

CLOSED RANGE WEIGHTED COMPOSITION OPERATORS BETWEEN L^p -SPACES

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Abstract. We characterize the closedness of ranges of weighted composition operators between L^p -spaces, where $1 \leq p \leq \infty$. When the L^p -spaces are weighted sequence spaces, several corollaries about this class of operators are also deduced.

Keywords: weighted composition operator, Lebesgue space, closed range.

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1. INTRODUCTION

Let (X, Σ, μ) and (Y, Γ, ν) be two σ -finite and complete measure spaces. Suppose $\varphi : Y \rightarrow X$ is a point mapping such that $\varphi^{-1}(E) \in \Gamma$ for all $E \in \Sigma$. Assume that φ is also non-singular, which means the measure defined by $\nu\varphi^{-1}(E) := \nu(\varphi^{-1}(E))$ for $E \in \Sigma$, is absolutely continuous with respect to μ . Let $u : Y \rightarrow \mathbb{C}$ be a Γ -measurable function. The functions u and φ induce a *weighted composition operator* uC_φ from $L^p(\mu)$ ($1 \leq p \leq \infty$) into the linear space of all Γ -measurable functions on Y by

$$(uC_\varphi f)(y) := u(y)f(\varphi(y)) \quad \text{for every } f \in L^p(\mu) \text{ and } y \in Y.$$

The non-singularity of φ guarantees that uC_φ is a well-defined mapping of equivalence classes of functions. When $u \equiv 1$ (resp. $(X, \Sigma, \mu) = (Y, \Gamma, \nu)$ and $\varphi(x) = x$ for all $x \in X$), the corresponding operator, denoted by C_φ (resp. by M_u), is called a *composition operator* (resp. a *multiplication operator*). Observe that $uC_\varphi = M_u \circ C_\varphi$. Suppose uC_φ maps $L^p(\mu)$ into $L^q(\nu)$, where $1 \leq p, q \leq \infty$. Since norm convergence in the L^p -norm implies pointwise convergence a.e. of a subsequence, it follows from the closed graph theorem that uC_φ is also bounded.

Operator-theoretic properties of weighted composition maps from $L^p(\mu)$ into itself were obtained in [1, 10, 12] and such operators acting between two distinct

L^p -spaces in [4–7, 9]. In this paper, we investigate closed range weighted composition operators between L^p -spaces. Singh and Kumar [11] characterized multiplication and composition operators on $L^2(\mu)$ with closed ranges. Takagi and Yokouchi [13, 14] not only extended the results of Singh and Kumar to a general L^p -space, but also obtained characterizations for the closedness of ranges of such operators between distinct L^p -spaces. Our results generalize their findings to the weighted case. While this question has been considered in [6], the results and proofs therein are sketchy and incomplete. We shall provide proofs that are somewhat different from those in [6], deduce interesting consequences when the underlying spaces are weighted sequence spaces and illustrate the results with examples.

2. PRELIMINARIES

Let f be a complex-valued Σ -measurable function on X . Its support, written as $\text{supp } f$, is defined by

$$\text{supp } f := \{x \in X : f(x) \neq 0\}.$$

For $1 \leq p \leq \infty$, we define

$$\|f\|_p := \begin{cases} (\int_X |f|^p d\mu)^{1/p} & \text{if } 1 \leq p < \infty, \\ \inf\{M > 0 : \mu(\{x \in X : |f(x)| > M\}) = 0\} & \text{if } p = \infty. \end{cases}$$

The Lebesgue space, denoted by $L^p(\mu)$, consists of all (equivalence classes of) Σ -measurable functions f on X for which $\|f\|_p < \infty$. It is a Banach space under the norm $\|\cdot\|_p$ and is written as $\|\cdot\|_{L^p(\mu)}$. The functions in $L^\infty(\mu)$ are said to be essentially bounded.

Let $w := \{w_n\}_{n \in \mathbb{N}}$ be a sequence of positive real numbers. If we take $X = \mathbb{N}$, $\Sigma = \mathcal{P}(\mathbb{N})$ (the power set of \mathbb{N}) and $\mu(E) = \sum_{n \in E} w_n$ for every $E \in \mathcal{P}(\mathbb{N})$, then $L^p(\mu)$ is the weighted sequence space $l^p(w)$ for $1 \leq p < \infty$. If $w_n = 1$ for all $n \in \mathbb{N}$, $l^p(w)$ is just the classical sequence space l^p . When $p = \infty$, we define $l^\infty(w)$ (or simply l^∞) as the space of all bounded sequences of complex numbers.

Analogously, we may define $L^p(\nu)$ and $\|\cdot\|_{L^p(\nu)}$ for $1 \leq p \leq \infty$. Moreover, we define $\text{supp } g$ similarly for a complex-valued Γ -measurable function g on Y .

In the sequel, we adopt the following decomposition of (X, Σ, μ) :

$$X = \left(\bigcup_{i=1}^\infty A_i \right) \cup B,$$

where $\{A_i\}_{i=1}^\infty$ is a countable collection of pairwise disjoint atoms and B , being disjoint from each A_i , is nonatomic. This decomposition is unique in the sense that equality of two Σ -measurable sets interpreted as their symmetric difference is of zero μ -measure. The σ -finiteness of (X, Σ, μ) ensures that $\mu(A_i) < \infty$ for every $i \in \mathbb{N}$. Moreover, if $X = \bigcup_{i=1}^\infty A_i$ (resp. $X = B$), then (X, Σ, μ) is said to be atomic (resp. nonatomic).

The two facts below will be useful.

- (a) Let E be a nonatomic set in Σ with $\mu(E) > 0$. For every real number α satisfying $0 < \alpha < \mu(E)$, there is a set $E_\alpha \in \Sigma$ with $E_\alpha \subset E$ and $\mu(E_\alpha) = \alpha$.
- (b) A Σ -measurable function $f : X \rightarrow \mathbb{C}$ is constant μ -a.e. on an atom M in (X, Σ, μ) . Consequently, we view an atom M as a ‘point’ at which f takes a constant value denoted by $f(M)$.

For $1 \leq q < \infty$, the measure μ_q defined by

$$\mu_q(E) = \int_{\varphi^{-1}(E)} |u|^q d\nu \quad \text{for every } E \in \Sigma$$

is absolutely continuous with respect to μ . The corresponding Radon–Nikodym derivative, denoted by $[d\mu_q/d\mu]$, satisfies an important property.

Lemma 2.1 ([7, Lemma 2]). *If uC_φ is a weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$, where $1 \leq p, q < \infty$, then*

$$\|uC_\varphi f\|_{L^q(\nu)}^q = \int_X \left[\frac{d\mu_q}{d\mu} \right] |f|^q d\mu \quad \text{for every } f \in L^p(\mu).$$

Let $(B_1, \|\cdot\|_{B_1})$ and $(B_2, \|\cdot\|_{B_2})$ be two Banach spaces. For a bounded linear operator T from B_1 into B_2 , we denote its kernel and range by $\ker T$ and $T(B_1)$ respectively. The following result, which can be found in [2], will be used frequently in the subsequent sections. We state it for quick reference:

There exists a constant $c > 0$ such that $\|Tx\|_{B_2} \geq c\|x\|_{B_1}$ for all $x \in B_1$ (2.1)
 if and only if $\ker T = \{0\}$ and $T(B_1)$ is closed in B_2 .

The condition “ $\|Tx\|_{B_2} \geq c\|x\|_{B_1}$ for all $x \in B_1$ ” is usually referred to as “ T is bounded below”.

3. MAIN RESULTS: $1 \leq p, q < \infty$

In this section, we characterize the closedness of ranges of weighted composition operators from $L^p(\mu)$ into $L^q(\nu)$, where $1 \leq p, q < \infty$. For the sake of completeness, we state without proof the following result which characterizes weighted composition maps from $L^p(\mu)$ to $L^p(\nu)$ that have closed ranges. To this end, let \mathfrak{C} be the collection of all Σ -measurable sets E such that

- (a) $\mu(E) < \infty$, and
- (b) whenever $G \in \Sigma$ satisfies $G \subset E$ and $\int_{\varphi^{-1}(G)} |u|^p d\nu = 0$, $\mu(G) = 0$.

Theorem 3.1 ([6, Theorem 3.4]). *Let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^p(\nu)$, where $1 \leq p < \infty$. The following statements are equivalent.*

- (i) *The operator uC_φ has closed range.*
- (ii) *There is a constant $c > 0$ such that $\left[\frac{d\mu_p}{d\mu}\right] \geq c$ μ -a.e. on $\text{supp}\left[\frac{d\mu_p}{d\mu}\right]$.*
- (iii) *There is a constant $\alpha > 0$ such that $\int_{\varphi^{-1}(E)} |u|^p d\nu \geq \alpha\mu(E)$ for all $E \in \mathfrak{E}$.*

Here is an interesting corollary of Theorem 3.1.

Corollary 3.2. *Suppose that $1 \leq p < \infty$ and $0 < \inf_{n \in \mathbb{N}} w_n \leq \sup_{n \in \mathbb{N}} w_n < \infty$.*

- (a) *If uC_φ is a weighted composition operator from $l^p(w)$ to $l^p(w)$ such that $\inf_{n \in \text{supp } u} |u(n)| > 0$, then uC_φ has closed range.*
- (b) *Every composition operator from $l^p(w)$ to $l^p(w)$ has closed range. In particular, all composition operators from l^p to l^p have closed ranges.*

Proof. It suffices to prove (a) only, since (b) follows from (a) directly.

Put $\alpha := \inf_{n \in \mathbb{N}} w_n$, $\beta := \sup_{n \in \mathbb{N}} w_n$ and $\gamma := \inf_{n \in \text{supp } u} |u(n)|$ ($0 < \alpha, \beta, \gamma < \infty$). In view of Theorem 3.1, we are to show that there is a constant $c > 0$ such that $[d\mu_p/d\mu](n) \geq c$ for every $n \in \text{supp}[d\mu_p/d\mu]$.

Note that we have

$$\begin{aligned} \left[\frac{d\mu_p}{d\mu}\right](n)\mu(\{n\}) &= \int_{\{n\}} \left[\frac{d\mu_p}{d\mu}\right] d\mu \\ &= \int_{\varphi^{-1}(\{n\}) \cap \text{supp } u} |u|^p d\mu \\ &= \sum_{i \in \varphi^{-1}(\{n\}) \cap \text{supp } u} |u(i)|^p \mu(\{i\}) \end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N}$. From (3.1), we see that for each $n \in \text{supp}[d\mu_p/d\mu]$, there is some $j \in \text{supp } u$ with $\varphi(j) = n$. Then

$$\left[\frac{d\mu_p}{d\mu}\right](n) = \frac{\sum_{i \in \varphi^{-1}(\{n\}) \cap \text{supp } u} |u(i)|^p \mu(\{i\})}{\mu(\{n\})} \geq \frac{|u(j)|^p \mu(\{j\})}{\mu(\{n\})} \geq \frac{\gamma^p \alpha}{\beta}.$$

By taking $c = (\gamma^p \alpha)/\beta$, we obtain the desired result. □

Example 3.3. Let $X = [0, \infty)$ be equipped with the Lebesgue measure μ on the σ -algebra Σ of Borel sets in $[0, \infty)$. If

$$\varphi(x) := 2x\chi_{[0,1)}(x) + \left(\frac{8}{3}x - \frac{8}{3}\right)\chi_{[1,\frac{7}{2})}(x) + \left(\frac{3}{8}x + \frac{11}{16}\right)\chi_{[\frac{7}{2},\infty)}(x) \quad \text{for } x \in X,$$

it follows from [3, Proposition 2.1] that C_φ is a composition operator from $L^p(\mu)$ ($1 \leq p < \infty$) into itself with

$$\left[\frac{d\mu_p}{d\mu}\right](x) = \frac{7}{8}\chi_{[0,2)}(x) + \frac{73}{24}\chi_{[2,\frac{20}{3})}(x) + \frac{8}{3}\chi_{[\frac{20}{3},\infty)}(x) \quad \mu\text{-a.e. on } X.$$

Using Theorem 3.1, we see that the range of C_φ is closed.

We now consider weighted maps between distinct L^p -spaces that have closed ranges. A few notations are in order. Let $E \in \Sigma$ with $\mu(X \setminus E) > 0$, we define

$$L^p|_E(\mu) = \{f \in L^p(\mu) : f = 0 \text{ } \mu\text{-a.e. on } X \setminus E\}.$$

Then, $L^p|_E(\mu)$ (equipped with the $\|\cdot\|_{L^p(\mu)}$ norm) is a closed subspace of $L^p(\mu)$ and is thus a Banach space. The restriction of uC_φ to $L^p|_E(\mu)$ is denoted by $uC_\varphi|_E$.

Theorem 3.4. *Let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$, where $1 \leq q < p < \infty$. The following statements are equivalent.*

- (i) *The operator uC_φ has closed range.*
- (ii) *The operator uC_φ has finite rank.*
- (iii) $\left[\frac{d\mu_q}{d\mu}\right] = 0$ μ -a.e. on B and the set $\left\{i \in \mathbb{N} : \left[\frac{d\mu_q}{d\mu}\right](A_i) > 0\right\}$ is finite.
- (iv) $\nu(\varphi^{-1}(B) \cap \text{supp } u) = 0$ and the set $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \text{supp } u) > 0\}$ is finite.

Proof. (ii) \Rightarrow (i) This follows from the fact that every finite-dimensional normed space is closed.

(iii) \Leftrightarrow (iv) Note that

$$\int_E \left[\frac{d\mu_q}{d\mu}\right] d\mu = \int_{\varphi^{-1}(E)} |u|^q d\nu = \int_{\varphi^{-1}(E) \cap \text{supp } u} |u|^q d\nu$$

for every $E \in \Sigma$. Thus, $[d\mu_q/d\mu]$ vanishes μ -a.e. on B if and only if

$$\int_{\varphi^{-1}(B) \cap \text{supp } u} |u|^q d\nu = 0,$$

which in turn is equivalent to $\nu(\varphi^{-1}(B) \cap \text{supp } u) = 0$. For each $i \in \mathbb{N}$, we have

$$\left[\frac{d\mu_q}{d\mu}\right](A_i)\mu(A_i) = \int_{A_i} \left[\frac{d\mu_q}{d\mu}\right] d\mu = \int_{\varphi^{-1}(A_i) \cap \text{supp } u} |u|^q d\nu.$$

Therefore, $[d\mu_q/d\mu](A_i) > 0$ if and only if $\nu(\varphi^{-1}(A_i) \cap \text{supp } u) > 0$. The equivalence of (iii) and (iv) has been established.

(iii) \Rightarrow (ii) Suppose (iii) holds. To show that uC_φ has finite rank, it suffices to prove the set

$$S := \{g \in uC_\varphi(L^p(\mu)) : \|g\|_{L^q(\nu)} \leq 1\}$$

is compact in $L^q(\nu)$.

If the set $\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$ is empty, then uC_φ is just the zero operator and (ii) trivially holds. Otherwise, we may assume (upon a suitable re-indexing of atoms if necessary) there exists some $k \in \mathbb{N}$ with

$$\left[\frac{d\mu_q}{d\mu}\right](A_i) \begin{cases} > 0 & \text{if } i = 1, 2, \dots, k, \\ = 0 & \text{if } i = k + 1, k + 2, \dots \end{cases}$$

Take an arbitrary sequence $\{uC_\varphi f_n\}_{n \in \mathbb{N}}$ in S . Since $[d\mu_q/d\mu] = 0$ μ -a.e. on B , it follows that

$$\begin{aligned} \|uC_\varphi f_n\|_{L^q(\nu)}^q &= \int_{\bigcup_{i \in \mathbb{N}} A_i} \left[\frac{d\mu_q}{d\mu} \right] |f_n|^q d\mu + \int_B \left[\frac{d\mu_q}{d\mu} \right] |f_n|^q d\mu \\ &= \sum_{i=1}^k \left[\frac{d\mu_q}{d\mu} \right] (A_i) |f_n(A_i)|^q \mu(A_i) \\ &= \sum_{i=1}^k \alpha_i |f_n(A_i)|^q, \end{aligned}$$

where

$$\alpha_i := [d\mu_q/d\mu](A_i)\mu(A_i) > 0.$$

With $\|uC_\varphi f_n\|_{L^q(\nu)} \leq 1$ for all $n \in \mathbb{N}$, we have $|f_n(A_i)| \leq (1/\alpha_i)^{1/q}$ for $i = 1, 2, \dots, k$ and $n \in \mathbb{N}$. By the Bolzano–Weierstrass theorem, there is a subsequence of natural numbers $\{n_j\}_{j \in \mathbb{N}}$ such that for each fixed $i = 1, 2, \dots, k$, the sequence $\{f_{n_j}(A_i)\}_{j \in \mathbb{N}}$ converges. Suppose $\lim_{j \rightarrow \infty} f_{n_j}(A_i) = \zeta_i$, say, and define $f = \sum_{i=1}^k \zeta_i \chi_{A_i}$. Then, $f \in L^p(\mu)$ and

$$\|uC_\varphi f\|_{L^q(\nu)}^q = \sum_{i=1}^k \alpha_i |\zeta_i|^q \leq 1,$$

i.e. $uC_\varphi f \in S$. Moreover,

$$\begin{aligned} \|uC_\varphi f_{n_j} - uC_\varphi f\|_{L^q(\nu)}^q &= \sum_{i=1}^k \alpha_i |f_{n_j}(A_i) - f(A_i)|^q \\ &= \sum_{i=1}^k \alpha_i |f_{n_j}(A_i) - \zeta_i|^q \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Hence the set S is compact in $L^q(\nu)$.

(i) \Rightarrow (iii) We first prove that if uC_φ has closed range, then $[d\mu_q/d\mu] = 0$ μ -a.e. on B . Suppose, on the contrary,

$$\mu(\{x \in B : [d\mu_q/d\mu](x) > 0\}) > 0.$$

Then there is a constant $\delta > 0$ for which the set

$$G := \{x \in B : [d\mu_q/d\mu](x) \geq \delta\}$$

has positive μ -measure. We may assume $\mu(G) < \infty$. As G is nonatomic, we can further assume that $\mu(X \setminus G) > 0$.

Consider the operator $uC_\varphi|_G$ defined on $L^p|_G(\mu)$. We claim that:

- (a) $uC_\varphi|_G(L^p|_G(\mu))$ is closed in $L^q(\nu)$ (under the assumption uC_φ has closed range),
- (b) $\ker uC_\varphi|_G = \{0\}$.

To prove (a), take any convergent sequence $\{uC_\varphi|_G f_n\}_{n \in \mathbb{N}}$ in $uC_\varphi|_G(L^p|_G(\mu))$ and let $g \in L^q(\nu)$ satisfy $\|uC_\varphi|_G f_n - g\|_{L^q(\nu)} \rightarrow 0$ as $n \rightarrow \infty$. Note that $\{uC_\varphi|_G f_n\}_{n \in \mathbb{N}}$ can be regarded as a sequence in $uC_\varphi(L^p(\mu))$. The closedness of range of uC_φ yields a function $f \in L^p(\mu)$ with $g = uC_\varphi f$ ν -a.e. on Y . Since

$$\begin{aligned} \|uC_\varphi|_G f_n - g\|_{L^q(\nu)}^q &= \|uC_\varphi|_G f_n - uC_\varphi f\|_{L^q(\nu)}^q \\ &= \int_G \left[\frac{d\mu_q}{d\mu} \right] |f_n - f|^q d\mu + \int_{X \setminus G} \left[\frac{d\mu_q}{d\mu} \right] |f_n - f|^q d\mu \\ &= \int_G \left[\frac{d\mu_q}{d\mu} \right] |f_n - f|^q d\mu + \int_{X \setminus G} \left[\frac{d\mu_q}{d\mu} \right] |f|^q d\mu \end{aligned}$$

for every $n \in \mathbb{N}$, it follows that $\int_{X \setminus G} [d\mu_q/d\mu] |f|^q d\mu = 0$. Now,

$$\|uC_\varphi|_G f \chi_G - uC_\varphi f\|_{L^q(\nu)}^q = \int_{X \setminus G} \left[\frac{d\mu_q}{d\mu} \right] |f|^q d\mu = 0.$$

Thus, $g = uC_\varphi|_G f \chi_G$ ν -a.e. on Y , where $f \chi_G \in L^p|_G(\mu)$.

To prove (b), we take any $f \in \ker uC_\varphi|_G$. Then $uC_\varphi|_G f = 0$ ν -a.e. on Y , or

$$\int_G [d\mu_q/d\mu] |f|^q d\mu = \|uC_\varphi|_G f\|_{L^q(\nu)}^q = 0.$$

This, together with the inequalities

$$0 \leq \delta \int_G |f|^q d\mu \leq \int_G \left[\frac{d\mu_q}{d\mu} \right] |f|^q d\mu,$$

implies that $f = 0$ μ -a.e. on G . With $f \in L^p|_G(\mu)$, (b) follows.

By (2.1), there exists a constant $c > 0$ such that

$$\|uC_\varphi|_G f\|_{L^q(\nu)} \geq c \|f\|_{L^p(\mu)}$$

for all $f \in L^p|_G(\mu)$. We claim that this is impossible by showing that for each $\alpha > 0$, there is some $f_\alpha \in L^p|_G(\mu)$ satisfying

$$\|uC_\varphi|_G f_\alpha\|_{L^q(\nu)} < \alpha \|f_\alpha\|_{L^p(\mu)}.$$

For every $n \in \mathbb{N}$, define

$$G_n = \{x \in G : n - 1 \leq [d\mu_q/d\mu](x) < n\}.$$

Then

$$G = \left(\bigcup_{n=1}^{\infty} G_n \right) \cup \left\{ x \in G : \left[\frac{d\mu_q}{d\mu} \right] (x) = \infty \right\}.$$

Since uC_φ is a bounded operator from $L^p(\mu)$ into $L^q(\nu)$, it follows from [5, Theorem 2.3] that $[d\mu_q/d\mu] \in L^{p/(p-q)}(\mu)$. In particular, $[d\mu_q/d\mu]$ is finite-valued μ -a.e. on X and

$$\mu \left(\left\{ x \in G : \left[\frac{d\mu_q}{d\mu} \right] (x) = \infty \right\} \right) = 0.$$

With $\mu(G) > 0$, we see that $\mu(G_N) > 0$ for some $N \in \mathbb{N}$. Fix any $\alpha > 0$. By the nonatomicity of G_N , we may choose a set $E_\alpha \in \Sigma$ with $E_\alpha \subset G_N$ and $\mu(E_\alpha) = \mu(G_N)/K$, where $K \in \mathbb{N}$ and

$$K > (N^{\frac{p}{p-q}} \mu(G_N)) / \alpha^{\frac{pq}{p-q}}.$$

Take $f_\alpha := \chi_{E_\alpha}$. Then, $f_\alpha \in L^p|_G(\mu)$ and $\|f_\alpha\|_{L^p(\mu)} = (\mu(G_N)/K)^{1/p}$. Moreover,

$$\begin{aligned} \|uC_\varphi|_G f_\alpha\|_{L^q(\nu)} &= \left(\int_{E_\alpha} \left[\frac{d\mu_q}{d\mu} \right] d\mu \right)^{1/q} \\ &\leq (N\mu(E_\alpha))^{1/q} = \frac{N^{\frac{1}{q}} \mu(G_N)^{\frac{1}{p} + \frac{p-q}{pq}}}{K^{\frac{1}{p} + \frac{p-q}{pq}}} < \alpha \|f_\alpha\|_{L^p(\mu)}. \end{aligned}$$

This proves our claim. Therefore, we must have $[d\mu_q/d\mu] = 0$ μ -a.e. on B .

It remains to show that if uC_φ has closed range, then the set

$$\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$$

is finite.

We also argue by contradiction. Suppose, on the contrary, the set

$$\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$$

is infinite. Without losing generality, we assume that $[d\mu_q/d\mu](A_i) > 0$ for all $i \in \mathbb{N}$. Let $A := \bigcup_{i \in \mathbb{N}} A_i$ and assume $\mu(X \setminus A) > 0$. This time, we consider the operator $uC_\varphi|_A$ on $L^p|_A(\mu)$ (if $\mu(X \setminus A) = 0$, then (X, Σ, μ) is atomic. Upon replacing $uC_\varphi|_A$ and $L^p|_A(\mu)$ by uC_φ and $L^p(\mu)$ respectively, the proof below is still valid).

Similarly, one may show that $uC_\varphi|_A(L^p|_A(\mu))$ is a closed subspace of $L^q(\nu)$ and $\ker uC_\varphi|_A = \{0\}$. By (2.1), there is a constant $d > 0$ with

$$\|uC_\varphi|_A f\|_{L^q(\nu)} \geq d \|f\|_{L^p(\mu)}$$

for all $f \in L^p|_A(\mu)$. Then

$$\left[\frac{d\mu_q}{d\mu} \right] (A_i) \mu(A_i) = \|uC_\varphi|_A \chi_{A_i}\|_{L^q(\nu)}^q \geq d^q \|\chi_{A_i}\|_{L^p(\mu)}^q = d^q \mu(A_i)^{q/p},$$

so that

$$\left[\frac{d\mu_q}{d\mu} \right] (A_i)^{\frac{p}{p-q}} \mu(A_i) \geq d^{\frac{pq}{p-q}}$$

for all $i \in \mathbb{N}$. From the above inequality,

$$\sum_{i \in \mathbb{N}} d^{\frac{pq}{p-q}} \leq \sum_{i \in \mathbb{N}} \left[\frac{d\mu_q}{d\mu} \right] (A_i)^{\frac{p}{p-q}} \mu(A_i) = \|uC_\varphi\|_{\frac{pq}{p-q}} < \infty.$$

This is absurd, since $\sum_{i \in \mathbb{N}} d^{\frac{pq}{p-q}} = \infty$.

Hence $[d\mu_q/d\mu](A_i) > 0$ for finitely many $i \in \mathbb{N}$ only. The proof of the implication (i) \Rightarrow (iii) is now complete. \square

Theorem 3.5. *Let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$, where $1 \leq p < q < \infty$. The following statements are equivalent.*

- (i) *The operator uC_φ has closed range.*
- (ii) *The operator uC_φ has finite rank.*
- (iii) *The set $\left\{ i \in \mathbb{N} : \left[\frac{d\mu_q}{d\mu} \right] (A_i) > 0 \right\}$ is finite.*
- (iv) *The set $\{ i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \text{supp } u) > 0 \}$ is finite.*

Proof. The proofs of (iii) \Leftrightarrow (iv), (ii) \Rightarrow (i) and (iii) \Rightarrow (ii) are the same as those in Theorem 3.4. It suffices to prove (i) \Rightarrow (iii) only.

Suppose, on the contrary, the set $\{ i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0 \}$ is infinite. We may further assume that $[d\mu_q/d\mu] > 0$ on A , where $A := \bigcup_{i \in \mathbb{N}} A_i$. By [4, Theorem 2.5], $[d\mu_q/d\mu] < \infty$ on A . Let $d > 0$ be a constant such that

$$\|uC_\varphi|_A f\|_{L^q(\nu)} \geq d \|f\|_{L^p(\mu)} \quad \text{for all } f \in L^p|_A(\mu).$$

Consider the operator M on $L^p|_A(\mu)$ defined by

$$Mf = \left[\frac{d\mu_q}{d\mu} \right]^{1/q} f \quad \text{for } f \in L^p|_A(\mu).$$

Since $\|Mf\|_{L^q(\mu)} = \|uC_\varphi|_A f\|_{L^q(\nu)}$, M maps $L^p|_A(\mu)$ into $L^q|_A(\mu)$ and M is bounded below. We shall show that $M(L^p|_A(\mu))$ is dense in $L^q|_A(\mu)$, for this implies that M is a surjective map between $L^p|_A(\mu)$ and $L^q|_A(\mu)$.

Take any $g \in L^q|_A(\mu)$ and let $g_n := \sum_{i=1}^n g(A_i)\chi_{A_i}$ for each $n \in \mathbb{N}$. Then

$$\|g_n - g\|_{L^q(\mu)}^q = \|g\|_{L^q(\mu)}^q - \sum_{i=1}^n |g(A_i)|^q \mu(A_i) \rightarrow 0$$

as $n \rightarrow \infty$. We also define

$$f_n = \begin{cases} [d\mu_q/d\mu]^{-1/q} g_n & \text{on } A, \\ 0 & \text{on } X \setminus A. \end{cases}$$

Note that $f_n \in L^p|_A(\mu)$ (with $\|f_n\|_{L^p(\mu)}^p = \sum_{i=1}^n \frac{|g_n(A_i)|^p \mu(A_i)}{[d\mu_q/d\mu](A_i)^{p/q}} < \infty$) and

$$\|Mf_n - g\|_{L^q(\mu)} = \|g_n - g\|_{L^q(\mu)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus, $M(L^p|_A(\mu))$ is dense in $L^q|_A(\mu)$.

The inverse operator of M , denoted by M^{-1} , is given by $g \mapsto pg$ for $g \in L^q|_A(\mu)$, where

$$p := \begin{cases} [d\mu_q/d\mu]^{-1/q} & \text{on } A, \\ 0 & \text{on } X \setminus A. \end{cases}$$

By the bounded inverse theorem, M^{-1} is also bounded. Applying [5, Theorem 2.3], we obtain

$$\int_A \frac{1}{[d\mu_q/d\mu]^{p/(q-p)}} d\mu < \infty. \tag{3.2}$$

Furthermore, according to [4, Theorem 2.5], we have

$$(0 <) \|uC_\varphi\|^q = \sup_{i \in \mathbb{N}} \frac{[d\mu_q/d\mu](A_i)}{\mu(A_i)^{(q-p)/p}}.$$

This, together with (3.2), yields

$$\sum_{i \in \mathbb{N}} \frac{1}{\|uC_\varphi\|^{pq/(q-p)}} \leq \sum_{i \in \mathbb{N}} \frac{\mu(A_i)}{[d\mu_q/d\mu](A_i)^{p/(q-p)}} = \int_A \frac{1}{[d\mu_q/d\mu]^{p/(q-p)}} d\mu < \infty.$$

This is absurd because

$$\sum_{i \in \mathbb{N}} 1/\|uC_\varphi\|^{pq/(q-p)} = \infty.$$

Hence the set $\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$ is finite if uC_φ has closed range. □

In Corollary 3.2, we showed that every composition operator from l^p to itself has closed range. This is *not* necessarily true, when the domain of the operator differs from the co-domain. The following example demonstrates this.

Example 3.6. Let $T : l^1 \rightarrow l^2$ be defined by

$$T(x_1, x_2, \dots) = (x_2, x_3, \dots) \quad \text{for every } \{x_n\}_{n \in \mathbb{N}} \in l^1.$$

The map T is a composition operator (also known as the left shift operator) induced by $\varphi(n) = n + 1$ for $n \in \mathbb{N}$. Since

$$[d\mu_2/d\mu](n) = \begin{cases} 0 & \text{if } n = 1, \\ 1 & \text{if } n = 2, 3, \dots, \end{cases}$$

an appeal to Theorem 3.5 shows that T does not have closed range. This fact can also be established by a direct argument: for each $n \in \mathbb{N}$, let $f_n (= \{x_j^n\}_{j \in \mathbb{N}})$ be the sequence

$$x_j^n := \begin{cases} 1/j & \text{if } j = 1, 2, \dots, n, \\ 0 & \text{if } j = n + 1, n + 2, \dots \end{cases}$$

Then as $n \rightarrow \infty$, $\|Tf_n - g\|_{l^2} \rightarrow 0$, where $g := (1/2, 1/3, 1/4, \dots) \in l^2$. However, $g \notin T(l^1)$.

4. MAIN RESULTS: $p = \infty$ OR $q = \infty$

In this section, we study the closedness of ranges of weighted composition maps from $L^p(\mu)$ into $L^q(\nu)$, where $p = \infty$ or $q = \infty$. When $1 \leq p, q < \infty$, we rely on the Radon–Nikodym derivative $[d\mu_q/d\mu]$ in stating characterizations and constructing proofs. This is no longer viable if $q = \infty$. It is necessary to utilize properties intrinsic to uC_φ and the measure $\nu\varphi^{-1}$ instead. The following notations will be adopted in this section.

Let $S \in \Gamma$. The restriction of a Γ -measurable function $g : Y \rightarrow \mathbb{C}$ to S is denoted by $g|_S$. We also define the σ -algebra Γ_S and the measure ν_S on Γ_S by

$$\Gamma_S = \{F \cap S : F \in \Gamma\} \text{ and } \nu_S(F) = \nu(F) \text{ for } F \in \Gamma_S$$

respectively. Then (S, Γ_S, ν_S) is a σ -finite and complete measure space. For $1 \leq p \leq \infty$ and a Γ_S -measurable function $h : S \rightarrow \mathbb{C}$, let

$$\|h\|_p := \begin{cases} \left(\int_S |h|^p d\nu\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \inf\{M > 0 : \nu(\{y \in S : |h(y)| > M\}) = 0\} & \text{if } p = \infty. \end{cases}$$

The set $L^p(\nu_S)$ is the Banach space of all (equivalence classes of) Γ_S -measurable functions h on S such that $\|h\|_p < \infty$. We denote the norm $\|\cdot\|_p$ by $\|\cdot\|_{L^p(\nu_S)}$.

If $S \in \Sigma$, then $L^p(\mu_S)$ and $\|\cdot\|_{L^p(\mu_S)}$ are defined similarly.

When (X, Σ, μ) is atomic, we have the following sufficient condition for the closedness of range of uC_φ from $L^\infty(\mu)$ to $L^\infty(\nu)$. A general characterization, however, has not yet been obtained.

Theorem 4.1. *Suppose that (X, Σ, μ) is atomic. Let uC_φ be a weighted composition operator from $L^\infty(\mu)$ into $L^\infty(\nu)$. If there is a constant $\delta > 0$ such that $|u(y)| \geq \delta$ ν -a.e. on $\text{supp } u$, then uC_φ has closed range.*

Proof. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $L^\infty(\mu)$ and g be a function in $L^\infty(\nu)$ with $\|uC_\varphi f_n - g\|_{L^\infty(\nu)} \rightarrow 0$ as $n \rightarrow \infty$. Then there is a set $F \in \Gamma$ for which $\nu(Y \setminus F) = 0$ and $uC_\varphi f_n$ converges to g uniformly on F . Put $S := \text{supp } u$ and $\psi := \varphi|_S$. Consider the composition operator $C_\psi : L^\infty(\mu) \rightarrow L^\infty(\nu_S)$ defined by $C_\psi f = f \circ \psi$ for $f \in L^\infty(\mu)$. Since

$$\left|C_\psi f_n - \frac{g}{u}\right|_S = \left|\frac{1}{u}\right| |uC_\psi f_n - g| \leq \frac{1}{\delta} |uC_\varphi f_n - g| \text{ on } S$$

and $\nu(S \setminus (F \cap S)) = 0$, it follows that $\|C_\psi f_n - (g/u)|_S\|_{L^\infty(\nu_S)} \rightarrow 0$ as $n \rightarrow \infty$, i.e. the sequence $\{C_\psi f_n\}_{n \in \mathbb{N}}$ converges to $(g/u)|_S \in L^\infty(\nu_S)$ uniformly on $F \cap S$.

We claim that $C_\psi(L^\infty(\mu))$ is closed. To prove this, let $T : L^\infty(\mu)/\ker C_\psi \rightarrow L^\infty(\nu_S)$ be defined as

$$T(f + \ker C_\psi) = C_\psi f,$$

with $f \in L^\infty(\mu)$ and $L^\infty(\mu)/\ker C_\psi$ being a Banach space equipped with the quotient norm given by

$$\|f + \ker C_\psi\| := \inf\{\|f + h\|_{L^\infty(\mu)} : h \in \ker C_\psi\}.$$

The map T is well-defined and linear. Fix any $f \in L^\infty(\mu)$. For every $h \in \ker C_\psi$,

$$\|f + h\|_{L^\infty(\mu)} \geq \|C_\psi(f + h)\|_{L^\infty(\nu_S)} = \|T(f + \ker C_\psi)\|_{L^\infty(\nu_S)}.$$

Thus,

$$\|f + \ker C_\psi\| \geq \|T(f + \ker C_\psi)\|_{L^\infty(\nu_S)}. \tag{4.1}$$

If we let

$$s := - \sum_{i \in I} f(A_i)\chi_{A_i},$$

where

$$I := \{i \in \mathbb{N} : \nu\psi^{-1}(A_i) = 0\},$$

then $s \in \ker C_\psi$ and

$$\|C_\psi f\|_{L^\infty(\nu_S)} = \|f + s\|_{L^\infty(\mu)} (= \sup_{i \notin I} |f(A_i)|).$$

Hence

$$\begin{aligned} \|T(f + \ker C_\psi)\|_{L^\infty(\nu_S)} &= \|C_\psi f\|_{L^\infty(\nu_S)} \\ &= \|f + s\|_{L^\infty(\mu)} \\ &\geq \|f + \ker C_\psi\|. \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), T is indeed an isometry. Therefore, the range of C_ψ is closed.

In view of the claim, we obtain a function f in $L^\infty(\mu)$ such that $C_\psi f = (g/u)|_S$ ν -a.e. on S . With $g = 0$ ν -a.e. on $Y \setminus S$, we conclude that $uC_\psi f = g$ ν -a.e. on Y . This completes the proof. \square

We digress momentarily to consider invertible weighted composition operators from $L^p(\mu)$ onto $L^p(\nu)$. When $1 \leq p < \infty$ and (X, Σ, μ) is nonatomic, characterizations for this class of operators were obtained by the authors in [8, Theorem 1.2]. We now completely characterize these maps for $p = \infty$, without assuming the nonatomicity of (X, Σ, μ) .

Theorem 4.2. *Let uC_φ be a weighted composition operator from $L^\infty(\mu)$ into $L^\infty(\nu)$. Then it is invertible if and only if the following conditions are all satisfied:*

- (i) *There exists a constant $\delta > 0$ such that $|u| \geq \delta$ ν -a.e. on Y .*
- (ii) *μ is absolutely continuous with respect to $\nu\varphi^{-1}$.*
- (iii) *For each set $F \in \Gamma$, there is a set $E \in \Sigma$ such that $\varphi^{-1}(E) = F$.*

Proof. Assume uC_φ is invertible. By (2.1), there exists a constant $c > 0$ with

$$\|uC_\varphi f\|_{L^\infty(\nu)} \geq c\|f\|_{L^\infty(\mu)} \quad \text{for all } f \in L^\infty(\mu).$$

We claim that $|u| \geq c/2$ ν -a.e. on Y . Otherwise, there is a set $G \in \Gamma$ such that $\nu(G) > 0$ and $|u| < c/2$ on G . Let $g \in L^\infty(\mu)$ be the function with $uC_\varphi g = \chi_G$. Then

$$|g| \leq \|g\|_{L^\infty(\mu)} \leq \frac{1}{c}\|uC_\varphi g\|_{L^\infty(\nu)} = \frac{1}{c}\|\chi_G\|_{L^\infty(\nu)} = \frac{1}{c} \quad \mu\text{-a.e. on } X.$$

It follows that

$$1 = |\chi_G| = |uC_\varphi g| = |u||g \circ \varphi| < \frac{c}{2} \cdot \frac{1}{c} = \frac{1}{2} \quad \nu\text{-a.e. on } G.$$

This absurdity justifies our claim. To prove (ii), suppose $\nu\varphi^{-1}(E) = 0$ for $E \in \Sigma$. Since $uC_\varphi\chi_E = 0$, the injectivity of uC_φ implies that $\chi_E = 0$, i.e. $\mu(E) = 0$.

It remains to prove (iii). Choose any set $F \in \Gamma$ with $\nu(F) < \infty$. Let $g \in L^\infty(\mu)$ be the function with $uC_\varphi g = \chi_F$, or $C_\varphi g = (1/u)\chi_F$. Let $\mathcal{E} := \{\varphi^{-1}(E) : E \in \Sigma\}$. Since $C_\varphi g$ is \mathcal{E} -measurable, so is $(1/u)\chi_F$. By writing $Y = \bigcup_{i=1}^\infty F_i$, where $\{F_i\}_{i=1}^\infty$ is an increasing sequence of Γ -measurable sets with finite ν -measures, we have $1/u = \lim_{i \rightarrow \infty} (1/u)\chi_{F_i}$ on Y . It follows that $1/u$ is \mathcal{E} -measurable. Hence χ_F is also \mathcal{E} -measurable for all $F \in \Gamma$ satisfying $\nu(F) < \infty$.

Conversely, assume all the three conditions hold. The equality $\varphi^{-1}(E) = F$ in (iii) may be expressed as $C_\varphi\chi_E = \chi_F$. In view of this and the fact that simple functions (not necessarily with supports of finite measures) are dense in $L^\infty(\mu)$ (or $L^\infty(\nu)$), it suffices to show that

$$\|C_\varphi f\|_{L^\infty(\nu)} = \|f\|_{L^\infty(\mu)} \tag{4.3}$$

for every simple function $f \in L^\infty(\mu)$. This will establish the injectivity of C_φ . Together with the inclusion $(1/u)L^\infty(\nu) \subset L^\infty(\nu)$, this shows that uC_φ is invertible as well. To verify (4.3), write $f = \sum_{i=1}^n c_i\chi_{E_i}$, where the E_i 's are pairwise disjoint and $\mu(E_i) > 0$ for all i . By (ii), we have $\nu\varphi^{-1}(E_i) > 0$ and so

$$\|C_\varphi f\|_{L^\infty(\nu)} = \left\| \sum_{i=1}^n c_i\chi_{\varphi^{-1}(E_i)} \right\|_{L^\infty(\nu)} = \max_{i=1,2,\dots,n} |c_i| = \|f\|_{L^\infty(\mu)}.$$

□

The proof of the following result is analogous to that of Theorem 3.4 and is thus omitted.

Theorem 4.3. *Let uC_φ be a weighted composition operator from $L^\infty(\mu)$ into $L^q(\nu)$, where $1 \leq q < \infty$. The following statements are equivalent.*

- (i) *The operator uC_φ has closed range.*
- (ii) *The operator uC_φ has finite rank.*
- (iii) *$\nu(\varphi^{-1}(B) \cap \text{supp } u) = 0$ and the set $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \text{supp } u) > 0\}$ is finite.*

Corollary 4.4. *Suppose that $1 \leq q < p \leq \infty$ and (X, Σ, μ) is nonatomic. If uC_φ is a weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$ with closed range, then it is the zero operator.*

Theorem 4.5. *Let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^\infty(\nu)$, where $1 \leq p < \infty$. The following statements are equivalent.*

- (i) *The operator uC_φ has closed range.*
- (ii) *The operator uC_φ has finite rank.*
- (iii) *The set $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \text{supp } u) > 0\}$ is finite.*

Proof. (ii) \Rightarrow (i) Since $uC_\varphi(L^p(\mu))$ is finite-dimensional, it is also closed in $L^\infty(\nu)$.

(iii) \Rightarrow (ii) By the boundedness of uC_φ , we have $\nu(\varphi^{-1}(B) \cap \text{supp } u) = 0$ [7, Theorem 4]. Put

$$I := \{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \text{supp } u) > 0\}.$$

If (iii) holds, then it follows that

$$uC_\varphi(L^p(\mu)) = \text{span}_{i \in I} uC_\varphi \chi_{A_i}.$$

Thus, $\dim(uC_\varphi(L^p(\mu))) < \infty$ and (ii) follows.

(i) \Rightarrow (iii) We prove the result by contradiction. Suppose, on the contrary, the set $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \text{supp } u) > 0\}$ is infinite. Without losing generality, we may assume $\nu(\varphi^{-1}(A_i) \cap \text{supp } u) > 0$ for all $i \in \mathbb{N}$. Let $S := \bigcup_{i \in \mathbb{N}} S_i$, where $S_i := \varphi^{-1}(A_i) \cap \text{supp } u$, and put $A := \bigcup_{i \in \mathbb{N}} A_i$. Consider the weighted composition operator

$$wC_\psi : L^p(\mu_A) \rightarrow L^\infty(\nu_S)$$

induced by $\psi := \varphi|_S$ with the weight $w := u|_S$. Since wC_ψ is injective and has closed range, it follows from (2.1) that there exists some constant $c > 0$ with

$$\|wC_\psi f\|_{L^\infty(\nu_S)} \geq c \|f\|_{L^p(\mu_A)}$$

for every $f \in L^p(\mu_A)$.

For each $i \in \mathbb{N}$, we now let

$$\|w\|_{S_i, \infty} := \inf\{M > 0 : \nu(\{y \in S_i : |w(y)| > M\}) = 0\}.$$

Note that $0 < \|w\|_{S_i, \infty} < \infty$. Let

$$\Theta := \sum_{i \in \mathbb{N}} \|w\|_{S_i, \infty} \chi_{A_i}$$

and define the operator M_Θ on $L^p(\mu_A)$ by

$$M_\Theta f = \Theta f \quad \text{for } f \in L^p(\mu_A).$$

According to [7, Theorem 4] again, M_Θ is a bounded operator from $L^p(\mu_A)$ into $L^\infty(\mu_A)$ with

$$\|M_\Theta\|^p = \sup_{i \in \mathbb{N}} \|w\|_{S_i, \infty}^p / \mu(A_i) (> 0).$$

It is also plain that

$$\|M_{\Theta}f\|_{L^\infty(\mu_A)} = \alpha,$$

where

$$\alpha := \sup_{i \in \mathbb{N}} \|w\|_{S_i, \infty} |f(A_i)|.$$

We claim that $\|wC_\psi f\|_{L^\infty(\nu_S)} = \alpha$ as well.

If $y \in S_i$, then

$$|wC_\psi f(y)| = |w(y)f(\varphi(y))| \leq \|w\|_{S_i, \infty} |f(A_i)| \leq \alpha$$

ν -a.e. on S_i . Since this inequality holds for every $i \in \mathbb{N}$, we have $\|wC_\psi f\|_{L^\infty(\nu_S)} \leq \alpha$. With $\|w\|_{S_i, \infty} |f(A_i)| \leq \|wC_\psi f\|_{L^\infty(\nu_S)}$ for each $i \in \mathbb{N}$, we also have $\alpha \leq \|wC_\psi f\|_{L^\infty(\nu_S)}$. Our claim now follows.

From the preceding paragraphs, we have $\|M_{\Theta}f\|_{L^\infty(\mu_A)} \geq c\|f\|_{L^p(\mu_A)}$ for $f \in L^p(\mu_A)$. Let

$$f_n := \sum_{i=1}^n \|w\|_{S_i, \infty}^{-1} \chi_{A_i} \quad \text{for } n \in \mathbb{N}.$$

Then,

$$\|f_n\|_{L^p(\mu_A)}^p = \sum_{i=1}^n \mu(A_i) / \|w\|_{S_i, \infty}^p \quad \text{and} \quad \|M_{\Theta}f_n\|_{L^\infty(\mu_A)} = 1.$$

Hence

$$\sum_{i=1}^n \frac{\mu(A_i)}{\|w\|_{S_i, \infty}^p} \leq \frac{1}{c^p} \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ yields $\sum_{i=1}^\infty \mu(A_i) / \|w\|_{S_i, \infty}^p \leq 1/c^p < \infty$. On the other hand, with $\|w\|_{S_i, \infty}^p / \mu(A_i) \leq \|M_{\Theta}\|^p$ for each $i \in \mathbb{N}$, we have $\sum_{i=1}^\infty \mu(A_i) / \|w\|_{S_i, \infty}^p = \infty$. This is a contradiction. \square

From Example 3.6, not every composition operator from l^p into l^q , $p \neq q$, has closed range. Actually every uC_φ such that the image of $\text{supp } u$ under φ is infinite does not have closed range.

Corollary 4.6. *Let uC_φ be a weighted composition operator from $l^p(w)$ into $l^q(w)$, where $1 \leq p, q \leq \infty$ and $p \neq q$. Then uC_φ has closed range if and only if $\varphi(\text{supp } u)$ is finite.*

Let uC_φ be a weighted composition map from $L^p(\mu)$ to $L^q(\nu)$, where $1 \leq p, q \leq \infty$ and $p \neq q$. An immediate consequence of Theorems 3.4, 3.5, 4.3 and 4.5 is that if uC_φ has closed range, then uC_φ is compact. The converse, however, is *not* necessarily true. For example, let $T : l^p \rightarrow l^q$, $1 \leq p < q \leq \infty$, be defined by

$$T(x_1, x_2, x_3, \dots) = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots \right) \quad \text{for every } \{x_n\}_{n \in \mathbb{N}} \in l^p.$$


The operator T does not have closed range, yet it is compact by [9, Theorems 3.3 and 3.7].

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