n)\right) = \Pi\left((q_1 + h_1, \ \ldots, \ q_n + h_n)\right)$$

by (5.2)

$$= ([q_1 + h_1], \ \ldots, \ [q_n + h_n])$$

by (5.8)
$$= ([q_1] + [h_1],\ \ldots,\ [q_n] + [h_n])$$
by (2.9)
$$= ([q_1],\ \ldots,\ [q_n]) + ([h_1],\ \ldots,\ [h_n])$$
by (5.6)
$$= \Pi\left((q_1,\ \ldots,\ q_n)\right) + \Pi\left((h_1,\ \ldots,\ h_n)\right),$$

for all $(q_1, \ldots, q_n), (h_1, \ldots, h_n) \in \mathbb{H}^n$. i.e., the addition on $\mathbb{H}^n$ is preserved to be that on $\mathcal{H}_2^n$ by Π.

Observe now that

$$\Pi\left(q(q_1,\ \ldots,\ q_n)\right) = \Pi\left((qq_1,\ \ldots,\ qq_n)\right)$$

by (5.3)
$$= ([qq_1],\ \ldots,\ [qq_n])$$
by (5.8)
$$= ([q][q_1],\ \ldots,\ [q][q_n])$$
by (2.9)
$$= q\left([q_1],\ \ldots,\ [q_n]\right)$$
by (5.6)
$$= q\Pi\left((q_1,\ \ldots,\ q_n)\right),$$
and similarly,
$$\Pi\left((q_1, \ldots, q_n)\, q\right) = \Pi\left((q_1, \ldots, q_n)\right) q,$$

for all $q \in \mathbb{H}$, and $(q_1, \ldots, q_n) \in \mathbb{H}^n$, i.e., the scalar product on $\mathbb{H}^n$ is preserved to be that on $\mathcal{H}_2^n$ by Π over $\mathbb{H}$.

Therefore, this bijection Π is a $\mathbb{H}$-linear transformation, and hence, it is a $\mathbb{H}$-isomorphism over $\mathbb{H}$ by (5.4). So, the isomorphic relation (5.7) holds. □

By the relation (5.7), two $\mathbb{H}$-vector spaces $\mathbb{H}^n$ and $\mathcal{H}_2^n$ are regarded as a same $\mathbb{H}$-vector space over $\mathbb{H}$. So, if needed, we use them alternatively as our n-dimensional $\mathbb{H}$-vector space.

In the rest of this section, we consider certain weaker concepts of $\mathbb{H}$-linear transformations.

Definition 5.6. Let V be a $\mathbb{H}$-vector space, and let W be a R-vector space (i.e., a vector space over a ring R), where R is a subring of $\mathbb{H}$. If a function $T : V \to W$ satisfies

$$T\left(\mathbf{v} + \mathbf{w}\right) = T\left(\mathbf{v}\right) + T\left(\mathbf{w}\right),$$

and

$$T\left(r\mathbf{v}\right) = rT\left(\mathbf{v}\right), \quad T\left(\mathbf{v}r\right) = T\left(\mathbf{v}\right) r,$$

in W, for all $\boldsymbol{v}, \mathbf{w} \in V$ and $r \in R$, then this function T is called a linear transformation over R (or, in short, a R-linear transformation). If a R-linear transformation T is bijective, then it is said to be an R-isomorphism; and, in such a case, V and W are said to be R-isomorphic.

By the above definition, one can obtain the following result.

Theorem 5.7. *The n-dimensional $\mathbb{H}$-vector space $\mathbb{H}^n$ is $\mathbb{R}$-isomorphic to $\mathbb{C}^{2n}$. And it is also $\mathbb{R}$-isomorphic to $\mathbb{R}^4$, i.e.,*

$$\mathbb{H}^n \overset{real\text{-}iso}{=} \mathbb{C}^{2n}, \quad \text{and} \quad \mathbb{H}^n \overset{real\text{-}iso}{=} \mathbb{R}^{4n}, \tag{5.9}$$

where "$\overset{real\text{-}iso}{=}$" means "being $\mathbb{R}$-isomorphic".

Proof. Define a function $T : \mathbb{H}^n \to \mathbb{C}^{2n}$ by

$$T\left(q_1, q_2 \ldots, q_n\right) = \left(a_1, b_1, a_2, b_2, \ldots, a_n, b_n\right),$$

for all $(q_1, \ldots, q_n) \in \mathbb{H}^n$, where $q_l = a_l + b_l j$, with $a_l, b_l \in \mathbb{C}$, for all $l=1, \ldots, n$. Then it is not difficult to show this function T is a $\mathbb{R}$-isomorphism, i.e., the first $\mathbb{R}$-isomorphic relation of (5.9) holds.

Now, define a function $S : \mathbb{H}^n \to \mathbb{R}^{4n}$ by

$$S\left(q_1, \ldots, q_n\right) = \left(x_1, y_1, u_1, v_1, \ldots, x_n, y_n, u_n, v_n\right),$$

for all $(q_1, \ldots, q_n) \in \mathbb{H}^n$, with

$$q_l = x_l + y_l i + u_l j + v_l k \in \mathbb{H}, \quad \text{with } x_l, y_l, u_l, v_l \in \mathbb{R},$$

for all $l = 1, \ldots, n$. Then, similarly, it is shown that it is a $\mathbb{R}$-isomorphism, i.e., the second $\mathbb{R}$-isomorphic relation of (5.9) holds, too. □

The above theorem, itself, seems trivial by constructions, however, it is meaningful to demonstrate that the study of $\mathbb{R}$-linear transformations on $\mathbb{H}^n$ (as functions on $\mathbb{H}^n$) has close connections with the usual functional analysis (including matrix theory, operator theory, and noncommutative geometry, etc.).

5.2. THE FUNCTION $\sum^n : \mathbb{H}^n \to \mathcal{H}_2^n$

In Section 5.1, we considered $\mathbb{H}$-vector spaces $\mathbb{H}^k$ and $\mathbb{H}$-isomorphic vector spaces $\mathcal{H}_2^k$, for $k \in \mathbb{N}$. In this section, motivated by Section 4, we fix $n \in \mathbb{N}$, and study a certain function

$$\Sigma^n : \mathbb{H}^n \to \mathcal{H}_2^n,$$

implying our spectral analytic considerations in Sections 2 and 3.

Let $\sigma : \mathbb{H} \to \mathbb{H}$ be the q-spectralization (3.39), i.e., for all $x + yi + uj + vk \in \mathbb{H}$,

$$\sigma\left(x + yi + uj + vk\right) = x + i\sqrt{y^2 + u^2 + v^2},$$

if either $u \neq 0$ or $v \neq 0$, and

$$\sigma\left(x + yi + 0j + 0k\right) = x + yi.$$

Then this morphism σ induces the q-spectral forms (3.6), or (3.7) of quaternions, i.e., the following diagram assigns a function

$$q \overset{\sigma}{\longmapsto} \sigma(q) \overset{\pi}{\longmapsto} \mathbf{q} = \begin{pmatrix} \sigma(q) & 0 \\ 0 & \overline{\sigma(q)} \end{pmatrix},$$

in $\mathcal{H}_2$. Define a function $\sum : \mathbb{H} \to \mathcal{H}_2$ by

$$\sum \overset{def}{=} \pi \circ \sigma,$$

i.e.,

$$\sum(q) = \begin{pmatrix} \sigma(q) & 0 \\ 0 & \overline{\sigma(q)} \end{pmatrix} = [\sigma(q)], \tag{5.10}$$

for all $q \in \mathbb{H}$. Since $\sigma : \mathbb{H} \to \mathbb{H}$ is a well-defined function whose range is $\mathbb{C}$ by (3.40), and the action $\pi : \mathbb{H} \to \mathcal{H}_2$ is a well-defined bijection, the function $\sum$ of (5.10) is well-defined.

Observe that if $q_1 = 2 + i - j + k$ and $q_2 = 2 - i + j + k$ are quaternions, then

$$\sum (q_1) = \begin{pmatrix} 2 + \sqrt{3}i & 0 \\ 0 & 2 - \sqrt{3}i \end{pmatrix} = \sum (q_2),$$

in $\mathcal{H}_2$. It shows that the function $\sum$ is not injective. Moreover, since $\sigma(\mathbb{H}) = \mathbb{C}$ in $\mathbb{H}$, this function $\sum$ is not surjective either, i.e., this function $\sum$ is neither injective nor surjective from $\mathbb{H}$ to $\mathcal{H}_2$.

Also, consider that if

$$q_1 = 1 + 0i + j + 0k,$$

and

$$q_2 = 2 - i + j + 0k,$$

in $\mathbb{H}$, then

$$\sum (q_1) = \begin{pmatrix} 1 + i & 0 \\ 0 & 1 - i \end{pmatrix},$$

respectively,

$$\sum (q_2) = \begin{pmatrix} 2 + \sqrt{2}i & 0 \\ 0 & 2 - \sqrt{2}i \end{pmatrix}, \tag{5.11}$$

in $\mathcal{H}_2$. If q_1 and q_2 are as above, then

$$q_1 + q_2 = 3 - i + 2j + 0k,$$

satisfying

$$\sum (q_1 + q_2) = \begin{pmatrix} 3 + \sqrt{5}i & 0 \\ 0 & 3 - \sqrt{5}i \end{pmatrix}. \tag{5.12}$$

The formulas (5.11) and (5.12) shows that

$$\sum (q_1 + q_2) \neq \sum (q_1) + \sum (q_2),$$

in $\mathcal{H}_2$. It implies that the function $\sum$ is not linear either.

Lemma 5.8. *The function* $\sum$ *of* (5.10) *is a function from* $\mathbb{H}$ *into* $\mathcal{H}_2$ *"over* $\mathbb{R}$*", in the sense that: it is a well-defined function from* $\mathbb{H}$ *into* $\mathcal{H}_2$ *satisfying*

$$\sum(tq) = t\sum(q) = \sum(q)\,t = \sum(qt)\,, \tag{5.13}$$

for all $q \in \mathbb{H}$, *and "*$t \in \mathbb{R}$*", where* $zA \in \mathcal{H}_2$ *means*

$$(zI_2)\,(A) \text{ in } \mathcal{H}_2 \subset M_2\,(\mathbb{C})\,,$$

for all $z \in \mathbb{C}$ *and* $A \in M_2\,(\mathbb{C})$.

Proof. Let $\sum : \mathbb{H} \to \mathcal{H}_2$ be the function (5.10). In the very above paragraphs, we showed that $\sum = \pi \circ \sigma$ is a well-defined function, but it is not linear. But this function is over $\mathbb{R}$ in the sense of (5.13). Indeed, if $t \in \mathbb{R}$ and $q \in \mathbb{H}$, then

$$\sigma\,(tq) = t\sigma\,(q) = \sigma\,(q)\,t = \sigma\,(qt) \quad \text{in } \mathbb{H},$$

by the q-spectral mapping theorem (3.38). Since π is a well-defined bijective action of $\mathbb{H}$,

$$\pi\,(\sigma\,(tq)) = \pi\,(t\sigma\,(q)) = t\,[\sigma\,(q)] = [\sigma\,(q)]\,t = \pi\,(\sigma\,(qt))\,,$$

in $\mathcal{H}_2$. Remark that the second and the third equalities hold because $[\sigma\,(q)]$ is a diagonal matrix in $\mathcal{H}_2$ and $t \in \mathbb{R}$. □

Definition 5.9. Let V_1 and V_2 be $\mathbb{H}$-vector spaces. A function $f : V_1 \to V_2$ is said to be over $\mathbb{R}$, if

$$f\,(tw) = tf(w), \quad \text{and} \quad f\,(wt) = f\,(w)\,t,$$

in V_2, for all $w \in V_1$ and $t \in \mathbb{R}$. Similarly, f is said to be over $\mathbb{C}$, or over $\mathbb{H}$, if the above equalities hold for all $t \in \mathbb{C}$, respectively, for all $t \in \mathbb{H}$.

Now, let $\mathbb{H}^n$ be the n-dimensional $\mathbb{H}$-vector space, and let $\mathcal{H}_2^n$ be an isomorphic $\mathbb{H}$-vector space of $\mathbb{H}^n$ (by (5.7)). Define a function,

$$\sum^n : \mathbb{H}^n \to \mathcal{H}_2^n$$

by

$$\sum^n ((q_1, \ldots, q_n)) = \left(\sum(q_1), \ \ldots, \ \sum(q_n)\right) \text{ in } \mathcal{H}_2^n, \tag{5.14}$$

for all $(q_1, \ldots, q_n) \in \mathbb{H}^n$.

In the above lemma, we showed that $\sum^1 = \sum$ is a well-defined function over $\mathbb{R}$. So, one can verify that the function $\sum^n$ of (5.14) is over $\mathbb{R}$, too, in the sense of Definition 5.9:

$$\Sigma^n\,(tw) = t\Sigma^n\,(w) = \Sigma^n\,(w)\,t = \Sigma^n\,(wt)\,,$$

in $\mathcal{H}_2^n$, for all $w \in \mathbb{H}^n$, and $t \in \mathbb{R}$. Indeed, in $\mathbb{H}^n$,

$$tw = wt \quad \text{in } \mathbb{H}^n, \text{ for all } w \in \mathbb{H}^n,\ t \in \mathbb{R},$$

and hence, the above second and the third equalities hold automatically. Thus, by Lemma 5.8, one obtains the following result.

Lemma 5.10. *The function* $\sum^n : \mathbb{H}^n \to \mathcal{H}_2^n$ *of* (5.14) *is a function over* $\mathbb{R}$.

Proof. Since a function $\sum : \mathbb{H} \to \mathcal{H}_2$ of (5.10) is well-defined, the morphism $\sum^n$ is a well-defined function from $\mathbb{H}^n$ to $\mathcal{H}_2^n$ by (5.14).

Now, let $t \in \mathbb{R}$, and $w = (q_1, \ldots, q_n) \in \mathbb{H}^n$. Then

$$\sum^n (tw) = \sum^n ((tq_1, \ldots, tq_n))$$

by (5.3)

$$= \left(\sum(tq_1), \ldots, \sum(tq_n)\right)$$

by (5.14)

$$= \left(t\sum(q_1), \ldots, t\sum(q_n)\right) \tag{5.15}$$

by (5.13)

$$= t\left(\sum(q_1), \ldots, \sum(q_n)\right)$$
$$= t\sum^n ((q_1, \ldots, q_n)) = t\sum^n(w).$$

So, the function $\sum^n$ is a function over $\mathbb{R}$ by (5.15). □

5.3. THE QUATERNION-SPECTRALIZATION ON $\mathbb{H}^n$

In Section 5.2, we showed that there exists a well-defined function $\sum^n : \mathbb{H}^n \to \mathcal{H}_2^n$ over $\mathbb{R}$, where

$$\sum^n = \prod_{k=1}^n \sum, \quad \text{with } \sum = \pi \circ \sigma,$$

by (5.10) and (5.14), for $n \in \mathbb{N}$, where π is the action of $\mathbb{H}$ on $\mathcal{H}_2$, and $\sigma : \mathbb{H} \to \mathbb{C}$ is the q-spectralization.

Throughout this section, we fix $n \in \mathbb{N}$. Define a function,

$$\sigma^n : \mathbb{H}^n \to \mathbb{H}^n$$

by

$$\sigma^n ((q_1, \ldots, q_n)) = (\sigma(q_1), \ldots, \sigma(q_n)) \quad \text{in } \mathbb{C}^n \subset \mathbb{H}^n, \tag{5.16}$$

for all $(q_1, \ldots, q_n) \in \mathbb{C}^n$. The well-definedness of this function σ^n of (5.16) is guaranteed by that of $\sum^n$ by (4.3). By the definition (5.16) and (4.3), it is verified canonically that

$$\sigma^n (\mathbb{H}^n) = \mathbb{C}^n \quad \text{in } \mathbb{H}^n.$$

Note that, since the q-spectralization σ is not ($\mathbb{H}$-, or $\mathbb{R}$-)linear, the function σ^n on $\mathbb{H}^n$ is not linear either by (5.16). But it is over $\mathbb{R}$ in the sense of Definition 5.9.

Lemma 5.11. *The function $\sigma^n : \mathbb{H}^n \to \mathbb{H}^n$ of (5.16) is a function over $\mathbb{R}$.*

Proof. The function σ^n is over $\mathbb{R}$ because the function $\sum^n$ is over $\mathbb{R}$. Indeed, for any $t \in \mathbb{R}$, and $v = (q_1, \ldots, q_n) \in \mathbb{H}^n$, one has that

$$\sigma^n(tv) = \sigma^n\left((tq_1, \ldots, tq_n)\right) = (t\sigma(q_1), \ldots, t\sigma(q_n))$$

by (3.38)

$$= t\left(\sigma(q_1), \ldots, \sigma(q_n)\right) = t\sigma^n\left((q_1, \ldots, q_n)\right). \tag{5.17}$$

Therefore, the function σ^n is over $\mathbb{R}$ by (5.17). ☐

Definition 5.12. For any $n \in \mathbb{N}$, the function $\sigma^n : \mathbb{H}^n \to \mathbb{H}^n$ of (5.16) over $\mathbb{R}$ is called the (n-)quaternion-spectralization (in short, the q-spectralization) on $\mathbb{H}^n$. Our q-spectralization σ of (3.39) is covered by this generalized definition, i.e., $\sigma = \sigma^1$ on $\mathbb{H} = \mathbb{H}^1$.

6. NONCOMMUTATIVE UNITAL RINGS $\sum_n(\mathbb{H})$ OVER $\mathbb{R}$

In Section 5, we introduced two types of functions acting on the $\mathbb{H}$-vector space $\mathbb{H}^n$, for $n \in \mathbb{N}$. The first one is the function

$$\Sigma^n : \mathbb{H}^n \to \mathcal{H}_2^n$$

of (5.14), and the second one is the function

$$\sigma^n : \mathbb{H}^n \to \mathbb{H}^n$$

called the q-spectrailization. These functions are not $\mathbb{H}$-linear, but they are over $\mathbb{R}$ in the sense of Definition 5.9.

Since $\mathcal{H}_2^n$ and $\mathbb{H}^n$ are $\mathbb{H}$-isomorphic, and since $\sum = \pi \circ \sigma$, the functional properties of $\sum^n$ and σ^n would be same up to $\mathbb{H}$-isomorphic relation. So, we may identify $\sum^n$ and σ^n as a same function, called the q-spectralization on $\mathbb{H}^n$.

Also, by (4.3), the range of σ^n is $\mathbb{C}^n$, i.e.,

$$\sigma^n\left(\mathbb{H}^n\right) = \mathbb{C}^n,$$

set-theoretically.

Theorem 6.1. *The q-spectralization σ^n is idempotent in the sense that*

$$\sigma^n \circ \sigma^n = \sigma^n \text{ on } \mathbb{H}^n. \tag{6.1}$$

Proof. Let σ^n be our q-spectralization on $\mathbb{H}^n$. For any $\mathbb{H}$-vector $w = (q_1, \ldots, q_n)$,

$$\begin{aligned}(\sigma^n \circ \sigma^n)(w) &= \sigma^n\left(\sigma^n\left((q_1, \ldots, q_n)\right)\right) \\ &= \sigma^n\left((\sigma(q_1), \ldots, \sigma(q_n))\right) \\ &= \left(\sigma\left(\sigma(q_1), \ldots, \sigma\left(\sigma(q_n)\right)\right)\right).\end{aligned} \tag{6.2}$$

Observe that, since $\sigma(q_l)$ are the q-spectral values of quaternions q_l, they are $\mathbb{C}$-quantities in $\mathbb{H}$, for all $l = 1, \ldots, n$. By Definition 3.2, if

$$q = \lambda + 0j + 0k, \quad \text{with } \lambda \in \mathbb{C},$$

then

$$\sigma(q) = \lambda = q \quad \text{in } \mathbb{C} \subset \mathbb{H}. \tag{6.3}$$

So, since $\sigma(q_l) \in \mathbb{C} \subset \mathbb{H}$, we have

$$\sigma(\sigma(q_l)) = \sigma(q_l), \quad \text{for all } l = 1, \ldots, n,$$

in (6.2), by (6.3).

Thus the equality (6.2) satisfies that

$$\sigma^n(\sigma^n(w)) = (\sigma(q_1), \ldots, \sigma(q_n)) = \sigma^n(w),$$

in $\mathbb{C}^n \subset \mathbb{H}^n$. Since $w \in \mathbb{H}^n$ is arbitrary,

$$\sigma^n \circ \sigma^n = \sigma^n \quad \text{on } \mathbb{H}^n,$$

i.e., the relation (6.1) holds. □

The above theorem shows the idempotence of our q-spectralizations σ^k on $\mathbb{H}^k$, for all $k \in \mathbb{N}$, by (6.1).

Now, let $M_n(\mathbb{C})$ be the matricial ring acting on $\mathbb{C}^n$. Define now a set $\sum_n(\mathbb{H})$ by

$$\sum_n(\mathbb{H}) \overset{def}{=} \{\alpha \circ \sigma^n : \alpha \in M_n(\mathbb{C})\}, \tag{6.4}$$

where $(\circ)$ is the functional composition. By the definition (6.4) of $\sum_n(\mathbb{H})$, all elements $\alpha \circ \sigma^n$ are the well-defined functions on $\mathbb{H}^n$ over $\mathbb{R}$ by (6.1).

Remark that, if

$$w_1 = (q_1, \ldots, q_n), \quad w_2 = (h_1, \ldots, h_n) \in \mathbb{H}^n, \quad \text{and} \quad t \in \mathbb{R},$$

then

$$\begin{aligned} \alpha \circ \sigma^n(w_1 + w_2) &= \alpha(\sigma^n((q_1 + h_1, \ldots, q_n + h_n))) \\ &= \alpha((\sigma(q_1 + h_1),\ \ldots,\ \sigma(q_n + h_n))) \\ &\neq \alpha(\sigma^n((q_1, \ldots, q_n))) + \alpha(\sigma^n((h_1, \ldots, h_n))) \\ &= \alpha \circ \sigma^n(v) + \alpha \circ \sigma^n(w), \end{aligned} \tag{6.5}$$

in general, because

$$\sigma(q_l + h_l) \neq \sigma(q_l) + \sigma(h_l),$$

in general, for $l = 1, \ldots, n$.

However,

$$\begin{aligned}\alpha \circ \sigma^n (tw_1) &= \alpha\left(\sigma^n\left((tq_1, \dots, tq_n)\right)\right) \\ &= \alpha\left((t\sigma(q_1),\ \dots,\ t\sigma(q_n))\right) \\ &= \alpha\left(t\sigma^n\left((q_1,\ \dots,\ q_n)\right)\right) \\ &= t\left(\alpha \circ \sigma^n\right)\left((q_1, \dots, q_n)\right) \\ &= (t\alpha) \circ \sigma^n(w_1) = \alpha \circ \left(t\sigma^n\right)(w_1),\end{aligned} \tag{6.6}$$

for all $t \in \mathbb{R}$, since $\alpha \in M_n(\mathbb{C})$ is (linear, and hence, it is) over $\mathbb{R}$, too.

Lemma 6.2. *Every element $\alpha \circ \sigma^n \in \Sigma_n(\mathbb{H})$ is a function over $\mathbb{R}$.*

Proof. Each element $\alpha \circ \sigma^n \in \sum_n(\mathbb{H})$ is not linear by (6.5), but it is over $\mathbb{R}$ by (6.6). □

From below, we denote $\alpha \circ \sigma^n \in \sum_n(\mathbb{H})$ simply by $\alpha^{(n)}$.

On the set $\sum_n(\mathbb{H})$ of (6.4), define the operations $(+)$ and $(\cdot)$ by

$$\begin{aligned}\alpha_1^{(n)} + \alpha_2^{(n)} &\overset{def}{=} (\alpha_1 + \alpha_2) \circ \sigma^{(n)} = (\alpha_1 + \alpha_2)^{(n)}, \\ \alpha_1^{(n)}\alpha_2^{(n)} &\overset{def}{=} (\alpha_1\alpha_2) \circ \sigma^{(n)} = (\alpha_1\alpha_2)^{(n)},\end{aligned} \tag{6.7}$$

for all $\alpha_1^{(n)}, \alpha_2^{(n)} \in \sum_n(\mathbb{H})$, respectively, where the addition $(+)$ and the multiplication $(\cdot)$ on the far right-hand sides of (6.7) are the usual matricial addition, and matricial multiplication on $M_n(\mathbb{C})$, respectively. These operations of (6.7) are well-defined (or closed) on $\sum_n(\mathbb{H})$ by (6.1), i.e., the triple,

$$\sum_n(\mathbb{H}) \overset{\text{denote}}{=} \left(\sum_n(\mathbb{H}),\ +,\ \cdot\right) \tag{6.8}$$

forms an algebraic structure, where $(+)$ and $(\cdot)$ of (6.8) are in the sense of (6.7).

Note that the operations of (6.7) can be well-defined by (3.40) and (6.1). It is trivial that the addition $(+)$ of (6.7) is well-defined. Observe now that if

$$w = (q_1, \dots, q_n) \in \mathbb{H}^n,$$

then

$$\left(\alpha_1^{(n)}\alpha_2^{(n)}\right)(w) = \alpha^{(1)}\left(\alpha_2\left(\sigma(q_1), \dots, \sigma(q_n)\right)\right) = \alpha_1\left(\sigma^n\left(z_1, \dots, z_n\right)\right),$$

where $(z_1, \dots, z_n) = \alpha_2\left(\sigma(q_1), \dots, \sigma(q_n)\right) \in \mathbb{C}^n(\text{in } \mathbb{H}^n)$,

$$= \alpha_1\left(z_1, \dots, z_n\right) = (\alpha_1\alpha_2)\left(\sigma(q_1), \dots, \sigma(q_n)\right)$$

by (6.1)

$$= \left(\alpha_1\alpha_2 \circ \sigma^n\right)(w) = (\alpha_1\alpha_2)^{(n)}(w),$$

in $\mathbb{H}^n$.

Since $w \in \mathbb{H}^n$ is arbitrary, indeed, we have

$$\alpha_1^{(n)} \alpha_2^{(n)} = (\alpha_1 \alpha_2)^{(n)} \quad \text{in } \Sigma_n(\mathbb{H}),$$

i.e., the multiplication $(\cdot)$ of (6.7) is well-defined, too.

Moreover, one can define a $\mathbb{R}$-scalar product on $\sum_n(\mathbb{H})$ by

$$t \cdot \alpha^{(n)} = t \cdot \left(\alpha \circ \sigma^{(n)}\right) = t\alpha \circ \sigma^n = \alpha \circ t\sigma^n, \tag{6.9}$$

for all $t \in \mathbb{R}$, and $\alpha^{(n)} \in \sum_n(\mathbb{H})$. Note that the scalar product (6.9) is well-defined over $\mathbb{R}$ by (6.6).

Theorem 6.3. *The triple $\sum_n(\mathbb{H})$ of* (6.8) *is a noncommutative unital ring over $\mathbb{R}$, in the sense that:* (i) *it is a noncommutative ring with its unity, and* (ii) *there is a well-defined $\mathbb{R}$-scalar product on $\sum_n(\mathbb{H})$.*

Proof. Let $\sum_n(\mathbb{H})$ be the algebraic triple (6.8). For the addition $(+)$ of (6.7),

$$\begin{aligned}\left(\alpha_1^{(n)} + \alpha_2^{(n)}\right) + \alpha_3^{(n)} &= (\alpha_1 + \alpha_2)^{(n)} + \alpha_3^{(n)} \\ &= ((\alpha_1 + \alpha_2) + \alpha_3)^{(n)} = (\alpha_1 + (\alpha_2 + \alpha_3))^{(n)} \\ &= \alpha_1^{(n)} + (\alpha_2 + \alpha_3)^{(n)} = \alpha_1^{(n)} + \left(\alpha_2^{(n)} + \alpha_3^{(n)}\right),\end{aligned}$$

for $\alpha_l^{(n)} \in \sum_n(\mathbb{H})$, for all $l = 1, 2, 3$. Also, there exists the zero matrix O_n of $M_n(\mathbb{C})$ inducing

$$0_n = O_n \circ \sigma^n \in \Sigma_n(\mathbb{H}),$$

such that

$$\alpha^{(n)} + 0^{(n)} = (\alpha + O_n)^{(n)} = \alpha^{(n)} = (O_n + \alpha)^{(n)} = 0^{(n)} + \alpha^{(n)},$$

for all $\alpha^{(n)} \in \sum_n(\mathbb{H})$.

For the $(+)$-identity $0^{(n)}$ and arbitrary $\alpha^{(n)} \in \sum_n(\mathbb{H})$, there exists a unique $(-\alpha)^{(n)} \in \sum_n(\mathbb{H})$, such that

$$\alpha^{(n)} + (-\alpha)^{(n)} = (\alpha - \alpha)^{(n)} = 0^{(n)} = (-\alpha + \alpha)^{(n)} = (-\alpha)^{(n)} + \alpha^{(n)},$$

in $\sum_n(\mathbb{H})$. Finally,

$$\alpha_1^{(n)} + \alpha_2^{(n)} = (\alpha_1 + \alpha_2)^{(n)} = (\alpha_2 + \alpha_1)^{(n)} = \alpha_2^{(n)} + \alpha_1^{(n)},$$

for all $\alpha_1^{(n)}, \alpha_2^{(n)} \in \sum_n(\mathbb{H})$. Thus, the pair $(\sum_n(\mathbb{H}),\ +)$ is an abelian group.

For the multiplication $(\cdot)$, one has that

$$\left(\alpha_1^{(n)} \alpha_2^{(n)}\right) \alpha_3^{(n)} = ((\alpha_1 \alpha_2)\alpha_3)^{(n)} = (\alpha_1(\alpha_2 \alpha_3))^{(n)} = \alpha_1^{(n)} \left(\alpha_2^{(n)} \alpha_3^{(n)}\right),$$

for $\alpha_l^{(n)} \in \sum_n(\mathbb{H})$, for all $l = 1, 2, 3$.

Since the matricial ring $M_n(\mathbb{C})$ is noncommutative, the above well-defined associative multiplication is noncommutative. So, the pair $(\sum_n(\mathbb{H}), \cdot)$ forms a noncommutative semigroup. Moreover, it has its unity (or the multiplication-identity),

$$1^{(n)} = I_n \circ \sigma^n \in \Sigma_n(\mathbb{H}),$$

where I_n is the identity matrix of $M_n(\mathbb{C})$, satisfying

$$\alpha^{(n)} \cdot 1^{(n)} = (\alpha \circ I_n)^{(n)} = \alpha^{(n)} = (I_n \circ \alpha)^{(n)} = 1^{(n)} \cdot \alpha^{(n)},$$

for all $\alpha^{(n)} \in \sum_n(\mathbb{H})$.

It is not hard to check the left-and-right distributiveness of the operations $(+)$ and $(\cdot)$ of (6.7) on $\sum_n(\mathbb{H})$, by those of matricial addition and multiplication on $M_2(\mathbb{C})$. So, the triple $\sum_n(\mathbb{H})$ of (6.8) is a unital ring.

Finally, the $\mathbb{R}$-scalar product (6.9) is well-defined on $\sum_n(\mathbb{H})$, because all matrices of $M_2(\mathbb{C})$ and our q-spectralization σ^n are over $\mathbb{R}$. Therefore, this ring is well-determined over $\mathbb{R}$. □

The above theorem characterizes the algebraic triple $\sum_n(\mathbb{H})$ of (6.8) as a noncommutative unital ring over $\mathbb{R}$.

Let $\alpha^{(n)} = \alpha \circ \sigma^n \in \sum_n(\mathbb{H})$, with $\alpha \in M_n(\mathbb{C})$. If α is invertible in $M_n(\mathbb{C})$, then $\alpha^{(n)}$ is invertible in $\sum_n(\mathbb{H})$, in the sense that: there exists a unique element $\beta^{(n)} = \beta \circ \sigma^n \in \sum_n(\mathbb{H})$, such that

$$\alpha^{(n)}\beta^{(n)} = 1^{(n)} = \beta^{(n)}\alpha^{(n)}, \tag{6.10}$$

in $\sum_n(\mathbb{H})$, where $1^{(n)} = I_n \circ \sigma^n$ is the unity of $\sum_n(\mathbb{H})$. Indeed, one can take

$$\beta = \alpha^{-1}, \quad \text{the inverse of } \alpha \text{ in } M_n(\mathbb{C}),$$

satisfying that

$$\alpha^{(n)} \left(\alpha^{-1}\right)^{(n)} = \left(\alpha\alpha^{-1}\right) \circ \sigma^n = 1^{(n)} = (\alpha^{-1}\alpha) \circ \sigma^n = \left(\alpha^{-1}\right)^{(n)} \alpha^{(n)},$$

in $\sum_n(\mathbb{H})$, i.e., the invertibility of α on $M_n(\mathbb{C})$ implies the invertibility of $\alpha^{(n)}$ on $\sum_n(\mathbb{H})$.

Remark 6.4. Remark that "$\alpha^{(n)}$ is invertible in $\sum_n(\mathbb{H})$" does not mean $\alpha^{(n)}$ is invertible "on $\mathbb{H}^n$", as a function. Indeed, by definition, $\alpha^{(n)}$ is neither injective nor surjective on $\mathbb{H}^n$, implying that it cannot be invertible on $\mathbb{H}^n$ "as a function", i.e., the invertibility (6.10) on $\sum_n(\mathbb{H})$ is pure-algebraic invertibility which does not imply the functional-invertibility of its elements on $\mathbb{H}^n$.

By the (pure-algebraic) invertibility on $\sum_n(\mathbb{H})$, one can define the subring $S_n(\mathbb{H})$ by

$$S_n(\mathbb{H}) \stackrel{def}{=} \left\{ \alpha^{(n)} \in \sum_n(\mathbb{H}) : \alpha^{(n)} \text{ is invertible in } \sum_n(\mathbb{H}) \right\}. \tag{6.11}$$

It is not difficult to check that the sub-structure $(S_n(\mathbb{H}), \ \cdot)$ of this subring $S_n(\mathbb{H})$ of (6.11) forms a non-abelian group, where $(\cdot)$ is the multiplication (6.7).

Definition 6.5. We call the noncommutative unital ring $\sum_n(\mathbb{H})$ of (6.8), the quaternion-spectral matricial ring acting on $\mathbb{H}^n$ (in short, the q-spectral ring). The subring $S_n(\mathbb{H})$ of (6.11) is called the invertible quaternion-spectral matricial ring (in short, the invertible q-spectral ring). In particular, the algebraic pair $(S_n(\mathbb{H}), \cdot)$ is said to be the invertible q(uaternion)-spectral group.

7. CERTAIN FUNCTIONAL EQUATIONS

In this section, we consider certain equations induced by elements of our q-spectral ring $\sum_n(\mathbb{H})$, which is a noncommutative unital ring over $\mathbb{R}$. By definition, every element $\alpha^{(n)}$ of $\sum_n(\mathbb{H})$ assigns $\mathbb{H}$-vectors to $\mathbb{C}$-vectors (embedded in $\mathbb{H}^n$), since $\alpha^{(n)}(\mathbb{H}^n) = \mathbb{C}^n$. So, one can naturally define an equation,

$$\alpha^{(n)}(\mathbf{v}) = w, \tag{7.1}$$

for a fixed $w \in \mathbb{C}^n$, where $\mathbf{v} \in \mathbb{H}^n$ is understood to be a $\mathbb{H}$-vector variable. We here study the solvability of an equation (7.1).

7.1. CERTAIN FUNCTIONAL EQUATIONS ON $\mathbb{H}$ INDUCED BY σ

In this section, before considering the equations (7.1), we study how to solve certain equations induced by the q-spectralization σ. Let $h \in \mathbb{H}$ be a $\mathbb{H}$-variable, and let

$$t \in \mathbb{R}^{\times} = \mathbb{R} \setminus \{0\}, \quad \text{and} \quad s \in \mathbb{R}$$

be arbitrarily fixed real numbers. Construct an equation for a fixed $z_0 \in \mathbb{C}$,

$$t\sigma(h) + s = z_0 \overset{let}{=} \lambda_1 + \lambda_2 i. \tag{7.2}$$

Suppose $h_0 = x + yi + uj + vk \in \mathbb{H}$ is a solution of (7.2). Assume first that

$$h_0 = x + yi + 0j + 0k \in \mathbb{H},$$

and hence,

$$\sigma(h_0) = x + yi \quad \text{in } \mathbb{H}. \tag{7.3}$$

Then it satisfies that

$$t\sigma(h_0) + s = z_0$$

$$\Longleftrightarrow$$

$$t(x + yi) + s = \lambda_1 + \lambda_2 i$$

$$\Longleftrightarrow$$

$$(tx + s) + tyi = \lambda_1 + \lambda_2 i$$

$$\Longleftrightarrow$$

$$x = \frac{\lambda_1 - s}{t}, \quad \text{and} \quad y = \frac{\lambda_2}{t} \quad \text{in } \mathbb{R}$$

$\Longleftrightarrow$

$$\sigma\left(h_0\right)=\left(\frac{\lambda_1-s}{t}\right)+\left(\frac{\lambda_2}{t}\right)i \quad \text{in } \mathbb{C}$$

$\Longleftrightarrow$

$$h_0=\left(\frac{\lambda_1-s}{t}\right)+\left(\frac{\lambda_2}{t}\right)i+0j+0k \quad \text{in } \mathbb{H}. \tag{7.4}$$

Assume now that

$$h_0=x+yi+uj+vk\in\mathbb{H}, \quad \text{with } u\in\mathbb{R}^{\times}, \text{ or } v\in\mathbb{R}^{\times},$$

and hence,

$$\sigma(h_0)=x+i\sqrt{y^2+u^2+v^2} \quad \text{in } \mathbb{C}. \tag{7.5}$$

If such a quaternion h_0 of (7.5) is a solution of (7.2), then

$$t\sigma(h_0)+s=z_0$$

$\Longleftrightarrow$

$$t\left(x+i\sqrt{y^2+u^2+v^2}\right)+s=\lambda_1+\lambda_2 i$$

$\Longleftrightarrow$

$$(tx+s)+\left(t\sqrt{y^2+u^2+v^2}\right)i=\lambda_1+\lambda_2 i$$

$\Longleftrightarrow$

$$x=\frac{\lambda_1-s}{t}, \quad \text{and} \quad \sqrt{y^2+u^2+v^2}=\frac{\lambda_2}{t} \quad \text{in } \mathbb{R}$$

$\Longleftrightarrow$

$$\sigma(h_0)=\left(\frac{\lambda_1-s}{t}\right)+\left(\frac{\lambda_2}{t}\right)i \quad \text{in } \mathbb{C}. \tag{7.6}$$

Lemma 7.1. *A quaternion $h_0=x+yi+uj+vk\in\mathbb{H}$ is a solution of* (7.2)*, if and only if*

$$x=\frac{\lambda_1-s}{t}, \quad \textit{and} \quad y^2+u^2+v^2=\left(\frac{\lambda_2}{t}\right)^2, \tag{7.7}$$

in $\mathbb{R}$. As a special case, a quaternion $h_0=x+yi+0j+0k$ is a solution, if and only if

$$h_0=\left(\frac{\lambda_1-s}{t}\right)+\left(\frac{\lambda_2}{t}\right)i+0j+0k \quad \textit{in } \mathbb{H}.$$

Proof. It is shown by (7.5) and (7.6), i.e., $h_0\in\mathbb{H}$ is a solution of (7.2), if and only if

$$h_0=x+yi+uj+uk\in\mathbb{H},$$

satisfying the condition (7.7). The special case is obtained by (7.3) and (7.4). $\square$

More general to (7.7), assume now that

$$t \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}, \quad \text{and} \quad s \in \mathbb{C}, \tag{7.8}$$

in the linear equation (7.2) under the generalized condition (7.8). Then, similar to (7.7), if

$$\begin{aligned} t &= t_1 + t_2 i \in \mathbb{C}, \quad \text{with } t_1, t_2 \in \mathbb{R}, \\ s &= s_1 + s_2 i \in \mathbb{C}, \quad \text{with } s_1, s_2 \in \mathbb{R}, \end{aligned}$$

then a quaternion

$$h_0 = x + yi + uj + vk \in \mathbb{H}$$

is a solution, if and only if

$$(t_1 + t_2 i)\left(x + i\sqrt{y^2 + u^2 + v^2}\right) + (s_1 + s_2 i) = \lambda_1 + \lambda_2 i$$

$$\Longleftrightarrow$$

$$(t_1 x - t_2 \Delta + s_1) + (t_2 x + t_1 \Delta + s_2)\, i = \lambda_1 + \lambda_2 i$$

where

$$\Delta = \sqrt{y^2 + u^2 + v^2} \in \mathbb{R}$$

$$\Longleftrightarrow$$

$$\begin{cases} t_1 x - t_2 \Delta + s_1 = \lambda_1, \\ t_2 x + t_1 \Delta + s_2 = \lambda_2 \end{cases}$$

$$\Longleftrightarrow$$

$$\begin{cases} t_1 x - t_2 \Delta = \lambda_1 - s_1, \\ t_2 x + t_1 \Delta = \lambda_2 - s_2. \end{cases} \tag{7.9}$$

If we solve the linear system (7.9), then

$$x = \frac{t_1\lambda_1 - t_1 s_1 + t_2 \lambda_2 - t_2 s_2}{t_1^2 + t_2^2},$$

and

$$\Delta^2 = \left(\frac{t_1\lambda_2 + t_2 s_1 - t_2\lambda_1 - t_1 s_2}{t_1^2 + t_2^2}\right)^2. \tag{7.10}$$

Theorem 7.2. *Let* $t = t_1 + t_2 i \in \mathbb{C}^{\times}$, *and* $s = s_1 + s_2 i, z_0 = \lambda_1 + \lambda_2 i \in \mathbb{C}$, *and* h, *a* $\mathbb{H}$*-variable, and let*

$$t\sigma(h) + s = z_0$$

be an equation (7.2) *on* $\mathbb{C}$. *A quaternion*

$$h_0 = x + yi + uj + vk \in \mathbb{H}$$

is a solution, if and only if

$$x = \frac{t_1\lambda_1 - t_1 s_1 + t_2 \lambda_2 - t_2 s_2}{t_1^2 + t_2^2},$$

and

$$y^2 + u^2 + v^2 = \left(\frac{t_1\lambda_2 + t_2 s_1 - t_2\lambda_1 - t_1 s_2}{t_1^2 + t_2^2}\right)^2, \tag{7.11}$$

in $\mathbb{R}$.

Proof. The relation (7.11) is proven by (7.9) and (7.10). □

Remark 7.3. Note that, in (7.10) and (7.11), the reason why we put

$$\Delta^2 = \left(\frac{t_1\lambda_2 + t_2 s_1 - t_2\lambda_1 - t_1 s_2}{t_1^2 + t_2^2}\right)^2,$$

where $\Delta = y^2 + u^2 + v^2$, instead of putting

$$\Delta = \frac{t_1\lambda_2 + t_2 s_1 - t_2\lambda_1 - t_1 s_2}{t_1^2 + t_2^2},$$

is that, in fact, there are two possible cases:

(i) if either $u \neq 0$, or $v \neq 0$, then

$$\sigma\left(x + yi + uj + vk\right) = x + i\sqrt{y^2 + u^2 + v^2} \in \mathbf{H}_+,$$

in $\mathbb{C}$, where $\mathbf{H}_+$ is the upper half plane of $\mathbb{C}$;

(ii) meanwhile,

$$\sigma\left(x + yi + 0j + 0k\right) = x + yi \in \mathbb{C},$$

by (3.39). Indeed, if a solution $h_0 \in \mathbb{H}$ of (7.11) is of the form

$$h_0 = x + yi + uj + vk, \quad \text{with } u \neq 0, \text{ or } v \neq 0,$$

then

$$\triangle = \frac{t_1\lambda_2 + t_2 s_1 - t_2\lambda_1 - t_1 s_2}{t_1^2 + t_2^2},$$

meanwhile, if h_0 is of the form,

$$h_0 = x + yi + 0j + 0k,$$

then

$$\triangle = y = \pm\left(\frac{t_1\lambda_2 + t_2 s_1 - t_2\lambda_1 - t_1 s_2}{t_1^2 + t_2^2}\right).$$

One can easily verify that the relation (7.11) implies the relation (7.7). Indeed, if $t \in \mathbb{R}^\times$, and $s \in \mathbb{R}$ in $\mathbb{C}$, then

$$t_1 = t, \quad s_1 = s, \quad \text{and} \quad t_2 = 0 = s_2, \quad \text{in } \mathbb{R}.$$

So, in such a case, the relation (7.11) goes to

$$x = \frac{t_1\lambda_1 - t_1 s}{t_1^2} = \frac{\lambda_1 - s}{t},$$

and

$$y^2 + u^2 + v^2 = \left(\frac{t_1\lambda_2}{t_1^2}\right)^2 = \left(\frac{\lambda_2}{t}\right)^2,$$

which is nothing but the relation (7.7).

7.2. EQUATIONS INDUCED BY $\sum_n(\mathbb{H})$

Fix $n \in \mathbb{N}$, and let $\sum_n(\mathbb{H})$ be the q-spectral ring (6.8) over $\mathbb{R}$. For $\alpha^{(n)} \in \sum_n(\mathbb{H})$, define a canonical equation

$$\alpha^{(n)}(w) = \mathbf{v}_0, \tag{7.12}$$

where

$$w = (h_1, \ldots, h_n)$$

is a $\mathbb{H}$-vector variable on $\mathbb{H}^n$, and

$$\mathbf{v}_0 = (a_1, \ldots, a_n) \in \mathbb{C}^n \subset \mathbb{H}^n$$

is an arbitrarily $\mathbb{H}$-vector. In fact, if there exists at least one $l_0 \in \{1, \ldots, n\}$ such that $a_{l_0} \in \mathbb{H} \setminus \mathbb{C}$, equivalently, if $\mathbf{v}_0 \in \mathbb{H}^n \setminus \mathbb{C}^n$, then the above equation (7.12) is not solvable, since

$$\sigma^n(\mathbb{H}^n) = \mathbb{C}^n, \quad \text{and} \quad \alpha \in M_n(\mathbb{C}),$$

implying that

$$\alpha^{(n)}(\mathbb{H}^n) \subset \mathbb{C}^n \quad \text{in } \mathbb{H}^n.$$

For convenience, we let

$$\alpha = [z_{ij}]_{n\times n} = \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix} \in M_n(\mathbb{C}). \tag{7.13}$$

If $\alpha \in M_n(\mathbb{C})$ is in the sense of (7.13), then the above equation (7.12) can be re-stated by

$$\alpha^{(n)}(w) = \mathbf{v}_0$$

$$\Longleftrightarrow \quad \begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix} \begin{pmatrix} \sigma(h_1) \\ \sigma(h_2) \\ \vdots \\ \sigma(h_n) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$\Longleftrightarrow \quad \begin{pmatrix} \sum_{l=1}^{n} z_{1l}\sigma(h_l) \\ \sum_{l=1}^{n} z_{2l}\sigma(h_l) \\ \vdots \\ \sum_{l=1}^{n} z_{nl}\sigma(h_l) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \tag{7.14}$$

i.e., as in the usual matrix theory, solving an equation (7.12) is equivalent to solve the linear system (7.14).

Suppose $\alpha^{(n)}$ is contained in the invertible q-spectral ring $S_n(\mathbb{H})$ in $\sum_n(\mathbb{H})$, equivalently, $\alpha \in M_n(\mathbb{C})$ is invertible with its inverse $\alpha^{-1} \in M_n(\mathbb{C})$. Then the equation (7.12) satisfies that

$$\alpha^{(n)}(w) = \mathbf{v}_0 \Longleftrightarrow (\alpha \circ \sigma^n)(w) = \mathbf{v}_0$$

$\Longleftrightarrow$

$$\alpha(\sigma^n(w)) = \mathbf{v}_0 \Longleftrightarrow \sigma^n(w) = \alpha^{-1}(\mathbf{v}_0). \tag{7.15}$$

So, if a matrix $\alpha = [z_{ij}]_{n\times n} \in M_n(\mathbb{C})$ of (7.13) is invertible with its inverse

$$\alpha^{-1} = [\lambda_{ij}]_{n\times n} \in M_n(\mathbb{C}), \tag{7.16}$$

and hence, $\alpha^{(n)} \in S_n(\mathbb{H})$ in $\sum_n(\mathbb{H})$, then the equation (7.12) is equivalent to

$$\begin{pmatrix}\sigma(h_1)\\ \sigma(h_2)\\ \vdots \\ \sigma(h_n)\end{pmatrix} = \alpha^{-1}(\mathbf{v}_0) = [\lambda_{ij}]_{n\times n}\begin{pmatrix}a_1\\ a_2\\ \vdots\\ a_n\end{pmatrix}$$

$\Longleftrightarrow$

$$\begin{pmatrix}\sigma(h_1)\\ \sigma(h_2)\\ \vdots \\ \sigma(h_n)\end{pmatrix} = \begin{pmatrix}\sum_{l=1}^{n}\lambda_{1l}a_l\\ \sum_{l=1}^{n}\lambda_{2l}a_l\\ \vdots\\ \sum_{l=1}^{n}\lambda_{nl}a_l\end{pmatrix}, \tag{7.17}$$

by (7.15) and (7.16).

Theorem 7.4. *Let $\alpha^{(n)}(w) = \mathbf{v}_0$ be an equation* (7.12). *Then it is solvable, if and only if* $\det(\alpha) \neq 0$, *where* $\det(\alpha)$ *is the usual determinant on* $M_n(\mathbb{C})$. *Moreover, a $\mathbb{H}^n$-vector*

$$w_0 = (q_1, \ldots, q_n) \in \mathbb{H}^n$$

with

$$q_l = x_l + y_l i + u_l j + v_l k \in \mathbb{H},$$

for all $l = 1, \ldots, n$, is a solution, if and only if

$$x_l = \operatorname{Re}\left(\sum_{k=1}^{n}\lambda_{lk}a_k\right),$$

and

$$y_l^2 + u_l^2 + v_l^2 = \left(\operatorname{Im}\left(\sum_{k=1}^{n}\lambda_{lk}a_k\right)\right)^2, \tag{7.18}$$

for all $l = 1, \ldots, n$, where $\{\lambda_{ij}\}_{i,j\in\{1,\ldots,n\}}$ are the entries of the inverse,

$$\alpha^{-1} = [\lambda_{ij}]_{n\times n} \in M_n(\mathbb{C}),$$

where $\operatorname{Re}(z)$ *and* $\operatorname{Im}(z)$ *are the real parts, respectively, the imaginary parts of $z \in \mathbb{C}$.*

Proof. Let $\alpha^{(n)}(w) = \mathbf{v}_0$ be the given equation. By (7.14), (7.15) and (7.17), if $\alpha \in M_n(\mathbb{C})$ is invertible, then this equation is solvable; conversely, if α is not invertible in $M_n(\mathbb{C})$, then this equation cannot have a solution, i.e., $\alpha \in M_n(\mathbb{C})$ is invertible, if and only if the equation is solvable. It is well-known that $\alpha \in M_n(\mathbb{C})$ is invertible, if and only if $\det(\alpha) \neq 0$. Therefore, equivalently, the equation (7.12) is solvable, if and only if $\det(\alpha) \neq 0$.

Suppose $\det(\alpha) \neq 0$, and hence, there exists a solution,

$$w_0 = (q_1, \ldots, q_n) \in \mathbb{H}^n,$$

satisfying $\alpha^{(n)}(w_0) = \mathbf{v}_0$, where

$$q_l = x_l + y_l i + u_l j + v_l k \in \mathbb{H},$$

for $l = 1, \ldots, n$. Then, by (7.17),

$$\sigma(q_l) = \sum_{k=1}^{n} \lambda_{lk} a_k \overset{\text{denote}}{=} \omega_l \in \mathbb{C},$$

for all $l = 1, \ldots, n$, implying that

$$x_l = \operatorname{Re}(\omega_l), \text{ and } y_l^2 + u_l^2 + v_l^2 = (\operatorname{Im}(\omega_l))^2,$$

for all $l = 1, \ldots, n$, by (7.17) (or (7.11)).

Therefore, the relation (7.18) holds, whenever the equation is solvable. □

8. CERTAIN EQUATIONS INDUCED BY QUATERNION-SPECTRAL FUNCTIONS

In this section, we fix $n \in \mathbb{N}$, and study our q-spectral ring $\sum_n(\mathbb{H})$ more in detail. As in Sections 6 and 7,

$$\alpha^{(n)} = \alpha \circ \sigma^{(n)} \in \Sigma_n(\mathbb{H}), \quad \text{with } \alpha \in M_n(\mathbb{C}),$$

is a function on $\mathbb{H}^n$ over $\mathbb{R}$.

Definition 8.1. All functions of $\sum_n(\mathbb{C})$ are said to be quaternion-spectral matricial functions (in short, q-spectral functions).

Note that the unital ring, the q-spectral ring $\sum_n(\mathbb{H})$, is over $\mathbb{R}$, equipped with the $\mathbb{R}$-scalar product (6.9);

$$\mathbb{R} \times \Sigma_n(\mathbb{H}) \ni \left(t, \alpha^{(n)}\right) \to (t\alpha)^{(n)} \in \Sigma_n(\mathbb{H}),$$

where

$$(t\alpha)^{(n)} = t\alpha \circ \sigma^n \quad \text{in } \Sigma_n(\mathbb{H}). \tag{8.1}$$

Motivated by (8.1), observe that, for any $t \in \mathbb{R}$ and $\alpha^{(n)} \in \sum_n (\mathbb{H})$

$$\alpha^{(n)} - t \cdot 1^{(n)} = \alpha \circ \sigma^n - t\,(I_n \circ \sigma^n)$$

where $1^{(n)} = I_n \circ \sigma^n \in \sum_n (\mathbb{H})$ is the unity induced by the identity matrix $I_n \in M_n(\mathbb{C})$,

$$= \alpha \circ \sigma^n + ((-t)I_n \circ \sigma^n)$$

by (8.1)

$$= (\alpha - tI_n) \circ \sigma^n = (\alpha - tI_n)^{(n)} \tag{8.2}$$

in $\sum_n (\mathbb{H})$ by (6.10), i.e.,

$$\alpha^{(n)} - t \cdot 1^{(n)} = (\alpha - tI_n)^{(n)},$$

in $\sum_n (\mathbb{H})$ by (8.2), for $t \in \mathbb{R}$.

Therefore,

$$\alpha^{(n)} - t \cdot 1^{(n)} = 0^{(n)} \text{ in } \Sigma_n (\mathbb{H}),$$

where

$$0^{(n)} = O_n \circ \sigma^n \in \Sigma_n (\mathbb{H})$$

is the zero element with the zero matrix $O_n \in M_n(\mathbb{C})$, if and only if

$$\alpha - tI_n = O_n \text{ in } M_n(\mathbb{C}). \tag{8.3}$$

Lemma 8.2. *Let $\alpha^{(n)} \in \Sigma_n (\mathbb{H})$, and let $t \in \mathbb{R}$. Then*

$$\alpha^{(n)} - t \cdot 1^{(n)} = 0^{(n)} \quad \textit{in } \Sigma_n (\mathbb{H}),$$

if and only if

$$\alpha - tI_n = O_n \quad \textit{in } M_n(\mathbb{C}). \tag{8.4}$$

Proof. The relation (8.4) holds by (8.2) and (8.3). □

Now, consider the equality

$$\alpha - tI_n = O_n \quad \text{in } M_n(\mathbb{C}), \tag{8.5}$$

in both global and local senses, where $\alpha \in M_n(\mathbb{C})$ and $t \in \mathbb{R}$ in $\mathbb{C}$. Note again that $t = t + 0i$ is a $\mathbb{R}$-quantity in $\mathbb{C}$ in (8.5).

To satisfy (8.5), one can have the universal (or, global) case, and the restricted (or, local) case as in spectral theory, i.e., the universal case is the case satisfying

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(\mathbf{v}) = 0_n, \quad \text{for all } \mathbf{v} \in \mathbb{H}^n,$$

and the restricted case is the case where there exists a subset $\mathcal{E} \subset \mathbb{H}^n$, such that

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(\mathbf{v}) = 0_n, \quad \text{for all } \mathbf{v} \in \mathcal{E}, \text{ in } \mathbb{H}^n,$$

where $0_n = (0, \ldots, 0)$ is the zero $\mathbb{H}$-vector of $\mathbb{H}^n$.

By regarding $\alpha - tI_n$ as a (usual) matrix acting on the n-dimensional vector space $\mathbb{C}^n$, the universal case is of course that

$$\alpha = tI_n \quad \text{in } M_n(\mathbb{C}),$$

i.e., globally, "for all $\mathbf{v} \in \mathbb{C}^n$", one has

$$\alpha(\mathbf{v}) = t\mathbf{v} \quad \text{in } \mathbb{C}^n.$$

Lemma 8.3. *Let $\alpha^{(n)} \in \Sigma_n(\mathbb{H})$, and let $t \in \mathbb{R}$. If $\alpha = tI_n$ in $M_n(\mathbb{C})$, then*

$$\alpha^{(n)} - t \cdot 1^{(n)} = 0^{(n)} \quad \text{in } \Sigma_n(\mathbb{H}).$$

Proof. Clearly, if $\alpha = tI_n$ in $M_n(\mathbb{C})$, then $\alpha - tI_n = O_n$ in $M_n(\mathbb{C})$, and hence,

$$\alpha^{(n)} - t \cdot 1^{(n)} = 0^{(n)} \quad \text{in } \Sigma_n(\mathbb{H}),$$

by (8.4). □

Meanwhile, the equality (8.5) can holds "locally" on $\mathbb{C}^n$, i.e., there exists a subspace $\mathcal{E}$ of $\mathbb{C}^n$ such that

$$\alpha - tI_n = O_n \quad \text{"on } \mathcal{E}\text{," in } \mathbb{C}^n,$$

if and only if

$$\alpha(\mathbf{w}) = t\mathbf{w}, \quad \text{for all } \mathbf{w} \in \mathcal{E}.$$

This is the case where $t \in \mathbb{R}$ is contained in the spectrum of α,

$$spec(\alpha) = \{\lambda \in \mathbb{C} : \text{there exists } w \in \mathbb{C}^n \text{ such that } \alpha(\mathbf{w}) = \lambda\mathbf{w}\},$$

or

$$spec(\alpha) = \{\lambda \in \mathbb{C} : \alpha - \lambda I_n \text{ is not invertible on } \mathbb{C}^n\},$$

in $\mathbb{C}$, i.e., the case where t is an eigenvalue of α, where the invertibility here means the usual invertibility on the matricial ring $M_n(\mathbb{C})$. In this case, if $\mathcal{E}_t \subset \mathbb{C}^n$ is the eigenspace of the eigenvalue $t \in spec(\alpha)$, then

$$\alpha(\mathbf{w}) = t\mathbf{w} \Longleftrightarrow (\alpha - tI_n)(\mathbf{w}) = 0_n, \quad \text{for all } \mathbf{w} \in \mathcal{E}_t,$$

if and only if

$$\alpha - tI_n = O_n \quad \text{"on } \mathcal{E}_t'', \text{ in } \mathbb{C}^n, \tag{8.6}$$

locally.

Lemma 8.4. *Let $\alpha^{(n)} \in \Sigma_n(\mathbb{H})$, and let $t \in \mathbb{R}$ and $w \in \mathbb{H}^n$. If*

$$t \in spec(\alpha), \text{ the spectrum of } \alpha \in M_n(\mathbb{C}),$$

and

$$\sigma^n(w) \in \mathcal{E}_t, \text{ the eigenspace of } t \text{ in } \mathbb{C}^n,$$

then

$$\left(\alpha^{(n)} - t \circ 1^{(n)}\right)(w) = 0_n \quad in\ \mathbb{H}^n.$$

Proof. Suppose $t \in spec(\alpha) \cap \mathbb{R}$ is an eigenvalue of $\alpha \in M_n(\mathbb{C})$, and $\mathcal{E}_t$ is the corresponding eigenspace of t in $\mathbb{C}^n$. Then

$$\alpha(\mathbf{w}) = t\mathbf{w} \Longleftrightarrow (\alpha - tI_n)(\mathbf{w}) = 0_n,$$

for all $\mathbf{w} \in \mathcal{E}_\mathbf{t}$ by (8.6). So, if there exists $w \in \mathbb{H}^n$, such that $\sigma^n(w) \in \mathcal{E}_t$, then

$$(\alpha - tI_n)(\sigma^n(w)) = 0_n,$$

if and only if

$$(\alpha - tI_n)^{(n)}(w) = 0_n,$$

if and only if

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(w) = 0_n \quad \text{in } \mathbb{H}^n,$$

by (8.2). Therefore, the relation (8.4) holds. □

By Lemmas 8.3 and 8.4, we obtain the following result.

Theorem 8.5. *Let $\alpha^{(n)} \in \sum_n(\mathbb{H})$ be a q-spectral function and $t \in \mathbb{R}$. Then there exists "non-zero" $w \in \mathbb{H}^n$ such that*

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(w) = 0_n \quad in\ \mathbb{H}^n,$$

if and only if

$$t \in spec(\alpha), \quad and \quad \sigma^n(w) \in \mathcal{E}_t,$$

where $\mathcal{E}_t$ is the eigenspace of an eigenvalue t.

Proof. ($\Leftarrow$) By (8.2) and Lemma 40, if $\alpha = tI_n$ in $M_n(\mathbb{C})$, then the identity

$$\alpha^{(n)} - t \cdot 1^{(n)} = (\alpha - tI_n)^{(n)} = 0^{(n)},$$

holds in the q-spectral ring $\sum_n(\mathbb{H})$ (globally), i.e., for all $w \in \mathbb{H}^n$, the equality of (8.5) holds. Remark that if $\alpha = tI_n$ in $M_n(\mathbb{C})$, then

$$spec(\alpha) = \{t\}, \quad \text{and} \quad \mathcal{E}_t = \mathbb{C}^n.$$

Thus, the above equality of (8.5) holds.

And, by (8.4), if $t \in spec(\alpha)$ and $\sigma^n(w) \in \mathcal{E}_t$, then the equality of (8.5) holds true (locally), too. Therefore, if

$$t \in spec(\alpha), \quad \text{and} \quad \sigma^n(w) \in \mathcal{E}_t,$$

then the equality of (8.5) holds.

($\Rightarrow$) Now, assume that either

$$t \notin spec(\alpha), \quad \text{or} \quad \sigma^n(w) \notin \mathcal{E}_t.$$

First, assume that $t \notin spec(\alpha)$. Then there does not exist $\mathbf{w} \in \mathbb{C}^{\mathbf{n}}$, such that

$$\alpha(\mathbf{w}) = t\mathbf{w} \Longleftrightarrow (\alpha - tI_n)(\mathbf{w}) = 0_n,$$

implying that

$$(\alpha - tI_n)^{(n)}(w) \neq 0_n, \quad \text{for all } w \in \mathbb{H}^n,$$

by (8.2), since $\sigma^n(w) \in \mathbb{C}^n$ in $\mathbb{H}^n$, for all $w \in \mathbb{H}^n$, i.e., if $t \notin spec(\alpha)$, then the equality of (8.5) does not hold.

Suppose now that $\mathbf{w} \overset{\text{denote}}{=} \sigma^n(w) \notin \mathcal{E}_t$, for $w \in \mathbb{H}^n$. Then

$$\alpha(\mathbf{w}) \neq t\mathbf{w} \Longleftrightarrow (\alpha - tI_n)(\mathbf{w}) \neq 0_n,$$

in $\mathbb{C}^n$, if and only if

$$(\alpha - tI_n)^{(n)}(w) = \left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(w) \neq 0_n,$$

in $\mathbb{H}^n$, because $w \in \mathbb{H}^n$ is assumed to be non-zero, i.e., if $\sigma^n(w) \notin \mathcal{E}_t$, then the equality of (8.5) does not hold, i.e., if the equality of (8.5) holds for non-zero $w \in \mathbb{H}^n$, then

$$t \in spec(\alpha), \quad \text{and} \quad \sigma^n(w) \in \mathcal{E}_t. \qquad \square$$

The above theorem characterizes the conditions where

$$\alpha^{(n)} - t \cdot 1^{(n)} = 0^{(n)} \quad \text{on } \mathbb{H}^n,$$

globally, or locally, for $\alpha^{(n)} \in \Sigma_n(\mathbb{H})$ and $t \in \mathbb{R}$. It says there exists $w \in \mathbb{H}^n$ such that

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(w) = 0_n \quad \text{in } \mathbb{H}^n,$$

if and only if

$$t \in spec(\alpha), \quad \text{and} \quad \sigma^n(w) \in \mathcal{E}_t.$$

Theorem 8.6. *Let $\alpha^{(n)} \in \Sigma_n(\mathbb{H})$ be a q-spectral function, and let $t \in \mathbb{R}$. Then $\alpha - tI_n$ is invertible (in the usual sense) in $M_n(\mathbb{C})$, if and only if*

$$\alpha^{(n)} - t \cdot 1^{(n)} \neq 0^{(n)} \quad \text{in } \Sigma_n(\mathbb{H}). \tag{8.7}$$

Proof. ($\Rightarrow$) Suppose the matrix $\alpha - tI_n$ is invertible in $M_n(\mathbb{C})$ in the usual sense. Then this matrix is nonzero, and the kernel of it satisfies

$$ker(\alpha - tI_n) = \{0_n\} \quad \text{in } \mathbb{C}^n,$$

implying that the zero vector $0_n \in \mathbb{C}^n$ is the only vector satisfying

$$(\alpha - tI_n)(0_n) = 0_n \quad \text{in } \mathbb{C}^n,$$

i.e., there does not exists any non-zero vector $\mathbf{w} \in \mathbb{C}^n \setminus \{0_n\}$, such that

$$(\alpha - tI_n)(\mathbf{w}) = 0_n \Longleftrightarrow \alpha(\mathbf{w}) = t\mathbf{w},$$

in $\mathbb{C}^n$. Thus, by (8.2),

$$\alpha^{(n)} - t \cdot 1^{(n)} \neq 0^{(n)} \quad \text{in } \Sigma_n(\mathbb{H}),$$

implying that there "does not" exist non-zero $w \in \mathbb{H}^n$ satisfying

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(w) = 0_n \quad \text{in } \mathbb{C}^n \subset \mathbb{H}^n.$$

i.e., the non-equality (8.7) holds.

($\Leftarrow$) Suppose now that $\alpha - tI_n$ is not invertible in $M_n(\mathbb{C})$. Then, $t \in spec(\alpha)$. So, the relation (8.5) holds, whenever $\sigma^n(w) \in \mathcal{E}_t$, for $w \in \mathbb{H}^n$. By the definition of the q-spectralization σ^n, such $\mathbb{H}^n$-vectors do exist since $\sigma^n(\mathbb{H}^n) = \mathbb{C}^n$. Therefore, if $\alpha - tI_n$ is not invertible in $M_n(\mathbb{C})$, then there exists at least one non-zero $w \in \mathbb{H}^n$, such that

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(w) = 0_n \quad \text{in } \mathbb{H}^n.$$

Equivalently, if the non-equality (8.7) holds, then $\alpha - tI_n$ is invertible in $M_n(\mathbb{C})$. □

The following corollary is obtained by Definition 6.5, and Theorem 8.5.

Corollary 8.7. *Let $\alpha^{(n)} \in \sum_n(\mathbb{H})$ and $t \in \mathbb{R}$. If $\alpha^{(n)} - t \cdot 1^{(n)} \in \Sigma_n(\mathbb{H})$ is contained in the invertible q-spectral ring $S_n(\mathbb{H})$ in the sense of Definition 6.5, then it is non-zero element of $\sum_n(\mathbb{H})$, i.e.,*

$$\alpha^{(n)} - t \cdot 1^{(n)} \in S_n(\mathbb{H}) \ \ in \sum_n(\mathbb{H}) \Longleftrightarrow t \notin spec(\alpha). \tag{8.8}$$

Proof. Observe that the q-spectral function $\alpha^{(n)} - t \cdot 1^{(n)}$ is contained in the invertible q-spectral ring $S_n(\mathbb{H})$, if and only if $(\alpha - tI_n)^{(n)} \in S_n(\mathbb{H})$ by (8.2), if and only if $\alpha - tI_n$ is invertible (in the usual sense) in $M_n(\mathbb{C})$ (by Definition 6.5), if and only if it is non-zero in $\sum_n(\mathbb{H})$ by (8.7). Therefore, the relation (8.8) holds by (8.5). □

Our main results, Theorems 8.5, 8.6, and Corollary 8.7, illustrate that the existence of $t \in \mathbb{R}$ and $w \in \mathbb{H}^n$ satisfying

$$\left(\alpha^{(n)} - t \cdot 1^{(n)}\right)(w) = 0_n \quad \text{in } \mathbb{H}^n,$$

for a q-spectral function $\alpha^{(n)} - t \cdot 1^{(n)} \in \Sigma_n(\mathbb{H})$ is highly related to the usual spectral theory on $M_n(\mathbb{C})$ by (8.8). In particular,

$$t \in spec\,(\alpha)\,, \quad \text{and} \quad \sigma^n(w) \in \mathcal{E}_t,$$

if and only if the above equality holds, by (8.5) and (8.7).

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