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Dedicated to late Prof. A.S. Vasudeva Murthy,
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CERTAIN PROPERTIES OF CONTINUOUS FRACTIONAL WAVELET TRANSFORM ON HARDY SPACE AND MORREY SPACE

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Abstract. In this paper we define a new class of continuous fractional wavelet transform (CFrWT) and study its properties in Hardy space and Morrey space. The theory developed generalize and complement some of already existing results.

Keywords: fractional Fourier transform, continuous fractional wavelet transform, Hardy space, Morrey space.

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1. INTRODUCTION

Even though the classical wavelet transform (WT) serves as a powerful tool in signal processing and analysis, its analyzing capability is limited to the time-frequency plane. Fractional Fourier transform (FrFT) ([1, 16, 20]) gives the fractional Fourier domain (FrFD) frequency content of the signal, but it fails in giving the local information of the signal. Mendlovic *et al.* ([17]), first introduced the FrWT to deal with the optical signals. They first derive the fractional spectrum of the signal by using the FrFT and performed the WT of the fractional spectrum. But the transform defined in such a way, fails in giving the information about the local property of the signal, since the FrFT gives the fractional frequency of the signal during the entire duration of the signal rather than for a particular time, and the fractional spectrum of the signal cannot be ascertained when those fractional frequencies exist.

The novel fractional wavelet transform (FrWT) based on fractional convolution was proposed by Shi *et al.* ([34]). They studied basic properties of the FrWT like inner product theorem, Parseval's relation and inversion formula for the function in $L^2(\mathbb{R})$.

Prasad *et al.* ([21]) studied some properties of FrFT such as the Riemann–Lebesgue lemma. Also, they extended the inner product theorem of the CFrWT, studied in [34], in the context of two fractional wavelets. Dai *et al.* ([5]) proposed a new type of FrWT and obtained the associated multiresolution analysis (MRA). This is more general than the transforms defined in [21] and [34]. It displays the time and FrFD-frequency information jointly in the time-FrFD-frequency plane.

Luchko *et al.* ([14]) introduced a new FrFT and implemented this theory on the Lizorkin space, and also discussed many important results involving fractional derivatives. To know more about the FrFT reader may follow [12, 36]. In [35, 37], authors studied the new theory of FrWT, associated with the FrFT given in [12,14,36], and obtained some of its properties like inner product relation, inversion formula, etc. They also discussed MRA associated with this FrWT, along with the construction of the orthogonal fractional wavelets. This theory can also be used in the study of quantum mechanics, signal processing and other areas of science and engineering.

Several important function spaces like Besov, Sobolev, Holder, Zygmund, BMO, etc. are given characterization in terms of wavelets involved in the classical WT ([6,18]). In [3, 13, 15, 32, 33] the wavelet characterization and decomposition of the function spaces like Anisotropic Hardy space, Besov–Morrey spaces, Triebel–Lizorkin–Morrey spaces and Besov–Triebel–Lizorkin–Morrey spaces are studied. WT has also been studied in various function spaces and the spaces of distributions ([22, 24, 31]). Chuong *et al.* [4] studied the boundedness of the WT on the Besov, BMO and Hardy spaces. Furthermore, for the compactly supported basic wavelet, the boundedness of the WT is also established on the weighted Besov space and weighted BMO space associated with the tempered weight function. In the recent years, Prasad and Kumar ([26–28]) discussed the CFrWT on the generalized weighted Sobolev spaces and some function spaces and obtained its boundedness. Not only that, the WT and CFrWT have also been studied by many authors on some spaces of test functions, like Gelfand–Shilov spaces ([23, 25, 29, 30]). Based on the convolution of linear canonical transform (LCT) ([38]), Guo *et al.* ([8]) proposed a linear canonical wavelet transform (LCWT), which is a generalization of the transform studied in [21]. The authors also proved the continuity of this transform on some space of test functions and the generalized Sobolev space. To know more about the literature, reader can read the references and the references therein.

Motivated by above works we have studied the CFrWT. We complement the theory of CFrWT studied in [35, 37] by adding some new results and studying its properties in Hardy and Morrey spaces. We present the orthogonality relation which helps us to conclude that the images of the CFWTs associated with two different fractional wavelets are orthogonal if the respective argument functions are orthogonal. Also, we establish the reconstruction formula and the characterization of the range of CFrWT based on two fractional wavelets. Moreover, we derive the formulas for the CFrWT associated with the convolution and correlation of two functions. Furthermore, we study the boundedness of the CFrWT on Hardy space $H^1(\mathbb{R})$ and on a subspace of Morrey space $L_M^{1,\nu}(\mathbb{R})$ and also determining the $H^1(\mathbb{R})$ and $L_M^{1,\nu}(\mathbb{R})$-distance of two CFrWTs.

The organization of the paper is as follows: In Section 2, we recall some basic definitions and results. In Section 3, we have derived the orthogonality relation, the reconstruction formula and characterized the range of the transform in the context of two fractional wavelets. Also, we have derived the formulas for the CFrWT when the argument function or fractional wavelet is a convolution or correlation of two functions. Section 4 is further divided into two subsections. In each of these two subsections the boundedness of CFrWT on $H^1(\mathbb{R})$ and on a subspace of $L_M^{1,\nu}(\mathbb{R})$ along with its approximation properties are studied. Finally, we end this paper by the conclusions in Section 5.

2. PRELIMINARIES

In this section we recall some existing definitions and results that will be used in this paper.

Definition 2.1. The convolution of complex-valued measurable functions f and g defined on $\mathbb{R}$ is given by

$$(f \star g)(x) = \int_{\mathbb{R}} f(u)g(x-u)du, \;\; x \in \mathbb{R} \tag{2.1}$$

whenever the integral is well-defined.

Definition 2.2. The correlation of complex-valued measurable functions f and g defined on $\mathbb{R}$ is given by

$$(f \circ g)(x) = \int_{\mathbb{R}} \overline{f(u)}g(x+u)du, \;\; x \in \mathbb{R} \tag{2.2}$$

whenever the integral is well-defined.

Definition 2.3 ([35])**.** The fractional Fourier transform (FrFT), of real order θ $(0 < \theta \leq 1)$, of a function $f \in L^2(\mathbb{R})$ is defined by

$$(\mathfrak{F}_\theta f)(\xi) = \int_{\mathbb{R}} e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}}t} f(t)dt, \quad \xi \in \mathbb{R}. \tag{2.3}$$

For $\theta = 1$, the fractional Fourier transform defined in (2.3) reduces to the classical Fourier transform.

The corresponding inverse fractional Fourier transform is defined as follows:

$$f(t) = \frac{1}{2\pi\theta} \int_{\mathbb{R}} e^{i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}}t} (\mathfrak{F}_\theta f)(\xi)|\xi|^{\frac{1}{\theta}-1}d\xi.$$

Lemma 2.4. *Let* $\psi \in L^2(\mathbb{R})$, *then*

$$(\mathfrak{F}_\theta \psi_{a,b,\theta})(\xi) = |a|^{\frac{1}{2\theta}} e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}} b} (\mathfrak{F}_\theta \psi)(a\xi), \tag{2.4}$$

where

$$\psi_{a,b,\theta}(t) = \frac{1}{|a|^{\frac{1}{2\theta}}} \psi\left(\frac{t-b}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right), \quad a, b \in \mathbb{R}. \tag{2.5}$$

Proof. Using equation (2.5), we have

$$\begin{aligned}
(\mathfrak{F}_\theta \psi_{a,b,\theta})(\xi) &= \int_{\mathbb{R}} e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}} t} \frac{1}{|a|^{\frac{1}{2\theta}}} \psi\left(\frac{t-b}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right) dt \\
&= e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}} b} \int_{\mathbb{R}} e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}} t} \frac{1}{|a|^{\frac{1}{2\theta}}} \psi\left(\frac{t}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right) dt \\
&= e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}} b} \frac{1}{|a|^{\frac{1}{2\theta}}} \int_{\mathbb{R}} e^{-i(\operatorname{sgn}(a\xi))|a\xi|^{\frac{1}{\theta}} \frac{t}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}} \psi\left(\frac{t}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right) dt \\
&= e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}} b} |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} e^{-i(\operatorname{sgn}(a\xi))|a\xi|^{\frac{1}{\theta}} t} \psi(t)\, dt.
\end{aligned}$$

Therefore, we have

$$(\mathfrak{F}_\theta \psi_{a,b,\theta})(\xi) = |a|^{\frac{1}{2\theta}} e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}} b} (\mathfrak{F}_\theta \psi)(a\xi).$$

This completes the proof. □

3. CFRWT

Before we begin with the definition of the CFrWT we recall the definition of the fractional wavelet given by Srivastava *et al.* ([35]). We then prove a theorem that helps in constructing a family of fractional wavelets from a given one.

Definition 3.1. A fractional wavelet is a non-zero function $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, satisfying

$$C_{\psi,\theta} := \int_{\mathbb{R}} \frac{|\mathfrak{F}_\theta \psi(\xi)|^2}{|\xi|} d\xi < \infty. \tag{3.1}$$

Now, we prove the following theorem which indicate the construction of a family of fractional wavelets from a given one.

Theorem 3.2. *Let* ψ *be a fractional wavelet and* ϕ *be a function in* $L^1(\mathbb{R})$, *then* $\psi \star \phi$ *and* $\psi \circ \phi$ *are also fractional wavelets.*

Proof. Since $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\phi \in L^1(\mathbb{R})$, $\psi \star \phi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Now,

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\mathfrak{F}_\theta(\psi \star \phi)(\xi)|^2}{|\xi|} d\xi &= \int_{\mathbb{R}} \frac{|(\mathfrak{F}_\theta \psi)(\xi)|^2 |(\mathfrak{F}_\theta \phi)(\xi)|^2}{|\xi|} d\xi \\ &\leq \|\phi\|^2_{L^1(\mathbb{R})} \int_{\mathbb{R}} \frac{|(\mathfrak{F}_\theta \psi)(\xi)|^2}{|\xi|} d\xi \\ &= \|\phi\|^2_{L^1(\mathbb{R})} C_{\psi,\theta},\end{aligned}$$

because

$$\mathfrak{F}_\theta(\psi \star \phi)(\xi) = (\mathfrak{F}_\theta \psi)(\xi)(\mathfrak{F}_\theta \phi)(\xi) \quad \text{and} \quad \|\mathfrak{F}_\theta \phi\|_{L^\infty(\mathbb{R})} \leq \|\phi\|_{L^1(\mathbb{R})}.$$

Since ψ is a fractional wavelet and $\phi \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \frac{|\mathfrak{F}_\theta(\psi \star \phi)(\xi)|^2}{|\xi|} d\xi < \infty.$$

Hence by Definition 3.1, $\psi \star \phi$ is a fractional wavelet. Similarly, it can be shown that $\psi \circ \phi$ is also a fractional wavelet. This completes the proof. □

Definition 3.3 ([35])**.** The CFrWT of f with respect to a fractional wavelet ψ is defined by

$$\left(W_\psi^\theta f\right)(b,a) = \int_{\mathbb{R}} f(t)\overline{\psi_{a,b,\theta}(t)} dt, \quad a, b \in \mathbb{R}, \tag{3.2}$$

provided the integral is well-defined. Here $\psi_{a,b,\theta}$ is given by equation (2.5).

We derive some new results of the CFrWT and also generalized some existing results in the context of two fractional wavelets. If $f,\ g \in L^2(\mathbb{R})$ are orthogonal then the image $W_\psi^\theta f$ and $W_\psi^\theta g$ are also orthogonal in $L^2\left(\mathbb{R} \times \mathbb{R}, \frac{dbda}{|a|^{\frac{1}{\theta}+1}}\right)$. This fact is observed by the orthogonality relation for the CFrWT given in [35]. But this relation is not enough to conclude the orthogonality of $W_\psi^\theta f$ and $W_\phi^\theta g$ for two different fractional wavelets ψ and ϕ. So in this regard we introduce a more general version of orthogonality relation. We also derive reconstruction formula and characterized its range. For the case $\psi = \phi$, our results coincide with the results in [35].

Theorem 3.4 (Orthogonality relation)**.** *If the fractional wavelets ϕ and ψ satisfy*

$$\int_{\mathbb{R}} |(\mathfrak{F}_\theta \phi)(u)|\ |(\mathfrak{F}_\theta \psi)(u)| \frac{1}{|u|} du < \infty, \tag{3.3}$$

then for $f, g \in L^2(\mathbb{R})$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(W_\phi^\theta f\right)(b,a) \overline{\left(W_\psi^\theta g\right)(b,a)} \frac{dbda}{|a|^{\frac{1}{\theta}+1}} = C_{\phi,\psi,\theta} \langle f, g\rangle_{L^2(\mathbb{R})},$$

where

$$C_{\phi,\psi,\theta} = \int_{\mathbb{R}} \overline{(\mathfrak{F}_\theta \phi)(u)} (\mathfrak{F}_\theta \psi)(u) \frac{1}{|u|} du. \tag{3.4}$$

Proof. We have

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left(W_\phi^\theta f\right)(b,a)\overline{\left(W_\psi^\theta g\right)(b,a)}\frac{dbda}{|a|^{\frac{1}{\theta}+1}}\\
&=\frac{1}{2\pi\theta}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}\left(\mathfrak{F}_{\theta,b}\left\{\left(W_\phi^\theta f\right)(b,a)\right\}\right)(\xi)\right.\\
&\qquad\qquad \left.\times\overline{\left(\mathfrak{F}_{\theta,b}\left\{\left(W_\psi^\theta g\right)(b,a)\right\}\right)(\xi)}d\xi\right)\frac{da}{|a|^{\frac{1}{\theta}+1}}\\
&=\frac{1}{2\pi\theta}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}|a|^{\frac{1}{\theta}}(\mathfrak{F}_\theta f)(\xi)\overline{(\mathfrak{F}_\theta\phi)(a\xi)}\ \overline{(\mathfrak{F}_\theta g)(\xi)}(\mathfrak{F}_\theta\psi)(a\xi)d\xi\right)\frac{da}{|a|^{\frac{1}{\theta}+1}}\\
&=\frac{1}{2\pi\theta}\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta f)(\xi)\overline{(\mathfrak{F}_\theta g)(\xi)}\left(\int_{\mathbb{R}}\overline{(\mathfrak{F}_\theta\phi)(a\xi)}(\mathfrak{F}_\theta\psi)(a\xi)\frac{1}{|a|}da\right)d\xi. \qquad (3.5)
\end{aligned}
$$

Substituting $a\xi = u$, in the integral in the parenthesis of equation (3.5), we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \left(W_\phi^\theta f\right)(b,a)\overline{\left(W_\psi^\theta g\right)(b,a)}\frac{dbda}{|a|^{\frac{1}{\theta}+1}}\\
&=\frac{1}{2\pi\theta}\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta f)(\xi)\overline{(\mathfrak{F}_\theta g)(\xi)}\left(\int_{\mathbb{R}}\overline{(\mathfrak{F}_\theta\phi)(u)}(\mathfrak{F}_\theta\psi)(u)\frac{1}{|u|}du\right)d\xi\\
&=\frac{C_{\phi,\psi,\theta}}{2\pi\theta}\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta f)(\xi)\overline{(\mathfrak{F}_\theta g)(\xi)}d\xi\\
&=C_{\phi,\psi,\theta}\langle f,g\rangle_{L^2(\mathbb{R})}.
\end{aligned}
$$

This completes the proof. □

The following corollary follows from Theorem 3.4.

Corollary 3.5. *Let ϕ,ψ be two fractional wavelets and are such that they satisfy the hypothesis of Theorem 3.4. If further $C_{\phi,\psi,\theta}=0$, where $C_{\phi,\psi,\theta}$ is given by equation* (3.4)*, then $W_\phi^\theta\left(L^2(\mathbb{R})\right)$ and $W_\psi^\theta\left(L^2(\mathbb{R})\right)$ are orthogonal.*

Theorem 3.6 (Reconstruction formula)**.** *Let $f\in L^2(\mathbb{R})$ and ϕ,ψ be two fractional wavelets satisfying* (3.3) *and $C_{\phi,\psi,\theta}$, as defined in* (3.4)*, is non-zero. Then*

$$
f(t)=\frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}}\int_{\mathbb{R}}\psi_{a,b,\theta}(t)\left(W_\phi^\theta f\right)(b,a)\frac{dbda}{|a|^{\frac{1}{\theta}+1}}.
$$

Proof. We have

$$
\begin{aligned}
&\frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}}\int_{\mathbb{R}}\psi_{a,b,\theta}(t)\left(W_\phi^\theta f\right)(b,a)\frac{dbda}{|a|^{\frac{1}{\theta}+1}}\\
&=\frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}}\frac{1}{2\pi\theta}\Bigg(\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}\overline{\left(\mathfrak{F}_{\theta,b}\left\{\overline{\psi_{a,b,\theta}(t)}\right\}\right)(\xi)}\\
&\qquad\qquad\times\left(\mathfrak{F}_{\theta,b}\left\{\left(W_\phi^\theta f\right)(b,a)\right\}\right)(\xi)d\xi\Bigg)\frac{da}{|a|^{\frac{1}{\theta}+1}}\\
&=\frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}}\frac{1}{2\pi\theta}\Bigg(\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}\frac{1}{|a|^{\frac{1}{2\theta}}}|a|^{\frac{1}{\theta}}\overline{e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}}t}\overline{(\mathfrak{F}_\theta\psi)(a\xi)}}\\
&\qquad\qquad\times|a|^{\frac{1}{2\theta}}(\mathfrak{F}_\theta f)(\xi)\overline{(\mathfrak{F}_\theta\phi)(a\xi)}d\xi\Bigg)\frac{da}{|a|^{\frac{1}{\theta}+1}}\\
&=\frac{1}{2\pi\theta C_{\phi,\psi,\theta}}\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}e^{i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}}t}(\mathfrak{F}_\theta f)(\xi)\left(\int_{\mathbb{R}}(\mathfrak{F}_\theta\psi)(a\xi)\overline{(\mathfrak{F}_\theta\phi)(a\xi)}\frac{1}{|a|}da\right)d\xi\\
&=\frac{1}{2\pi\theta}\int_{\mathbb{R}}|\xi|^{\frac{1}{\theta}-1}e^{i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}}t}(\mathfrak{F}_\theta f)(\xi)d\xi\\
&=f(t).
\end{aligned}
$$

Thus the theorem follows. □

Theorem 3.7 (Characterization of the range). *Let $C_{\phi,\psi,\theta}$ as defined in* (3.4)*, for two fractional wavelets satisfying* (3.3)*, is non-zero. Then $F\in L^2\left(\mathbb{R}\times\mathbb{R},\frac{dbda}{|a|^{\frac{1}{\theta}+1}}\right)$ is a CFrWT, with respect to ϕ, of some $f\in L^2(\mathbb{R})$ iff*

$$
F(b_0,a_0)=\int_{\mathbb{R}}\int_{\mathbb{R}}F(b,a)K_{\phi,\psi,\theta}(b_0,a_0;b,a)\frac{dbda}{|a|^{\frac{1}{\theta}+1}},\quad (b_0,a_0)\in\mathbb{R}\times\mathbb{R}, \tag{3.6}
$$

where $K_{\phi,\psi,\theta}$ is the kernel given by

$$
K_{\phi,\psi,\theta}(b_0,a_0;b,a)=\frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}}\psi_{a,b,\theta}(t)\overline{\phi_{a_0,b_0,\theta}(t)}dt. \tag{3.7}
$$

Moreover, in such a case the kernel is pointwise bounded:

$$
|K_{\phi,\psi,\theta}(b_0,a_0;b,a)|\leq\frac{1}{C_{\phi,\psi,\theta}}\|\phi\|_{L^2(\mathbb{R})}\|\psi\|_{L^2(\mathbb{R})}.
$$

Proof. Let $f \in L^2(\mathbb{R})$ such that $W_\phi^\theta f = F$, then

$$\begin{aligned}
F(b_0,a_0) &= \left(W_\phi^\theta f\right)(b_0,a_0)\\
&= \int_{\mathbb{R}} f(t)\overline{\phi_{a_0,b_0,\theta}(t)}dt\\
&= \int_{\mathbb{R}} \frac{1}{C_{\phi,\psi,\theta}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \psi_{a,b,\theta}(t)\left(W_\phi^\theta f\right)(b,a)\frac{dbda}{|a|^{\frac{1}{\theta}+1}}\right)\overline{\phi_{a_0,b_0,\theta}(t)}dt\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(W_\phi^\theta f\right)(b,a)\left(\frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}} \psi_{a,b,\theta}(t)\overline{\phi_{a_0,b_0,\theta}(t)}dt\right)\frac{dbda}{|a|^{\frac{1}{\theta}+1}}\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} F(b,a)K_{\phi,\psi,\theta}(b_0,a_0;b,a)\frac{dbda}{|a|^{\frac{1}{\theta}+1}},
\end{aligned}$$

where

$$K_{\phi,\psi,\theta}(b_0,a_0;b,a) = \frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}} \psi_{a,b,\theta}(t)\overline{\phi_{a_0,b_0,\theta}(t)}dt.$$

Conversely, let for the given $F \in L^2\left(\mathbb{R}\times\mathbb{R}, \frac{dbda}{|a|^{\frac{1}{\theta}+1}}\right)$ equation (3.6) holds. Then the required f is given by

$$\frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}}\int_{\mathbb{R}} F(b,a)\psi_{a,b,\theta}(t)\frac{dbda}{|a|^{\frac{1}{\theta}+1}}.$$

Again,

$$\begin{aligned}
|K_{\phi,\psi,\theta}(b_0,a_0;b,a)| &\le \frac{1}{C_{\phi,\psi,\theta}}\int_{\mathbb{R}} |\psi_{a,b,\theta}(t)\overline{\phi_{a_0,b_0,\theta}(t)}|dt.\\
&\le \frac{1}{C_{\phi,\psi,\theta}}\|\psi_{a,b,\theta}\|_{L^2(\mathbb{R})}\|\phi_{a_0,b_0,\theta}\|_{L^2(\mathbb{R})}\\
&= \frac{1}{C_{\phi,\psi,\theta}}\|\psi\|_{L^2(\mathbb{R})}\|\phi\|_{L^2(\mathbb{R})}.
\end{aligned}$$

This completes the proof. □

Now, we prove the theorem that gives the formula for the wavelet transform of the convolution and correlation of two functions.

Theorem 3.8. *Let $f \in L^1(\mathbb{R})$, $g \in L^2(\mathbb{R})$ and ψ be a fractional wavelet, then*

$$\left(W_\psi^\theta(f\star g)\right)(b,a) = \left(f(\cdot)\star(W_\psi^\theta g)(\cdot,a)\right)(b)$$

and

$$\left(W_\psi^\theta(f\circ g)\right)(b,a) = \left(f(\cdot)\circ(W_\psi^\theta g)(\cdot,a)\right)(b).$$

Proof. We have

$$\begin{aligned}\left(W_\psi^\theta(f\star g)\right)(b,a) &= \int_{\mathbb{R}} (f\star g)(t)\overline{\psi_{a,b,\theta}(t)}dt \\ &= \int_{\mathbb{R}} \left\{\int_{\mathbb{R}} f(y)g(t-y)dy\right\} \frac{1}{|a|^{\frac{1}{2\theta}}}\overline{\psi\left(\frac{t-b}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)}dt, \\ &= \int_{\mathbb{R}} f(y)\left\{\int_{\mathbb{R}} g(t)\overline{\psi_{a,b-y,\theta}}dt\right\}dy \\ &= \int_{\mathbb{R}} f(y)(W_\psi^\theta g)(b-y,a)dy,\end{aligned}$$

using Definition 2.1. Therefore,

$$\left(W_\psi^\theta(f\star g)\right)(b,a) = \left(f(\cdot)\star(W_\psi^\theta g)(\cdot,a)\right)(b).$$

Similarly, it can be shown that

$$\left(W_\psi^\theta(f\circ g)\right)(b,a) = \left(f(\cdot)\circ(W_\psi^\theta g)(\cdot,a)\right)(b).$$

This completes the proof. □

The following theorem gives the expression of the CFrWT when the fractional wavelet associated with the transform is the convolution or the correlation of two functions.

Theorem 3.9. *Let $f\in L^1(\mathbb{R})$ $g\in L^2(\mathbb{R})$ and ψ be a fractional wavelet, then*

$$\left(W_{f\star\psi}^\theta g\right)(b,a) = \frac{1}{|a|^{\frac{1}{\theta}}}\left(f\left(\frac{\cdot}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)\circ(W_\psi^\theta g)(\cdot,a)\right)(b)$$

and

$$\left(W_{f\circ\psi}^\theta g\right)(b,a) = \frac{1}{|a|^{\frac{1}{\theta}}}\left(f\left(\frac{\cdot}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)\star(W_\psi^\theta g)(\cdot,a)\right)(b).$$

Proof. Since $f \in L^1(\mathbb{R})$ and ψ is a fractional wavelet, by Theorem 3.2, $f \star \psi$ is a wavelet. Now,

$$\begin{aligned}\left(W^\theta_{f\star\psi} g\right)(b,a) &= \int_{\mathbb{R}} g(t)\overline{(f\star\psi)_{a,b,\theta}(t)}dt \\ &= \int_{\mathbb{R}} g(t)\left\{\frac{1}{|a|^{\frac{1}{2\theta}}}\overline{\int_{\mathbb{R}} f(y)\psi\left(\frac{t-b}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}-y\right)dy}\right\}dt \\ &= \int_{\mathbb{R}} \overline{f(y)}\left\{\int_{\mathbb{R}} g(t)\frac{1}{|a|^{\frac{1}{2\theta}}}\overline{\psi\left(\frac{t-(b+y(\operatorname{sgn} a)|a|^{\frac{1}{\theta}})}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)}dt\right\}dy \\ &= \int_{\mathbb{R}} \overline{f(y)}(W^\theta_\psi g)(b+(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}y,a)dy.\end{aligned}$$

Therefore,

$$\left(W^\theta_{f\star\psi} g\right)(b,a) = \frac{1}{|a|^{\frac{1}{\theta}}}\left(f\left(\frac{\cdot}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)\circ(W^\theta_\psi g)(\cdot,a)\right)(b).$$

Again by Theorem 3.2, $f \circ \psi$ is a wavelet. Proceeding similarly as above it can be shown that

$$\left(W^\theta_{f\circ\psi} g\right)(b,a) = \frac{1}{|a|^{\frac{1}{\theta}}}\left(f\left(\frac{\cdot}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)\star(W^\theta_\psi g)(\cdot,a)\right)(b).$$

This completes the proof. □

Theorem 3.10. *Let $f, g \in L^2(\mathbb{R})$ and ϕ, ψ be two fractional wavelets, then*

$$\int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1}(W^\theta_\phi f)(b,a)\overline{(W^\theta_\psi g)(b,a)}db = \frac{|a|^{\frac{1}{\theta}}}{4\pi^2\theta^2}\langle P_\theta, Q_\theta\rangle_{L^2(\mathbb{R})},$$

where

$$P_\theta(\xi) = |\xi|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta f)(\xi)\overline{(\mathfrak{F}_\theta\phi)(a\xi)}$$

and

$$Q_\theta(\xi) = |\xi|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta g)(\xi)\overline{(\mathfrak{F}_\theta\psi)(a\xi)}.$$

Proof. Note that

$$\int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1}(W^\theta_\phi f)(b,a)\overline{(W^\theta_\psi g)(b,a)}db = \int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1}\langle f,\phi_{a,b,\theta}\rangle_{L^2(\mathbb{R})}\overline{\langle g,\psi_{a,b,\theta}\rangle_{L^2(\mathbb{R})}}db.$$

Using Parseval's formula ([35, Theorem 1]), we have

$$\begin{aligned}
&\int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1}(W_\phi^\theta f)(b,a)\overline{(W_\psi^\theta g)(b,a)}db \\
&= \left(\frac{1}{2\pi\theta}\right)^2 \int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1} \left\langle |\cdot|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta f)(\cdot), (\mathfrak{F}_\theta \phi_{a,b,\theta})(\cdot)\right\rangle_{L^2(\mathbb{R})} \\
&\qquad \times \overline{\left\langle |\cdot|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta g)(\cdot), (\mathfrak{F}_\theta \psi_{a,b,\theta})(\cdot)\right\rangle_{L^2(\mathbb{R})}} db \\
&= \left(\frac{1}{2\pi\theta}\right)^2 \int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1} \left(\int_{\mathbb{R}} |\xi|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta f)(\xi)\overline{(\mathfrak{F}_\theta \phi_{a,b,\theta})(\xi)}d\xi\right) \\
&\qquad \times \overline{\left(\int_{\mathbb{R}} |\omega|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta f)(\omega)\overline{(\mathfrak{F}_\theta \psi_{a,b,\theta})(\omega)}d\omega\right)} db \\
&= \left(\frac{1}{2\pi\theta}\right)^2 |a|^{\frac{1}{\theta}} \int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1} \left(\int_{\mathbb{R}} |\xi|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta f)(\xi)\overline{e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}}b}(\mathfrak{F}_\theta \phi)(a\xi)}d\xi\right) \\
&\qquad \times \overline{\left(\int_{\mathbb{R}} |\omega|^{\frac{1}{\theta}-1}(\mathfrak{F}_\theta g)(\omega)\overline{e^{-i(\operatorname{sgn}(\omega))|\omega|^{\frac{1}{\theta}}b}(\mathfrak{F}_\theta \psi)(a\omega)}d\omega\right)} db \\
&= \left(\frac{1}{2\pi\theta}\right)^2 |a|^{\frac{1}{\theta}} \int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1} \overline{\left(\int_{\mathbb{R}} e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}}b}|\xi|^{\frac{1}{\theta}-1}\overline{(\mathfrak{F}_\theta f)(\xi)}(\mathfrak{F}_\theta \phi)(a\xi)d\xi\right)} \\
&\qquad \times \left(\int_{\mathbb{R}} e^{-i(\operatorname{sgn}(\omega))|\omega|^{\frac{1}{\theta}}b}|\omega|^{\frac{1}{\theta}-1}\overline{(\mathfrak{F}_\theta g)(\omega)}(\mathfrak{F}_\theta \psi)(a\omega)d\omega\right) db \\
&= \left(\frac{1}{2\pi\theta}\right)^2 |a|^{\frac{1}{\theta}} \int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1} \overline{\left(\int_{\mathbb{R}} e^{-i(\operatorname{sgn}\xi)|\xi|^{\frac{1}{\theta}}b}\overline{P_\theta(\xi)}d\xi\right)} \\
&\qquad \times \left(\int_{\mathbb{R}} e^{-i(\operatorname{sgn}(\omega))|\omega|^{\frac{1}{\theta}}b}\overline{Q_\theta(\omega)}d\omega\right) db.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1}(W_\phi^\theta f)(b,a)\overline{(W_\psi^\theta g)(b,a)}db \\
&= \left(\frac{1}{2\pi\theta}\right)^2 |a|^{\frac{1}{\theta}} \int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1}\overline{\left(\mathfrak{F}_\theta \overline{P_\theta}\right)(b)}\left(\mathfrak{F}_\theta \overline{Q_\theta}\right)(b)db.
\end{aligned} \tag{3.8}$$

Using [35, Theorem 1] in equation (3.8), we get

$$\begin{aligned}
&\int_{\mathbb{R}} |b|^{\frac{1}{\theta}-1}(W_\phi^\theta f)(b,a)\overline{(W_\psi^\theta g)(b,a)}db \\
&= \left(\frac{1}{2\pi\theta}\right)^2 |a|^{\frac{1}{\theta}} \left\langle |\cdot|^{\frac{1}{\theta}-1}\left(\mathfrak{F}_\theta\overline{Q_\theta}\right)(\cdot), \left(\mathfrak{F}_\theta\overline{P_\theta}\right)(\cdot)\right\rangle_{L^2(\mathbb{R})} \\
&= \left(\frac{1}{2\pi\theta}\right)^2 |a|^{\frac{1}{\theta}} \left\langle \overline{Q_\theta}, \overline{P_\theta}\right\rangle_{L^2(\mathbb{R})} \\
&= \left(\frac{1}{2\pi\theta}\right)^2 |a|^{\frac{1}{\theta}} \left\langle P_\theta, Q_\theta\right\rangle_{L^2(\mathbb{R})}.
\end{aligned}$$

This completes the proof. □

4. CFRWT ON HARDY SPACE & MORREY SPACES

In this section we study some of the properties of CErWT on Hardy space and Morrey space. The purpose of this section is to establish the boundedness of the CFrWT on these spaces and to give the $H^1(\mathbb{R})$ and $L_M^{1,\nu}(\mathbb{R})$-distance estimate of two CFrWTs.

4.1. HARDY SPACE

Definition 4.1 ([4])**.** The Hardy space $H^1(\mathbb{R})$, defined by

$$H^1(\mathbb{R}) = \left\{ f \in L^1(\mathbb{R}) : \int_{\mathbb{R}} \sup_{t>0} |(f \star \eta_t)(x)| dx < \infty \right\},$$

is a Banach space normed by

$$\|f\|_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \sup_{t>0} |(f \star \eta_t)(x)| dx, \tag{4.1}$$

where η is a function in the Schwartz space such that $\int_{\mathbb{R}} \eta(x)dx \neq 0$ and $\eta_t(x) = \frac{1}{t}\eta(\frac{x}{t})$, $t > 0$, $x \in \mathbb{R}$.

We now study some properties of the CFrWT on the Hardy space $H^1(\mathbb{R})$. To establish the boundedness of the CFrWT on the Hardy space we need to prove the following lemma, which gives the boundedness on $L^1(\mathbb{R})$ of the CFrWT.

Lemma 4.2. *Let $a \in \mathbb{R} - \{0\}$, $f \in L^1(\mathbb{R})$ and ψ is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $(W_\psi^\theta f)(\cdot, a) \in L^1(\mathbb{R})$, where $W_\psi^\theta f$ denotes the CFrWT of f.*

Proof. For a fixed $a \in \mathbb{R} - \{0\}$, $(W_\psi^\theta f)(b,a)$ is a function of b and is such that

$$\begin{aligned}
|(W_\psi^\theta f)(b,a)| &\le \int_{\mathbb{R}} |f(u)||\psi_{a,b,\theta}(u)|du \\
&= \int_{\mathbb{R}} |f(u)|\frac{1}{|a|^{\frac{1}{2\theta}}}\left|\psi\left(\frac{u-b}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)\right|du \\
&= |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} \left|f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}x+b\right)\right| |\psi(x)|dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}} |(W_\psi^\theta f)(b,a)|db &\le |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} |\psi(x)| \left(\int_{\mathbb{R}} \left|f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}x+b\right)\right| db\right) dx \\
&= |a|^{\frac{1}{2\theta}} \|\psi\|_{L^1(\mathbb{R})}\|f\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Hence, it follows that $(W_\psi^\theta f)(\cdot,a) \in L^1(\mathbb{R})$. □

Theorem 4.3. *Let $a \in \mathbb{R} - \{0\}$ and ψ is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the operator $W_\psi^\theta : H^1(\mathbb{R}) \to H^1(\mathbb{R})$ defined by $f \mapsto (W_\psi^\theta f)(\cdot,a)$ is bounded. Furthermore,*

$$\|(W_\psi^\theta f)(\cdot,a)\|_{H^1(\mathbb{R})} \le |a|^{\frac{1}{2\theta}} \|\psi\|_{L^1(\mathbb{R})}\|f\|_{H^1(\mathbb{R})}.$$

Proof. From the definition of W_ψ^θ we get

$$(W_\psi^\theta f)(b,a) = |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}x+b\right)\overline{\psi(x)}dx.$$

Now,

$$\begin{aligned}
&((W_\psi^\theta f)(\cdot,a) \star \eta_t(\cdot))(b) \\
&= \int_{\mathbb{R}} (W_\psi^\theta f)(b-y,a)\eta_t(y)dy \\
&= \int_{\mathbb{R}} |a|^{\frac{1}{2\theta}} \left(\int_{\mathbb{R}} f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}x+b-y\right)\overline{\psi(x)}dx\right)\eta_t(y)dy \\
&= |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} \overline{\psi(x)} \left(\int_{\mathbb{R}} f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}x+b-y\right)\eta_t(y)dy\right) dx \\
&= |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} (f \star \eta_t)\left(\operatorname{sgn} a|a|^{\frac{1}{\theta}}x+b\right)\overline{\psi(x)}dx.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\|(W_\psi^\theta f)(\cdot,a)\|_{H^1(\mathbb{R})}\\
&= \int_{\mathbb{R}} \sup_{t>0} |\left((W_\psi^\theta f)(\cdot,a)\star\eta_t(\cdot)\right)(b)|db\\
&\leq |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} |\overline{\psi(x)}| \left(\int_{\mathbb{R}} \sup_{t>0} |(f\star\eta_t)((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}x+b)|db\right) dx.\\
&= |a|^{\frac{1}{2\theta}} \|\psi\|_{L^1(\mathbb{R})} \|f\|_{H^1(\mathbb{R})}.
\end{aligned}
$$

This completes the proof. □

Corollary 4.4. *If $a\in\mathbb{R}-\{0\}$, $f\in H^1(\mathbb{R})$ and ψ is in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$, then*

$$
\|(W_\psi^\theta f)(\cdot,a)\|_{H^1(\mathbb{R})} = O\left(|a|^{\frac{1}{2\theta}}\right).
$$

Proof. By using Theorem 4.3, we get the result. □

We will now determine the $H^1(\mathbb{R})$-distance of two CFrWTs.

Theorem 4.5. *Let $f,g\in H^1(\mathbb{R})$ and $\phi,\psi\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, then for $a\in\mathbb{R}-\{0\}$,*

$$
\begin{aligned}
&\|(W_\phi^\theta f)(\cdot,a) - (W_\psi^\theta g)(\cdot,a)\|_{H^1(\mathbb{R})}\\
&\leq |a|^{\frac{1}{2\theta}} \left(\|f\|_{H^1(\mathbb{R})} \|\phi-\psi\|_{L^1(\mathbb{R})} + \|f-g\|_{H^1(\mathbb{R})} \|\psi\|_{L^1(\mathbb{R})}\right).
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\|(W_\phi^\theta f)(\cdot,a) - (W_\psi^\theta g)(\cdot,a)\|_{H^1(\mathbb{R})}\\
&\leq \|(W_\phi^\theta f)(\cdot,a) - (W_\psi^\theta f)(\cdot,a)\|_{H^1(\mathbb{R})} + \|(W_\psi^\theta f)(\cdot,a) - (W_\psi^\theta g)(\cdot,a)\|_{H^1(\mathbb{R})}.
\end{aligned} \tag{4.2}
$$

Now,

$$
\begin{aligned}
&(W_\phi^\theta f)(b,a) - (W_\psi^\theta f)(b,a)\\
&= \frac{1}{|a|^{\frac{1}{2\theta}}} \int_{\mathbb{R}} f(t) \left\{\overline{\phi\left(\frac{t-b}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)} - \overline{\psi\left(\frac{t-b}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)}\right\} dt\\
&= |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}x+b\right) \left(\overline{\phi(x)-\psi(x)}\right) dx.
\end{aligned}
$$

Observe that

$$\begin{aligned}
&\left\{\left((W_\phi^\theta f)(\cdot,a) - (W_\phi^\theta f)(\cdot,a)\right) \star \eta_t(\cdot)\right\}(b)\\
&= \int_{\mathbb{R}} \left\{\left((W_\phi^\theta f)(b-y,a) - (W_\phi^\theta f)(b-y,a)\right)\right\} \eta_t(y)dy\\
&= \int_{\mathbb{R}} \left(|a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}} x + b - y\right) \left(\overline{\phi(x) - \psi(x)}\right) dx \right) \eta_t(y)dy\\
&= |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} \left(\overline{\phi(x) - \psi(x)}\right) \left(\int_{\mathbb{R}} f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}} x + b - y\right) \eta_t(y)dy \right) dx\\
&= |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} (f \star \eta_t)\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}} x + b\right) \left(\overline{\phi(x) - \psi(x)}\right) dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|(W_\phi^\theta f)(\cdot,a) - (W_\psi^\theta f)(\cdot,a)\|_{H^1(\mathbb{R})}\\
&= \int_{\mathbb{R}} \sup_{t>0} \left|\left\{\left((W_\phi^\theta f)(\cdot,a) - (W_\phi^\theta f)(\cdot,a)\right) \star \eta_t(\cdot)\right\}(b)\right| db\\
&\leq |a|^{\frac{1}{2\theta}} \int_{\mathbb{R}} |\phi(x) - \psi(x)| \left(\int_{\mathbb{R}} \sup_{t>0} \left|(f \star \eta_t)\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}} x + b\right)\right| db \right) dx\\
&= |a|^{\frac{1}{2\theta}} \|f\|_{H^1(\mathbb{R})} \|\phi - \psi\|_{L^1(\mathbb{R})}. \qquad (4.3)
\end{aligned}$$

Also, using Theorem 4.3, we have

$$\|(W_\psi^\theta f)(\cdot,a) - (W_\psi^\theta g)(\cdot,a)\|_{H^1(\mathbb{R})} \leq |a|^{\frac{1}{2\theta}} \|f - g\|_{H^1(\mathbb{R})} \|\psi\|_{L^1(\mathbb{R})}. \qquad (4.4)$$

The result follows from equations (4.2), (4.3) and (4.4). □

Remark 4.6. For $\theta = 1$, Theorems 4.3 and 4.5 coincide with those studied in [4].

4.2. MORREY SPACE

Morrey in [19] introduced the Morrey space to study the local behaviour of solutions of second order elliptic partial differential equation. Recently, considerable attention is given to study the boundedness of operators on Morrey-type spaces. In [9], Gurbuz studied the boundedness results for a large class of pseudo-differential operators with smooth symbols on weighted Morrey and weighted fractional Sobolev–Morrey spaces. Gurbuz [10] studied the boundedness of sublinear operators with rough kernel generated by Calderon–Zygmund operators and their commutators on generalized Morrey spaces. Gurbuz [11] also studied the boundedness of Marcinkiewicz integral with rough kernel associated with Schrödinger operators and their commutators on generalized weighted Morrey spaces. In this subsection we study the properties of the CFrWT on classical Morrey space. We recall the definition of classical Morrey spaces ([2, 10, 19]).

Definition 4.7. The Morrey space $L^{p,\nu}_M(\mathbb{R})$, with $1 \le p < \infty$ and $0 \le \nu \le 1$, defined by

$$L^{p,\nu}_M(\mathbb{R}) = \left\{ f \in L^p_{loc}(\mathbb{R}) : \sup_{\substack{x\in\mathbb{R}\\ r>0}} \left(\frac{1}{r^\nu} \int\limits_{B(x,r)} |f(t)|^p dt \right) < \infty \right\},$$

is a Banach space normed by

$$\|f\|_{L^{p,\nu}_M(\mathbb{R})} = \sup_{\substack{x\in\mathbb{R}\\ r>0}} \left(\frac{1}{r^\nu} \int\limits_{B(x,r)} |f(t)|^p dt \right)^{\frac{1}{p}}.$$

Since the Schwartz class $\mathcal{S}(\mathbb{R})$ is not dense in Morrey space, so we consider the space $\overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L^{1,\nu}_M(\mathbb{R})}}$, which is the closure of $\mathcal{S}(\mathbb{R})$ in $L^{1,\nu}_M(\mathbb{R})$. We now study some properties of the CFrWT on $\overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L^{1,\nu}_M(\mathbb{R})}}$.

Theorem 4.8. *Let $a \in \mathbb{R} - \{0\}$ and $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the operator $W^\theta_\psi : \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L^{1,\nu}_M(\mathbb{R})}} \to \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L^{1,\nu}_M(\mathbb{R})}}$ defined by $f \mapsto (W^\theta_\psi f)(\cdot, a)$ is bounded. Furthermore,*

$$\|(W^\theta_\psi f)(\cdot, a)\|_{L^{1,\nu}_M(\mathbb{R})} \le |a|^{\frac{1}{2\theta}} \|\psi\|_{L^1(\mathbb{R})} \|f\|_{L^{1,\nu}_M(\mathbb{R})}, \quad \text{for all } f \in \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L^{1,\nu}_M(\mathbb{R})}}.$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$, then by Lemma 4.2, we get

$$(W^\theta_\psi f)(\cdot, a) \in L^1(\mathbb{R}) = \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L^1(\mathbb{R})}} \subset \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L^{1,\nu}_M(\mathbb{R})}}.$$

We have

$$\|(W^\theta_\psi f)(\cdot, a)\|_{L^{1,\nu}_M(\mathbb{R})} = \sup_{\substack{x\in\mathbb{R}\\ r>0}} \left(\frac{1}{r^\nu} \int\limits_{B(x,r)} |(W^\theta_\psi f)(b, a)| db \right). \tag{4.5}$$

Now,

$$\begin{aligned}
&\frac{1}{r^{\nu}} \int\limits_{B(x,r)} |(W_{\psi}^{\theta} f)(b,a)| db \\
&\leq \frac{|a|^{\frac{1}{2\theta}}}{r^{\nu}} \int\limits_{B(x,r)} \left(\int\limits_{\mathbb{R}} \left| f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}} u + b \right) \right| |\psi(u)| du \right) db \\
&= \frac{|a|^{\frac{1}{2\theta}}}{r^{\nu}} \int\limits_{\mathbb{R}} |\psi(u)| \left(\int\limits_{B(x,r)} \left| f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}} u + b \right) \right| db \right) du \\
&= |a|^{\frac{1}{2\theta}} \int\limits_{\mathbb{R}} |\psi(u)| \left(\frac{1}{r^{\nu}} \int\limits_{B\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}} u + x, r \right)} |f(\boldsymbol{z})| d\boldsymbol{z} \right) du. \qquad (4.6)
\end{aligned}$$

Also

$$\frac{1}{r^{\nu}} \int\limits_{B\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}} u + x, r \right)} |f(\boldsymbol{z})| d\boldsymbol{z} \leq \|f\|_{L_M^{1,\nu}(\mathbb{R})}. \qquad (4.7)$$

From equations (4.6) and (4.7) we have

$$\frac{1}{r^{\nu}} \int\limits_{B(x,r)} |(W_{\psi}^{\theta} f)(b,a)| db \leq |a|^{\frac{1}{2\theta}} \|f\|_{L_M^{1,\nu}(\mathbb{R})} \|\psi\|_{L^1(\mathbb{R})},$$

which gives

$$\sup_{\substack{x \in \mathbb{R} \\ r > 0}} \left(\frac{1}{r^{\nu}} \int\limits_{B(x,r)} |(W_{\psi}^{\theta} f)(b,a)| db \right) \leq |a|^{\frac{1}{2\theta}} \|f\|_{L_M^{1,\nu}(\mathbb{R})} \|\psi\|_{L^1(\mathbb{R})}. \qquad (4.8)$$

From (4.5) and (4.8) we have

$$\|(W_{\psi}^{\theta} f)(\cdot, a)\|_{L_M^{1,\nu}(\mathbb{R})} \leq |a|^{\frac{1}{2\theta}} \|\psi\|_{L^1(\mathbb{R})} \|f\|_{L_M^{1,\nu}(\mathbb{R})} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}).$$

Now by the density principle ([7, p. 329]), since $\mathcal{S}(\mathbb{R})$ is dense in $\overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}}$ and W_{ψ}^{θ} is a bounded linear operator from $\mathcal{S}(\mathbb{R})$ into the Banach space $\overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}}$,

W_ψ^θ can be extended uniquely to a bounded linear operator

$$W_\psi^\theta : \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}} \to \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}}.$$

Moreover,

$$\|(W_\psi^\theta f)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})} \leq |a|^{\frac{1}{2\theta}} \|\psi\|_{L^1(\mathbb{R})} \|f\|_{L_M^{1,\nu}(\mathbb{R})} \text{ for all } f \in \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}}.$$

This completes the proof. □

The following corollary is a consequence of Theorem 4.8.

Corollary 4.9. *Let $a \in \mathbb{R} - \{0\}$, $f \in \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}}$ and $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then*

$$\|(W_\psi^\theta f)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})} = O\left(|a|^{\frac{1}{2\theta}}\right).$$

We will now determine the $L_M^{1,\nu}(\mathbb{R})$-distance of two CFrWTs.

Theorem 4.10. *Let $f, g \in \overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}}$ and $\phi, \psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then for $a \in \mathbb{R} - \{0\}$,*

$$\begin{aligned}&\|(W_\phi^\theta f)(\cdot,a) - (W_\psi^\theta g)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})}\\ &\leq |a|^{\frac{1}{2\theta}} \left(\|f\|_{L_M^{1,\nu}(\mathbb{R})} \|\phi-\psi\|_{L^1(\mathbb{R})} + \|f-g\|_{L_M^{1,\nu}(\mathbb{R})} \|\psi\|_{L^1(\mathbb{R})}\right).\end{aligned}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R})$. We have

$$\begin{aligned}&\|(W_\phi^\theta f)(\cdot,a) - (W_\psi^\theta f)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})}\\ &= \sup_{\substack{x\in\mathbb{R}\\ r>0}} \left(\frac{1}{r^\nu} \int_{B(x,r)} \left| \int_{\mathbb{R}} \left(f(y)\overline{\phi_{a,t,\theta}(y)} - f(y)\overline{\psi_{a,t,\theta}(y)}\right) dy \right| dt \right). \end{aligned} \tag{4.9}$$

Now,

$$\begin{aligned}
&\frac{1}{r^{\nu}}\int\limits_{B(x,r)}\left|\int\limits_{\mathbb{R}}\left(f(y)\overline{\phi_{a,t,\theta}(y)}-f(y)\overline{\psi_{a,t,\theta}(y)}\right)dy\right|dt\\
&\leq \frac{1}{r^{\nu}}\int\limits_{B(x,r)}\int\limits_{\mathbb{R}}|f(y)|\left|(\phi_{a,t,\theta}(y)-\psi_{a,t,\theta}(y))\right|dydt\\
&=\frac{1}{r^{\nu}}\int\limits_{B(x,r)}\left(\int\limits_{\mathbb{R}}\frac{1}{|a|^{\frac{1}{2\theta}}}|f(y)|\left|\phi\left(\frac{y-t}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)-\psi\left(\frac{y-t}{(\operatorname{sgn} a)|a|^{\frac{1}{\theta}}}\right)\right|dy\right)dt\\
&=\frac{|a|^{\frac{1}{2\theta}}}{r^{\nu}}\int\limits_{B(x,r)}\left(\int\limits_{\mathbb{R}}\left|f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}z+t\right)\right|\left|\phi(z)-\psi(z)\right|dz\right)dt\\
&=\frac{|a|^{\frac{1}{2\theta}}}{r^{\nu}}\int\limits_{\mathbb{R}}|\phi(z)-\psi(z)|\left(\int\limits_{B(x,r)}\left|f\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}z+t\right)\right|dt\right)dz\\
&=|a|^{\frac{1}{2\theta}}\int\limits_{\mathbb{R}}|\phi(z)-\psi(z)|\left(\frac{1}{r^{\nu}}\int\limits_{B\left((\operatorname{sgn} a)|a|^{\frac{1}{\theta}}z+x,r\right)}|f(u)|du\right)dz.
\end{aligned}$$

Using equation (4.7), we get

$$\begin{aligned}
&\frac{1}{r^{\nu}}\int\limits_{B(x,r)}\left|\int\limits_{\mathbb{R}}\left(f(y)\overline{\phi_{a,t,\theta}(y)}-f(y)\overline{\psi_{a,t,\theta}(y)}\right)dy\right|dt\\
&\leq |a|^{\frac{1}{2\theta}}\|f\|_{L_M^{1,\nu}(\mathbb{R})}\int\limits_{\mathbb{R}}|\phi(z)-\psi(z)|\,dz.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sup_{\substack{x\in\mathbb{R}\\ r>0}}\left(\frac{1}{r^{\nu}}\int\limits_{B(x,r)}\left|\int\limits_{\mathbb{R}}\left(f(y)\overline{\phi_{a,t,\theta}(y)}-f(y)\overline{\psi_{a,t,\theta}(y)}\right)dy\right|dt\right)\\
&\leq |a|^{\frac{1}{2\theta}}\|f\|_{L_M^{1,\nu}(\mathbb{R})}\|\phi-\psi\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{4.10}$$

From equations (4.9) and (4.10) it follows that

$$\|(W_\phi^\theta f)(\cdot,a)-(W_\psi^\theta f)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})}\leq |a|^{\frac{1}{2\theta}}\|f\|_{L_M^{1,\nu}(\mathbb{R})}\|\phi-\psi\|_{L^1(\mathbb{R})}$$

for all $f\in\mathcal{S}(\mathbb{R})$. Thus using the density principle ([7]), we have

$$\|(W_\phi^\theta f)(\cdot,a)-(W_\psi^\theta f)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})}\leq |a|^{\frac{1}{2\theta}}\|f\|_{L_M^{1,\nu}(\mathbb{R})}\|\phi-\psi\|_{L^1(\mathbb{R})} \tag{4.11}$$

for all $f\in\overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}}$. Also, using Theorem 4.8 we have

$$\|(W_\psi^\theta f)(\cdot,a)-(W_\psi^\theta g)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})}\leq |a|^{\frac{1}{2\theta}}\|f-g\|_{L_M^{1,\nu}(\mathbb{R})}\|\psi\|_{L^1(\mathbb{R})} \tag{4.12}$$

for all $f,g\in\overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{L_M^{1,\nu}(\mathbb{R})}}$. Now

$$\begin{aligned}&\|(W_\phi^\theta f)(\cdot,a)-(W_\psi^\theta g)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})}\\&\leq \|(W_\phi^\theta f)(\cdot,a)-(W_\psi^\theta f)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})}+\|(W_\psi^\theta f)(\cdot,a)-(W_\psi^\theta g)(\cdot,a)\|_{L_M^{1,\nu}(\mathbb{R})}.\end{aligned} \tag{4.13}$$

From equations (4.11), (4.12) and (4.13) the theorem follows immediately. □

5. CONCLUSIONS

In this paper, we have studied CFrWT which as a generalization of the classical wavelet transform, reduces to classical wavelet transform for $\theta=1$. In Section 2, we have introduced some basic definitions and results. In Section 3, we have generalized the existing result like orthogonality relation, reconstruction formula and the range theorem, in [35, 37], in the context of two fractional wavelets. Also we have derived the formulas for the CFrWT when the argument function or fractional wavelet is a convolution or correlation of two functions. Lastly, in section 4, the boundedness of CFrWT on Hardy space $H^1(\mathbb{R})$ and on a subspace of Morrey space $L_M^{1,\nu}(\mathbb{R})$ along with its approximation property are established.

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