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CORONA THEOREM FOR STRICTLY PSEUDOCONVEX DOMAINS

Sebastian Gwizdek

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Abstract. Nearly 60 years have passed since Lennart Carleson gave his proof of Corona Theorem for unit disc in the complex plane. It was only recently that M. Kosiek and K. Rudol obtained the first positive result for Corona Theorem in multidimensional case. Using duality methods for uniform algebras the authors proved "abstract" Corona Theorem which allowed to solve Corona Problem for a wide class of regular domains. In this paper we expand Corona Theorem to strictly pseudoconvex domains with smooth boundaries.

Keywords: Corona theorem, Banach algebra, uniform algebra, Arens product, Gleason part, band of measures, representing measure.

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1. INTRODUCTION

Let Ω be a domain in $\mathbb{C}^d$. By $H^\infty(\Omega)$ denote a Banach algebra of bounded analytic functions on Ω. Spectrum of $H^\infty(\Omega)$, the set of all nonzero linear and multiplicative functionals on $H^\infty(\Omega)$, will be denoted by $\mathrm{Sp}(H^\infty(\Omega))$. We endow $\mathrm{Sp}(H^\infty(\Omega))$ with Gelfand topology, which is induced by the weak-star topology of the dual of $H^\infty(\Omega)$. There is a natural embedding of Ω into this spectrum. Namely, a mapping $\Omega \ni z \mapsto \delta_z \in \mathrm{Sp}(H^\infty(\Omega))$, where δ_z is the evaluation functional at z: $\delta_z(f) = f(z)$ for $f \in H^\infty(\Omega)$. In this sense we treat Ω as a subset of $\mathrm{Sp}(H^\infty(\Omega))$.

Corona Theorem states that (under suitable hypothesis on Ω) the Gelfand closure of Ω is equal to spectrum, i.e., that Ω is dense in $\mathrm{Sp}(H^\infty(\Omega))$. This is equivalent to the existence of solutions $\mathbf{g} = (g_1, \ldots, g_n) \in H^\infty(\Omega)^n$ of the Bézout equation

$$f_1(z)g_1(z) + \ldots + f_n(z)g_n(z) = 1, \quad z \in \Omega$$

for given any $\mathbf{f} = (f_1, \ldots, f_n) \in H^\infty(\Omega)^n$ satisfying the uniform estimate

$$|f_1(z)| + \ldots + |f_n(z)| \geq c > 0, \quad z \in \Omega.$$

The above equivalence can be proved as follows. Assume that there is some $\phi \in \mathrm{Sp}(H^\infty(\Omega))$ not belonging to the Gelfand closure of $\{\delta_z : z \in \Omega\}$ denoted as $\overline{\Omega}^\sigma$. By compactness of the spectrum there is n-tuple of Gelfand's transforms $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_n)$ such that $\hat{\mathbf{f}}(\phi) = \mathbf{0}$ while

$$\mathbf{0} \notin \hat{\mathbf{f}}(\overline{\Omega}^\sigma) = \overline{\mathbf{f}(\Omega)}.$$

It follows that $\mathbf{f}$ satisfies uniform estimate but one can't find $\mathbf{g}$ solving Bézout equation for $\mathbf{f}$ since $\phi(1) = 1$. Conversely, if Ω is dense in spectrum, then for any $\mathbf{f}$ satisfying uniform estimate

$$(0, \ldots, 0) \notin \overline{\mathbf{f}(\Omega)} = \hat{\mathbf{f}}(\mathrm{Sp}(H^\infty(\Omega)) = \sigma(f_1, \ldots, f_n),$$

so the ideal generated by $\mathbf{f}$ cannot be proper, hence it contains 1, yielding a solution to Bézout equation.

In the case of the unit disk $\mathbb{D}$ in the complex plane, Corona Theorem was proved by Lennart Carleson in [1] using interpolation and hard analysis methods. Since then, the result has been extended to some classes of plane domains, but only counterexamples were known for domains in higher dimensions and there was no positive result for any regular domain G in $\mathbb{C}^d$, when $d > 1$. Solutions of the Bezout equation were obtained in some wider classes like the intersection of all $H^p(G)$ with $p < \infty$. Many attempts to employ operator theory methods have ended without success. The first positive result in the multidimensional case came only recently in [7]. The authors were using duality theory for uniform algebras obtaining abstract counterparts of Corona Theorem. To apply them for concrete domains some assumptions on the domain regularity were needed. Apart from the strict pseudoconvexity or (alternatively) being a polydomain, the authors assumed that G is strongly starlike. We will show, how to get rid of the latter assumption.

2. UNIFORM ALGEBRAS

Let X be a compact Hausdorff space. A *uniform algebra* denoted by A is a closed unital subalgebra of $C(X)$ separating the points of X. Uniform algebra A is called *natural* if $\mathrm{Sp}(A) = X$, so that any nonzero linear and multiplicative functional on A is an evaluation functional δ_x for some $x \in X$.

Let A be a Banach algebra. For $\lambda \in A^*$, define $a \cdot \lambda$ and $\lambda \cdot a$ by duality

$$\langle b, a \cdot \lambda \rangle = \langle ba, \lambda \rangle, \quad \langle b, \lambda \cdot a \rangle = \langle ab, \lambda \rangle \quad (a, b \in A).$$

Now, for $\lambda \in A^*$ and $M \in A^{**}$, define $\lambda \cdot M$ and $M \cdot \lambda$ by

$$\langle a, \lambda \cdot M \rangle = \langle M, a \cdot \lambda \rangle, \quad \langle a, M \cdot \lambda \rangle = \langle M, \lambda \cdot a \rangle \quad (a \in A).$$

Finally, for $M, N \in A^{**}$, define

$$\langle M \square N, \lambda \rangle = \langle M, N \cdot \lambda \rangle, \quad \langle M \Diamond N, \lambda \rangle = \langle N, \lambda \cdot M \rangle \quad (\lambda \in A^*).$$

The products $\Box$ and $\Diamond$ are called, respectively, the *first* and *second Arens products* on A^{**}. A bidual of A is Banach algebra with respect to Arens products. The natural embedding of A into its bidual identifies A as a closed subalgebra of both $(A^{**}, \Box)$ and $(A^{**}, \Diamond)$

A Banach algebra A is called *Arens regular* if the two products $\Box$ and $\Diamond$ agree on A^{**}. All C^*-algebras are Arens regular. Closed subalgebras of Arens regular algebras are Arens regular, hence all uniform algebras are Arens regular. What is more, bidual of uniform algebra is again uniform algebra with respect to Arens products. For further information about Arens products we refer to [3].

Let A be a uniform algebra. There is an equivalence relation on $\mathrm{Sp}(A)$ given by

$$\|\phi - \psi\| < 2,$$

with $\|\cdot\|$ denoting the norm in A^* for ϕ, $\psi \in \mathrm{Sp}(A)$. The equivalence classes under the above relation are called *Gleason parts*.

In our case A will be the algebra $A(G)$ of those analytic functions on a regular domain $G \subset \mathbb{C}^d$, which have continuous extensions to the Euclidean closure $\overline{G}$. By connectedness and Harnack's inequalities, the entire set G is a single Gleason part. Trivial (one-point) parts correspond to the boundary points, since these are peak points for G in the strictly pseudoconvex case (see [8]).

The space of complex, regular Borel measures on X, with total variation norm will be denoted by $M(X)$. The Riesz Representation Theorem yields $M(X) = C(X)^*$. For a set $E \subset C(X)$, the *annihilator* of E denoted by $E^\perp$ is the set of $\mu \in M(X)$ satisfying $\int f d\mu = 0$ for all $f \in E$. A closed subspace $\mathcal{M}$ of $M(X)$ is called a *band of measures*, if for any measure $\nu \in \mathcal{M}$ it contains all measures absolutely continuous with respect to $|\nu|$. By $\mathcal{M}^s$ denote the set of all measures singular to any measure in $\mathcal{M}$. The set $\mathcal{M}^s$ is a band of measures called *singular band* to $\mathcal{M}$. Additionally, $\mathcal{M}^{ss} = \mathcal{M}$ for any band $\mathcal{M}$. Given $\mu \in M(X)$ there is Lebesgue-type decomposition

$$\mu = \mu^{\mathcal{M}} + \mu^s, \quad \mu^{\mathcal{M}} \in \mathcal{M}, \mu^s \in \mathcal{M}^s.$$

It follows that $M(X) = \mathcal{M} \oplus_1 \mathcal{M}^s$, where $\oplus_1$ denotes the Banach space ℓ^1-direct sum (that is, $\|\nu \oplus \mu\| = \|\nu\| + \|\mu\|$). A band is *reducing* if $\mu^{\mathcal{M}} \in A^\perp$ for any $\mu \in A^\perp$.

A *representing* (resp. *complex representing*) *measure* for $\phi \in \mathrm{Sp}(A)$ is a nonnegative (resp. complex) measure $\mu \in M(X)$ such that

$$\phi(f) = \int_X f d\mu \quad \text{for any} \quad f \in A.$$

It is well known that for any functional $\phi \in \mathrm{Sp}(A)$ there exists at least one representing measure. The reader who is interested in more detailed exposition of uniform algebras might be advised to read the work [4].

By $\mathcal{M}_G$ denote the smallest band containing all representing measures for a (non-trivial) Gleason part G. Such band always exists because intersection of nonempty collection of bands in $M(X)$ is again a band.

3. CORONA THEOREM

Recent results originated from the idea of studying some dual algebras isomorphic to $H^\infty(G)$ and their spectra with aid of duality methods. In many cases bounded analytic functions on G are pointwise limits of uniformly bounded sequences from the algebra $A(G)$. Such sequences converge weak-star in the suitable L^∞- type algebras. Hence $H^\infty(G)$ can be considered as the weak-star closure of $A(G)$ in some dual space. One of such dual spaces is a quotient space of the second dual $A(G)^{**}$, the other, $H^\infty(\mathcal{M}_G)$ is described below. The study of the spectrum $\mathrm{Sp}(H^\infty(G))$ can be based on the properties of the spectrum of an abstract bidual algebra A^{**} and of its Gleason parts. Some of them were investigated in [6].

Let G ba a nontrivial Gleason part for a uniform algebra A. By $H^\infty(\mathcal{M}_G)$ denote the weak-star closure of A in $\mathcal{M}_G^*$, here $\mathcal{M}_G$ is considered as a Banach space. This "abstract" algebra $H^\infty(\mathcal{M}_G)$ will be useful for studying properties of its "classical" counterpart $H^\infty(G)$. The following results were obtained by M. Kosiek and K. Rudol in [7].

Theorem 3.1 ([7]). *The algebra $H^\infty(\mathcal{M}_G)$ is isometrically and algebraically isomorphic to $A^{**}/\mathcal{M}_G^\perp \cap A^{**}$.*

For a wide class of domains $G \subset \mathbb{C}^d$ the authors show in [7] the assumptions denoted there by (8.1) under which the following general result holds.

Theorem 3.2 ([7]). *Let G be an open Gleason part in $Sp(A)$ satisfying the assumptions (8.1) of [7], on which the Gelfand and norm topologies are equal. Then the canonical image of G is dense in the spectrum of $H^\infty(\mathcal{M}_G)$ in its Gelfand topology.*

Roughly speaking, the assumptions (8.1) guarantee that the Gleason part G does not "grow" when embedded into bidual space. The equality of Gelfand and norm topologies on G is shown in [7, Theorem 10.1] for strictly pseudoconvex domains with C^2 boundary and for polydomain. What is more, the assumptions (8.1) are satisfied for these domains.

In the next theorem of that paper $G \subset \mathbb{C}^d$ is a bounded domain of holomorphy such that $\overline{G} = \mathrm{Sp}(A(G))$. For this it suffices to assume either that $\overline{G}$ is an intersection of a sequence of domains of holomorphy, that is, $\overline{G}$ has a Stein neighbourhood basis (see [9]), or that it has a smooth boundary (see [5]).

A domain G is a *strongly starlike domain* if there exists a point $x_0 \in G$ such that for the translate $G_{x_0} := G - x_0$ we have $\overline{G_{x_0}} \subset \rho \cdot G_{x_0}$, for $\rho > 1$. Note that convex domains are strongly starlike with respect to any of their points x_0, since for any $x_1 \in \partial G$ the open segment joining these points is contained in G.

Theorem 3.3 ([7]). *If G is a strongly starlike domain in $\mathbb{C}^d$ satisfying the above assumptions, then the algebras $H^\infty(G)$ and $H^\infty(\mathcal{M}_G)$ are isometrically isomorphic. The associated homeomorphism of spectra preserves the points of G.*

These results suffice to obtain a classical Corona Theorem for such domains. However the "strong star-likeness" assumption (although easy to show for convex domains) is a bit awkward. It turns out that it can be omitted.

The following result by B.J. Cole and R.M. Range will be crucial in the proof of our main theorem.

Theorem 3.4 ([2])**.** *Let f be a bounded analytic function on a strictly pseudoconvex domain domain $G \subset \mathbb{C}^d$ with C^2-boundary. There exist functions $f_n \in A(G)$ such that $\|f_n\| \leq \|f\|$ and $f_n \to f$ pointwise.*

Let $z \in G$ and ν_z be its representing measure with respect to the algebra $A(G)$. For $f \in A(G)$ we have $f(z) = \langle \nu_z, f\rangle = \int f d\nu_z$. Since $A(G)$ is weak-star dense in $H^\infty(\mathcal{M}_G)$ we also have $f(z) = \langle f, \nu_z\rangle$ for $f \in H^\infty(\mathcal{M}_G)$. We are ready to state our result.

Theorem 3.5 (Corona theorem)**.** *If a domain $G \subset \mathbb{C}^d$ is strictly pseudoconvex with C^2-boundary, then Corona theorem holds true in $H^\infty(G)$: its spectrum is the Gelfand closure of* (*the canonical image of*) *G.*

Proof. It suffices to show that algebras $H^\infty(G)$ and $H^\infty(\mathcal{M}_G)$ are isometrically isomorphic. Given $f \in H^\infty(\mathcal{M}_G)$ the mapping

$$f\big|_G \colon G \ni z \mapsto f(z) = \langle f, \nu_z\rangle = \int f d\nu_z,$$

where ν_z is a representing measure for $z \in G$, defines a bounded analytic function ([7, Proposition 11.2]). We will prove that the mapping

$$\Phi \colon H^\infty(\mathcal{M}_G) \ni f \mapsto f\big|_G \in H^\infty(G),$$

is a desired isomorphism. Clearly Φ is injective, linear and multiplicative. It remains to show that Φ is onto. Let $g \in H^\infty(G)$, by Theorem 3.4 there exists a uniformly bounded sequence of functions $g_n \in A(G)$ converging pointwise to g in G. As a consequence of Lebesgue dominated convergence theorem the sequence (g_n) has an adherent point $h \in H^\infty(\mathcal{M}_G)$ such that $\langle \mu, g_n\rangle \to \langle h, \mu\rangle$ for any measure $\mu \in \mathcal{M}_G$. In particular $\langle \nu_z, g_n\rangle \to \langle h, \nu_z\rangle$ for any measure ν_z representing $z \in G$. On the other hand $\langle \nu_z, g_n\rangle = g_n(z) \to g(z)$. Whence $h\big|_G = g$ on G and Φ is surjective. Isomorphisms in Banach algebras preserve the spectrum and the norm in uniform algebras equals the spectral radius, hence Φ is isometric. Since algebras $H^\infty(G)$ and $H^\infty(\mathcal{M}_G)$ are isometrically isomorphic their spectra are homeomorphic. Applying Theorem 3.2 we obtain the result. □

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Sebastian Gwizdek
gwizdek@agh.edu.pl
https://orcid.org/0000-0002-9149-8507

AGH University of Science and Technology
Faculty of Applied Mathematics AGH
al. Mickiewicza 30, 30-059 Kraków, Poland

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