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OSCILLATION CRITERIA FOR LINEAR DIFFERENCE EQUATIONS WITH SEVERAL VARIABLE DELAYS

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Abstract. We obtain new sufficient criteria for the oscillation of all solutions of linear delay difference equations with several (variable) finite delays. Our results relax numerous well-known limes inferior-type oscillation criteria from the literature by letting the limes inferior be replaced by the limes superior under some additional assumptions related to slow variation. On the other hand, our findings generalize an oscillation criterion recently given for the case of a constant, single delay.

Keywords: oscillation, difference equations, several delays, non-monotone argument, slowly varying function.

Mathematics Subject Classification: 39A21, 39A10.

1. INTRODUCTION

We consider the following linear difference equations with constant and variable delays:

$$\Delta x(n) + \sum_{i=1}^{k} p_i(n)x(n - l_i) = 0 \tag{1.1}$$

and

$$\Delta x(n) + \sum_{i=1}^{k} p_i(n)x(\tau_i(n)) = 0, \tag{1.2}$$

where $p_i(n)\colon \mathbb{N} \to [0,\infty)$, $l_i \in \mathbb{N}$ and we assume that there exists a positive integer N such that $0 < l_1 < l_2 < \ldots < l_k \le N$ holds in the constant delay case, and for variable delays the retarded arguments $\tau_i\colon \mathbb{N} \to \mathbb{Z}$ satisfy $n - N \le \tau_i(n) \le n - 1$ for all $1 \le i \le k$ and $n \in \mathbb{N}$. These hypotheses are assumed throughout the paper.

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Here Δ denotes the forward difference operator, i.e.,

$$\Delta x(n) = x(n+1) - x(n)$$

and $\mathbb{N}$ denotes the set of nonnegative integers.

Clearly, equation (1.1) is a special case of equation (1.2) with

$$\tau_i(n) = n - l_i.$$

On the other hand, due to finiteness of the delay, equation (1.2) can also be rewritten in the form of (1.1) (with possibly different coefficient functions p_i and by choosing $k = N$). As both formulations have their advantages, we will deal with both. However, our proofs will only focus on the variable delay case.

By a solution of the difference equation (1.2) we mean a sequence of real numbers $(x(n))_{n=n_0}^{\infty}$ (with $n_0 \geq -N$), which satisfies equation (1.2) for all integers $n \geq n_0 + N$. A solution $(x(n))_{n=n_0}^{\infty}$ of the difference equation (1.2) is said to be *oscillatory* if $(x(n))_{n=n_0}^{\infty}$ is neither eventually positive nor eventually negative. Otherwise, the solution $(x(n))_{n=n_0}^{\infty}$ is said to be *nonoscillatory.*

Oscillation criteria for linear difference equations with delays have been the subject of many studies in the last thirty years. For an introduction to the topic we recommend the interested reader the monographs [1] and [10]. For an overview of the most recent results we refer to [11], the survey paper [16], and the references therein.

In the following we recall some results from the literature that will be relevant for our study.

Recently, Chatzarakis, Pinelas, and Stavroulakis [6] (corrected version of [5]) obtained the following oscillation criterion for equation (1.2).

Theorem 1.1 ([6, Theorem 2.2])**.** *Suppose that the functions $\tau_i(\cdot)$, $1 \leq i \leq k$, are nondecreasing for all $1 \leq i \leq k$. Moreover, assume that*

$$\limsup_{n\to\infty} \sum_{i=1}^{k} p_i(n) > 0 \quad \textit{and} \quad \liminf_{n\to\infty} \sum_{h=1}^{k} \sum_{j=\tau_i(n)}^{n-1} p_h(j) > \frac{1}{e} \tag{1.3}$$

hold for all $1 \leq i \leq k$. Then, all solutions of (1.2) oscillate.

Remark 1.2. It is easy to see that the assumption $\limsup_{n\to\infty} \sum_{i=1}^{k} p_i(n) > 0$ in (1.3) is actually redundant. On the other hand, although monotonicity of the retarded arguments τ_i is assumed, it is nowhere used throughout the proof in [6], so both of these assumptions can be omitted.

Clearly, Theorem 1.1 applies for the case of constant delay as well, that is, to equation (1.1).

The following result was essentially obtained by Yan, Meng, and Yan in 2006 [18, Theorem 1], however, as Karpuz and Stavroulakis recently pointed out [11], some corrections were necessary.

Theorem 1.3 ([11, Theorem G]). *Assume that*

$$\liminf_{n\to\infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j) > 0 \quad \textit{holds for some } 1 \le i \le k \tag{1.4}$$

and

$$\liminf_{n\to\infty} \sum_{j=\tau_{\max}(n)}^{n-1} \sum_{h=1}^{k} p_h(j) \left(\frac{j-\tau_h(j)+1}{j-\tau_h(j)} \right)^{j-\tau_h(j)+1} > 1, \tag{1.5}$$

where $\tau_{\max}(n) := \max_{1\le i\le k} \tau_i(n)$ *for* $n \in \mathbb{N}$. *Then every solution of* (1.2) *oscillates.*

For the case of constant delays the following sharper result was obtained in 1999 by Tang and Yu [17].

Theorem 1.4 ([17, Corollary 4]). *Assume that*

$$\liminf_{n\to\infty} \sum_{i=1}^{k} \left(\frac{l_i+1}{l_i} \right)^{l_i+1} \sum_{j=n+1}^{n+l_i} p_i(j) > 1.$$

Then every solution of equation (1.1) *is oscillatory.*

The aim of this paper is to improve the results listed above by replacing the lower limit with the upper limit in the main conditions under some additional assumptions that are related to slowly varying functions.

In accordance with [2] a sequence $(a(n))_{n\in\mathbb{N}}$ is called *slowly varying* (*at infinity*) if

$$\lim_{n\to\infty} (a(n+m) - a(n)) = 0$$

holds for all $m \in \mathbb{N}$. It is worth noting that it suffices if the above equality holds for $m = 1$. For a thorough description of slowly varying functions the reader is referred to the monograph [15], in which, however, a different but related notion of slowly varying functions is treated. For a discussion on the connection of these two notions see [15, Chapter 1].

The idea of using slowly varying functions to obtain sharp oscillation criteria for linear delay differential equations (with a single constant delay) originates from Pituk [14]. This first result was then successfully generalized in a series of papers [7–9] for the case of several (variable) delays. This concept was first utilized for difference equations by three of the current authors in a recent paper [3].

Our work, on the one hand, offers discrete analogues of the results obtained in [8] and, on the other hand, generalizes the result of [3], where equation (1.1) was treated with a single delay.

In the next section we state and prove our main theorems, then, in Section 3, we illustrate the applicability of them by means of two examples.

2. RESULTS

We shall need the following auxiliary result. To the best of our knowledge, this lemma is only available in the literature under the additional assumption that the delayed arguments τ_i are nondecreasing [12]. However, as we will see, finiteness of the delays (i.e., $n - \tau_i(n) \le N$) – which is assumed in our case anyway – also guarantees such a result.

Lemma 2.1. *Let us assume that*

$$\liminf_{n\to\infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j) > 0 \quad \textit{holds for all } 1 \le i \le k. \tag{2.1}$$

If $(x(n))_{n=n_0}^{\infty}$ *is an eventually positive solution of* (1.2), *then there exist* $K > 0$ *and* $N_1 \in \mathbb{N}$ *such that*

$$\frac{x(\tau_i(n))}{x(n+1)} \le K \quad \textit{for all } n \ge N_1 \textit{ and } 1 \le i \le k.$$

Proof. The proof is based on the main ideas of Theorem 2 of [13] and Lemma 2.2 of [4].

First, let us fix an arbitrary index $1 \le i \le k$ and define

$$\delta_i(n) \coloneqq \max\{\tau_i(\ell) : 0 \le \ell \le n\}$$

and

$$\mu_i(n) \coloneqq \max\{\ell \in \mathbb{N} : \tau_i(\ell) = \delta_i(n) \text{ and } \ell \le n\}$$

for all $n \in \mathbb{N}$. Note that $(\delta_i(n))_{n\in\mathbb{N}}$ is a monotone nondecreasing sequence.

According to (2.1) there exists

$$c_i \coloneqq \liminf_{n\to\infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j) > 0. \tag{2.2}$$

We claim that this implies

$$\tilde{c}_i \coloneqq \liminf_{n\to\infty} \sum_{j=\delta_i(n)}^{n-1} p_i(j) = c_i.$$

To see this, first note that the definition of δ_i yields that $n - N \le \tau_i(n) \le \delta_i(n) \le n - 1$ holds for all $n \in \mathbb{N}$, which in turn implies that $\tilde{c}_i \le c_i$.

It remains to prove that $\tilde{c}_i \ge c_i$. Now, set a subsequence $(n_m)_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \sum_{j=\delta_i(n_m)}^{n_m-1} p_i(j) = \tilde{c}_i.$$

From the definitions of δ_i and μ_i we obtain that

$$\delta_i(n) = \tau_i(\mu_i(n)) = \delta_i(\mu_i(n))$$

holds for all $n \in \mathbb{N}$. Hence, using that $\mu_i(n) \leq n$ for all $n \in \mathbb{N}$, and that $\lim_{n\to\infty} \mu_i(n) = \infty$ for all $1 \leq i \leq k$, we have

$$\begin{aligned}\tilde{c}_i &= \lim_{m\to\infty} \sum_{j=\delta_i(n_m)}^{n_m-1} p_i(j) \geq \lim_{m\to\infty} \sum_{j=\delta_i(n_m)}^{\mu_i(n_m)-1} p_i(j) \\ &= \lim_{m\to\infty} \sum_{j=\tau_i(\mu_i(n_m))}^{\mu_i(n_m)-1} p_i(j) \geq \liminf_{n\to\infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j) = c_i,\end{aligned}$$

which proves our claim.

Thus we may suppose that for any $\varepsilon \in (0, c_i)$ the estimate

$$\sum_{j=\delta_i(n)}^{n-1} p_i(j) \geq c_i - \varepsilon \tag{2.3}$$

holds for all sufficiently large n.

Observe that $(x(n))_{n=n_0}^{\infty}$ is an eventually positive solution, then, thanks to (1.2) and the positiveness of the coefficients, it is clearly eventually monotone nonincreasing.

Furthermore, from inequality (2.3) we infer that for any large enough $n \in \mathbb{N}$ there exists an $n^* \geq n$ such that

$$\sum_{j=n}^{n^*-1} p_i(j) < \frac{c_i - \varepsilon}{2} \quad \text{and} \quad \sum_{j=n}^{n^*} p_i(j) \geq \frac{c_i - \varepsilon}{2}, \tag{2.4}$$

where the first sum is zero by definition in case $n^* = n$ (i.e., when $p_i(n) \geq \frac{c_i-\varepsilon}{2}$).

It is easy to see that $\delta_i(n^*) \leq n-1$. Indeed, for $n^* = n$, $\delta_i(n^*) = \delta_i(n) \leq n-1$ clearly holds. Otherwise, if $n^* > n$, then $\delta_i(n^*) \geq n$ would imply by (2.4) the inequalities

$$\sum_{j=\delta_i(n^*)}^{n^*-1} p_i(j) \leq \sum_{j=n}^{n^*-1} p_i(j) < \frac{c_i - \varepsilon}{2},$$

a contradiction to (2.3).

Thus, using inequalities (2.3) and (2.4) we have

$$\sum_{j=\delta_i(n^*)}^{n-1} p_i(j) = \sum_{j=\delta_i(n^*)}^{n^*-1} p_i(j) - \sum_{j=n}^{n^*-1} p_i(j) \geq \frac{c_i - \varepsilon}{2}. \tag{2.5}$$

Now, the second inequality of (2.4) in combination with equation (1.2), the positivity of x and the coefficients p_h $(1 \le h \le k)$, and the monotonicity of x and δ_i yields

$$\begin{aligned} x(n) \ge x(n) - x(n^*+1) &= \sum_{\ell=n}^{n^*}(x(\ell)-x(\ell+1)) = \sum_{\ell=n}^{n^*}\sum_{h=1}^{k} p_h(\ell)x(\tau_h(\ell)) \\ &\ge \sum_{\ell=n}^{n^*} p_i(\ell)x(\tau_i(\ell)) \ge x(\delta_i(n^*))\sum_{\ell=n}^{n^*} p_i(\ell) \ge x(\delta_i(n^*))\frac{c_i-\varepsilon}{2}. \end{aligned} \tag{2.6}$$

In a similar fashion (using (2.5)) we obtain the estimates

$$\begin{aligned} x(\delta_i(n^*)) \ge x(\delta_i(n^*)) - x(n) &\ge \sum_{\ell=\delta_i(n^*)}^{n-1} p_i(\ell)x(\tau_i(\ell)) \\ &\ge x(\delta_i(n-1))\sum_{\ell=\delta_i(n^*)}^{n-1} p_i(\ell) \ge x(\delta_i(n-1))\frac{c_i-\varepsilon}{2}. \end{aligned} \tag{2.7}$$

From the combination of inequalities (2.6) and (2.7) together with the monotonicity of the solution x and the fact that $\delta_i(n) \le n-1$ we obtain that

$$\frac{4}{(c_i-\varepsilon)^2} \ge \frac{x(\delta_i(n))}{x(n+1)} \ge \frac{x(n-1)}{x(n+1)}$$

holds for all large enough n.

By iteration, and bearing in mind that $n - N \le \tau_i(n)$, we conclude that there exists a constant $K > 0$ such that

$$\frac{x(\tau_i(n))}{x(n+1)} \le K$$

holds for all sufficiently large n. □

Our first main result in the next theorem improves Theorem 1.1.

Theorem 2.2. *Suppose that condition* (2.1) *is satisfied and that there exists a positive constant M such that $0 \le p_i(n) \le M$ holds for all $1 \le i \le k$. Assume further that there exists a sequence $(\tau^*(n))_{n\in\mathbb{N}}$ such that $\tau_i(n) \le \tau^*(n) \le n-1$ holds for all $1 \le i \le k$ and $n \in \mathbb{N}$, and that the function*

$$A(n) \coloneqq \sum_{h=1}^{k}\sum_{j=\tau^*(n)}^{n-1} p_h(j)$$

is slowly varying and the inequality

$$\limsup_{n\to\infty} A(n) = \limsup_{n\to\infty} \sum_{h=1}^{k}\sum_{j=\tau^*(n)}^{n-1} p_h(j) > \frac{1}{e} \tag{2.8}$$

is fulfilled. Then, all solutions of (1.2) *oscillate.*

Proof. Assume to the contrary that there exists a nonoscillatory solution $(x(n))_{n=-N}^{\infty}$ of (1.2). Without loss of generality we may assume that $x(n) > 0$ for all $n \geq -N$. We also note that since

$$x(n+1) - x(n) = -\sum_{i=1}^{k} p_i(n)x(\tau_i(n)) \leq 0,$$

the solution $x(n)$ is nonincreasing for all $n \in \mathbb{N}$.

Firstly, by Lemma 2.1, there exists some $K > 0$ such that

$$\frac{x(\tau_i(n))}{x(n+1)} \leq K \tag{2.9}$$

holds for all $n \in \mathbb{N}$ and $1 \leq i \leq k$. By (2.8) we may consider a strictly increasing sequence $\delta(n)$ of natural numbers such that

$$\lim_{n\to\infty} A(\delta(n)) = \limsup_{n\to\infty} A(n) > \frac{1}{e}$$

and, for each $n, m \in \mathbb{N}$, we set

$$\begin{aligned} y_m(n) &= \frac{x(\delta(n)+m)}{x(\delta(n)+1)}, \\ q_{m,i}(n) &= p_i(\delta(n)+m), \\ \tau_{m,i}(n) &= \tau_i(\delta(n)+m) - \delta(n). \end{aligned} \tag{2.10}$$

Note that $y_m(n) \leq 1$ for $m \geq 1$, as $x(n)$ is nonincreasing. Furthermore, since $x(n)$ is a solution of (1.2) we have

$$x(\delta(n)+m+1) - x(\delta(n)+m) + \sum_{i=1}^{k} p_i(\delta(n)+m)x(\tau_i(\delta(n)+m)) = 0,$$

for each $n, m \in \mathbb{N}$. Dividing both sides of the last equation by $x(\delta(n)+1)$ yields

$$y_{m+1}(n) - y_m(n) + \sum_{i=1}^{k} q_{m,i}(n) y_{\tau_{m,i}(n)}(n) = 0. \tag{2.11}$$

Our aim now is to pass to a limiting equation as $n \to \infty$ and apply Theorem 1.1 to arrive at a contradiction. In order to do so, we first need to show that such a limiting equation exists.

From inequality (2.9) we obtain that for all $m, n \in \mathbb{N}$

$$\begin{aligned} &x(\delta(n)+m) - x(\delta(n)+m-1) \\ &\quad = -\sum_{i=1}^{k} p_i(\delta(n)+m-1)\frac{x(\tau_i(\delta(n)+m-1))}{x(\delta(n)+m)} x(\delta(n)+m) \\ &\quad \geq -kMKx(\delta(n)+m) \end{aligned}$$

holds, which in turn implies $x(\delta(n)+m) \geq \frac{x(\delta(n)+m-1)}{1+kMK}$.

By iteration we obtain the inequality

$$x(\delta(n)+m) \geq \frac{x(\delta(n)+1)}{(1+kMK)^{m-1}}.$$

After division by $x(\delta(n)+1)$ we may conclude that the estimate

$$\frac{1}{(1+kMK)^{m-1}} \leq y_m(n) \leq 1 \tag{2.12}$$

holds for all $m \geq 1$ and $n \in \mathbb{N}$.

Our assumptions and the definitions of the functions $q_{m,i}$ and $\tau_{m,i}$ yield that $m-N \leq \tau_{m,i}(n) \leq m-1$ and $0 \leq q_{m,i}(n) \leq M$ hold for all $m,n \in \mathbb{N}$ and $1 \leq i \leq k$. These combined with (2.12), the Bolzano–Weierstrass theorem, and Cantor's diagonal argument imply that there exists a strictly increasing sequence of natural numbers $(s_n)_{n\in\mathbb{N}}$ such that for each $m \in \mathbb{N}$

$$\lim_{n\to\infty} y_m(s_n) =: y(m), \qquad \lim_{n\to\infty} q_{m,i}(s_n) =: q_i(m), \tag{2.13}$$

and for each $1 \leq i \leq k$ and $m \in \mathbb{N}$ there is an integer $m-N \leq \sigma_i(m) \leq m-1$ such that

$$\tau_{m,i}(s_n) = \sigma_i(m) \tag{2.14}$$

holds for any large enough $n \in \mathbb{N}$.

Now, by virtue of (2.11),

$$y_{m+1}(s_n) - y_m(s_n) + \sum_{i=1}^{k} q_{m,i}(s_n) y_{\tau_{m,i}(s_n)}(s_n) = 0$$

holds for all $m,n \in \mathbb{N}$. Letting $n \to \infty$ we obtain

$$y(m+1) - y(m) + \sum_{i=1}^{k} q_i(m) y(\sigma_i(m)) = 0 \quad \text{for all } m \in \mathbb{N}. \tag{2.15}$$

Thus, in view of the inequality (2.12), $y(m)$ is a nonoscillatory solution of the equation (2.15). Note that thanks to the definition of $\tau_{m,i}$ the chain of inequalities $m-N \leq \sigma_i(m) \leq m-1$ also holds for all $m \in \mathbb{N}$ and $1 \leq i \leq k$.

In order to apply Theorem 1.1 it remains to show that the conditions in (1.3) are also satisfied for equation (2.15).

In view of equations (2.13) and (2.14) and using that A is slowly varying and inequality (2.8) holds, we infer that

$$\begin{aligned}
\sum_{h=1}^{k}\sum_{j=\sigma_i(m)}^{m-1} q_h(j) &= \sum_{h=1}^{k}\sum_{j=\sigma_i(m)}^{m-1} \lim_{n\to\infty} q_{j,h}(s_n) \\
&= \lim_{n\to\infty}\sum_{h=1}^{k}\sum_{j=\tau_{m,i}(s_n)}^{m-1} q_{j,h}(s_n) \\
&= \lim_{n\to\infty}\sum_{h=1}^{k}\sum_{j=\tau_{m,i}(s_n)}^{m-1} p_h(\delta(s_n)+j) \\
&= \lim_{n\to\infty}\sum_{h=1}^{k}\sum_{j=\delta(s_n)+\tau_{m,i}(s_n)}^{\delta(s_n)+m-1} p_h(j) \\
&= \lim_{n\to\infty}\sum_{h=1}^{k}\sum_{j=\tau_i(\delta(s_n)+m)}^{\delta(s_n)+m-1} p_h(j) \\
&\geq \lim_{n\to\infty}\sum_{h=1}^{k}\sum_{j=\tau^*(\delta(s_n)+m)}^{\delta(s_n)+m-1} p_h(j) \\
&= \lim_{n\to\infty} A(\delta(s_n)+m) = \lim_{n\to\infty} A(\delta(s_n)) \\
&= \limsup_{n\to\infty} A(n) > \frac{1}{e}
\end{aligned}$$

for any $m \in \mathbb{N}$ and $1 \leq i \leq k$.

Hence, bearing Remark 1.2 in mind, Theorem 1.1 can be applied to conclude that all solutions of (2.15) are oscillatory, which is a contradiction. The proof is completed. □

Since equation (1.1) is a special case of equation (1.2) with $\tau_i(n) = n - l_i$, we obtain the following result as an immediate corollary of Theorem 2.2.

Theorem 2.3. *Suppose that*

$$\liminf_{n\to\infty}\sum_{j=n-l_i}^{n-1} p_i(j) > 0 \quad \text{for all } 1 \leq i \leq k,$$

and there exist a positive constant M and a positive integer $l^* \leq l_1$ such that $0 \leq p_i(n) \leq M$ hold for all $1 \leq i \leq k$ and the function

$$\sum_{h=1}^{k}\sum_{j=n-l^*}^{n-1} p_h(j)$$

is slowly varying, and moreover, the inequality

$$\limsup_{n\to\infty} \sum_{h=1}^{k} \sum_{j=n-l^*}^{n-1} p_h(j) > \frac{1}{e} \tag{2.16}$$

is fulfilled. Then all solutions of (1.1) *oscillate.*

Remark 2.4. Observe that τ^* (resp. l^*) is intentionally not defined as $\tau^*(n) = \max_{1\le i\le k} \tau_i(n)$ (resp. $l^* = l_1$), since, as we will also see in Example 3.1, it can very well be that with this specific choice of τ^* (resp. l^*) our assumption on slow variation fails to be true, while a more appropriate choice of these parameters can satisfy all assumptions.

Analogously to Theorem 2.2 we can make Theorem 1.3 sharper under some additional assumptions as follows.

Theorem 2.5. *Suppose that condition* (2.1) *is satisfied and there exists a positive constant M such that $0 \le p_i(n) \le M$ holds for all $1 \le i \le k$. Assume further that there exist a sequence $(\tau^*(n))_{n\in\mathbb{N}}$ such that $\tau_i(n) \le \tau^*(n) \le n-1$ holds for all $1 \le i \le k$ and $n \in \mathbb{N}$, and that the function*

$$A(n) := \sum_{j=\tau^*(n)}^{n-1} \sum_{h=1}^{k} p_h(j) \left(\frac{j-\tau_h(j)+1}{j-\tau_h(j)} \right)^{j-\tau_h(j)+1} \tag{2.17}$$

is slowly varying and the inequality

$$\limsup_{n\to\infty} \sum_{j=\tau^*(n)}^{n-1} \sum_{h=1}^{k} p_h(j) \left(\frac{j-\tau_h(j)+1}{j-\tau_h(j)} \right)^{j-\tau_h(j)+1} > 1$$

is fulfilled. Then all solutions of equation (1.2) *are oscillating.*

Proof. Proceeding exactly as in the proof of Theorem 2.2 and using the same notations – with the only exception that now A is defined by (2.17) – we obtain a positive sequence $(y(m))_{m\in\mathbb{N}}$ that solves (2.15).

Arguing again indirectly, our aim is to apply Theorem 1.3 and to arrive at a contradiction. So let us show that (1.4) and (1.5) hold for equation (2.15). Bearing in mind equations (2.13) and (2.14) and using that A is slowly varying we obtain that

$$\begin{aligned}
&\sum_{j=\sigma_i(m)}^{m-1} \sum_{h=1}^{k} q_h(j) \left(\frac{j-\sigma_h(j)+1}{j-\sigma_h(j)} \right)^{j-\sigma_h(j)+1} \\
&= \lim_{n\to\infty} \sum_{j=\tau_{m,i}(s_n)}^{m-1} \sum_{h=1}^{k} q_{j,h}(s_n) \left(\frac{j-\tau_{j,h}(s_n)+1}{j-\tau_{j,h}(s_n)} \right)^{j-\tau_{j,h}(s_n)+1} \\
&= \lim_{n\to\infty} \sum_{j=\tau_i(m+\delta(s_n))-\delta(s_n)}^{m-1} \Bigg[\sum_{h=1}^{k} p_h(\delta(s_n)+j) \\
&\qquad \times \left(\frac{j-\tau_h(\delta(s_n)+j)+\delta(s_n)+1}{j-\tau_h(\delta(s_n)+j)+\delta(s_n)} \right)^{j-\tau_h(\delta(s_n)+j)+\delta(s_n)+1} \Bigg] \\
&= \lim_{n\to\infty} \sum_{j=\tau_i(m+\delta(s_n))}^{m+\delta(s_n)-1} \sum_{h=1}^{k} p_h(j) \left(\frac{j-\tau_h(j)+1}{j-\tau_h(j)} \right)^{j-\tau_h(j)+1} \\
&\geq \lim_{n\to\infty} \sum_{j=\tau^*(m+\delta(s_n))}^{m+\delta(s_n)-1} \sum_{h=1}^{k} p_h(j) \left(\frac{j-\tau_h(j)+1}{j-\tau_h(j)} \right)^{j-\tau_h(j)+1} \\
&= \lim_{n\to\infty} A(m+\delta(s_n)) = \lim_{n\to\infty} A(\delta(s_n)) = \limsup_{n\to\infty} A(n) > 1
\end{aligned}$$

holds for all $1 \leq i \leq k$ and $m \in \mathbb{N}$. That means, in particular, that the inequality

$$\sum_{j=\sigma_{\max}(m)}^{m-1} \sum_{h=1}^{k} q_h(j) \left(\frac{j-\sigma_h(j)+1}{j-\sigma_h(j)} \right)^{j-\sigma_h(j)+1} > 1$$

is also fulfilled for all $m \in \mathbb{N}$, where

$$\sigma_{\max}(m) := \max_{1\leq i\leq k} \sigma_i(m).$$

That is, condition (1.5) holds for equation (2.15).

Finally, note that a similar argument and assumption (2.1) yield that

$$\liminf_{m\to\infty} \sum_{j=\sigma_i(m)}^{m-1} q_i(j) > 0$$

also holds for some $1 \leq i \leq k$. Thus all assumptions of Theorem 1.3 are satisfied by equation (2.15) and hence all solutions of equation (2.15) are oscillatory. This is a contradiction which concludes our proof. □

A similar argument can be applied to prove the following improvement of Theorem 1.4. The proof is left to the reader.

Theorem 2.6. *Suppose that*

$$\liminf_{n\to\infty} \sum_{j=n-l_i}^{n-1} p_i(j) > 0 \quad \text{for all } 1 \le i \le k, \tag{2.18}$$

and there exists $M > 0$ such that $0 \le p_i(n) \le M$ holds for all $1 \le i \le k$. Furthermore, assume that the function

$$\sum_{i=1}^{k} \left(\frac{l_i+1}{l_i}\right)^{l_i+1} \sum_{j=n+1}^{n+l_i} p_i(j)$$

is slowly varying, and the inequality

$$\limsup_{n\to\infty} \sum_{i=1}^{k} \left(\frac{l_i+1}{l_i}\right)^{l_i+1} \sum_{j=n+1}^{n+l_i} p_i(j) > 1$$

holds. Then every solution of equation (1.1) *is oscillatory.*

Observe that Theorem 2.1 of [3] is a special case of Theorems 2.5 and 2.6, i.e. with $k = 1$ (and with the choice $\tau^*(n) = \tau_1(n)$ in Theorem 2.5).

3. EXAMPLES

We provide two illustrative examples to demonstrate the applicability and significance of the results. First we compare Theorems 2.2 and 1.1.

Example 3.1. Let us consider equation (1.2) with $k = 2$ and $N = 3$. We suppose that $\tau_i(n) \in \{n-2, n-3\}$ for all $n \in \mathbb{N}$ and $i = 1, 2$, and let us not make further assumptions on τ_1 and τ_2 for now. Furthermore, let the coefficient functions be defined by

$$p_1(n) = c_1 + d_1 \cos^2\left(\sqrt{n} \cdot \pi/2\right) + (-1)^n \varepsilon, \quad \text{and} \quad p_2(n) = c_2$$

for all $n \in \mathbb{N}$, where c_1, c_2, d_1 and ε are positive parameters with $\varepsilon < c_1$. Then, by choosing $\tau^*(n) = n - 2$ for all $n \in \mathbb{N}$, it is elementary to show that the sequence

$$A(n) := \sum_{h=1}^{k} \sum_{j=\tau^*(n)}^{n-1} p_h(j) = 2c_1 + 2c_2 + d_1 \sum_{j=1}^{2} \cos^2\left(\sqrt{n-j} \cdot \pi/2\right)$$

is slowly varying at infinity and that

$$\limsup_{n\to\infty} A(n) = 2(c_1 + c_2 + d_1)$$

(consider the subsequence $\delta(n) = 4n^2 + 1$). On the other hand, the coefficient functions are clearly nonnegative and bounded, and moreover, condition (2.1) also holds. Hence the application of Theorem 2.2 implies that all solutions oscillate, whenever

$$2(c_1 + c_2 + d_1) > \frac{1}{e} \tag{3.1}$$

is fulfilled.

On the other hand, if, for example, $\tau_1(n) = n-2$ and $\tau_2(n) = n-3$ for all $n \in \mathbb{N}$, then it is not hard to check that Theorem 1.1 can only guarantee oscillation of all solutions if

$$\liminf_{n\to\infty} \sum_{h=1}^{k} \sum_{j=n-2}^{n-1} p_h(j) = 2c_1 + 2c_2 + d_1 \liminf_{n\to\infty} \sum_{j=1}^{2} \cos^2\Big(\sqrt{n-j}\cdot \pi/2\Big)$$
$$= 2(c_1+c_2) > \frac{1}{e},$$

which is clearly more restrictive than condition (3.1).

Observe that – as noted in Remark 2.4 – the flexibility that the definition of τ^* in Theorem 2.2 gives us can come handy here. Even in the constant (single) delay case with $\tau_1(n) = \tau_2(n) \equiv n-3$ we should choose $\tau^*(n) = n-2$ for these coefficient functions. This is because with the choice of $\tau^*(n) \equiv n-3$ we would get

$$A(n) = 3c_1 + 3c_2 + (-1)^{n-1}\varepsilon + d_1 \sum_{j=1}^{3} \cos^2\Big(\sqrt{n-j}\cdot \pi/2\Big),$$

resulting in $\lim_{n\to\infty} |A(n+1) - A(n)| = 2\varepsilon \neq 0$, meaning that $A(n)$ is not slowly varying and one could not apply Theorem 2.2 with this choice of τ^*.

We conclude this paper with an example for the application of Theorem 2.6.

Example 3.2. Consider the constant delay equation (1.1) with $l_1 = 1$ and $l_2 = 2$ and coefficient functions

$$p_1(n) = c_1 + d_1 \cos^2\big(\sqrt{n}\cdot \pi/2\big), \quad \text{and} \quad p_2(n) = c_2,$$

with positive constants c_1, c_2 and d_1. The coefficients are evidently nonnegative and bounded, moreover, condition (2.18) is satisfied.

Then the function

$$\sum_{i=1}^{k} \left(\frac{l_i+1}{l_i}\right)^{l_i+1} \sum_{j=n+1}^{n+l_i} p_i(j) = 4\left(c_1 + d_1\cos^2\big(\sqrt{n+1}\cdot\pi/2\big)\right) + \left(\frac{3}{2}\right)^3 2c_2$$

is slowly varying at infinity, so Theorem 2.6 can be applied to obtain that all solutions are oscillatory, provided

$$4\left(c_1 + d_1 \limsup_{n\to\infty} \cos^2\big(\sqrt{n+1}\cdot\pi/2\big)\right) + \frac{27}{4}c_2 = 4(c_1+d_1) + \frac{27}{4}c_2 > 1.$$

On the other hand, Theorem 1.4 can guarantee oscillation of all solutions only if the stronger condition

$$4c_1 + \frac{27}{4}c_2 > 1$$

is satisfied.

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