Opuscula Math. 41, no. 5 (2021), 629–648
https://doi.org/10.7494/OpMath.2021.41.5.629 Opuscula Mathematica

EXTREME POINTS OF THE BESICOVITCH–ORLICZ SPACE OF ALMOST PERIODIC FUNCTIONS EQUIPPED WITH ORLICZ NORM

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Communicated by P.A. Cojuhari

Abstract. In the present paper, we give criteria for the existence of extreme points of the Besicovitch-Orlicz space of almost periodic functions equipped with Orlicz norm. Some properties of the set of attainable points of the Amemiya norm in this space are also discussed.

Keywords: extreme points, strict convexity, almost periodic functions, Besicovitch–Orlicz spaces of almost periodic functions.

Mathematics Subject Classification: 46B20, 46B25, 46E30.

1. INTRODUCTION

Extreme points are without doubt the most basic concepts in the study of the behavior of balls in Banach spaces (see e.g. [5]). To justify the importance of this notion, we can cite the celebrated Krein–Milman theorem which states that any compact convex set of a Banach space is the convex hull of its extreme points set. In particular, this is what happens for the unit ball of L^p spaces when $p > 1$. The notion of extreme point is also connected with the strict convexity. More precisely, a Banach space $\mathbb{X}$ is said to be strictly convex (or rotund) if every point of its unit sphere $S(\mathbb{X})$ is an extreme point of its closed unit ball $B(\mathbb{X})$, i.e. $S(\mathbb{X}) = \operatorname{extr}[B(\mathbb{X})]$.

For the convenience of the reader, we recall that $f \in S(\mathbb{X})$ is said to be an extreme point of $B(\mathbb{X})$ if it can not be written as the arithmetic mean $\frac{1}{2}(g+h)$ of two distinct points $g, h \in B(\mathbb{X})$. Namely, if the following implication holds

$$g, h \in B(\mathbb{X}), \; f = \frac{g+h}{2} \Rightarrow g = h.$$

In [18], M.A. Picardello has used another definition: a vector f in $B(\mathbb{X})$ is an extreme point of $B(\mathbb{X})$ if and only if there does not exist $g \neq 0$ in $\mathbb{X}$ such that $f \pm g \in B(\mathbb{X})$. In reworking the arguments of [10], he studied the extreme points of unit balls of the Besicovitch–Marcinkiewicz Banach spaces (spaces of functions with bounded upper means), and he characterized the extreme points of unit balls of the Stepanov spaces defined in [20].

The criteria for extreme points and strict convexity in Orlicz spaces and Musielak–Orlicz spaces equipped with the Orlicz norm, the Luxemburg norm, and Amemiya norm, have been obtained earlier (see for instance [4, 7, 19, 21] and references therein).

In his celebrated paper [9], Hillmann has used a similar approach of Besicovitch [2] to obtain an extension of the Besicovitch almost periodicity in the context of Orlicz spaces. He introduced the Besicovitch–Orlicz spaces of almost periodic functions denoted $B^{\phi}_{a.p.}(\mathbb{R})$ (ϕ is an Orlicz function), and proved their completeness when they are endowed with the Luxemburg norm (2.3).

In [16], Morsli *et al.* have defined the Orlicz norm (2.5), and they proved that it is equivalent to the Luxemburg norm and equals to the Amemiya norm (see Proposition 2.4).

In the recent years, some geometrical properties of $B^{\phi}_{a.p.}(\mathbb{R})$ have been considered by Morsli and his collaborators in [1, 3, 11, 13, 15].

Morsli [11] has discussed the criteria of rotundity of $B^{\phi}_{a.p.}(\mathbb{R})$ equipped with Luxemburg norm. He proved that $B^{\phi}_{a.p.}(\mathbb{R})$ is strictly convex if and only if ϕ is strictly convex and has at most polynomial growth (ϕ satisfies the Δ_2-condition (2.1)).

In [13], Morsli *et al.* have characterized the rotundity of $B^{\phi}_{a.p.}(\mathbb{R})$, when it is endowed with the Orlicz norm. However, to our knowledge, the criteria for extreme points have not been discussed yet.

The paper [8] is the first work to look at the extreme points of the unit ball of $B^{\phi}_{a.p.}(\mathbb{R})$ equipped with the Luxemburg norm. Here we continue investigations started there.

The main goal of this paper is to characterize extreme points of the unit ball in $B^{\phi}_{a.p.}(\mathbb{R})$ equipped with the Orlicz norm and give some properties of the set $K(f)$ defined in Proposition 2.4.

2. PRELIMINARIES

In this section, we recall a sequence of definitions and results which will be used in what follows.

2.1. ORLICZ FUNCTIONS

A function $\phi : \mathbb{R} \to \mathbb{R}^+$ is said to be an Orlicz function (called also N-function) if it is even, convex, $\phi(x) = 0$, $\phi(x) > 0$ if and only if $x \neq 0$ and $\lim_{x\to+\infty} \frac{\phi(x)}{x} = +\infty$, $\lim_{x\to 0} \frac{\phi(x)}{x} = 0$.

From now on, we always denote by ϕ an Orlicz function.

ϕ is said to satisfy the Δ_2-condition for large values (we write $\phi \in \Delta_2$), when there exist constants $k > 0$ and $u_0 > 0$ such that

$$\phi(2u) \leq k\phi(u), \quad \forall |u| \geq u_0. \tag{2.1}$$

For every Orlicz function ϕ we define the complementary function ψ by the formula

$$\psi(y) = \sup\{x|y| - \phi(x), x \geq 0\}, \quad \forall y \in \mathbb{R}.$$

The complementary function ψ is also an Orlicz function. The pair (ϕ, ψ) satisfies the Young inequality

$$xy \leq \phi(x) + \psi(y), \quad x, y \in \mathbb{R}.$$

The function ϕ is called strictly convex on $\mathbb{R}$ if

$$\phi\left(\frac{u+v}{2}\right) < \frac{1}{2}(\phi(u) + \phi(v)), \quad \forall u, v \in \mathbb{R},\ u \neq v.$$

Let us recall that if ϕ is strictly convex on $\mathbb{R}$, then it is uniformly convex on any bounded interval (see [4, Proposition 1.4]). Namely, for any $l > 0$ and $\varepsilon > 0$, and $[c, d] \subset]0, 1[$ there exists $\delta > 0$ such

$$\phi(\lambda u + (1-\lambda)v) \leq (1-\delta)(\lambda\phi(u) + (1-\lambda)\phi(v)) \tag{2.2}$$

for any $\lambda \in [c, d]$ and all $u, v \in \mathbb{R}$ satisfying $|u| \leq l$, $|v| \leq l$ and $|u - v| \geq \varepsilon$.

Following [4], an interval $[a, b]$ is called a structural affine interval of an Orlicz function ϕ, provided that ϕ is affine on $[a, b]$ and it is not affine on either $[a - \varepsilon, b]$ or $[a, b + \varepsilon]$ for any $\varepsilon > 0$.

Let $\{[a_i, b_i]\}_i$ be all the structural affine intervals of ϕ. Let

$$S_\phi = \mathbb{R} \backslash \left[\bigcup_i]a_i, b_i[\right]$$

be the set of strictly convex points of ϕ. Clearly, if $u, v \in \mathbb{R}$, $\alpha \in]0, 1[$ and $\alpha u + (1-\alpha)v \in S_\phi$, then

$$\phi(\alpha u + (1-\alpha)v) < \alpha\phi(u) + (1-\alpha)\phi(v).$$

2.2. BESICOVITCH–ORLICZ SPACES OF ALMOST PERIODIC FUNCTIONS

Let $M(\mathbb{R})$ be the set of all real Lebesgue measurable functions defined on $\mathbb{R}$, $\Sigma(\mathbb{R})$ the σ-algebra of all Lebesgue-measurable subsets of $\mathbb{R}$ and μ the Lebesgue measure on $\mathbb{R}$.

We denote by $L^\phi_{loc}(\mathbb{R})$ the subspace of $M(\mathbb{R})$ such that for each bounded interval U there exists $\alpha > 0$ such that

$$\int_U \phi(\alpha|f(s)|)\, ds < \infty.$$

When $U = [0, 1]$, we get *the Orlicz space* $L^\phi([0, 1])$ (see [4]).

The Besicovitch–Orlicz pseudo modular ρ_{B^ϕ} is defined in [9] as follows:

$$\rho_{B^\phi} : L^\phi_{loc}(\mathbb{R}) \to \overline{\mathbb{R}}^+, \quad f \mapsto \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{-T}^{T} \phi\left(|f(t)|\right) dt.$$

Its associated modular space, called the Besicovitch–Orlicz space, is

$$\mathfrak{B}^\phi(\mathbb{R}) = \left\{ f \in L^\phi_{loc}(\mathbb{R}) : \rho_{B^\phi}(\lambda f) < +\infty \text{ for some } \lambda > 0 \right\}.$$

This space is endowed with the Luxemburg pseudonorm

$$\|f\|_{B^\phi} = \inf\left\{ k > 0 : \rho_{B^\phi}\left(\frac{f}{k}\right) \le 1 \right\}. \tag{2.3}$$

Let us consider the equivalence relation

$$\forall f, g \in \mathfrak{B}^\phi(\mathbb{R}) : \; f \sim_\phi g \Leftrightarrow \|f - g\|_{B^\phi} = 0.$$

We denote by $B^\phi(\mathbb{R}) := \mathfrak{B}^\phi(\mathbb{R}) / \sim_\phi$ the quotient space. Henceforth, we will not distinguish between an element of $\mathfrak{B}^\phi(\mathbb{R})$ and its equivalence class in $B^\phi(\mathbb{R})$.

Endowed with the Luxemburg norm $\|\cdot\|_{B^\phi}$, $B^\phi(\mathbb{R})$ is a Banach space.

In order to define the Besicovitch–Orlicz space of almost periodic functions, let us denote by $Trig(\mathbb{R})$ the linear set of all generalized trigonometric polynomials, i.e.

$$Trig(\mathbb{R}) = \left\{ P(t) = \sum_{j=1}^{n} \alpha_j \exp(i\lambda_j t) : \lambda_j \in \mathbb{R}, \alpha_j \in \mathbb{C}, j \in \mathbb{N} \right\}.$$

The Besicovitch–Orlicz space of almost periodic functions, $B^\phi_{a.p.}(\mathbb{R})$, is the closure of $Trig(\mathbb{R})$ in $B^\phi(\mathbb{R})$, with respect to the norm $\|.\|_{B^\phi}$, see [12]. More exactly,

$$\begin{aligned} B^\phi_{a.p.}(\mathbb{R}) &= \left\{ f \in B^\phi(\mathbb{R}) : \exists (P_n)_{n\ge1} \subset Trig(\mathbb{R}) \lim_{n\to\infty} \|f - P_n\|_{B^\phi} = 0 \right\} \\ &= \left\{ f \in B^\phi(\mathbb{R}) : \exists (P_n)_{n\ge1} \subset Trig(\mathbb{R}) \, \forall k > 0 \lim_{n\to\infty} \rho_{B^\phi}(k(f - P_n)) = 0 \right\}. \end{aligned}$$

We define the space $\widetilde{B}^\phi_{a.p.}(\mathbb{R})$ as the closure of $Trig(\mathbb{R})$ with respect to the modular convergence (see [11, 12]). Namely,

$$\widetilde{B}^\phi_{a.p.}(\mathbb{R}) = \overline{Trig(\mathbb{R})}^{\rho_{B^\phi}}.$$

More precisely,

$$\widetilde{B}^\phi_{a.p.}(\mathbb{R}) = \left\{ f \in B^\phi(\mathbb{R}) : \exists (P_n)_{n\ge1} \subset Trig(\mathbb{R}) \, \exists k > 0 \lim_{n\to\infty} \rho_{B^\phi}(k(f - P_n)) = 0 \right\}.$$

Remark 2.1.

(1) $B^{\phi}_{a.p.}(\mathbb{R})$ and $\widetilde{B}^{\phi}_{a.p.}(\mathbb{R})$ endowed with the Luxemburg norm (2.3) are Banach spaces. The equality between them holds if and only if $\phi \in \Delta_2$.
(2) If we denote by $AP(\mathbb{R})$ the Banach space of Bohr almost periodic functions, we have $AP(\mathbb{R}) \subset B^{\phi}_{a.p.}(\mathbb{R})$ (see [9]).
(3) From [11] we know that when $f \in B^{\phi}_{a.p.}(\mathbb{R})$ the limit exists and is finite in the expression of $\rho_{B^\phi}(f)$, i.e.

$$\rho_{B^\phi}(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \phi(|f(t)|)d\mu. \tag{2.4}$$

This fact is very useful in our computations.
(4) By the definition of the Luxemburg norm, for all $f \in B^{\phi}(\mathbb{R})$ we have

$$\rho_{B^\phi}(f) \le \|f\|_{B^\phi} \text{ whenever } \|f\|_{B^\phi} < 1.$$

The following lemma gives a relation between the Luxemburg norm $\|.\|_{B^\phi}$ and the modular ρ_{B^ϕ}. It is proved in [13] with the assumption that ϕ satisfies the Δ_2-condition.

Lemma 2.2 ([1])**.** *Let $f \in B^{\phi}_{a.p.}(\mathbb{R})$. Then:*

(1) $\|f\|_{B^\phi} \le 1$ *if and only if* $\rho_{B^\phi}(f) \le 1$,
(2) $\|f\|_{B^\phi} = 1$ *if and only if* $\rho_{B^\phi}(f) = 1$.

Hillmann [9] has defined the set function $\overline{\mu}_B$ on $\Sigma(\mathbb{R})$ as the following

$$\overline{\mu}_B(A) = \varlimsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \chi_A(t)d\mu = \varlimsup_{T\to\infty} \frac{1}{2T}\mu(A \cap [-T,T]),$$

where χ_A denotes the characteristic function of A.

It is clear that $\overline{\mu}_B$ is increasing, null on sets with μ-finite measure and it is not σ-additive.

Hereafter, using [11, Lemma 4], we give an example of a function in $\widetilde{B}^{\phi}_{a.p.}(\mathbb{R})$.

Example 2.3. Let $(a_n)_{n\ge1}$ and $(u_n)_{n\ge1}$ be two sequences defined by

$$a_n = \phi^{-1}\Big(\frac{1}{2^n}\Big) \quad \text{and} \quad u_n = \frac{1}{2^n} \quad \text{for every } n \ge 1.$$

Put

$$S_n = \sum_{k=1}^{n} u_k = 1 - \frac{1}{2^n}.$$

We define a set sequence $(A_n)_{n\ge1}$ by $A_n = [S_n, S_{n+1}[$. Then we have:

(i) $A_i \cap A_j = \emptyset$ for all i, j such that $i \ne j$, because $(S_n)_n$ is strictly increasing,
(ii) $\lim_{n\to+\infty} S_n = 1$ which implies that $\bigcup_{n\ge1} A_n \subset [0,1[$,
(iii) $\mu(A_n) = u_{n+1}$, $\phi(a_n) = \frac{1}{2^n}$ and $\sum_{n\ge1} \phi(a_n)\mu(A_n) = \frac{1}{2}\sum_{n\ge1} \frac{1}{4^n} < \infty$.

Consider the function defined on $[0,1]$ by

$$f = \sum_{n\geq 1} \phi^{-1}\left(\frac{1}{2^n}\right) \chi_{\left[1-\frac{1}{2^n},1-\frac{1}{2^{n+1}}\right[}.$$

Let $\tilde{f}$ be the periodic extension of f to the whole $\mathbb{R}$, with period $\tau = 1$. Then by [11, Lemma 4] we have $\tilde{f} \in \tilde{B}^{\phi}_{a.p.}(\mathbb{R})$.

Beside the Luxemburg norm Morsli *et al.* [16] have defined in $B^{\phi}_{a.p.}(\mathbb{R})$ another norm, called Orlicz norm, by the formula

$$\|f\|^{o}_{B^{\phi}} = \sup\left\{M(|fg|) : g \in B^{\psi}_{a.p.}(\mathbb{R}),\ \rho_{B^{\psi}}(g)\ \leq 1\right\}, \tag{2.5}$$

where ψ is the complementary function of ϕ and

$$M(f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{+T} f(t)d\mu.$$

The norm (2.5) is not easy to deal with. So Morsli *et al.* [13, 14, 16] expressed it by the Amemiya formula (2.6) which is far more convenient to make use.

Proposition 2.4 ([13, 14]). *Let $f \in B^{\phi}_{a.p.}(\mathbb{R})$, $\|f\|_{B^{\phi}} \neq 0$. Then the following assertions hold.*

(1) *The Orlicz norm and the Amemiya norm are equal, i.e.*

$$\|f\|^{o}_{B^{\phi}} = \inf_{k>0} \frac{1}{k}\left[1 + \rho_{B^{\phi}}(kf)\right]. \tag{2.6}$$

Moreover, there exists

$$k_0 \in K(f) = \left\{k > 0 : \|f\|^{o}_{B^{\phi}} = \frac{1}{k}\left[1 + \rho_{B^{\phi}}(kf)\right]\right\}.$$

(2)

$$\|f\|_{B^{\phi}} \leq \|f\|^{o}_{B^{\phi}} \leq 2\,\|f\|_{B^{\phi}}.$$

These two norms are equivalent, nevertheless, the corresponding geometric properties between them are different. So their extreme points need not be the same.

The next lemma will be very useful in the proofs of our results.

Lemma 2.5 ([11]). *Let $f \in B^{\phi}_{a.p.}(\mathbb{R})$ such that $\|f\|_{B^{\phi}} > 0$. Then there exist real numbers $0 < \alpha < \beta$ and $\theta \in]0,1[$ such that $\overline{\mu}_B(G) \geq \theta$, where*

$$G = \{t \in \mathbb{R} : \alpha \leq |f(t)| \leq \beta\}.$$

3. MAIN RESULTS

First, we prove the following auxiliary lemmas.

Lemma 3.1. *Functions $f \in B^{\phi}_{a.p.}(\mathbb{R})$ are absolutely ϕ-integrable in the $\bar{\mu}_B$ sense.*

Before giving the proof of this Lemma, we need to give the definition of the absolutely ϕ-integrable function.

Definition 3.2. A function $f \in B^{\phi}(\mathbb{R})$ is said to be absolutely ϕ-integrable in $\bar{\mu}_B$ sense, if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that for every measurable subset $A \in \Sigma(\mathbb{R})$ with $\bar{\mu}_B(A) < \delta$ we have

$$\|f\chi_A\|_{B^\phi} \leq \varepsilon.$$

Proof. We use the same arguments as used in [6]. First, let us show that bounded functions are absolutely ϕ-integrable in $\bar{\mu}_B$ sense. Let $\varepsilon > 0$, $A \in \Sigma(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ be a bounded function. Put

$$\mathcal{C} = \sup_{t\in\mathbb{R}} |f(t)|.$$

Here, we exclude for simplicity the trivial case, when $\bar{\mu}_B(A) = 0$. Clearly, we have

$$\|\chi_A\|_{B^\phi} = \frac{1}{\phi^{-1}\left(\frac{1}{\bar{\mu}_B(A)}\right)} \quad \text{and} \quad \|f\chi_A\|_{B^\phi} \leq \mathcal{C}\,\|\chi_A\|_{B^\phi}. \tag{3.1}$$

Since the function $t \mapsto \left(\phi^{-1}(1/t)\right)^{-1}$ is continuous and increasing on $]0, +\infty[$, we deduce that there exists $\delta := \left(\phi\left(\frac{\mathcal{C}}{\varepsilon}\right)\right)^{-1}$ such that $\|f\chi_A\|_{B^\phi} \leq \varepsilon$, whenever $\bar{\mu}_B(A) < \delta$.

Now, let us assume that $f \in B^{\phi}_{a.p}(\mathbb{R})$. There exists a trigonometric polynomial P_ε such that

$$\|f - P_\varepsilon\|_{B^\phi} \leq \frac{\varepsilon}{2}.$$

Since P_ε is absolutely ϕ-integrable in the $\bar{\mu}_B$ sense, there exists $\delta > 0$ such that $\|P_\varepsilon\chi_A\|_{B^\phi} \leq \frac{\varepsilon}{2}$ whenever $\bar{\mu}_B(A) < \delta$. For such δ, we have

$$\|f\chi_A\|_{B^\phi} \leq \|(f - P_\varepsilon)\chi_A\|_{B^\phi} + \|P_\varepsilon\chi_A\|_{B^\phi} \leq \|f - P_\varepsilon\|_{B^\phi} + \|P_\varepsilon\chi_A\|_{B^\phi} \leq \varepsilon.$$

This completes the proof of the lemma. □

Lemma 3.3. *Let f be a function in $B^{\phi}_{a.p.}(\mathbb{R})$, then there exists $\delta > 0$ such that*

$$f\chi_{E^c} \in B^{\phi}_{a.p.}(\mathbb{R}),$$

for any $E \in \Sigma(\mathbb{R})$ with $\bar{\mu}_B(E) < \delta$, where E^c is the complementary of E. Consequently, $f\chi_E \in B^{\phi}_{a.p.}(\mathbb{R})$.

Proof. Let $\varepsilon > 0$. There exists a trigonometric polynomial P_ε such that

$$\|f - P_\varepsilon\|_{B^\phi} \leq \frac{\varepsilon}{2}.$$

Using Lemma 3.1, there exists $\delta > 0$ such that $\|P_\varepsilon \chi_E\|_{B^\phi} \leq \frac{\varepsilon}{2}$, for every measurable subset $E \in \Sigma(\mathbb{R})$ with $\bar{\mu}_B(E) < \delta$.

For the above P_ε, E and δ, we have

$$\begin{aligned}\|f\chi_{E^c} - P_\varepsilon\|_{B^\phi} &= \|f\chi_{E^c} - P_\varepsilon\chi_{E^c} - P_\varepsilon\chi_E\|_{B^\phi} \\ &\leq \|(f - P_\varepsilon)\chi_{E^c}\|_{B^\phi} + \|P_\varepsilon\chi_E\|_{B^\phi} \\ &\leq \|f - P_\varepsilon\|_{B^\phi} + \|P_\varepsilon\chi_E\|_{B^\phi} \leq \varepsilon.\end{aligned}$$

This show that $f\chi_{E^c} \in B^\phi_{a.p.}(\mathbb{R})$. Hence, as the space $B^\phi_{a.p.}(\mathbb{R})$ is linear, we get $f\chi_E \in B^\phi_{a.p.}(\mathbb{R})$. □

Now we prove some basic properties of the set $K(f)$ when $f \in B^\phi_{a.p.}(\mathbb{R})$.

Lemma 3.4. *Let $f \in B^\phi_{a.p.}(\mathbb{R})$. Then*

$$K(f) = \left\{k > 0 : \|f\|^0_{B^\phi} = \frac{1}{k}(1 + \rho_{B^\phi}(kf))\right\}$$

is an interval.

Proof. Let $s, m \in K(f)$ be such that $s \neq m$. We will prove that $[s, m] \subseteq K(f)$. Let $a \in [0, 1]$. By the convexity of ϕ, we get

$$\rho_{B^\phi}((as + (1 - a)m)f) \leq a\rho_{B^\phi}(sf) + (1 - a)\rho_{B^\phi}(mf).$$

It follows that

$$\begin{aligned}1 + \rho_{B^\phi}((as + (1 - a)m)f) &\leq 1 + a\rho_{B^\phi}(sf) + (1 - a)\rho_{B^\phi}(mf) \\ &= a(1 + \rho_{B^\phi}(sf)) + (1 - a)(1 + \rho_{B^\phi}(mf)).\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{as + (1 - a)m}\left[1 + \rho_{B^\phi}((as + (1 - a)m)f)\right] \\ &\leq \frac{a\,(1 + a\rho_{B^\phi}(sf))}{as + (1 - a)m} + \frac{(1 - a)(1 + \rho_{B^\phi}(mf))}{as + (1 - a)m} \\ &= \frac{as}{as + (1 - a)m}\Big(\frac{1}{s}(1 + \rho_{B^\phi}(sf))\Big) + \frac{(1 - a)m}{as + (1 - a)m}\Big(\frac{1}{m}(1 + \rho_{B^\phi}(mf))\Big) \\ &= \frac{as}{as + (1 - a)m}\|f\|^0_{B^\phi} + \frac{(1 - a)m}{as + (1 - a)m}\|f\|^0_{B^\phi} \\ &= \|f\|^0_{B^\phi}.\end{aligned} \tag{3.2}$$

On the other hand, by Proposition 2.4(1) we have

$$\|f\|^0_{B^\phi} = \inf_{k>0}\frac{1}{k}\left[1 + \rho_{B^\phi}(kf)\right] \leq \frac{1}{as + (1 - a)m}\left[1 + \rho_{B^\phi}(as + (1 - a)m)f\right]. \tag{3.3}$$

Combining (3.2) and (3.3) we get

$$\|f\|^0_{B^\phi} = \frac{1}{as+(1-a)m}(1+\rho_{B^\phi}((as+(1-a)m)f),$$

and so $as+(1-a)m \in K(f)$ for all $a \in [0,1]$. This means that $[s,m] \subseteq K(f)$. □

Now, let us introduce this notation. For $k > 0$, $f \in B^\phi_{a.p.}(\mathbb{R})$, define the following set:

$$\overline{S}_\phi(f,k) = \{t \in \mathbb{R} : kf(t) \notin S_\phi\},$$

and $S_\phi(f,k)$ its complementary.

Lemma 3.5. *Let $f \in B^\phi_{a.p.}(\mathbb{R})$, $\|f\|_{B^\phi} \neq 0$. We suppose that $\mu(\overline{S}_\phi(f,k)) = 0$ for any $k \in K(f)$. Then $K(f)$consists exactly of one element from $]0,+\infty[$.*

Proof. Suppose that there exists $m, s \in K(f)$ such that $s < m$. Then $\|(s-m)f\|^0_{B^\phi} > 0$ because $\|f\|^0_{B^\phi} \neq 0$. By Lemma 2.5, there exist reals numbers $0 < \alpha < \beta$, and $\theta \in]0,1[$ such that $\overline{\mu}_B(G) \geq \theta$, where

$$G = \{t \in \mathbb{R} : \alpha \leq |(s-m)f(t)| \leq \beta\}.$$

We know that $\mu(\overline{S}_\phi(f,k)) = 0$ with $k \in K(f)$. This implies that

$$\overline{\mu}_B(S_\phi(f,k) \cap G) = \overline{\mu}_B(G). \tag{3.4}$$

We just write

$$G = (G \cap S_\phi(f,k)) \cup (G \cap \overline{S}_\phi(f,k)).$$

By the convexity of ϕ and Proposition 2.4(1), we have

$$\begin{aligned}
\|f\|^0_{B^\phi} &\leq \frac{2}{s+m}\left[1+\rho_{B^\phi}\left(\frac{s+m}{2}f\right)\right] \\
&\leq \frac{2}{s+m}\left[1+\frac{1}{2}\rho_{B^\phi}(sf)+\frac{1}{2}\rho_{B^\phi}(mf)\right] \\
&\leq \frac{2}{s+m}\left[\frac{1}{2}(1+\rho_{B^\phi}(sf))+\frac{1}{2}(1+\rho_{B^\phi}(mf))\right] \\
&= \frac{2}{s+m}\left[\frac{s}{2}\left(\frac{1}{s}[1+\rho_{B^\phi}(sf)]\right)+\frac{m}{2}\left(\frac{1}{m}[1+\rho_{B^\phi}(mf)]\right)\right] \\
&= \frac{2}{s+m}\left[\frac{s}{2}\|f\|^0_{B^\phi}+\frac{m}{2}\|f\|^0_{B^\phi}\right] = \|f\|^0_{B^\phi}.
\end{aligned}$$

So all the inequalities in the above formulae are, in fact, equalities. Therefore $\frac{s+m}{2} \in K(f)$, and

$$\rho_{B^\phi}\Big(\frac{sf+mf}{2}\Big) = \frac{1}{2}\left[\rho_{B^\phi}(sf)+\rho_{B^\phi}(mf)\right]. \tag{3.5}$$

Let $a = \|mf\|^0_{B^\phi}$. Choose η such that $\min(a\eta, \eta) > 1$. Put

$$A = \{t \in \mathbb{R} : |mf(t)| > a\eta\}.$$

Using Proposition 2.4(2) we get

$$a = \|mf\|^0_{B^\phi} \geq \|mf\|_{B^\phi} \geq \|mf\chi_A\|_{B^\phi} \geq a\eta\|\chi_A\|_{B^\phi}$$

which implies that $\|\chi_A\|_{B^\phi} \leq \frac{1}{\eta} < 1$, and then, by Remark 2.1, $\rho_{B^\phi}(\chi_A) \leq \frac{1}{\eta}$.

In view of the following implication (see, e.g. [17, Lemma 1])

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall A \in \Sigma(\mathbb{R}) : \rho_{B^\phi}(\chi_A) \leq \delta \Rightarrow \overline{\mu}_B(A) < \varepsilon, \tag{3.6}$$

we get that $\overline{\mu}_B(A) < \frac{\theta}{4}$.

Let

$$F_1(u,v) = \frac{2\phi(\frac{u+v}{2})}{\phi(u) + \phi(v)} \quad \text{for each } (u,v) \in \mathbb{R}^2 \setminus \{(0,0)\}.$$

We have $F_1(u,v) < 1$ for all $(u,v) \in Q_1$, where

$$Q_1 = \Big\{(u,v) \in \mathbb{R}^2 : |u| \leq a\eta, |v| \leq a\eta, |u-v| > \alpha, \frac{u+v}{2} \in S_\phi\Big\}.$$

Then using the continuity of F_1 on the compact set Q_1, it follows that there exists $0 < \delta < 1$ such that

$$\sup_{(u,v) \in Q_1} F_1(u,v) = 1 - \delta.$$

More precisely, we have

$$\phi\Big(\frac{u+v}{2}\Big) \leq (1-\delta)\frac{\phi(u) + \phi(v)}{2}, \quad \forall (u,v) \in Q_1.$$

Let now $t \in (G \cap S_\phi(f,k)) \setminus A$. Then $(sf(t), mf(t)) \in Q_1$.

On the other hand, we have

$$\overline{\mu}_B((G \cap S_\phi(f,k)) \setminus A) \geq \overline{\mu}_B(G \cap S_\phi(f,k)) - \overline{\mu}_B(A) \geq \frac{3\theta}{4}.$$

Take $\overline{G} = (G \cap S_\phi(f,k)) \setminus A$. It follows that

$$\begin{aligned}
&\frac{1}{2T}\int_{-T}^{T} \phi\Big(\frac{|sf(t)+mf(t)|}{2}\Big)dt \\
&= \frac{1}{2T}\int_{[-T,T]\cap\overline{G}} \phi\Big(\frac{|sf(t)+mf(t)|}{2}\Big)dt + \frac{1}{2T}\int_{[-T,T]\cap\overline{G}^c} \phi\Big(\frac{|sf(t)+mf(t)|}{2}\Big)dt \\
&\le (1-\delta)\frac{1}{2T}\int_{[-T,T]\cap\overline{G}} \frac{\phi(|sf(t)|)+\phi(|mf|)}{2}dt \\
&\quad + \frac{1}{2T}\int_{[-T,T]\cap\overline{G}^c} \frac{\phi(|sf(t)|)+\phi(|mf(t)|)}{2}dt \\
&\le \frac{1}{2T}\int_{-T}^{T} \frac{\phi(|sf(t)|)+\phi(|mf(t)|)}{2}dt - \delta\frac{1}{2T}\int_{[-T,T]\cap\overline{G}} \frac{\phi(|sf(t)|)+\phi(|mf(t)|)}{2}dt \\
&\le \frac{1}{2}\left[\frac{1}{2T}\int_{-T}^{T}\phi(|sf(t)|dt) + \frac{1}{2T}\int_{-T}^{T}\phi(|mf(t)|)dt\right] \\
&\quad - \delta\frac{1}{2T}\int_{[-T,T]\cap\overline{G}} \phi\Big(\frac{|sf(t)-mf(t)|}{2}\Big)dt \\
&\le \frac{1}{2}\left[\frac{1}{2T}\int_{-T}^{T}\phi(|sf(t)|dt) + \frac{1}{2T}\int_{-T}^{T}\phi(|mf(t)|)\right]dt - \delta\phi\Big(\frac{\alpha}{2}\Big)\frac{\mu(\overline{G})}{2T}.
\end{aligned}$$

Letting T tend to infinity we get

$$\rho_{B^\phi}\Big(\frac{sf+mf}{2}\Big) \le \frac{1}{2}\left[\rho_{B^\phi}(sf)+\rho_{B^\phi}(mf)\right] - \delta\phi\Big(\frac{\alpha}{2}\Big)\overline{\mu}_B(\overline{G}).$$

Then we get

$$\frac{1}{2}(\rho_{B^\phi}(sf)+\rho_{B^\phi}(mf)) - \rho_{B^\phi}\Big(\frac{sf+mf}{2}\Big) \ge \delta\phi\Big(\frac{\alpha}{2}\Big)\overline{\mu}_B(\overline{G}) \ge \frac{3\theta}{4}\delta\phi\Big(\frac{\alpha}{2}\Big) > 0.$$

This contradicts equality (3.5). Then we necessarily have $s = m$. □

Remark 3.6. It follows from Lemma 3.5 that if the function ϕ is strictly convex, then the set $K(f)$ consists of exactly one element.

Now we characterize the extreme points of the unit ball of $B^\phi_{a.p.}(\mathbb{R})$ equipped with the Orlicz norm.

Theorem 3.7. *Let $f \in S(B^\phi_{a.p.}(\mathbb{R}))$. Then f is an extreme point of $B(B^\phi_{a.p.}(\mathbb{R}))$ if and only if $\mu\left(\overline{S}_\phi(f,k)\right) = 0$ for any $k \in K(f)$.*

Proof. Sufficiency: Suppose that $\mu\left(\overline{S}_{\phi}(f,k)\right) = 0$ for any $k \in K(f)$ and $f \notin \operatorname{extr}\left[B(B^{\phi}_{a.p.}(\mathbb{R}))\right]$. So there exists $g,h \in S(B^{\phi}_{a.p}(\mathbb{R}))$ such that

$$g \neq h \quad \text{and} \quad f = \frac{g+h}{2}.$$

By Proposition 2.4, we know that $K(g) \neq \emptyset$ and $K(h) \neq \emptyset$. For $k_1 \in K(g)$ and $k_2 \in K(h)$ we have $\|k_1 g - k_2 h\|^0_{B^\phi} > 0$.

By Lemma 2.5, there exist $\alpha, \beta > 0$ and $\theta \in]0,1[$ such that for the set

$$G = \{t \in \mathbb{R} : \alpha \leq |k_1 g(t) - k_2 h(t)| \leq \beta\},$$

we have $\overline{\mu}_B(G) > \theta$.

In order to simplify the notation we put $k = \frac{k_1 k_2}{k_1+k_2}$. By the convexity of ϕ, we get

$$\begin{aligned} 2 = \|g\|^0_{B^\phi} + \|h\|^0_{B^\phi} &= \frac{1}{k_1}\left[1+\rho_{B^\phi}(k_1 g)\right] + \frac{1}{k_2}\left[1+\rho_{B^\phi}(k_2 h)\right] \\ &= \frac{k_1+k_2}{k_1 k_2}\left[1 + \frac{k_2}{k_1+k_2}\rho_{B^\phi}(k_1 g) + \frac{k_1}{k_1+k_2}\rho_{B^\phi}(k_2 h)\right] \\ &\geq \frac{1}{k}\left[1+\rho_{B^\phi}(kg+kh)\right] = 2\frac{1}{2k}\left[1+\rho_{B^\phi}(2kf)\right] \\ &\geq 2\|f\|^0_{B^\phi} = 2. \end{aligned} \tag{3.7}$$

This implies that $2k \in K(f)$ and

$$\rho_{B^\phi}(2kf) = \frac{k_2}{k_1+k_2}\rho_{B^\phi}(k_1 g) + \frac{k_1}{k_1+k_2}\rho_{B^\phi}(k_2 h). \tag{3.8}$$

By hypothesis, $\mu\left(\overline{S}_{\phi}(f,2k)\right) = 0$. Then, as in (3.4) we get

$$\overline{\mu}_B(G) = \overline{\mu}_B(G \cap S_\phi(f,2k)).$$

Let $\xi > 1$. Define the sets

$$A_1 = \{t \in \mathbb{R} : |g(t)| \geq \xi\}, \quad A_2 = \{t \in \mathbb{R} : |h(t)| \geq \xi\}.$$

Our first claim is that

$$1 = \|g\|^0_{B^\phi} \geq \|g\|_{B^\phi} \geq \|g\chi_{A_1}\|_{B^\phi} \geq \xi\|\chi_{A_1}\|_{B^\phi}.$$

thus $\|\chi_{A_1}\|_{B^\phi} \leq \frac{1}{\xi}$. Similar computations lead to $\|\chi_{A_2}\|_{B^\phi} \leq \frac{1}{\xi}$. Then, by using (3.6), we get $\overline{\mu}_B(A_i) < \frac{\theta}{4}$ for $i = 1, 2$.

Now we choose $b = \max\{k_1, k_2\}$ and consider the set

$$Q = \left\{(u,v) \in \mathbb{R}^2 : |u| \leq b\xi + \beta, |v| \leq b\xi + \beta, |u-v| > \alpha, \frac{k_2}{k_1+k_2}u + \frac{k_1}{k_1+k_2}v \in S_\phi\right\},$$

and define the map F on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$F(u,v) = \frac{\phi(\frac{k_2}{k_1+k_2}u + \frac{k_1}{k_1+k_2}v)}{\frac{k_2}{k_1+k_2}\phi(u) + \frac{k_1}{k_1+k_2}\phi(v)}.$$

For all $t \in (S_\phi(f,2k)\cap G)\setminus(A_1\cup A_2)$, we have $(k_1g(t), k_2h(t)) \in Q$. Then, using same arguments as in the proof of Lemma 3.5, there exists $0<\delta<1$ such that

$$\begin{aligned}\phi\Big(\frac{k_1k_2}{k_1+k_2}(g(t)+h(t))\Big) &= \phi\Big(\frac{k_2}{k_1+k_2}(k_1g(t)) + \frac{k_1}{k_1+k_2}(k_2h(t))\Big)\\ &\le (1-\delta)\Big(\frac{k_2}{k_1+k_2}\phi(k_1g(t)) + \frac{k_1}{k_1+k_2}\phi(k_2h(t))\Big).\end{aligned}$$

Denote

$$\Theta = \frac{k_1+k_2}{k_1.k_2}\Big[\frac{k_2}{k_1+k_2}\rho_{B^\phi}(k_1g) + \frac{k_1}{k_1+k_1}\rho_{B^\phi}(k_2h) - \rho_{B^\phi}(2kf)\Big].$$

By (3.8), we have $\Theta = 0$.

On the other hand, if we denote

$$\overline{G} = [-T,T]\cap\Big((S_\phi(f,k)\cap G)\setminus(A_1\cup A_1)\Big),$$

we have

$$\begin{aligned}\Theta &\ge \frac{1}{k_1}\rho_{B^\phi}(k_1g\chi_{\overline{G}}) + \frac{1}{k_2}\rho_{B^\phi}(k_2h\chi_{\overline{G}})\\ &\quad - \frac{k_1+k_2}{k_1k_2}\Big[(1-\delta)\frac{k_2}{k_1+k_2}\rho_{B^\phi}(k_1g\chi_{\overline{G}}) + \frac{k_1}{k_1+k_2}\rho_{B^\phi}(k_2h\chi_{\overline{G}})\Big]\\ &\ge \frac{\delta}{k_1}\rho_{B^\phi}(k_1g\chi_{\overline{G}}) + \frac{\delta}{k_2}\rho_{B^\phi}(k_2h\chi_{\overline{G}})\\ &\ge \frac{2\delta}{b}\Big(\frac{\rho_{B^\phi}(k_1g\chi_{\overline{G}}) + \rho_{B^\phi}(k_2h\chi_{\overline{G}})}{2}\Big)\\ &\ge \frac{2\delta}{b}\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{\overline{G}}\frac{\phi(k_1|g(t)|)+\phi(k_2|h(t)|)}{2}dt\\ &\ge \frac{2\delta}{b}\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{\overline{G}}\phi\Big(\frac{|k_1g(t)-k_2h(t)|}{2}\Big)dt\\ &\ge \frac{2\delta}{b}\overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{\overline{G}}\phi\Big(\frac{|k_1g(t)-k_2h(t)|}{2}\Big)dt\\ &\ge \frac{2\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\overline{\mu}_B(\overline{G}) \ge \frac{\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\theta > 0.\end{aligned}$$

This is a contradiction with $\Theta = 0$. Therefore the sufficiency is proved.

Necessity: We will show that if $f \in \text{extr}\left[B\left(B^{\phi}_{a.p.}\right)\right]$, then we have

$$\mu\left(\overline{S}_{\phi}(f,k)\right) = 0 \quad \text{for any } k \in K(f).$$

We assume that there exists $k_0 \in K(f)$ such that $\mu\left(\overline{S}_{\phi}(f,k_0)\right) > 0$. Since $\mathbb{R} \setminus S_{\phi}$ is the union of at most countably many open intervals, there exists an interval $]a,b[$ such that for any $\varepsilon > 0$,

$$\mu\left(\{t \in \mathbb{R} : k_0 f(t) \in \,]a+\varepsilon, b-\varepsilon[\}\right) > 0,$$

and that ϕ is affine on $[a,b]$, i.e. for any $t \in [a,b]$, $\phi(t) = \alpha_1 t + \beta_1$, $\alpha_1 > 0$.

Let $\varepsilon > 0$. Take

$$H' = \{t \in \mathbb{R} : k_0 f(t) \in]a+\varepsilon, b-\varepsilon[\}.$$

Then there are two subsets A, B of H' such that $0 < \mu(A) < \infty$, $0 < \mu(B) < \infty$ and $A \cap B = \emptyset$. Indeed, let $\gamma > 0$. We know that $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [\gamma n, \gamma(n+1)[$. Then $H' = \bigcup_{n\in\mathbb{Z}} H'_n$, where $H'_n = H' \cap [\gamma n, \gamma(n+1)[$. We have

$$H'_i \cap H'_j = \emptyset, \forall i \neq j, \quad \mu(H'_n) \leq \mu\big([\gamma n, \gamma(n+1)[\big) = \gamma, \text{ and } \mu(H') = \sum_{n\in\mathbb{Z}} \mu(H'_n).$$

Since $\mu(H') > 0$, the following two cases may arise.

1. $\mu(H') < \infty$. Then there exists at least two sets H'_1, H'_2 such that

$$0 < \mu(H'_i) < \gamma \quad \text{for } i = 1,2 \left(\text{we choose } \gamma = \frac{\mu(H')}{2}\right).$$

2. $\mu(H') = \infty$. There exists infinitely many sets H'_n such that

$$0 < \mu(H'_n) < \gamma.$$

Now we put $H = A \cup B$. We define

$$\begin{cases} g(t) = f(t)\chi_{H^c}(t) + (f(t)-\varepsilon)\chi_A(t) + (f(t)+\varepsilon)\chi_B(t), \\ h(t) = f(t)\chi_{H^c}(t) + (f(t)+\varepsilon)\chi_A(t) + (f(t)-\varepsilon)\chi_B(t). \end{cases}$$

Then $g \neq h$ and $g + h = 2f$.

By Lemma 3.3, we get

$$f\chi_{H^c}, (f-\varepsilon)\chi_A, (f+\varepsilon)\chi_B \in B^{\phi}_{a.p.}(\mathbb{R})$$

which implies that $g, h \in B^{\phi}_{a.p.}(\mathbb{R})$.

Now we show that $\|g\|^0_{B^{\phi}} \leq 1$. Let

$$\rho_T(k_0 g) = \frac{1}{2T}\int_{-T}^{T} \phi(|k_0 g(t)|)dt.$$

Then, by using the fact that ϕ is affine on H we get

$$\begin{aligned}\rho_T(k_0 g) &= \rho_T(k_0 f\chi_{H^c}) + \rho_T(k_0 f\chi_A) + \rho_T(k_0 f\chi_B)\\ &= \rho_T(k_0 f\chi_{H^c})\\ &\quad + \frac{1}{2T}\int\limits_{[-T,T]\cap A}\phi\Big(k_0\big(|f(t)-\varepsilon|\big)\Big)dt + \frac{1}{2T}\int\limits_{[-T,T]\cap B}\phi\Big(k_0|f(t)+\varepsilon|)\Big)dt\\ &\le \rho_T(k_0 f\chi_{H^c})\\ &\quad + \frac{1}{2T}\int\limits_{[-T,T]\cap A}\phi\Big(k_0|f(t)|+k_0\varepsilon\Big)dt + \frac{1}{2T}\int\limits_{[-T,T]\cap B}\phi\Big(k_0|f(t)|+k_0\varepsilon\Big)dt\\ &\le \rho_T(k_0 f\chi_{H^c}) + \frac{1}{2T}\int\limits_{[-T,T]\cap A}\Big(\alpha_1 k_0|f(t)|+\beta_1+\alpha_1 k_0\varepsilon\Big)dt\\ &\quad + \frac{1}{2T}\int\limits_{[-T,T]\cap B}\Big(\alpha_1 k_0|f(t)|+k_0\beta_1+\alpha_1 k_0\varepsilon\Big)dt\\ &\le \rho_T(k_0 f\chi_{H^c}) + \rho_T(k_0 f\chi_H) + \alpha_1 k_0\varepsilon\frac{1}{2T}\mu(H\cap[-T,T])\\ &\le \rho_T(k_0 f) + \alpha_1 k_0\varepsilon\frac{1}{2T}\mu(H\cap[-T,T]).\end{aligned}$$

Then letting $T\to+\infty$ we get

$$\rho_{B^\phi}(k_0 g) \le \rho_{B^\phi}(k_0 f) + k_0\varepsilon\overline{\mu}_B(H).$$

Since $\overline{\mu}_B(H)=0$, we obtain

$$\rho_{B^\phi}(k_0 g) \le \rho_{B^\phi}(k_0 f) \quad \text{with } k_0\in K(f).$$

It follows that

$$\|g\|^0_{B^\phi} = \inf_{k>0}\frac{1}{k}\left[1+\rho_{B^\phi}(kg)\right] \le \frac{1}{k_0}\left[1+\rho_{B^\phi}(k_0 g)\right] \le \frac{1}{k_0}\left[1+\rho_{B^\phi}(k_0 f)\right] = 1.$$

Using same arguments, we get $\|h\|^0_{B^\phi}\le 1$.

Finally, we have $g,h\in B\left(B^\phi_{a.p.}(\mathbb{R})\right)$ with $2f=g+h$. This shows that $f\notin \operatorname{extr}\left[B\left(B^\phi_{a.p}(\mathbb{R})\right)\right]$. The proof is complete. □

Remark 3.8. A criterion for the strict convexity of $B^\phi_{a.p.}(\mathbb{R})$ endowed with the Orlicz norm is known (see [13, Theorem 4.1]), but we can easily deduce this result by our main Theorem 3.7.

Corollary 3.9. *Let $f\in S(B^\phi_{a.p.}(\mathbb{R}))$. If the set $K(f)$ consists of exactly one element $(K(f)=\left\{\overline{k}\right\},\ \overline{k}>0)$, and $\mu\left(\overline{S}_\phi(f,\overline{k})\right)<+\infty$, then $\mu\left(\overline{S}_\phi(f,\overline{k})\right)=0$.*

Proof. Assume that $0 < \mu\left(\overline{S}_\phi(f,\overline{k})\right) < +\infty$. Then by Theorem 3.7, we deduce that $f \notin \text{extr}\left[B(B^\phi_{a.p}(\mathbb{R}), \|\cdot\|^0_{B^\phi})\right]$.

It follows that there exist two functions g, h, $g \neq h$, such that

$$\|g\|^0_{B^\phi} = \|h\|^0_{B^\phi} = 1 \quad \text{and} \quad f = \frac{g+h}{2}.$$

Using same notations and arguments as in the proof of the necessity of Theorem 3.7, we get

$$\overline{\mu}_B(G) = \overline{\mu}_B(G \cap S_\phi(f,\overline{k})), \tag{3.9}$$

and $\overline{\mu}_B(A_i) < \frac{\theta}{4}$ for $i = 1, 2$.

We take

$$\Omega = (G \cap S_\phi(f,k)))\backslash(A_1 \cup A_2).$$

Then

$$\begin{aligned}\overline{\mu}_B(\Omega) &\geq \overline{\mu}_B(G \cap S_\phi(f,k)) - \overline{\mu}_B(A_1) - \overline{\mu}_B(A_2)\\ &\geq \overline{\mu}_B(G) - \overline{\mu}_B(A_1) - \overline{\mu}_B(A_2) \geq \frac{\theta}{2}.\end{aligned} \tag{3.10}$$

We have $k_1 \in K(g)$, $k_2 \in K(h)$ and $\|g\|^0_{B^\phi} = \|h\|^0_{B^\phi} = 1$, $f = \frac{g+h}{2}$ with $g \neq h$. By similar computations as in (3.7), we get

$$1 = \|f\|^0_{B^\phi} = \frac{1}{\frac{2k_1k_2}{k_1+k_2}}\left(1 + \rho_{B^\phi}\left(\frac{2k_1k_2}{k_1+k_2}f\right)\right).$$

Then

$$1 - \frac{k_1+k_2}{2k_1k_2} - \frac{k_1+k_2}{2k_1k_2}\rho_{B^\phi}\left(\frac{2k_1k_2}{k_1+k_2}f\right) = 0. \tag{3.11}$$

Put

$$\rho_T(f) = \frac{1}{2T}\int_{-T}^{T}\phi(|f(t)|)dt, \rho_T(g) = \frac{1}{2T}\int_{-T}^{T}\phi(|g(t)|)dt,$$
$$\rho_T(h) = \frac{1}{2T}\int_{-T}^{T}\phi(|h(t)|)dt,$$

Then

$$\begin{aligned}
\rho_T\Big(\frac{2k_1k_2}{k_1+k_2}f\Big) &= \frac{1}{2T}\int_{-T}^{T}\phi\left(\frac{2k_1k_2}{k_1+k_2}|f(t)|\right)dt \\
&= \frac{1}{2T}\int_{-T}^{T}\phi\left(\Big|\frac{k_2}{k_1+k_2}(k_1g(t))+\frac{k_1}{k_1+k_2}(k_2h(t))\Big|\right)dt \\
&= \frac{1}{2T}\int_{[-T,T]\cap\Omega}\phi\left(\Big|\frac{k_2}{k_1+k_2}(k_1g(t))+\frac{k_1}{k_1+k_2}(k_1h(t))\Big|\right)dt \\
&\quad+\frac{1}{2T}\int_{[-T,T]\cap\Omega^c}\phi\left(\Big|\frac{k_2}{k_1+k_2}(k_1g(t))+\frac{k_1}{k_1+k_2}(k_2h(t))\Big|\right)dt \\
&\le (1-\delta)\frac{1}{2T}\int_{[-T,T]\cap\Omega}\left[\frac{k_2}{k_1+k_2}\phi(k_1|g(t)|)+\frac{k_1}{k_1+k_2}\phi(k_2|h(t)|)\right]dt \\
&\quad+\frac{1}{2T}\int_{[-T,T]\cap\Omega^c}\left[\frac{k_2}{k_1+k_2}\phi(k_1|g(t)|)+\frac{k_1}{k_1+k_2}\phi(k_2|h(t)|)\right]dt \\
&\le \frac{1}{2T}\int_{-T}^{T}\left[\frac{k_2}{k_1+k_2}\phi(k_1|g(t)|)+\frac{k_1}{k_1+k_2}\phi(k_1|h(t)|)\right]dt \\
&\quad-\delta\frac{1}{2T}\int_{[-T,T]\cap\Omega}\left[\frac{k_2}{k_1+k_2}\phi(k_1|g(t)|)+\frac{k_1}{k_1+k_2}\phi(k_2|h(t)|)\right]dt.
\end{aligned}$$

We multiply both sides by $-\frac{k_1+k_2}{2k_1k_2}$. As a consequence, we get

$$\begin{aligned}
-\frac{k_1+k_2}{2k_1k_2}\rho_T\Big(\frac{2k_1k_2}{k_1+k_2}f\Big) &\ge \frac{-1}{2}\frac{1}{2T}\int_{-T}^{T}\left[\frac{1}{k_1}\phi(k_1|g(t)|)+\frac{1}{k_2}\phi(k_2|h(t)|)\right]dt \\
&\quad+\frac{\delta}{2}\frac{1}{2T}\int_{[-T,T]\cap\Omega}\left[\frac{1}{k_1}\phi(k_1|g(t)|)+\frac{1}{k_2}\phi(k_2|h(t)|)\right]dt.
\end{aligned}$$

Since $\frac{1}{k_1}>\frac{1}{b}$ and $\frac{1}{k_2}>\frac{1}{b}$, we obtain

$$\begin{aligned}
-\frac{k_1+k_2}{2k_1k_2}\rho_T\Big(\frac{2k_1k_2}{k_1+k_2}f\Big) &\ge \frac{-1}{2}\Big(\frac{1}{k_1}\rho_T(k_1g)+\frac{1}{k_2}\rho_T(k_1h)\Big) \\
&\quad+\frac{\delta}{b}\frac{1}{2T}\int_{[-T,T]\cap\Omega}\phi\Big(\frac{|k_1g(t)-k_2h(t)|}{2}\Big)dt \\
&\ge \frac{-1}{2}\Big(\frac{1}{k_1}\rho_T(k_1g)+\frac{1}{k_2}\rho_T(k_2h)\Big)+\frac{\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\frac{\mu([-T,T]\cap\Omega)}{2T}.
\end{aligned}$$

Letting T tend to infinity we have

$$\begin{aligned}
-\frac{k_1+k_2}{2k_1k_2}\rho_{B^\phi}\Big(\frac{2k_1k_2}{k_1+k_2}f\Big) &\geq \frac{-1}{2}\Big(\frac{1}{k_1}\rho_{B^\phi}(k_1g)+\frac{1}{k_2}\rho_{B^\phi}(k_2h)\Big)+\frac{\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\overline{\mu}_B(\Omega)\\
&\geq \frac{-1}{2}\Big(\frac{1}{k_1}(1+\rho_{B^\phi}(k_1g))\Big)\\
&\quad+\frac{1}{k_2}(1+\rho_{B^\phi}(k_2h))-\frac{1}{k_1}-\frac{1}{k_2}+\frac{\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\frac{\theta}{2}\\
&=\frac{-1}{2}\Big(\|g\|^0_{B^\phi}+\|h\|^0_{B^\phi}-\frac{k_1+k_2}{k_1k_2}\Big)+\frac{\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\frac{\theta}{2}\\
&=-1+\frac{k_1+k_2}{2k_1k_2}+\frac{\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\frac{\theta}{2}.
\end{aligned}$$

This implies

$$1-\frac{k_1+k_2}{2k_1k_2}-\frac{k_1+k_2}{2k_1k_2}\rho_T\Big(\frac{2k_1k_2}{k_1+k_2}f\Big)\geq\frac{\delta}{b}\phi\Big(\frac{\alpha}{2}\Big)\frac{\theta}{2}>0,$$

which contradicts the equality (3.11). This shows that $\mu(\overline{S}_\phi(f,\overline{k}))=0$. □

The following corollary is an immediate consequence of the previous results.

Corollary 3.10. *Let* $f\in S(B^\phi_{a.p}(\mathbb{R}),\|\cdot\|^o_{B^\phi})$ *and* $\mu\left(\overline{S}_\phi(f,k)\right)<\infty$ *for every* $k\in K(f)$. *Then the following statements are equivalent:*

(1) *the set* $K(f)$ *consists of exactly one element* $K(f)=\{\overline{k}\}$, $\overline{k}>0$,
(2) $\mu\left(\overline{S}_\phi(f,\overline{k})\right)=0$,
(3) $f\in\operatorname{extr}\left[B(B^\phi_{a.p.}(\mathbb{R}),\|\cdot\|^o_{B^\phi})\right]$.

Acknowledgements
This paper is supported by La Direction Générale de la Recherche Scientifique et du Développement Technologique DGRSDT/MESRS-Algeria, PRFU project: ID. C00L03UN060120180009.

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Received: September 4, 2020.
Revised: April 15, 2021.
Accepted: September 10, 2021.