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μ-HANKEL OPERATORS ON HILBERT SPACES

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Abstract. A class of operators is introduced (μ-Hankel operators, μ is a complex parameter), which generalizes the class of Hankel operators. Criteria for boundedness, compactness, nuclearity, and finite dimensionality are obtained for operators of this class, and for the case $|\mu| = 1$ their description in the Hardy space is given. Integral representations of μ-Hankel operators on the unit disk and on the Semi-Axis are also considered.

Keywords: Hankel operator, μ-Hankel operator, Hardy space, integral representation, nuclear operator, integral operator.

Mathematics Subject Classification: 47B25, 47B35.

1. INTRODUCTION

As is well known, classical Hankel operators form one of the most significant classes of operators in spaces of analytic functions. This class has a lot of applications to various parts of Analysis, Probability, Control Theory, etc. (see [11–13]). Therefore, it is not surprising that there are a large number of generalizations and analogues of operators of this class (see ibid, and also, e.g., [8], and the bibliography therein).

Hankel operators are significant, in particular, for an important class of Toeplitz operators (see [11,13]). On the other hand, an interesting generalization of Toeplitz operators (the "λ-Toeplitz operators") was given in [5] (see also [7]). In this paper, we consider a new class of operators in Hilbert spaces (the "μ-Hankel operators"; μ is a complex parameter), which are related to λ-Toeplitz operators as well as Hankel operators with Toeplitz ones. We give criteria for boundedness, compactness, nuclearity, and finite dimensionality for μ-Hankel operators. Operators of this class turned out to be associated with the Hankel ones in the case $|\mu| = 1$, but a new feature is the nuclearity of these operators for $|\mu| \neq 1$.

The last part of the paper can be considered as a contribution to the theory of integral operators. We shaw that some natural classes of integral operators are μ-Hankel and apply our results obtained in an abstract setting to these operators. In particular, μ-Hankel operators are closely related to the complex moment problem.

2. BOUNDEDNESS AND NUCLEARITY OF μ-HANKEL OPERATORS

Definition 2.1. Let μ, and ν be complex numbers, $\alpha = \{\alpha_j\}_{j\geq 0}$ be a sequence of complex numbers, and let $\mathcal{H}, \mathcal{H}'$ be separable Hilbert spaces. We call the operator $A_{(\mu,\nu),\alpha} : \mathcal{H} \to \mathcal{H}'$ (μ,ν)-*Hankel* if for some orthonormal bases $(e_k)_{k\geq 0} \subset \mathcal{H}$ and $(e'_j)_{j\geq 0} \subset \mathcal{H}'$ the matrix $(a_{jk})_{k,j\geq 0}$, of this operator (recall that $a_{jk} = \langle A_{(\mu,\nu),\alpha} e_k, e'_j\rangle$; here and below, the angle brackets denote the dot product) consists of elements of the form

$$a_{jk} = \mu^k \nu^j \alpha_{k+j}.$$

In particular, $A_{(1,1),\alpha}$ is a Hankel operator (for the latter see, e.g., [11, 13]).

Remark 2.2. For accuracy one can assume that the operator $A_{(\mu,\nu),\alpha}$ is initially defined on the linear span of the set $\{e_k : k \in \mathbb{Z}_+\}$.

Instead of $A_{(\mu,1),\alpha}$, we will further write $A_{\mu,\alpha}$ (or A_μ) and call such an operator μ-*Hankel*. To avoid the trivial case, for these operators we will assume that $\mu \neq 0$.

Thus, the matrix of a μ-Hankel operator has the form $(\mu^k \alpha_{k+j})_{k,j\geq 0}$, i. e.

$$(a_{jk})_{k,j\geq 0} = \begin{pmatrix} \alpha_0 & \mu\alpha_1 & \mu^2\alpha_2 & \mu^3\alpha_3 & \mu^4\alpha_4 & \dots \\ \alpha_1 & \mu\alpha_2 & \mu^2\alpha_3 & \mu^3\alpha_4 & \dots & \\ \alpha_2 & \mu\alpha_3 & \mu^2\alpha_4 & \dots & & \\ \alpha_3 & \mu\alpha_4 & \dots & & & \\ \alpha_4 & \dots & & & & \\ \vdots & & & & & \end{pmatrix}. \tag{2.1}$$

For what follows, it is useful to note that the adjoint operator $A^*_{(\mu,\nu),\alpha}$ has a matrix $\overline{\mu}^j \overline{\nu}^k \overline{\alpha_{k+j}}$ (the bar denotes complex conjugation), and, therefore, $A^*_{(\mu,\nu),\alpha} = A_{(\overline{\nu},\overline{\mu}),\overline{\alpha}}$. In particular, $A^*_{\mu,\alpha} = A_{(1,\overline{\mu}),\overline{\alpha}}$.

Since $\mu^k \nu^j \alpha_{k+j} = (\mu/\nu)^k (\nu^{k+j}\alpha_{k+j})$ for $\nu \neq 0$, we have $A_{(\mu,\nu),\alpha} = A_{\mu/\nu,\alpha'}$, where $\alpha'_k = \nu^k \alpha_k$, and thus the consideration of (μ,ν)-Hankel operators is reduced to the consideration of μ-Hankel operators, what we are going to do in this paper.

Like for Hankel operators, μ-Hankel operators can be characterized as operators satisfying some commuting relation.

Theorem 2.3. *A bounded operator A in the space $\ell^2(\mathbb{Z}_+)$ is μ-Hankel if and only if the following commuting relation is true:*

$$AS = \mu S^* A, \tag{2.2}$$

where S is a shift in $\ell^2(\mathbb{Z}_+)$.

Proof. Let the operator A be μ-Hankel. For all $k, j \in \mathbb{Z}_+$ we have

$$\begin{aligned} \langle ASe_k, e_j\rangle &= \langle Ae_{k+1}, e_j\rangle = \mu^{k+1}\alpha_{k+j+1} \\ &= \mu\langle Ae_k, e_{j+1}\rangle = \mu\langle Ae_k, Se_j\rangle \\ &= \langle \mu S^* Ae_k, e_j\rangle. \end{aligned}$$

Due to the boundedness of the operator A, this implies (2.2).

Conversely, let (2.2) be valid. Then for $k, j \in \mathbb{Z}_+$, $k > 0$ we have

$$\begin{aligned} a_{jk} := \langle Ae_k, e_j\rangle &= \langle ASe_{k-1}, e_j\rangle \\ &= \langle \mu S^* Ae_{k-1}, e_j\rangle = \mu\langle Ae_{k-1}, Se_j\rangle \\ &= \mu\langle Ae_{k-1}, e_{j+1}\rangle = \mu a_{(j+1),(k-1)}. \end{aligned}$$

Iterating this equality, we have $a_{jk} = \mu^k a_{(k+j),0}$. Since the last equality is obvious for $k = 0$, the operator A is μ-Hankel, and the theorem is proved. □

Corollary 2.4. *If a bounded operator A is μ-Hankel, then its kernel* Ker$A$ *is an invariant subspace of the shift operator. Therefore, if A is defined in the Hardy space $H^2 = H^2(\mathbb{T})$ ($\mathbb{T}$ is a unit circle), then* Ker$A$ *has the form θH^2, where θ is an inner function.*

The next theorem gives, in particular, criteria for boundedness of operators of the class under consideration. Below $\widehat{\psi}(n)$ denotes the nth Fourier coefficient of the function ψ.

Theorem 2.5. *Let $\mathcal{H}, \mathcal{H}'$ be a separable infinite-dimensional Hilbert spaces, and $A_\mu = A_{\mu,\alpha} : \mathcal{H} \to \mathcal{H}'$ a μ-Hankel operator. The following statements are true.*

(1) *Let $|\mu| < 1$. The operator A_μ is bounded if and only if $(\alpha_k) \in \ell^2(\mathbb{Z}_+)$. In this case, A_μ is nuclear with the Hilbert–Schmidt norm*

$$\|A_\mu\|_{\mathrm{S}_2} = \left(\sum_{k=0}^{\infty} |\mu|^{2k} \sum_{n=k}^{\infty} |\alpha_n|^2\right)^{1/2} \tag{2.3}$$

and with a trace

$$\mathrm{tr} A_\mu = \sum_{n=0}^{\infty} \mu^n \alpha_{2n}. \tag{2.4}$$

(2) *Let $|\mu| > 1$. The operator A_μ is bounded if and only if $(\mu^k \alpha_k) \in \ell^2(\mathbb{Z}_+)$. Moreover, in this case A_μ is nuclear, and its trace is represented by the formula* (2.4).

(3) *Let $|\mu| = 1$. Then $A_\mu = V_\mu \Gamma_\mu$, where $\Gamma_\mu : \mathcal{H} \to \mathcal{H}$ is Hankel with matrix $(\mu^{k+j}\alpha_{k+j})$, and $V_\mu : \mathcal{H} \to \mathcal{H}'$ is a unitary operator.*
In particular, the operator A_μ is bounded if and only if there is such a function $\psi \in L^\infty(\mathbb{T})$ that $\mu^n \alpha_n = \widehat{\psi}(n)$ for $n \in \mathbb{Z}_+$. In addition,

$$\|A_\mu\| = \inf\{\|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \alpha_n = \widehat{\psi}(n) \forall n \in \mathbb{Z}_+\}.$$

Proof. (1) Let $|\mu| < 1$. Notice that

$$A_\mu e_0 = \sum_j a_{j0} e'_j = \sum_j \alpha_j e'_j.$$

Thus, if A_μ is bounded then $\sum_j |\alpha_j|^2 = \|A_\mu e_0\|^2 \le \|A_\mu\|^2$, which proves the necessity.

Now let $\alpha \in \ell^2(\mathbb{Z}_+)$. We shall show that A_μ belongs to the Hilbert–Schmidt class $\mathbf{S}_2$ (and thus it is bounded). It is known (see, e.g., [14, p.152, Th. 6.22]), that for an operator A with matrix (a_{jk}) to belong to the Hilbert–Schmidt class $\mathbf{S}_2$, it suffices that

$$C^2 := \sum_j \sum_k |a_{jk}|^2 < \infty,$$

and its Hilbert–Schmidt norm is $\|A\|_{\mathbf{S}_2} = C$. In our case, $\alpha \in \ell^2(\mathbb{Z}_+)$, and therefore

$$\sum_j \sum_k |a_{jk}|^2 = \sum_j \sum_k |\mu|^{2k} |\alpha_{j+k}|^2 = \sum_k |\mu|^{2k} \sum_j |\alpha_{j+k}|^2$$

$$= \sum_{k=0}^{\infty} |\mu|^{2k} \sum_{n \geq k}^{2k} |\alpha_n|^2 \leq \|\alpha\|_{l^2}^2 \sum_{k=0}^{\infty} (|\mu|^2)^k = \frac{\|\alpha\|_{l^2}^2}{1-|\mu|^2} < \infty.$$

Hence, $A_\mu \in \mathbf{S}_2$ and

$$\|A_\mu\|_{\mathrm{S}_2} = \left(\sum_{k=0}^{\infty} |\mu|^{2k} \sum_{n=k}^{\infty} |\alpha_n|^2 \right)^{1/2}.$$

Moreover, the operator A_μ is nuclear, since under our assumptions

$$\mathrm{tr} A_\mu = \sum_{n=0}^{\infty} a_{nn} = \sum_{n=0}^{\infty} \mu^n \alpha_{2n} < \infty$$

(see, e.g., [4, Theorem 8.1]).

(2) Let $|\mu| > 1$. We denote $\alpha'_k := \mu^k \alpha_k$. As proved above the $1/\overline{\mu}$-Hankel operator $A_{1/\overline{\mu},\alpha'}$ is bounded if and only if $(\mu^k \alpha_k) = (\alpha'_n) \in \ell^2(\mathbb{Z}_+)$. The operator $A_{\mu,\alpha}$ is conjugate to the operator $A_{1/\overline{\mu},\alpha'}$ since the matrices of these operators in the bases (e_k) and (e'_j) coincide. This, in turn, means that the operator $A_{\mu,\alpha}$ is bounded if and only if $(\mu^k \alpha_k) \in \ell^2(\mathbb{Z}_+)$. Moreover, this operator is nuclear together with $A_{1/\overline{\mu},\alpha'}$ (see, e.g., [4]) and by the formula for the trace, proved in paragraph 1),

$$\mathrm{tr} A_{\mu,\alpha} = \overline{\mathrm{tr} A_{1/\overline{\mu},\alpha'}} = \overline{\sum_{n=0}^{\infty} \frac{1}{\overline{\mu}^n} \overline{\alpha}'_{2n}} = \sum_{n=0}^{\infty} \mu^n \alpha_{2n}.$$

(3) Let $|\mu| = 1$. Consider the operator $V_\mu : \mathcal{H} \to \mathcal{H}'$, $V_\mu x := \sum_{j=0}^{\infty} \overline{\mu}^j x_j e'_j$ for $x \in \mathcal{H}$, $x = \sum_{j=0}^{\infty} x_j e_j$. Since $|\mu| = 1$, the operator V_μ is unitary. Then the operator $\Gamma_\mu := V_\mu^{-1} A_\mu$ is Hankel with the matrix $(\alpha'_{k+j})_{j,k \geq 0} = (\mu^{j+k} \alpha_{k+j})_{j,k \geq 0}$, because in view of $A_\mu e_k = \sum_{j=0}^{\infty} \mu^k \alpha_{k+j} e_j$ we have $\Gamma_\mu e_k = V_\mu^{-1} A_\mu e_k = \sum_{j=0}^{\infty} \mu^{j+k} \alpha_{k+j} e_j$. But $A_{\mu,\alpha}$ is bounded if and only if Γ_μ is bounded, and by the Nehari Theorem (see, e.g., [13, Theorem 1.1.1]) this is equivalent to the fact that there exists a function $\psi \in L^\infty(\mathbb{T})$, such that $\widehat{\psi}(n) = \alpha'_n = \mu^n \alpha_n$ for $n \in \mathbb{Z}_+$. Moreover, by virtue of the same theorem

$$\|A_{\mu,\alpha}\| = \|\Gamma_\mu\| = \inf\{\|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \mu^n \alpha_n = \widehat{\psi}(n) \forall n \in \mathbb{Z}_+\},$$

which completes the proof. □

Example 2.6 (the generalized Hilbert matrix)**.** Let $\alpha_n = \frac{1}{n+1}$, $n \in \mathbb{Z}_+$. The corresponding μ-Hankel operator in $\ell^2(\mathbb{Z}_+)$ will be denoted by H_μ. (Thus, the operator H_1 is classical and has the Hilbert matrix, see, e.g., [13, p. 6]). According to Theorem 2.5 three cases are possible.

(1) $|\mu| < 1$. Then H_μ is nuclear and

$$\mathrm{tr} H_\mu = \sum_{n=0}^{\infty} \frac{\mu^n}{2n+1} = \frac{1}{2\sqrt{\mu}} \log \frac{1+\sqrt{\mu}}{1-\sqrt{\mu}}.$$

(2) $|\mu| > 1$. In this case H_μ is unbounded in $\ell^2(\mathbb{Z}_+)$.

(3) $|\mu| = 1$, $\mu = e^{i\theta}$. In this case, $H_\mu = V_\mu \Gamma_\mu$, where Γ_μ is Hankel in $\ell^2(\mathbb{Z}_+)$ with matrix $(\mu^{k+j}/k+j+1)$, and V_μ is unitary in $\ell^2(\mathbb{Z}_+)$. Consider the bounded function ψ_θ on $\mathbb{T}$ defined by

$$\psi_\theta(e^{it}) = ie^{-i(t-\theta)}(\pi - (t-\theta)), t \in [0, 2\pi).$$

It is easy to see that $\widehat{\psi_\theta}(n) = e^{in\theta}\widehat{\psi_0}(n) = \frac{\mu^n}{n+1} = \mu^n \alpha_n$ for $n \in \mathbb{Z}_+$. Thus, the operator H_μ is bounded in $\ell^2(\mathbb{Z}_+)$ and $\|H_\mu\| = \|\Gamma_\mu\|$.

Corollary 2.7. *The operator $A_{\mu,\alpha}$ is not left-Fredholm provided it is bounded.*

Proof. In cases (1) and (2) of Theorem 2.5 this follows from the compactness of this operator. In case (3) the failure of left-Fredholmness for $A_{\mu,\alpha}$ follows from the failure of left-Fredholmness for Hankel operators (see, e.g., [13]). □

Recall the definition of the degree of a rational function $R = P/Q$ (P and Q are polynomials of degree $\deg P$ and $\deg Q$). If the fraction P/Q is irreducible, then the value $\deg R = \max\{\deg P, \deg Q\}$ is called the *degree of the function* R. It is equal to the sum of the multiplicities of the poles R (taking into account the possible pole at infinity).

Following [13], we associate with the sequence of complex numbers $(\alpha_k)_{k\geq 0}$ the formal power series

$$\alpha(z) := \sum_{k=0}^{\infty} \alpha_k z^k. \tag{2.5}$$

Kronecker's theorem for Hankel operators readily implies

Theorem 2.8. *The matrix* (2.1) *of the operator $A_{\mu,\alpha}$ has finite rank if and only if the series* (2.5) *defines a rational function. Moreover, the rank of the matrix* (2.1) *is* $\deg(z\alpha(z))$.

Proof. From the form of the matrix (2.1) it immediately follows that the number of its linearly independent columns is equal to the number of linearly independent columns of the matrix $(\alpha_{j+k})_{j,k\geq 0}$. It remains to apply Kronecker's theorem in the formulation proposed in [13, Theorem 1.3.1]. □

3. μ-HANKEL OPERATORS IN THE HARDY SPACE

As is known (see, e.g., [11,13]) *Hardy space* consists of functions f analytic in the unit disk $\mathbb{D}$ for which $(\widehat{f}(n))_{n\geq 0} \in \ell^2(\mathbb{Z}_+)$, where $\widehat{f}(n)$ denotes the nth Taylor coefficient of a function f, $n \in \mathbb{Z}_+$. Moreover, $\|f\|_{H^2} = \|(\widehat{f}(n))_{n\geq 0}\|_{\ell^2}$. Thus, the mapping $f \mapsto (\widehat{f}(n))_{n\geq 0}$ is an isomorphism of the Hilbert spaces $H^2(\mathbb{D})$ and $\ell^2(\mathbb{Z}_+)$. Equivalently, $H^2(\mathbb{D})$ consists of functions f analytic in $\mathbb{D}$ and satisfying the condition

$$\|f\|^2_{H^2} := \sup_{0<r<1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

The space $H^2(\mathbb{D})$ can also be identified with the following subspace of the space $L^2(\mathbb{T})$:

$$\{f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0 \text{ for all } n < 0\},$$

where $\widehat{f}(n)$ denotes the nth Fourier coefficient of the function f, $n \in \mathbb{Z}$ (see, e.g., [11]). The functions $\chi_n(z) := z^n$ $(n \in \mathbb{Z}_+)$ form a standard orthonormal basis of H^2. We also put $H^2_- := L^2(\mathbb{T}) \ominus H^2$. The functions $\overline{\chi_{n+1}}$ $(n \in \mathbb{Z}_+)$ form the standard orthonormal basis of this space.

The next theorem describes bounded μ-Hankel operators in Hardy space for $|\mu| = 1$.

Theorem 3.1. *Let $|\mu| = 1$. For the operator $A : H^2 \to H^2_-$ the following statements are equivalent:*

(1) *A has a μ-Hankel matrix in standard bases and is bounded.*
(2) *$A = V_\mu H_\varphi$, where the operator $V_\mu f(z) := f(\mu z)$ is unitary in H^2_-, operator $H_\varphi : H^2 \to H^2_-$ is Hankel with a symbol $\varphi \in L^\infty(\mathbb{T})$, and $\widehat{\varphi}(-n) = \mu^n \alpha_n$ $(n \in \mathbb{Z}_+)$.*
(3) *Operator A is bounded and satisfies the generalized Hankel equation*

$$\mu P_- \mathcal{S} A = AS,$$

where $Sf(z) := zf(z)$ is the unilateral shift in H^2, $(\mathcal{S}g)(t) = tg(t)$ $(t \in \mathbb{T})$ is the bilateral shift in $L^2(\mathbb{T})$, and $P_- : L^2 \to H^2_-$ is the orthogonal projection.
(4) *$A = H_\psi U_\mu$, where $(U_\mu f)(z) = f(\mu z)$ is unitary in H^2, and $H_\psi : H^2 \to H^2_-$ is Hankel with a symbol $\psi \in L^\infty(\mathbb{T})$ and $\widehat{\psi}(-n) = \alpha_n$ $(n \in \mathbb{Z}_+)$.*

Proof. The proof will be carried out according to the scheme (1)⇒(2)⇒ (3)⇒(4)⇒ (1).

(1)⇒(2) Evidently, V_μ is unitary in H^2_-. We shall consider the operator $B := V_\mu^{-1} A : H^2 \to H^2_-$ and compute its matrix in standard bases. As $V_\mu \overline{\chi_{j+1}}(z) = \overline{\mu}^{j+1} \overline{\chi_{j+1}(z)}$, we have

$$\begin{aligned}\langle B\chi_k, \chi_{j+1}\rangle &= \langle V_\mu^{-1} A\chi_k, \overline{\chi_{j+1}}\rangle = \langle A\chi_k, V_\mu \overline{\chi_{j+1}}\rangle = \mu^{j+1}\langle A\chi_k, \chi_{j+1}\rangle \\ &= \mu^{j+1}\mu^k \alpha_{k+j+1} = \mu^{k+j}(\mu\alpha_{k+j+1}).\end{aligned}$$

This expression depends only on $k + j$, that is, the operator B is Hankel, $B = H_\varphi$, $\varphi \in L^\infty$.

Finally,

$$\widehat{\varphi}(-n) = \langle H_\varphi \chi_n, 1\rangle = \langle V_\mu^* A\chi_n, 1\rangle = \langle A\chi_n, V_\mu 1\rangle = \langle A\chi_n, \chi_0\rangle = \mu^n \alpha_n$$

for $n \in \mathbb{Z}_+$.

(2)⇒(3) If $A = V_\mu H_\varphi$, then A is bounded and, since $S\chi_k = \chi_{k+1}$ and $V_\mu^{-1}\overline{\chi_{j+1}} = \mu^{j+1}\overline{\chi_{j+1}}$, we have

$$\begin{aligned}\langle AS\chi_k, \overline{\chi_{j+1}}\rangle &= \langle V_\mu H_\varphi \chi_{k+1}, \overline{\chi_{j+1}}\rangle = \langle H_\varphi \chi_{k+1}, V_\mu^{-1}\overline{\chi_{j+1}}\rangle \\ &= \langle H_\varphi \chi_{k+1}, \mu^{j+1}\overline{\chi_{j+1}}\rangle = \overline{\mu}^{j+1}\langle H_\varphi \chi_{k+1}, \overline{\chi_{j+1}}\rangle = \overline{\mu}^{j+1}\widehat{\varphi}(-k-j-2).\end{aligned}$$

On the other hand, since $P_-\overline{\chi_{j+1}} = \overline{\chi_{j+1}}$ and $S^*\overline{\chi_{j+1}} = \overline{\chi_{j+2}}$, we have

$$\begin{aligned}\langle \mu P_- \mathcal{S} A\chi_k, \overline{\chi_{j+1}}\rangle &= \mu\langle \mathcal{S} A\chi_k, \overline{\chi_{j+1}}\rangle = \mu\langle \mathcal{S} V_\mu H_\varphi \chi_k, \overline{\chi_{j+1}}\rangle \\ &= \mu\langle V_\mu H_\varphi \chi_k, \overline{\chi_{j+2}}\rangle = \mu\langle H_\varphi \chi_k, V_\mu^{-1}\overline{\chi_{j+2}}\rangle \\ &= \mu\overline{\mu}^{j+2}\langle H_\varphi \chi_k, \overline{\chi_{j+2}}\rangle = \overline{\mu}^{j+1}\widehat{\varphi}(-k-j-2).\end{aligned}$$

Since both sides of the generalized Hankel equation contain bounded operators, they coincide on the entire space H^2.

(3)⇒(4) Let the operator A be bounded and satisfy the generalized Hankel equation. We consider the operator $H := AU_\mu^{-1} = AU_{\overline{\mu}}$ and show that it satisfies the Hankel equation $P_-\mathcal{S}H = HS$ (for the latter see, e.g., [13, Theorem 1.1.8]).

Indeed, by virtue of the generalized Hankel equation $P_-\mathcal{S}AU_{\overline{\mu}} = \overline{\mu}ASU_{\overline{\mu}}$ and thus

$$\langle P_-\mathcal{S}H\chi_k, \overline{\chi_{j+1}}\rangle = \overline{\mu}\langle ASU_{\overline{\mu}}\chi_k, \overline{\chi_{j+1}}\rangle. \tag{3.1}$$

But $SU_{\overline{\mu}}\chi_k = \overline{\mu}^k S\chi_k = \overline{\mu}^k \chi_{k+1}$, and $V_{\overline{\mu}}S\chi_k = V_{\overline{\mu}}\chi_{k+1} = \overline{\mu}^{k+1}\chi_{k+1}$. Therefore, $SU_{\overline{\mu}}\chi_k = \mu U_{\overline{\mu}}S\chi_k$. Substituting the last expression in (3.1), we get that for all $k, j \in \mathbb{Z}_+$

$$\langle P_-\mathcal{S}H\chi_k, \overline{\chi_{j+1}}\rangle = \overline{\mu}\mu\langle AU_{\overline{\mu}}S\chi_k, \overline{\chi_{j+1}}\rangle = \langle HS\chi_k, \overline{\chi_{j+1}}\rangle.$$

It follows by the continuity that $P_-\mathcal{S}H = HS$, and therefore (see, e.g., [13, Theorem 1.1.8]) H is Hankel, $H = H_\psi$, $\psi \in L^\infty$.

Moreover,

$$\widehat{\psi}(-n) = \langle H_\psi \chi_n, 1\rangle = \langle AU_{\overline{\mu}}\chi_n, 1\rangle = \overline{\mu}^n\langle A\chi_n, \chi_0\rangle = \alpha_n$$

for $n \in \mathbb{Z}_+$.

(4)⇒(1) If $A = H_\psi U_\mu$, then it is obvious that the operator A is bounded. Let us find its matrix in standard bases. Taking into account that $U_\mu \chi_k = \mu^k \chi_k$, we have

$$\langle H_\psi U_\mu \chi_k, \overline{\chi_{j+1}}\rangle = \mu^k\langle H_\psi \chi_k, \overline{\chi_{j+1}}\rangle = \mu^k\widehat{\psi}(-k-j-1)$$

and (1) follows. □

The previous theorem allows, in the case $|\mu| = 1$, to derive a number of properties of μ-Hankel operators from the corresponding properties of Hankel ones. Here are some examples.

Corollary 3.2. *Let $|\mu| = 1$, and A is μ-Hankel, $A : H^2 \to H^2_-$. Operator A is compact if and only if it can be represented in the form $A = H_\psi U_\mu$, where $\psi \in C(\mathbb{T}) + H^\infty(\mathbb{T})$.*

Proof. By Theorem 3.1, $A = H_\psi U_\mu$. Since U_μ is unitary, operator A is compact if and only if H_ψ is compact. So, the corollary follows from the Hartman's Theorem (see, e.g., [13, Theorem 1.5.5]). □

Corollary 3.3. *Let $|\mu| = 1$, and A is μ-Hankel, $A : H^2 \to H^2_-$. Then A is an operator of finite rank if and only if $A = H_\psi U_\mu$ and $P_-\psi$ is a rational function. Moreover,* $\mathrm{rank}A = \mathrm{deg}P_-\psi$.

Proof. In the notation of Theorem 3.1 operator A is an operator of finite rank if and only if $H_\psi = AU_\mu^{-1}$ is of finite rank. This is equivalent to the fact that $P_-\psi$ is a rational function (see, e.g., [13, Corollary 1.3.2]) and $\mathrm{rank}A = \mathrm{rank}H_\psi = \mathrm{deg}P_-\psi$, as required. □

Corollary 3.4. *Let $|\mu| = 1$, and operators $A, B : H^2 \to H^2_-$ are μ-Hankel and bounded, $A = H_\varphi U_\mu$, $B = H_\psi U_\mu$, $\varphi, \psi \in L^\infty$. Then the operator A^*B is unitarily equivalent to the semi-commutator $[T_{\overline{\varphi}}, T_\psi) := T_{\overline{\varphi}\psi} - T_\varphi T_\psi$ of Toeplitz operators. In particular, the operator A^*B is compact if φ or ψ belongs to $H^\infty(\mathbb{T}) + C(\mathbb{T})$.*

Proof. We have

$$A^*B = U_\mu^* H_\varphi^* H_\psi U_\mu = U_\mu^{-1} H_\varphi^* H_\psi U_\mu, \ \varphi, \psi \in L^\infty.$$

It remains to use the fact that the operator $H_\varphi^* H_\psi$ is equal to $[T_{\overline{\varphi}}, T_\psi)$ (see, e.g., [13, p. 89, (1.5)]). The compactness statement now follows from the corresponding property of Toeplitz operators (see, e.g.,[11, p. 253]). □

Corollary 3.5. *Let $|\mu| = 1$, $\varphi \in L^\infty$. The following statements are equivalent for the operator $A = V_\mu H_\varphi$:*

(1) *A has a non-trivial kernel,*
(2) *the image* $\mathrm{Im}A$ *of the operator A is not dense in H^2_-,*
(3) *$\varphi = \theta\varphi_1$ for some inner function θ and function φ_1 from H^∞.*

Proof. By virtue of the unitarity of the operator V_μ, the fulfillment of properties (1) (or (2)) for the operator A is equivalent to the fulfillment of the corresponding property for the operator H_φ. The equivalence of statements (1)–(3) now follows from [13, Theorem 1.2.3]. □

Remark 3.6. Let $|\mu| = 1$. Consider the unitary operator $W_\mu f(z) := f(\mu z)$ in $L^2(\mathbb{T})$. The restrictions $U_\mu := W_\mu|H^2(\mathbb{T})$ and $V_\mu := W_\mu|H^2_-(\mathbb{T})$ are unitary in $H^2(\mathbb{T})$ and $H^2_-(\mathbb{T})$ respectively. Let $\varphi \in L^\infty(\mathbb{T})$ and T_φ denotes the Toeplitz operator with symbol φ. Then the operator $T_{\mu,\varphi} := U_\mu T_\varphi$ is called μ-Toeplitz ([5], [7, Theorem 2.5]). On the other hand, the operator $A_\mu = V_\mu H_\varphi$ is μ-Hankel by Theorem 4. It is easy to verify that

$$T_{\mu,\varphi} f + A_\mu f = W_\mu M_\varphi f, \quad f \in H^2(\mathbb{T}),$$

where M_φ stands for multiplication by φ on $L^2(\mathbb{T})$. This is the simplest relation between μ-Hankel and μ-Toeplitz operators.

4. INTEGRAL REPRESENTATIONS

In this section, we will consider two classes of integral operators that are μ-Hankel.

4.1. μ-HANKEL OPERATORS AS INTEGRAL OPERATORS ON THE UNIT DISK

Let $\mu \in \mathbb{C}$, $\mu \neq 0$, and σ is a bounded (generally speaking, complex) measure on the closed unit disk $\overline{\mathbb{D}}$ in complex plane. In the case $|\mu| < 1$, we will assume that σ is concentrated on the set $\{|\zeta| \leq |\mu|\}$. Consider the operator

$$\Gamma_{\mu,\sigma} f(z) := \mu \int\limits_{\overline{\mathbb{D}}} \frac{f(\zeta)}{\mu - \zeta z} d\sigma(\zeta) \quad (|z| < 1)$$

and the sequence of moments of the measure σ

$$\gamma_n := \int\limits_{\overline{\mathbb{D}}} \zeta^n d\sigma(\zeta) \quad (n \in \mathbb{Z}_+).$$

Theorem 4.1. *For the operator $\Gamma_{\mu,\sigma}$ to be bounded in the Hardy space $H^2(\mathbb{D})$, the condition*

$$\sup_{k \in \mathbb{Z}_+} \sum_{j=0}^{\infty} \left| \frac{\gamma_{k+j}}{\mu^j} \right|^2 < \infty \tag{4.1}$$

is necessary. Under this condition, this operator is μ-Hankel in $H^2(\mathbb{D})$, has the matrix $(\gamma_{k+j}/\mu^j)_{k,j \in \mathbb{Z}_+}$ with respect to the standard basis of this space, and the following statements are true.

(1) *Let $|\mu| < 1$. Then the operator $\Gamma_{\mu,\sigma}$ is nuclear and*

$$\mathrm{tr}\Gamma_{\mu,\sigma} = \sum_{n=0}^{\infty} \frac{\gamma_{2n}}{\mu^n} = \mu \int\limits_{\overline{\mathbb{D}}} \frac{d\sigma(\zeta)}{\mu - \zeta^2}. \tag{4.2}$$

(2) *Let $|\mu| > 1$. Then the operator $\Gamma_{\mu,\sigma}$ is bounded if and only if $(\gamma_n) \in \ell^2(\mathbb{Z}_+)$. Moreover, it is nuclear, and its trace is expressed by the formula (4.2).*

(3) *Let $|\mu| = 1$. Operator $\Gamma_{\mu,\sigma}$ is bounded if and only if there is such function $\psi \in L^\infty(\mathbb{T})$, that $\gamma_n = \widehat{\psi}(n)$ for $n \in \mathbb{Z}_+$. Moreover*

$$\|\Gamma_{\mu,\sigma}\| = \inf\{\|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \gamma_n = \widehat{\psi}(n) \quad \forall n \in \mathbb{Z}_+\}.$$

Proof. Consider the standard orthonormal basis $\chi_n(z) = z^n$ $(n \in \mathbb{Z}_+)$ in $H^2(\mathbb{D})$. Taking into account that $|\zeta| \leq |\mu|$ and $|z| < 1$, we have

$$\begin{aligned} \Gamma_{\mu,\sigma}\chi_k(z) &= \mu \int\limits_{\overline{\mathbb{D}}} \frac{\zeta^k}{\mu - \zeta z} d\sigma(\zeta) = \int\limits_{\overline{\mathbb{D}}} \frac{\zeta^k}{1 - \frac{\zeta z}{\mu}} d\sigma(\zeta) = \int\limits_{\overline{\mathbb{D}}} \zeta^k \sum_{j=0}^{\infty} \left(\frac{\zeta z}{\mu}\right)^j d\sigma(\zeta) \\ &= \sum_{j=0}^{\infty} \frac{z^j}{\mu^j} \int\limits_{\overline{\mathbb{D}}} \zeta^{k+j} d\sigma(\zeta) = \sum_{j=0}^{\infty} z^j \frac{\gamma_{k+j}}{\mu^j} = \sum_{j=0}^{\infty} \frac{\gamma_{k+j}}{\mu^j} \chi_j(z). \end{aligned} \tag{4.3}$$

(The term-by-term integration of the series is legal, since for all $k \in \mathbb{Z}_+$

$$\sum_{j=0}^{\infty} \int_{\overline{\mathbb{D}}} \left| z^j \frac{\zeta^{k+j}}{\mu^j} \right| d|\sigma|(\zeta) \leq \sum_{j=0}^{\infty} |z|^j \int_{\overline{\mathbb{D}}} d|\sigma|(\zeta) < \infty.)$$

Thus, in order for the operator $\Gamma_{\mu,\sigma}$ to act and to be bounded in $H^2(\mathbb{D})$, it is necessary for the sequence $(\gamma_{k+j}/\mu^j)_{j\in\mathbb{Z}_+}$ to belong to $\ell^2(\mathbb{Z}_+)$ for all $k \geq 0$ and the equality

$$\sum_{j=0}^{\infty} \left| \frac{\gamma_{k+j}}{\mu^j} \right|^2 = \|\Gamma_{\mu,\sigma}\chi_k\|^2 \leq \|\Gamma_{\mu,\sigma}\|^2.$$

to be valid. This proves the necessity of the condition (4.1). Under this condition, the operator $\Gamma_{\mu,\sigma}$ has a matrix

$$a_{jk} = \frac{\gamma_{k+j}}{\mu^j} = \mu^k \alpha_{k+j}, \text{ where } \alpha_n = \frac{\gamma_n}{\mu^n},$$

with respect to the basis $(\chi_n)_{n\geq 0}$, which proves the first assertion of the theorem.

Now, by virtue of Theorem 2.5, we can assert the following.

(1) If $|\mu| < 1$, then the operator $\Gamma_{\mu,\sigma}$ is bounded if (and only if) $(\alpha_n)_{n\geq 0} = (\gamma_n/\mu^n)_{n\geq 0} \in \ell^2(\mathbb{Z}_+)$, which is true by virtue of (4.1). Moreover, this operator is nuclear, and

$$\begin{aligned} \mathrm{tr}\Gamma_{\mu,\sigma} &= \sum_{n=0}^{\infty} \mu^n \alpha_{2n} = \sum_{n=0}^{\infty} \frac{\gamma_{2n}}{\mu^n} = \sum_{n=0}^{\infty} \frac{1}{\mu^n} \int_{\overline{\mathbb{D}}} \zeta^{2n} d\sigma(\zeta) \\ &= \int_{\{|\zeta|\leq|\mu|\}} \sum_{n=0}^{\infty} \left(\frac{\zeta^2}{\mu} \right)^n d\sigma(\zeta) = \mu \int_{\overline{\mathbb{D}}} \frac{d\sigma(\zeta)}{\mu - \zeta^2}. \end{aligned}$$

The term-by-term integration of the series is legal, since

$$\sum_{n=0}^{\infty} \int_{\{|\zeta|\leq|\mu|\}} \left| \frac{\zeta^2}{\mu} \right|^n d|\sigma|(\zeta) = \sum_{n=0}^{\infty} |\mu|^n \int_{\{|\zeta|\leq|\mu|\}} \left| \frac{\zeta}{\mu} \right|^{2n} d|\sigma|(\zeta) \leq \frac{|\sigma|(\overline{\mathbb{D}})}{1 - |\mu|} < \infty.$$

(2) If $|\mu| > 1$, then $\Gamma_{\mu,\sigma}$ is bounded if and only if $(\mu^n \alpha_n) = (\gamma_n) \in \ell^2(\mathbb{Z}_+)$. Moreover, this operator is nuclear, and

$$\mathrm{tr}\Gamma_{\mu,\sigma} = \sum_{n=0}^{\infty} \mu^n \alpha_{2n} = \sum_{n=0}^{\infty} \frac{\gamma_{2n}}{\mu^n} = \sum_{n=0}^{\infty} \frac{1}{\mu^n} \int_{\overline{\mathbb{D}}} \zeta^{2n} d\sigma(\zeta) = \mu \int_{\overline{\mathbb{D}}} \frac{d\sigma(\zeta)}{\mu - \zeta^2}.$$

(The term-by-term integration of the series is legal, since

$$\sum_{n=0}^{\infty} \int_{\overline{\mathbb{D}}} \left| \frac{\zeta^{2n}}{\mu^n} \right| d|\sigma|(\zeta) \leq \sum_{n=0}^{\infty} \left| \frac{1}{\mu} \right|^n \int_{\overline{\mathbb{D}}} d|\sigma|(\zeta) < \infty.)$$

(3) Since in our case $\alpha_n = \gamma_n/\mu^n$, this follows from part (3) of Theorem 2.5 with $A\mu = \Gamma_{\mu,\sigma}$, which completes the proof. $\square$

For $|\mu| = 1$, the sufficient boundedness condition gives the next

Corollary 4.2. *Let $|\mu| = 1$. If the function*

$$\varphi_\sigma(\zeta) := \int\limits_{\overline{\mathbb{D}}} \frac{1-|z|^2}{|z-\zeta|^2} d\sigma(z) \quad (\zeta \in \mathbb{T})$$

belongs to $L^\infty(\mathbb{T})$, the operator $\Gamma_{\mu,\sigma}$ is bounded in $H^2(\mathbb{D})$ and

$$\|\Gamma_{\mu,\sigma}\| \le \|\varphi_\sigma\|_{L^\infty}.$$

Proof. We check the fulfillment of the conditions of Theorem 4.1. It is shown in [11, p. 314] that $\gamma_n = \widehat{\varphi_\sigma}(-n)$ for $n \in \mathbb{Z}_+$. Therefore we have by Parseval's equality that for all $k \in \mathbb{Z}_+$

$$\sum_{j=0}^{\infty} \left| \frac{\gamma_{k+j}}{\mu^j} \right|^2 = \sum_{j=0}^{\infty} |\gamma_{k+j}|^2 = \sum_{j=0}^{\infty} |\widehat{\varphi_\sigma}(-k-j)|^2 \le \sum_{n=-\infty}^{\infty} |\widehat{\varphi_\sigma}(n)|^2 = \|\varphi_\sigma\|^2_{L^2},$$

and therefore the condition (4.1) is satisfied. Moreover, $\gamma_n = \widehat{\varphi^\flat_\sigma}(n)$ for $n \in \mathbb{Z}_+$, where $\varphi^\flat(\zeta) := \varphi(\zeta^{-1}) = \varphi(\overline{\zeta})$, and $\varphi^\flat_\sigma \in L^\infty(\mathbb{T})$. Now it remains to apply assertion (3) of Theorem 4.1. □

Corollary 4.3. *Let the condition* (4.1) *be satisfied. The operator $\Gamma_{\mu,\sigma}$ has finite rank if and only if the function $\Gamma_{\mu,\sigma}1$ is rational. In this case* $\operatorname{rank}\Gamma_{\mu,\sigma} = \deg(z(\Gamma_{\mu,\sigma}1)(z))$.

Proof. As was shown in the proof of the previous theorem, $\alpha_n = \gamma_n/\mu^n$ for the operator $\Gamma_{\mu,\sigma}$. By Theorem 2.8, this operator has finite rank if and only if the function

$$\sum_{j=0}^{\infty} \alpha_j z^j = \sum_{j=0}^{\infty} \frac{\gamma_j}{\mu^j} z^j = \int\limits_{\overline{\mathbb{D}}} \sum_{j=0}^{\infty} \left(\frac{\zeta z}{\mu}\right)^j d\sigma(\zeta) = \Gamma_{\mu,\sigma}1$$

is rational (the validity of the term-by-term integration of the series was substantiated in the proof of Theorem 4.1). The second statement of the corollary now also follows from Theorem 2.8. □

We are going to show that the μ-Hankel operators are related to the complex moment problem (for the latter see [2, p. 117]). The main result for the unit disk states ([2, p. 117, Theorem 4.4.12]) that for a sequence $\gamma : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ there is a bounded positive measure σ on $\overline{\mathbb{D}}$ such that

$$\gamma(n,m) = \int\limits_{\overline{\mathbb{D}}} \zeta^n \overline{\zeta}^m d\sigma(\zeta), \ (n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \tag{4.4}$$

if and only if γ is bounded and positive definite on $\mathbb{Z}_+ \times \mathbb{Z}_+$ (this means that quadratic forms $\sum_{j,k=1}^n c_j \overline{c_k} \gamma(s_j + s_k)$ are positive definite for all $n \in \mathbb{N}$, $\{s_1, \dots, s_n\} \subset \mathbb{Z}_+ \times \mathbb{Z}_+$ [2, p. 87]).

The following proposition is a partial converse of Theorem 4.1.

Theorem 4.4. *Let a sequence $\gamma : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{C}$ be bounded and positive definite, $\gamma_n = \gamma(n, 0)$, and $|\mu| \geq 1$. Then there is a bounded positive measure σ on $\mathbb{D}$ such that the matrix $(\gamma_{k+j}/\mu^j)_{k,j\in\mathbb{Z}_+}$ corresponds to the operator $\Gamma_{\mu,\sigma}$ in $H^2(\mathbb{D})$ with respect to the standard basis of this space.*

Proof. Indeed, formula (4.4) implies that there is a bounded positive measure σ on $\mathbb{D}$ such that

$$\gamma_n = \int\limits_{\overline{\mathbb{D}}} \zeta^n d\sigma(\zeta), \; n \in \mathbb{Z}_+.$$

Since $|\mu| \geq 1$, the calculations (4.3) for the corresponding operator $\Gamma_{\mu,\sigma}$ are valid. □

We finish this section with considering the adjoint operator for $\Gamma_{\mu,\sigma}$.

Lemma 4.5. *If the operator $\Gamma_{\mu,\sigma}$ is bounded in $H^2(\mathbb{D})$, then the adjoint operator has the form*

$$\Gamma^*_{\mu,\sigma} f(z) = \int\limits_{\overline{\mathbb{D}}} \frac{f(\zeta/\mu)}{1-\zeta z} d\sigma(\zeta) \quad (|z| < 1).$$

Proof. It was shown at the beginning of the proof of Theorem 4.1 that

$$\langle \Gamma_{\mu,\sigma}\chi_k, \chi_n \rangle = \gamma_{k+n}/\mu^n \quad k, n \in \mathbb{Z}_+.$$

On the other hand, for all $n \in \mathbb{Z}_+$,

$$\Gamma^*_{\mu,\sigma}\chi_n(z) = \int\limits_{\overline{\mathbb{D}}} \frac{(\zeta/\mu)^n}{1-\zeta z} d\sigma(\zeta) = \frac{1}{\mu^n} \int\limits_{\overline{\mathbb{D}}} \zeta^n \left(\sum_{j=0}^{\infty} \zeta^j z^j \right) d\sigma(\zeta)$$
$$= \sum_{j=0}^{\infty} \frac{1}{\mu^n} \left(\int\limits_{\overline{\mathbb{D}}} \zeta^{n+j} d\sigma(\zeta) \right) z^j = \sum_{j=0}^{\infty} \frac{\gamma_{n+j}}{\mu^n} \chi_j(z).$$

The validity of the term-by-term integration of the series here follows from the fact that for $|\zeta| \leq 1$, $|\zeta/\mu| \leq 1$, $|z| < 1$ we have the estimate

$$\sum_{j=0}^{\infty} \int\limits_{\overline{\mathbb{D}}} \frac{1}{|\mu|^n} |\zeta|^{n+j} |z|^j d|\sigma|(\zeta) \leq \sum_{j=0}^{\infty} |z|^j \int\limits_{\overline{\mathbb{D}}} |\sigma|(\zeta) < \infty.$$

Thus, $\langle \Gamma_{\mu,\sigma}\chi_k, \chi_n \rangle = \langle \chi_k, \Gamma^*_{\mu,\sigma}\chi_n \rangle$ for all $k, n \in \mathbb{Z}_+$. Since the operators $\Gamma_{\mu,\sigma}$ and $\Gamma^*_{\mu,\sigma}$ are bounded, the Lemma follows. □

Theorem 4.1 and the standard facts about the adjoint operator directly imply the following

Theorem 4.6. *For the adjoint operator $\Gamma^*_{\mu,\sigma}$ to be bounded in the Hardy space $H^2(\mathbb{D})$, it is necessary that condition (4.1) was met. Under this condition, this operator is $1/\overline{\mu}$-Hankel in $H^2(\mathbb{D})$ with matrix $(\overline{\gamma_{k+j}}/\overline{\mu}^k)_{k,j\in\mathbb{Z}_+}$ with respect to the basis $(\chi_n)_{n\geq 0}$, the standard basis of this space, and the following statements are true.*

(1) *Let* $|\mu| < 1$. *Then the operator* $\Gamma^*_{\mu,\sigma}$ *is nuclear and*

$$\operatorname{tr}\Gamma^*_{\mu,\sigma} = \sum_{n=0}^{\infty} \frac{\overline{\gamma_{2n}}}{\overline{\mu}^n} = \overline{\mu} \int\limits_{\overline{\mathbb{D}}} \frac{d\overline{\sigma}(\zeta)}{\overline{\mu} - \overline{\zeta}^2}. \tag{4.5}$$

(2) *Let* $|\mu| > 1$. *In this case, the operator* $\Gamma^*_{\mu,\sigma}$ *is bounded if and only if* $(\gamma_n) \in \ell^2(\mathbb{Z}_+)$. *Moreover, this operator is nuclear, and its trace is expressed by the formula* (4.5).

(3) *Let* $|\mu| = 1$. *The operator* $\Gamma^*_{\mu,\sigma}$ *is bounded if and only if there is such a function* $\psi \in L^\infty(\mathbb{T})$ *that* $\gamma_n = \widehat{\psi}(n)$ *for* $n \in \mathbb{Z}_+$. *In addition*

$$\|\Gamma^*_{\mu,\sigma}\| = \inf\{\|\psi\|_{L^\infty} : \psi \in L^\infty(\mathbb{T}), \gamma_n = \widehat{\psi}(n)\, \forall n \in \mathbb{Z}_+\}.$$

Corollary 4.7. *Let condition* (4.1) *be satisfied. Operator* $\Gamma^*_{\mu,\sigma}$ *has a finite rank if and only if the function* $\Gamma^*_{\mu,\sigma}1$ *is rational. Moreover,* $\operatorname{rank}\Gamma^*_{\mu,\sigma} = \deg(z(\Gamma^*_{\mu,\sigma}1)(z))$.

The proof is similar to that of Corollary 4.3.

Remark 4.8. For the case when the measure σ is concentrated on the segment $[0, 1]$, operators of the form $\Gamma_{\mu,\sigma}$ were considered in [9, 10].

4.2. μ-HANKEL OPERATORS AS INTEGRAL OPERATORS ON THE SEMI-AXIS

In this subsection, we consider a certain class of μ-Hankel integral operators in the space $L^2(\mathbb{R}_+)$.

It is known (see, e.g., [15, p. 193]) that the functions

$$l_n(t) = -\imath \mathrm{L}_n(t) e^{-t/2}, \quad n \in \mathbb{Z}_+,$$

where $\mathrm{L}_n = \mathrm{L}^0_n$ are Laguerre polynomials, form an orthonormal basis of the space $L^2(\mathbb{R}_+)$. It is also clear that the functions l_n are bounded on $\mathbb{R}_+$. In what follows, we will assume that $|\mu| < 1$ and put

$$k_\mu(x, t) := \sum_{n \in \mathbb{Z}_+} \mu^n l_n(t) \overline{l_n(x)}. \tag{4.6}$$

Since the series $\sum_{n\in\mathbb{Z}_+} \mu^n l_n(t)$ absolutely converges for $|\mu| < 1$ and each t (see, e.g., [3], [6, Chapter XI, B]), this definition is correct, the series (4.6) absolutely converges in the norm of $L^2(\mathbb{R}_+, dx)$ for every fixed t, and therefore the function k_μ belongs to $L^2(\mathbb{R}_+)$ for each variable separately.

For a function $a \in L^2(\mathbb{R}_+)$ (which we consider to be independent of μ) we put

$$K_\mu(x, t) := \int\limits_{\mathbb{R}_+} a(x + y) k_\mu(y, t) dy.$$

Then the integral operator

$$\mathbf{A}_\mu f(x) := \int\limits_{\mathbb{R}_+} K_\mu(x,t)f(t)dt$$

is μ-Hankel in $L^2(\mathbb{R}_+)$. In order to check this, consider the operator

$$\mathbf{H}f(x) := \int\limits_{\mathbb{R}_+} a(x+y)f(y)dy.$$

It is known (see, e.g., [13]), that it is Hankel in $L^2(\mathbb{R}_+)$. Moreover, from the results of the work [15] (see also [13, remark after the Theorem 1.8.9]) it follows that it has Hankel matrix with respect to the basis $(l_n)_{n\in\mathbb{Z}_+}$. Indeed, it is shown in [15, p. 200] that the operator $\mathbf{H}$ is unitarily equivalent to H in $\ell^2(\mathbb{Z}_+)$, $\mathbf{H} = \mathcal{L}H\mathcal{L}^*$, where the operator $\mathcal{L} : L^2(\mathbb{R}_+) \to \ell^2(\mathbb{Z}_+)$ is unitary, and the operator H has Hankel matrix with respect to the standard basis $(e_n)_{n\in\mathbb{Z}_+}$ of the space $\ell^2(\mathbb{Z}_+)$ [15, (1.1)]. Moreover, it follows from [15, (2.27)] that $\mathcal{L}l_n = e_n$. Hence the quantity $\langle \mathbf{H}l_m, l_n\rangle = \langle He_m, e_n\rangle$ depends on $m+n$ only (and does not depend on μ).

Further, the integral operator

$$U_\mu f(x) := \int\limits_{\mathbb{R}_+} k_\mu(x,t)f(t)dt$$

has the matrix $\mathrm{diag}(1,\mu,\mu^2,\ldots)$ with respect to the basis $(l_n)_{n\in\mathbb{Z}_+}$ of $L^2(\mathbb{R}_+)$, and thus this operator is defined correctly and bounded. Moreover, the next lemma holds.

Lemma 4.9. *If $a \in L^2(\mathbb{R}_+)$ and $|\mu| < 1$, then*

$$\mathbf{A}_\mu = \mathbf{H}U_\mu,$$

and this operator is μ-Hankel.

Proof. Note that the Cauchy–Schwartz–Bunyakovskii inequality implies that

$$\int\limits_{\mathbb{R}_+} |k_\mu(y,t)f(t)|dt \le \|k_\mu(y,\cdot)\|_{L^2}\|f\|_{L^2}.$$

for all $f \in L^2(\mathbb{R}_+)$. In turn,

$$\|k_\mu(y,\cdot)\|_{L^2} \le \sum_{n\in\mathbb{Z}_+} |\mu|^n \|l_n\|_{L^2}|l_n(y)| = \sum_{n\in\mathbb{Z}_+} |\mu|^n|l_n(y)|.$$

Therefore

$$\int\limits_{\mathbb{R}_+} |k_\mu(y,t)f(t)|dt \le \sum_{n\in\mathbb{Z}_+} |\mu|^n|l_n(y)|\|f\|_{L^2}.$$

So, again by the Cauchy–Schwartz–Bunyakovskii inequality, we have

$$\begin{aligned}\int\limits_{\mathbb{R}_+} |a(x+y)| \int\limits_{\mathbb{R}_+} |k_\mu(y,t)f(t)|dtdy &\le \int\limits_{\mathbb{R}_+} |a(x+y)| \left(\sum_{n\in\mathbb{Z}_+} |\mu|^n |l_n(y)|\right) dy\|f\|_{L^2} \\ &= \sum_{n\in\mathbb{Z}_+} |\mu|^n \int\limits_{\mathbb{R}_+} |a(x+y)||l_n(y)|dy\|f\|_{L^2} \\ &\le \sum_{n\in\mathbb{Z}_+} |\mu|^n \|a(x+\cdot)\|_{L^2}\|l_n\|_{L^2}\|f\|_{L^2} < \infty.\end{aligned}$$

This justifies the application of the Fubini theorem in the following calculations:

$$\begin{aligned}\mathbf{A}_\mu f(x) &= \int\limits_{\mathbb{R}_+} \left(\int\limits_{\mathbb{R}_+} a(x+y)k_\mu(y,t)dy\right) f(t)dt \\ &= \int\limits_{\mathbb{R}_+} a(x+y) \int\limits_{\mathbb{R}_+} k_\mu(y,t)f(t)dtdy = \mathbf{H}U_\mu f(x).\end{aligned}$$

By virtue of this equality, the matrix elements $\langle \mathbf{A}_\mu l_m, l_n\rangle = \mu^m\langle \mathbf{H} l_m, l_n\rangle$ of the operator $\mathbf{A}_\mu$ have the form $\mu^m\alpha_{m+n}$, where $\alpha_{m+n} = \langle He_m, e_n\rangle$, and thus this operator is μ-Hankel. This completes the proof. □

Theorem 4.10. *An operator $\mathbf{A}_\mu$ is nuclear in $L^2(\mathbb{R}_+)$ if, in addition to the conditions of Lemma 4.9, we require that $a = \mathcal{F}\kappa|(0,\infty)$ for some $\kappa \in L^\infty(\mathbb{R})$, where $\mathcal{F}$ denotes the Fourier transform in a sense of distributions. Moreover, its Hilbert–Schmidt norm and trace are given by the formulas*

$$\|\mathbf{A}_\mu\|_{\mathrm{S}_2} = \left(\int\limits_{\mathbb{R}_+}\int\limits_{\mathbb{R}_+} |K_\mu(t,s)|^2 dtds\right)^{1/2}, \ \mathrm{tr}\mathbf{A}_\mu = \int\limits_{\mathbb{R}_+} K_\mu(t,t)dt.$$

Furthermore, $\|\mathbf{A}_\mu\| \le \|\mathbf{H}\|$.

Proof. It follows from [13, Theorem 1.8.8], that the operator $\mathbf{H}$ is bounded, and thus the operator $\mathbf{A}_\mu$ is bounded by Lemma 4.9, as well. Since $|\mu| < 1$, the nuclearity of this operator immediately follows from Theorem 2.5. The formula for the Hilbert–Schmidt norm of such operators is classical (see, e.g., [1, Chapter II, item 32]).

Further,

$$\begin{aligned}\mathrm{tr}\mathbf{A}_\mu &= \sum_{n\in\mathbb{Z}_+} \langle \mathbf{A}_\mu l_n, l_n\rangle = \sum_{n\in\mathbb{Z}_+} \mu^n \langle \mathbf{H} l_n, l_n\rangle \\ &= \sum_{n\in\mathbb{Z}_+} \mu^n \int\limits_{\mathbb{R}_+} \left(\int\limits_{\mathbb{R}_+} a(t+y)l_n(y)dy\right) \overline{l_n(t)}dt.\end{aligned} \tag{4.7}$$

Since by the Cauchy–Schwartz–Bunyakovskii inequality

$$\begin{aligned}\sum_{n\in\mathbb{Z}_+}\int_{\mathbb{R}_+}|\mu|^n\left|\int_{\mathbb{R}_+}a(t+y)l_n(y)dy\right||l_n(t)|dt &= \sum_{n\in\mathbb{Z}_+}|\mu|^n\int_{\mathbb{R}_+}|(\mathbf{H}l_n)(t)||l_n(t)|dt\\ &\le \sum_{n\in\mathbb{Z}_+}|\mu|^n\|\mathbf{H}l_n\|_{L^2}\|l_n\|_{L^2}\\ &\le \sum_{n\in\mathbb{Z}_+}|\mu|^n\|\mathbf{H}\|<\infty,\end{aligned}$$

formula (4.7) implies that

$$\mathrm{tr}\mathbf{A}_\mu=\int_{\mathbb{R}_+}\left(\sum_{n\in\mathbb{Z}_+}\mu^n\int_{\mathbb{R}_+}a(t+y)l_n(y)dy\right)\overline{l_n(t)}dt. \tag{4.8}$$

In turn, since (again by the Cauchy–Schwartz–Bunyakovskii inequality)

$$\begin{aligned}\sum_{n\in\mathbb{Z}_+}|\mu|^n\int_{\mathbb{R}_+}|a(t+y)||l_n(y)|dy &\le \sum_{n\in\mathbb{Z}_+}|\mu|^n\|a(t+\cdot)\|_{L^2}\|l_n\|_{L^2}\\ &\le \sum_{n\in\mathbb{Z}_+}|\mu|^n\|a\|_{L^2}<\infty,\end{aligned}$$

formula (4.8) implies that

$$\begin{aligned}\mathrm{tr}\mathbf{A}_\mu &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+}a(t+y)\sum_{n\in\mathbb{Z}_+}\mu^n l_n(y)\overline{l_n(t)}dydt\\ &= \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}a(t+y)k_\mu(y,t)dy\right)dt=\int_{\mathbb{R}_+}K_\mu(t,t)dt.\end{aligned}$$

Finally, the last inequality follows from Lemma 4.9 and the obvious equality $\|U_\mu\|=1$. □

Corollary 4.11. *Let the operator* $\mathbf{H}$ *be bounded. If the function* a *has the form*

$$a(t)=\sum_{j=1}^{n}\sum_{l=0}^{n_j-1}c_{j,l}t^l e^{\lambda_j t},$$

where $\mathrm{Re}\lambda_j<0$, *then the operator* $\mathbf{A}_\mu$ *is finite-dimensional.*

Proof. It follows from [13, Theorem 1.8.13] that the operator $\mathbf{H}$ is finite-dimensional. To finish the proof it remains to use Lemma 4.9. □

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