

γ -PAIRED DOMINATING GRAPHS OF CYCLES

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Abstract. A paired dominating set of a graph G is a dominating set whose induced subgraph contains a perfect matching. The paired domination number, denoted by $\gamma_{pr}(G)$, is the minimum cardinality of a paired dominating set of G . A $\gamma_{pr}(G)$ -set is a paired dominating set of cardinality $\gamma_{pr}(G)$. The γ -paired dominating graph of G , denoted by $PD_{\gamma}(G)$, is the graph whose vertices are $\gamma_{pr}(G)$ -sets. Two $\gamma_{pr}(G)$ -sets D_1 and D_2 are adjacent in $PD_{\gamma}(G)$ if there exists a vertex $u \in D_1$ and a vertex $v \notin D_1$ such that $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$. In this paper, we present the γ -paired dominating graphs of cycles.

Keywords: paired dominating graph, paired dominating set, paired domination number.

Mathematics Subject Classification: 05C69, 05C38.

1. INTRODUCTION

For notation and terminology, we refer the reader to [9]. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $v \in V(G)$, the *open neighborhood* and *closed neighborhood* of v are denoted by $N(v)$ and $N[v]$, respectively. For a set $D \subseteq V(G)$, the *open neighborhood* of D is $N(D) = \bigcup_{v \in D} N(v)$, and the *closed neighborhood* of D is $N[D] = N(D) \cup D$. The subgraph of G induced by D is denoted by $G[D]$. The vertices in D *dominate* the vertices in $S \subseteq V(G)$ if $S \subseteq N[D]$. We denote the graph obtained from G by deleting all vertices in D and all edges incident with them by $G - D$. A *path*, a *cycle*, and a *complete graph* with n vertices are denoted by P_n , C_n , and K_n , respectively.

A set $D \subseteq V(G)$ is a *dominating set* of G if $N[D] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G . A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$ -*set*. For a detailed literature on domination, see [5, 6].

The *gamma graph* $\gamma.G$ of a graph G , defined by Lakshmanan and Vijayakumar [7], is the graph whose vertices are $\gamma(G)$ -sets, and $\gamma(G)$ -sets D_1 and D_2 are adjacent in $\gamma.G$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. In 2011, Fricke *et al.* [2]

also defined the *gamma graph* $G(\gamma)$ with different meaning. The only difference is that two $\gamma(G)$ -sets D_1 and D_2 are adjacent in $G(\gamma)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1, v \notin D_1$, and $uv \in E(G)$. Notice that $G(\gamma)$ is a subgraph of $\gamma.G$ with the same vertex set. In 2014, Haas and Seyffarth [3] introduced the *k-dominating graph* of G , denoted by $D_k(G)$, as the graph whose vertices are dominating sets of cardinality at most k . Two dominating sets D_1 and D_2 are adjacent in $D_k(G)$ if $D_2 = D_1 \cup \{v\}$ for some $v \notin D_1$. They gave conditions that ensure $D_k(G)$ is connected.

In 2017, Wongsriya and Trakultraipruk [10] defined the *γ -total dominating graph* of G , denoted by $TD_\gamma(G)$, as the graph whose vertices are $\gamma_t(G)$ -sets, which are total dominating sets of minimum cardinality. Two $\gamma_t(G)$ -sets D_1 and D_2 are adjacent in $TD_\gamma(G)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. They determined the γ -total dominating graphs of paths and cycles. In 2019, Samanmoo *et al.* [8] introduced the *γ -independent dominating graph* of G , denoted by $ID_\gamma(G)$, as the graph whose vertices are $\gamma_i(G)$ -sets, which are independent dominating sets of minimum cardinality. Two $\gamma_i(G)$ -sets D_1 and D_2 are adjacent in $ID_\gamma(G)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1, v \notin D_1$. They provided the γ -independent dominating graphs of paths and cycles.

A *matching* in G is a set of independent edges in G . A *perfect matching* M in G is a matching such that every vertex of G is incident to an edge of M . A set $D \subseteq V(G)$ is a *paired dominating set* of G if it is a dominating set and the induced subgraph $G[D]$ has a perfect matching. The set $\{u, v\} \subseteq D$ is called *paired* if uv is an edge in a perfect matching of $G[D]$. The *paired domination number* $\gamma_{pr}(G)$ is the minimum cardinality of a paired dominating set of G . A *$\gamma_{pr}(G)$ -set* is a paired dominating set of cardinality $\gamma_{pr}(G)$. Paired domination was introduced by Haynes and Slater [4] as a model for assigning backups to guards for security purposes.

In [1], we introduced the *γ -paired dominating graph* of G , denoted by $PD_\gamma(G)$, as the graph whose vertices are $\gamma_{pr}(G)$ -sets, and two $\gamma_{pr}(G)$ -sets D_1 and D_2 are adjacent in $PD_\gamma(G)$ if $D_2 = (D_1 \setminus \{u\}) \cup \{v\}$ for some $u \in D_1$ and $v \notin D_1$. We determined the γ -paired dominating graphs of paths. In this paper, we present the γ -paired dominating graphs of cycles. For example, the γ -paired dominating graphs of cycles $C_4 : v_0v_1v_2v_3v_0$ and $C_5 : v_0v_1v_2v_3v_4v_0$ are shown in Figure 1. We see that $PD_\gamma(C_4) \cong C_4$ and $PD_\gamma(C_5) \cong K_5$.

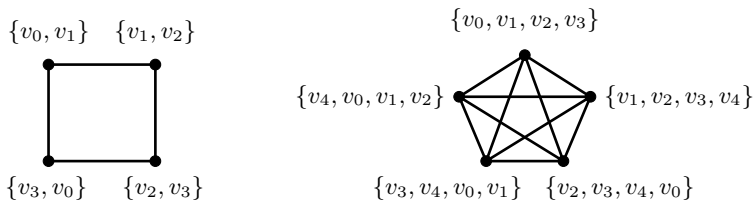


Fig. 1. The γ -paired dominating graphs of C_4 and C_5 , respectively

2. PRELIMINARY RESULTS

In this section, we recall some definitions, notations, and results used in the main results.

Haynes and Slater [4] established the following useful lemma.

Lemma 2.1. *For any integer $n \geq 3$, $\gamma_{pr}(P_n) = \gamma_{pr}(C_n) = 2\lceil \frac{n}{4} \rceil$.*

The *Cartesian product* of graphs G and H , denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ whose vertices (u, v) and (u', v') are adjacent if $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$.

For any positive integers p and q , let $P_p : u_1u_2u_3 \cdots u_p$ and $P_q : v_1v_2v_3 \cdots v_q$ be two paths with p and q vertices, respectively. Fricke *et al.* [2] defined a *stepgrid* $SG_{p,q}$ to be the subgraph of $P_p \square P_q$ induced by

$$\{(u_x, v_y) \in V(P_p \square P_q) : 1 \leq x \leq p, 1 \leq y \leq q, x - y \leq 1\}.$$

We call the vertex (u_x, v_y) in the stepgrid as the *vertex at the position* (x, y) . For example, the stepgrids $SG_{1,1}, SG_{2,2}$, and $SG_{4,3}$ are shown in Figure 2.

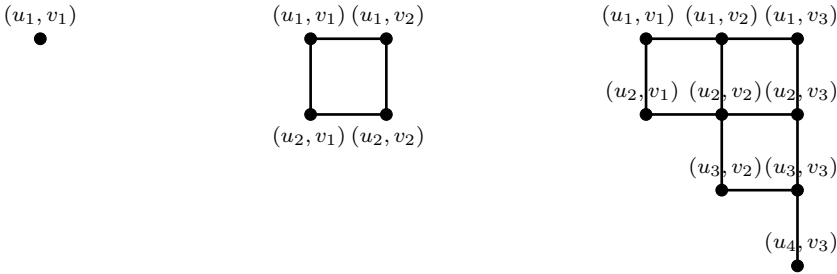


Fig. 2. The stepgrids $SG_{1,1}, SG_{2,2}$, and $SG_{4,3}$, respectively

For any positive integers p, q , and r , let $P_p : u_1u_2u_3 \cdots u_p$, $P_q : v_1v_2v_3 \cdots v_q$, and $P_r : w_1w_2w_3 \cdots w_r$ be three paths with p, q , and r vertices, respectively. We define a *stepgrid* $SG_{p,q,r}$ as the graph with the vertex set

$$V(SG_{p,q,r}) = \{(u_x, v_y, w_z) \in V(P_p \square P_q \square P_r) : 1 \leq x \leq p, 1 \leq y \leq q, 1 \leq z \leq r, x - y \leq 0, x - z \leq 1, y - z \geq 0\},$$

and the edge set

$$E(SG_{p,q,r}) = E(P_p \square P_q \square P_r) \cup \{(u_x, v_x, w_x)(u_{x+1}, v_{x+1}, w_x) : 1 \leq x \leq p - 1\}.$$

The vertex (u_x, v_y, w_z) is called the *vertex at the position* (x, y, z) in $SG(p, q, r)$. For example, the stepgrids $SG_{2,2,1}$ and $SG_{3,3,2}$ are shown in Figure 3, and the stepgrid $SG_{4,4,3}$ is shown in Figure 4, where we write (x, y, z) as (u_x, v_y, w_z) .

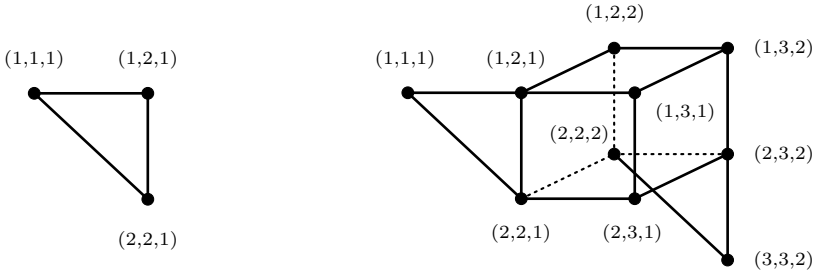


Fig. 3. The stepgrids $SG_{2,2,1}$ and $SG_{3,3,2}$, respectively

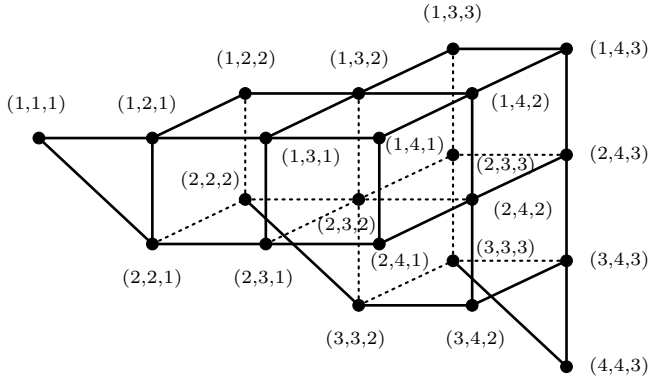


Fig. 4. The stepgrid $SG_{4,4,3}$

Let $P_n : v_1v_2v_3 \cdots v_n$ be a path with n vertices. In [1], we determined the γ -paired dominating graphs of paths, and gave the following results.

Lemma 2.2. *Let $k \geq 1$ be an integer. Then there is only one $\gamma_{pr}(P_{4k-1})$ -set containing the pair $\{v_{4k-2}, v_{4k-1}\}$, and there is only one $\gamma_{pr}(P_{4k-1})$ -set containing the pair $\{v_1, v_2\}$.*

Lemma 2.3. *Let $k \geq 2$ be an integer. Then all $\gamma_{pr}(P_{4k-2})$ -sets containing the pair $\{v_{4k-3}, v_{4k-2}\}$ form a path $A_1A_2 \cdots A_k$ with k vertices, where A_1 and A_k are of degree two, the others are of degree three, and A_k has a neighbor of degree two in $PD_\gamma(P_{4k-2})$. Moreover, A_k contains the pair $\{v_{4k-6}, v_{4k-5}\}$, and the others contain the pair $\{v_{4k-7}, v_{4k-6}\}$. The similar results also hold for the $\gamma_{pr}(P_{4k-2})$ -sets containing the pair $\{v_1, v_2\}$.*

Lemma 2.4. *Let $k \geq 3$ be an integer. Then all $\gamma_{pr}(P_{4k-3})$ -sets containing the pair $\{v_{4k-4}, v_{4k-3}\}$ form a stepgrid $SG_{k,k-1}$, where $B_{1,1}, B_{2,1}, B_{1,k-1}$ are of degree three, $B_{2,k-1}, B_{3,k-1}, \dots, B_{k-1,k-1}$ are of degree four, and $B_{k,k-1}$ is of degree two in $PD_\gamma(P_{4k-3})$. Moreover, $B_{1,k-1}, B_{2,k-1}, \dots, B_{k-1,k-1}$, contain the pair $\{v_{4k-7}, v_{4k-6}\}$, and $B_{k,k-1}$ contains the pair $\{v_{4k-6}, v_{4k-5}\}$ (see Figure 5). The similar results also hold for the $\gamma_{pr}(P_{4k-3})$ -sets containing the pair $\{v_1, v_2\}$.*

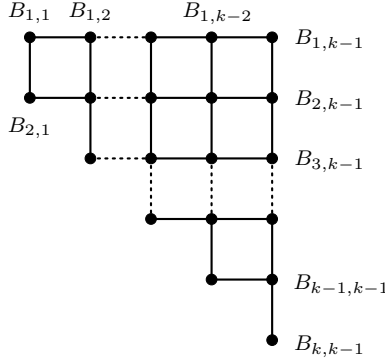


Fig. 5. The stepgrid $SG_{k,k-1}$

Theorem 2.5. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k}) \cong P_1$.*

Theorem 2.6. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k-1}) \cong P_{k+1}$.*

Theorem 2.7. *Let $k \geq 1$ be an integer. Then $PD_\gamma(P_{4k-2}) \cong SG_{k,k}$.*

Corollary 2.8. *Let $k \geq 2$ be an integer, and $A_{x,y}$ the $\gamma_{pr}(P_{4k-2})$ -set at the position (x, y) in $PD_\gamma(P_{4k-2}) \cong SG_{k,k}$ (see Figure 6) for all $x, y \in \{1, 2, \dots, k\}$ with $x - y \leq 1$. If $A_{x,k}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$, then we get the following properties.*

- (A1) *If $y = k$, then $A_{x,y}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$; otherwise, it contains the pair $\{v_{4k-4}, v_{4k-3}\}$.*
 - (A1.1) *$A_{x,k}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}, \{v_{4k-3}, v_{4k-2}\}$ for all $x \in \{1, 2, \dots, k-1\}$, and $A_{k,k}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}, \{v_{4k-3}, v_{4k-2}\}$.*
- (A2) *If $x = 1$, then $A_{x,y}$ contains the pair $\{v_1, v_2\}$; otherwise, it contains the pair $\{v_2, v_3\}$.*
 - (A2.1) *$A_{1,1}$ contains the pairs $\{v_1, v_2\}, \{v_4, v_5\}$, and $A_{1,y}$ contains the pairs $\{v_1, v_2\}, \{v_5, v_6\}$ for all $y \in \{2, 3, \dots, k\}$.*

Theorem 2.9. *Let $k \geq 2$ be an integer. Then $PD_\gamma(P_{4k-3}) \cong SG_{k,k,k-1}$.*

Corollary 2.10. *Let $k \geq 3$ be an integer and $B_{x,y,z}$ the $\gamma_{pr}(P_{4k-3})$ -set at the position (x, y, z) in $PD_\gamma(P_{4k-3}) \cong SG_{k,k,k-1}$ (see Figure 7) for all $x, y \in \{1, 2, \dots, k\}$, $z \in \{1, 2, \dots, k-1\}$ with $x - y \leq 0$, $x - z \leq 1$, $y - z \geq 0$. If $B_{x,k,z}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$, then we get the following properties.*

- (B1) If $y = k$, then $B_{x,y,z}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$; otherwise, it contains the pair $\{v_{4k-5}, v_{4k-4}\}$.
 - (B1.1) $B_{x,k,k-1}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}, \{v_{4k-4}, v_{4k-3}\}$ for all $x \in \{1, 2, \dots, k-1\}$, and $B_{k,k,k-1}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}, \{v_{4k-4}, v_{4k-3}\}$.
 - (B1.2) $B_{x,k,z}$ contains the pairs $\{v_{4k-8}, v_{4k-7}\}, \{v_{4k-4}, v_{4k-3}\}$ for all $z \neq k-1$.
- (B2) If $x = 1$, then $B_{x,y,z}$ contains the pair $\{v_1, v_2\}$; otherwise, it contains the pair $\{v_2, v_3\}$.
 - (B2.1) $B_{1,1,1}$ contains the pairs $\{v_1, v_2\}, \{v_3, v_4\}$, and $B_{1,y,1}$ contains the pairs $\{v_1, v_2\}, \{v_4, v_5\}$ for all $y \in \{2, 3, \dots, k\}$.
 - (B2.2) $B_{1,y,z}$ contains the pairs $\{v_1, v_2\}, \{v_5, v_6\}$ for all $z \neq 1$.

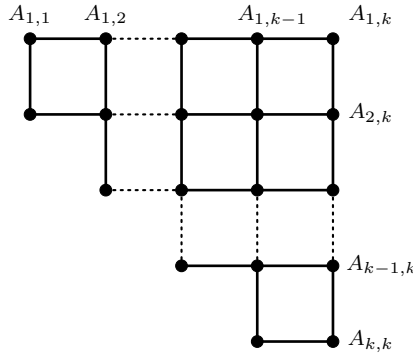


Fig. 6. The stepgrid $SG_{k,k}$

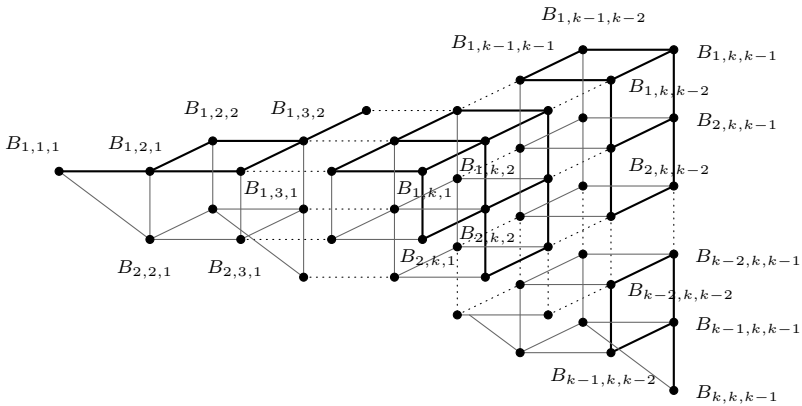


Fig. 7. The stepgrid $SG_{k,k,k-1}$

3. γ -PAIRED DOMINATING GRAPHS OF CYCLES

In this section, we present the γ -paired dominating graphs of cycles. We always let $C_n : v_0v_1v_2 \cdots v_{n-1}v_0$ be a cycle with n vertices. We first consider the γ -paired dominating graph of C_{4k} , as stated the following theorem.

Theorem 3.1. *Let $k \geq 1$ be an integer. Then*

$$PD_\gamma(C_{4k}) \cong \begin{cases} C_4 & \text{if } k = 1; \\ 4P_1 & \text{if } k \geq 2. \end{cases}$$

Proof. From Figure 1, we get that $PD_\gamma(C_4) \cong C_4$. Let $k \geq 2$. By Theorem 2.1, we have $\gamma_{pr}(C_{4k}) = 2k$. It is easy to check that

$$\begin{aligned} & \{v_0, v_1, v_4, v_5, \dots, v_{4k-4}, v_{4k-3}\}, & \{v_1, v_2, v_5, v_6, \dots, v_{4k-3}, v_{4k-2}\}, \\ & \{v_2, v_3, v_6, v_7, \dots, v_{4k-2}, v_{4k-1}\}, & \{v_3, v_4, v_7, v_8, \dots, v_{4k-1}, v_{4k}\} \end{aligned}$$

are the only $\gamma_{pr}(C_{4k})$ -sets. Thus, $PD_\gamma(C_{4k}) \cong 4P_1$. □

Before we prove the result on the γ -paired dominating graph of a cycle with $4k + 3$ vertices, we need the following lemma.

Lemma 3.2. *Let $k \geq 0$ be an integer and D a $\gamma_{pr}(C_{4k+3})$ -set. Then there is exactly one vertex not in D dominated by two vertices of D .*

Proof. We can easily get that the lemma holds for $k = 0$. Let $k \geq 1$. Note that $|D| = 2k + 2$, so we can write $D = \bigcup_{x=1}^{k+1} D_x$, where D_x 's are pairwise disjoint sets of paired vertices. Clearly, $|N[D_x]| = 4$ for all $x \in \{1, 2, 3, \dots, k + 1\}$, and $V(C_{4k+3}) = \bigcup_{x=1}^{k+1} N[D_x]$. If $N[D_x]$'s are pairwise disjoint sets, then

$$4k + 3 = |V(C_{4k+3})| = \sum_{x=1}^{k+1} |N[D_x]| = 4k + 4,$$

a contradiction. Therefore, without loss of generality, there are exactly two disjoint sets D_1 and D_2 such that $|N[D_1] \cap N[D_2]| = 1$. Thus, this common vertex is the only vertex not in D dominated by two vertices of D . □

Theorem 3.3. *Let $k \geq 0$ be an integer. Then $PD_\gamma(C_{4k+3}) \cong C_{4k+3}$.*

Proof. For convenience, we omit the modulo $4k+3$ in the subscript of each vertex. For example, we write v_{x+1} instead of $v_{(x+1) \pmod{4k+3}}$. For each $x \in \{0, 1, 2, \dots, 4k+2\}$, let

$$D_x = \{v_{x+4i+1}, v_{x+4i+2} : 0 \leq i \leq k\}$$

as shown in Figure 8, where D_x contains the black vertices. It is easy to check that D_x is a $\gamma_{pr}(C_{4k+3})$ -set such that $v_x \notin D_x$ is the only vertex dominated by two vertices of D_x . Hence, $D_0, D_1, D_2, \dots, D_{4k+2}$ are all distinct. Similarly, we omit the modulo $4k + 3$ in the subscript of each $\gamma_{pr}(C_{4k+3})$ -set.

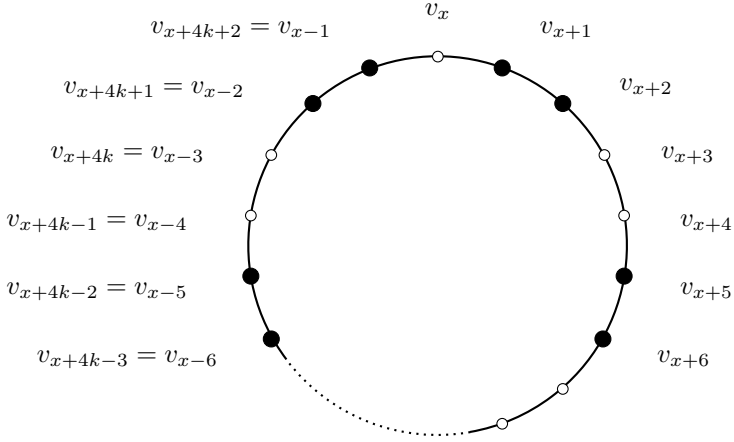


Fig. 8. The $\gamma_{pr}(C_{4k+3})$ -set D_x

Claim that $D_0, D_1, D_2, \dots, D_{4k+2}$ are the only $\gamma_{pr}(C_{4k+3})$ -sets. Let D be any $\gamma_{pr}(C_{4k+3})$ -set. By Lemma 3.2, there is a unique vertex $v_x \notin D$ for some $x \in \{0, 1, 2, \dots, 4k+2\}$, dominated by two vertices of D , so $D = D_x$.

Let $x \in \{0, 1, 2, \dots, 4k+2\}$. To find all neighbors of D_x in $PD_\gamma(C_{4k+3})$, we can only replace v_{x+1} by v_{x+3} , or v_{x-1} by v_{x-3} since v_x is the only vertex dominated by v_{x+1} and v_{x-1} of D_x . Thus, $(D_x \setminus \{v_{x+1}\}) \cup \{v_{x+3}\}$ and $(D_x \setminus \{v_{x-1}\}) \cup \{v_{x-3}\}$ are the only two neighbors of D_x in $PD_\gamma(C_{4k+3})$. Note that

$$(D_x \setminus \{v_{x+1}\}) \cup \{v_{x+3}\} = D_{x+4}$$

since v_{x+4} is the only vertex dominated by two dominating vertices. Similarly,

$$(D_x \setminus \{v_{x-1}\}) \cup \{v_{x-3}\} = D_{x-4}.$$

Therefore, $D_0, D_4, \dots, D_{4k-4}, D_{4k}, D_1, D_5, \dots, D_{4k-3}, D_{4k+1}, D_2, D_6, \dots, D_{4k-2}, D_{4k+2}, D_3, D_7, \dots, D_{4k-1}, D_0$ form a cycle with $4k+3$ vertices. This completes the proof. \square

Before we determine the γ -paired dominating graph of a cycle with $4k+2$ vertices, we define some notations and a new graph called a loopgrid.

For a nonnegative integer i , let $P_n(v_i : v_{i+n-1}) : v_i v_{i+1} v_{i+2} \cdots v_{i+n-1}$ be a path with n vertices.

For any positive integer k , let $G_1 : u_1 u_2 u_3 \cdots u_{2k-1}$ and $G_2 : v_1 v_2 v_3 \cdots v_{3k-1}$ be two paths with $2k-1$ and $3k-1$ vertices, respectively. We define a *loopgrid* of size k , denoted by LG_k , as the graph with the vertex set

$$V(LG_k) = \{(u_x, v_y) \in V(G_1 \square G_2) : 1 \leq x \leq 2k-1, 1 \leq y \leq 3k-1, 0 \leq y-x \leq k\},$$

and the edge set

$$E(LG_k) = E(G_1 \square G_2) \cup \{(u_1, v_y)(u_{2k-1}, v_{y+2k-1}) : 1 \leq y \leq k\}.$$

For example, Figure 9 illustrates the loopgrids LG_1 and LG_2 , where we use (x, y) as (u_x, v_y) .

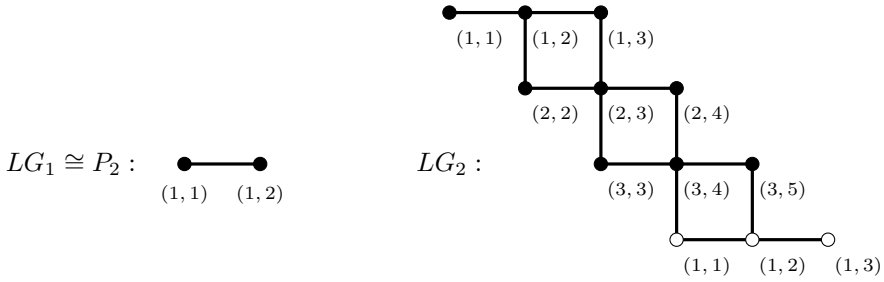


Fig. 9. The loopgrids LG_1 and LG_2 , respectively

Lemma 3.4. *Let $k \geq 2$ be an integer.*

- (1) *Each $\gamma_{pr}(C_{4k+2})$ -set cannot contain any six or more consecutive vertices.*
- (2) *For any fixed four consecutive vertices in C_{4k+2} , there is exactly one $\gamma_{pr}(C_{4k+2})$ -set containing them.*

Proof. We prove the first claim by contradiction. Suppose that there is a $\gamma_{pr}(C_{4k+2})$ -set D containing l consecutive vertices of C_{4k+2} , where $l \geq 6$ is an integer. Then these l vertices dominate $l + 2$ vertices in C_{4k+2} . Since $\gamma_{pr}(C_{4k+2}) = 2k + 2$, the other $2k + 2 - l$ vertices of D must dominate at least $4k + 2 - (l + 2) = 4k - l$ vertices in C_{4k+2} . We consider them as a path with $4k - l$ vertices. Note that the $2k + 2 - l$ remaining vertices of D can dominate at most $4k + 4 - 2l < 4k - l$ vertices in this path since $l \geq 6$. Thus, D cannot dominate all vertices in C_{4k+2} , a contradiction.

For the second claim, without loss of generality, we assume the four vertices are v_1, v_2, v_3, v_4 . We find all $\gamma_{pr}(C_{4k+2})$ -sets containing them. By the first claim, all such $\gamma_{pr}(C_{4k+2})$ -sets cannot contain v_0 and v_5 . The vertices v_1, v_2, v_3, v_4 dominate six vertices in C_{4k+2} . Note that $\gamma_{pr}(C_{4k+2}) = 2k + 2$, so the other $2k - 2$ vertices must dominate all vertices in $P_{4k-4}(v_6 : v_{4k+1})$. Since $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1})) = 2k - 2$, these $2k - 2$ vertices form a $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$ -set. Thus, each $\gamma_{pr}(C_{4k+2})$ -set containing v_1, v_2, v_3, v_4 is a union of a $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$ -set and $\{v_1, v_2, v_3, v_4\}$. By Theorem 2.5, there is a unique $\gamma_{pr}(P_{4k-4}(v_6 : v_{4k+1}))$ -set. The claim follows. \square

Theorem 3.5. *Let $k \geq 1$ be an integer. Then*

$$PD_\gamma(C_{4k+2}) \cong \begin{cases} C_3 \square C_3 & \text{if } k = 1, \\ LG_{k+1} & \text{if } k \geq 2. \end{cases}$$

Proof. Figure 10 shows that $PD_\gamma(C_6) \cong C_3 \square C_3$.

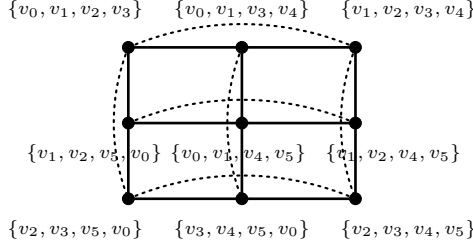


Fig. 10. The γ -paired dominating graph of C_6

Let $k \geq 2$. Since each $\gamma_{pr}(C_{4k+2})$ -set must dominate the vertex v_0 , we get it contains either the pair $\{v_{4k}, v_{4k+1}\}$, $\{v_{4k+1}, v_0\}$, $\{v_0, v_1\}$, or $\{v_1, v_2\}$. We first find all $\gamma_{pr}(C_{4k+2})$ -sets containing the pair $\{v_{4k}, v_{4k+1}\}$. By Lemma 3.4(1), such a $\gamma_{pr}(C_{4k+2})$ -set must satisfy one of the following:

- (i) it contains the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0 ,
- (ii) it contains the pairs $\{v_{4k-2}, v_{4k-1}\}$ and $\{v_{4k}, v_{4k+1}\}$,
- (iii) it contains the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_0, v_1\}$.

Note that each $\gamma_{pr}(C_{4k+2})$ -set containing the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0 is a union of a $\gamma_{pr}(P_{4k-2}(v_1 : v_{4k-2}))$ -set and $\{v_{4k}, v_{4k+1}\}$. By Theorem 2.7, we have

$$PD_\gamma(P_{4k-2}(v_1 : v_{4k-2})) \cong SG_{k,k}.$$

For all $x, y \in \{1, 2, \dots, k\}$ with $x - y \leq 1$, let $A_{x,y}^{(1)}$ be the $\gamma_{pr}(P_{4k-2}(v_1 : v_{4k-2}))$ -set at the position (x, y) in this stepgrid $SG_{k,k}$, and let

$$D_{x,y}^{(1)} = A_{x,y}^{(1)} \cup \{v_{4k}, v_{4k+1}\}.$$

Thus, $D_{x,y}^{(1)}$'s are the only $\gamma_{pr}(C_{4k+2})$ -sets containing the pair $\{v_{4k}, v_{4k+1}\}$, but not v_{4k-1}, v_0 , and they form a stepgrid $SG_{k,k}$ in $PD_\gamma(C_{4k+2})$. By Lemma 2.3, without loss of generality, we assume that $A_{x,k}$ contains the pair $\{v_{4k-3}, v_{4k-2}\}$ for each $x \in \{1, 2, \dots, k\}$. By Corollary 2.8 (A1.1), we have $A_{k,k}^{(1)}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}, \{v_{4k-3}, v_{4k-2}\}$. Let

$$D_{k+1,k}^{(1)} = (D_{k,k}^{(1)} \setminus \{v_{4k-3}\}) \cup \{v_{4k-1}\}.$$

By Lemma 3.4(2), the set $D_{k+1,k}^{(1)}$ is the only $\gamma_{pr}(C_{4k+2})$ -set containing the pairs $\{v_{4k-2}, v_{4k-1}\}$ and $\{v_{4k}, v_{4k+1}\}$. By Corollary 2.8 (A2.1), we get $A_{1,1}^{(1)}$ contains the pairs $\{v_1, v_2\}, \{v_4, v_5\}$. Let

$$D_{1,0}^{(1)} = (D_{1,1}^{(1)} \setminus \{v_2\}) \cup \{v_0\}.$$

By Lemma 3.4(2), the set $D_{1,0}^{(1)}$ is the only $\gamma_{pr}(C_{4k+2})$ -set containing the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_0, v_1\}$. Therefore, all $D_{x,y}^{(1)}$'s form the graph, named $D^{(1)}$, in $PD_\gamma(C_{4k+2})$ as shown in Figure 11.

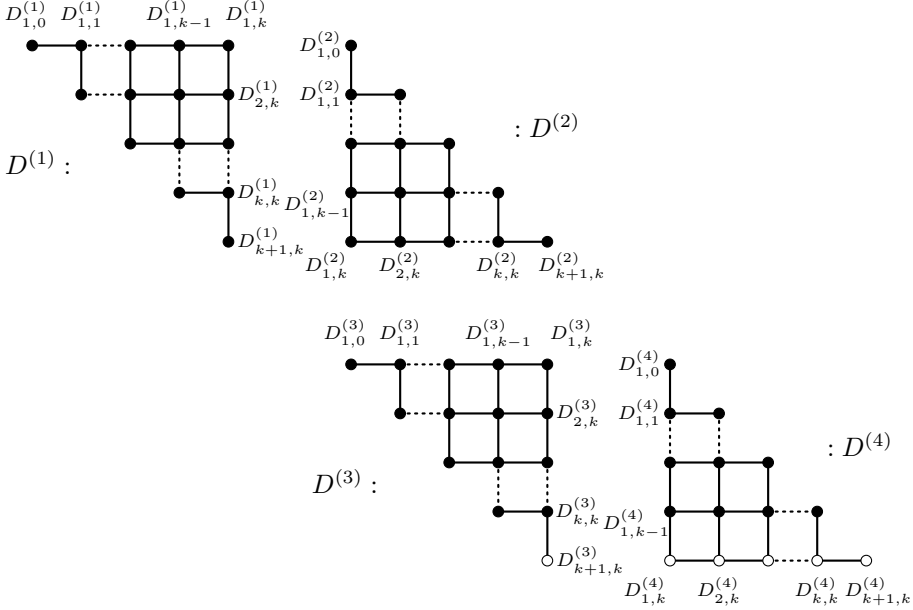


Fig. 11. The graphs $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, and $D^{(4)}$ in $PD_\gamma(C_{4k+2})$

Similarly, we can construct all $\gamma_{pr}(C_{4k+2})$ -sets as follows (the subscripts of all vertices are modulo $4k + 2$): for all $x, y \in \{1, 2, \dots, k\}$ with $x - y \leq 0$, and for each $i \in \{1, 2, 3, 4\}$,

$$D_{x,y}^{(i)} = A_{x,y}^{(i)} \cup \{v_{4k-1+i}, v_{4k+i}\}, \text{ where } A_{x,y}^{(i)} \text{ is a } \gamma_{pr}(P_{4k-2}(v_i : v_{4k-3+i}))\text{-set,}$$

$$D_{k+1,k}^{(i)} = (D_{k,k}^{(i)} \setminus \{v_{4k-4+i}\}) \cup \{v_{4k-2+i}\},$$

and

$$D_{1,0}^{(i)} = (D_{1,1}^{(i)} \setminus \{v_{i+1}\}) \cup \{v_{i-1}\}.$$

These $D_{x,y}^{(i)}$'s are the only $\gamma_{pr}(C_{4k+2})$ -sets containing the pair $\{v_{4k-1+i}, v_{4k+i}\}$, and they form the graph $D^{(i)}$ in $PD_\gamma(C_{4k+2})$ (see Figure 11). By Lemma 2.3, without loss of generality, we assume $A_{x,k}^{(i)}$ contains the pair $\{v_{4k-4+i}, v_{4k-3+i}\}$. For all $x, y \in \{1, 2, \dots, k\}$ with $x - y \leq 1$, we get the following properties.

- (A'1) If $y = k$, then $D_{x,y}^{(i)}$ contains the pairs $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$; otherwise, it contains the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$.
- (A'1.1) for all $x \in \{1, 2, \dots, k-1\}$, $D_{x,k}^{(i)}$ contains the pairs $\{v_{4k-8+i}, v_{4k-7+i}\}$, $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$, and $D_{k,k}^{(i)}$ contains the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$.
- (A'2) If $x = 1$, then $D_{x,y}^{(i)}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$; otherwise, it contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i+1}, v_{i+2}\}$.
- (A'2.1) $D_{1,1}^{(i)}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+3}, v_{i+4}\}$, and $D_{1,y}^{(i)}$ contains the pairs $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+4}, v_{i+5}\}$ for all $y \in \{2, 3, \dots, k\}$.
- (A'3) $D_{k+1,k}^{(i)}$ is the only $\gamma_{pr}(C_{4k+2})$ -set in $D^{(i)}$ containing the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-3+i}, v_{4k-2+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i+1}, v_{i+2}\}$.
- (A'4) $D_{1,0}^{(i)}$ is the only $\gamma_{pr}(C_{4k+2})$ -set in $D^{(i)}$ containing the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-1+i}, v_{4k+i}\}$, $\{v_{i-1}, v_i\}$, $\{v_{i+3}, v_{i+4}\}$.

Note that $D^{(1)}$ and $D^{(2)}$ cannot have any common vertices in $PD_\gamma(C_{4k+2})$ since otherwise there is a $\gamma_{pr}(C_{4k+2})$ -set containing the pairs $\{v_{4k}, v_{4k+1}\}$ and $\{v_{4k+1}, v_0\}$, which is impossible. Similarly, $D^{(i)}$ and $D^{(i+1)}$ do not share any vertices in $PD_\gamma(C_{4k+2})$ for all $i \in \{2, 3\}$.

We then consider all $\gamma_{pr}(C_{4k+2})$ -sets that are in both $D^{(1)}$ and $D^{(3)}$. Then these sets must contain the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$. By (A'4) and (A'3), $D_{1,0}^{(1)}$ and $D_{k+1,k}^{(3)}$ are the only $\gamma_{pr}(C_{4k+2})$ -sets in $D^{(1)}$ and $D^{(3)}$, respectively, containing the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$. By Lemma 3.4(2), we get $D_{1,0}^{(1)} = D_{k+1,k}^{(3)}$. Similarly, $D_{1,0}^{(2)} = D_{k+1,k}^{(4)}$ is the only $\gamma_{pr}(C_{4k+2})$ -set that is in both $D^{(2)}$ and $D^{(4)}$.

We next consider all $\gamma_{pr}(C_{4k+2})$ -sets, which are in both $D^{(1)}$ and $D^{(4)}$. These sets must contain the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$. By (A'2), $D_{1,1}^{(1)}, D_{1,2}^{(1)}, \dots, D_{1,k}^{(1)}$ are the only $\gamma_{pr}(C_{4k+2})$ -sets in $D^{(1)}$ containing the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and they form a path with k vertices. Then they also form a path in $D^{(4)}$. By (A'1), $D_{1,k}^{(4)}, D_{2,k}^{(4)}, \dots, D_{k,k}^{(4)}$ are the only $\gamma_{pr}(C_{4k+2})$ -sets in $D^{(4)}$ containing the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and they form a path with k vertices. To show that $D_{1,y}^{(1)} = D_{y,k}^{(4)}$ for each $y \in \{1, 2, \dots, k\}$, it suffices to show that $D_{1,1}^{(1)} = D_{1,k}^{(4)}$. By (A'2.1), $D_{1,1}^{(1)}$ contains the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, $\{v_4, v_5\}$. By (A'1) and (A'2), $D_{1,k}^{(4)}$ is the only $\gamma_{pr}(C_{4k+2})$ -set in $D^{(4)}$ containing these three pairs, and hence, $D_{1,1}^{(1)} = D_{1,k}^{(4)}$.

Next, we consider all edges between a set in $D^{(1)}$ and a set in $D^{(2)}$. We first show that $D_{1,0}^{(1)}$ has no neighbors in $D^{(2)}$. By (A'4), $D_{1,0}^{(1)}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$. Since each set in $D^{(2)}$ contains the pair $\{v_{4k+1}, v_0\}$, the set $D_{1,0}^{(1)}$ is adjacent to some set in $D^{(2)}$ if and only if $(D_{1,0}^{(1)} \setminus \{v_{4k}\}) \cup \{v_2\}$ or $(D_{1,0}^{(1)} \setminus \{v_1\}) \cup \{v_{4k-1}\}$ is a $\gamma_{pr}(C_{4k+2})$ -set. It is easy to check that $D_{1,0}^{(1)}$ is not adjacent to any sets in $D^{(2)}$. By (A'3),

$D_{k+1,k}^{(1)}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$, so $D_{k+1,k}^{(1)}$ is adjacent to some set in $D^{(2)}$ if and only if $(D_{k+1,k}^{(1)} \setminus \{v_{4k}\}) \cup \{v_0\}$ or $(D_{k+1,k}^{(1)} \setminus \{v_{4k-2}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$ -set. We have $(D_{k+1,k}^{(1)} \setminus \{v_{4k}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$ -set, but $(D_{k+1,k}^{(1)} \setminus \{v_{4k-2}\}) \cup \{v_0\}$ is not. We show that $D_{k+1,k}^{(1)}$ and $D_{1,k}^{(2)}$ are adjacent, i.e., $(D_{1,k}^{(2)} \setminus \{v_0\}) \cup \{v_{4k}\} = D_{k+1,k}^{(1)}$. By (A'1) and (A'2), $D_{1,k}^{(2)}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$, so $(D_{1,k}^{(2)} \setminus \{v_0\}) \cup \{v_{4k}\}$ is a $\gamma_{pr}(C_{4k+2})$ -set containing the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_{4k+1}\}$. Since $D_{k+1,k}^{(1)}$ also contains these two pairs, $(D_{1,k}^{(2)} \setminus \{v_0\}) \cup \{v_{4k}\} = D_{k+1,k}^{(1)}$ by Lemma 3.4(2).

We next find all neighbors in $D^{(2)}$ of the other $\gamma_{pr}(C_{4k+2})$ -sets in $D^{(1)}$. We show that $D_{x,k}^{(1)}$ is adjacent to $D_{1,x-1}^{(2)}$ for all $x \in \{1, 2, \dots, k\}$. Recall that for all $x, y \in \{1, 2, \dots, k\}$ with $x - y \leq 1$, $D_{x,y}^{(1)}$ contains the pair $\{v_{4k}, v_{4k+1}\}$ but not v_{4k-1}, v_0 . Note that $D_{x,y}^{(1)}$ is adjacent to some set in $D^{(2)}$ if and only if $(D_{x,y}^{(1)} \setminus \{v_{4k}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+2})$ -set. By (A'1), if $y \neq k$, then $D_{x,y}^{(1)}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k}, v_{4k+1}\}$, so $(D_{x,y}^{(1)} \setminus \{v_{4k}\}) \cup \{v_0\}$ is not a dominating set. By (A'1) and (A'2), $D_{1,k}^{(1)}$ contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, and $D_{2,k}^{(1)}, D_{3,k}^{(1)}, \dots, D_{k,k}^{(1)}$ contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$. For each $x \in \{1, 2, \dots, k\}$, let $D_x = (D_{x,k}^{(1)} \setminus \{v_{4k}\}) \cup \{v_0\}$, so D_x is a $\gamma_{pr}(C_{4k+2})$ -set, and these D_x 's form a path with k vertices in $D^{(2)}$. Note that D_1 contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_1, v_2\}$, and D_2, D_3, \dots, D_k contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$. By (A'4), $D_{1,0}^{(2)}$ is the only $\gamma_{pr}(C_{4k+2})$ -set in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_1, v_2\}$, and by (A'1), (A'2), $D_{1,1}^{(2)}, D_{1,2}^{(2)}, \dots, D_{1,k-1}^{(2)}$ are the only $\gamma_{pr}(C_{4k+2})$ -sets in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k+1}, v_0\}$, $\{v_2, v_3\}$, and they also form a path with k vertices in $D^{(2)}$. Then we can conclude that for all $x \in \{1, 2, \dots, k\}$, $D_x = D_{1,x-1}^{(2)}$, implying that $D_{x,k}^{(1)}$ is adjacent to $D_{1,x-1}^{(2)}$. To sum up, $D_{x,k}^{(1)}$ is adjacent to $D_{1,x-1}^{(2)}$ for all $x \in \{1, 2, \dots, k+1\}$. Similarly, for all $i \in \{2, 3\}$, we get $D_{x,k}^{(i)}$ is adjacent to $D_{1,x-1}^{(i+1)}$ for all $x \in \{1, 2, \dots, k+1\}$.

Now, all $\gamma_{pr}(C_{4k+2})$ -sets and edges form a loopgrid LG_{k+1} in $PD_\gamma(C_{4k+2})$. Then we only need to show that there is no more edge in $PD_\gamma(C_{4k+2})$. We first consider all edges between a set in

$$\widehat{D}^{(1)} = D^{(1)} - D_{1,0}^{(1)}$$

and a set in

$$\widehat{D}^{(3)} = D^{(3)} - D_{k+1,k}^{(3)}$$

since $D_{1,0}^{(1)} = D_{k+1,k}^{(3)}$. Note that each set in $\widehat{D}^{(1)}$ contains either the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_1, v_2\}$, or the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$ while every set in $\widehat{D}^{(3)}$ contains the pair $\{v_0, v_1\}$ but not $\{v_{4k}, v_{4k+1}\}$. Hence, there is no edge between a set in $\widehat{D}^{(1)}$ and a set in $\widehat{D}^{(3)}$. Similarly, there is no edge between a set in $D^{(2)} - D_{1,0}^{(2)}$ and a set in

$D^{(4)} - D_{k+1,k}^{(4)}$. Recall that $D_{1,y}^{(1)} = D_{y,k}^{(4)}$ for all $y \in \{1, 2, \dots, k\}$. Also, $D_{1,0}^{(1)} = D_{k+1,k}^{(3)}$, which has a neighbor in $D^{(4)}$. Thus, we consider all edges between a set in

$$\tilde{D}^{(1)} = D^{(1)} - \{D_{1,y}^{(1)} : 0 \leq y \leq k\}$$

and a set in

$$\tilde{D}^{(4)} = D^{(4)} - \{D_{y,k}^{(4)} : 1 \leq y \leq k\}.$$

Note that each set in $\tilde{D}^{(1)}$ contains the pairs $\{v_{4k}, v_{4k+1}\}$, $\{v_2, v_3\}$ while each set in $\tilde{D}^{(4)}$ contains the pair $\{v_1, v_2\}$ but not $\{v_{4k}, v_{4k+1}\}$. Hence, there is no edge between a set in $\tilde{D}^{(1)}$ and a set in $\tilde{D}^{(4)}$. This completes the proof. \square

For any positive integer k , let $G_1 : u_1 u_2 u_3 \cdots u_{2k}$, $G_2 : v_1 v_2 v_3 \cdots v_{2k}$, and $G_3 : w_1 w_2 w_3 \cdots w_{2k+1}$ be three paths with $2k$, $2k$, and $2k + 1$ vertices, respectively. We define a *loopbox* of size k , denoted by LB_k , as the graph with the vertex set

$$V(LB_k) = \{(u_x, v_y, w_z) \in V(G_1 \square G_2 \square G_3) : 1 \leq x, y \leq 2k, 1 \leq z \leq 2k + 1, \\ 0 \leq y - x \leq k, -1 \leq y - z \leq k - 1, 0 \leq z - x \leq k\},$$

and the edge set

$$E(LB_k) = E(G_1 \square G_2 \square G_3) \cup \{(u_x, v_{x+k-1}, w_x)(u_x, v_{x+k}, w_{x+1}) : 1 \leq x \leq k\} \\ \cup \{(u_1, v_1, w_1)(u_{k+1}, v_{2k}, w_{k+1})\} \\ \cup \{(u_x, v_x, w_{x+1})(u_{x+1}, v_{x+1}, w_{x+1}) : 1 \leq x \leq 2k - 1\} \\ \cup \{(u_1, v_k, w_{k+1})(u_{2k}, v_{2k}, w_{2k+1})\} \\ \cup \{(u_x, v_{x+k}, w_{x+k})(u_{x+1}, v_{x+k}, w_{x+k+1}) : 1 \leq x \leq k\} \\ \cup \{(u_1, v_y, w_z)(u_{z+k}, v_{2k}, w_{y+k+1}) : 1 \leq y, z \leq k, -1 \leq y - z \leq k - 1\}.$$

For example, the loopboxes of size 1, 2, and 3 are shown in Figures 12, 13, and 14, respectively, where we write (x, y, z) as (u_x, v_y, w_z) .

Lemma 3.6. *Let $k \geq 2$ be an integer.*

- (1) *Each $\gamma_{pr}(C_{4k+1})$ -set cannot contain any six or more consecutive vertices.*
- (2) *For any fixed four consecutive vertices in C_{4k+1} , there are k $\gamma_{pr}(C_{4k+1})$ -sets containing them, and each set is a union of a $\gamma_{pr}(P_{4k-5})$ -set and a set of these four vertices.*

Proof. Similar to 3.4(1), we can easily prove the first claim. Next, without loss of generality, we assume the four vertices are v_1, v_2, v_3, v_4 . Then these four vertices dominate six vertices in C_{4k+1} . Note that $\gamma_{pr}(C_{4k+1}) = 2k + 2$, so the other $2k - 2$ vertices must dominate all vertices in $P_{4k-5}(v_6 : v_{4k})$. Since

$$\gamma_{pr}(P_{4k-5}(v_6 : v_{4k})) = 2k - 2,$$

these $2k - 2$ vertices form a $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$ -set. Hence, each such $\gamma_{pr}(C_{4k+1})$ -set is a union of a $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$ -set and $\{v_1, v_2, v_3, v_4\}$. By Theorem 2.6, there are k $\gamma_{pr}(P_{4k-5}(v_6 : v_{4k}))$ -sets, so the claim follows. \square

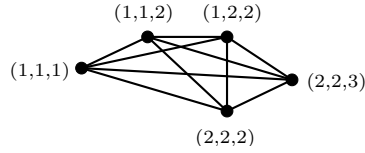


Fig. 12. The loopbox of size 1

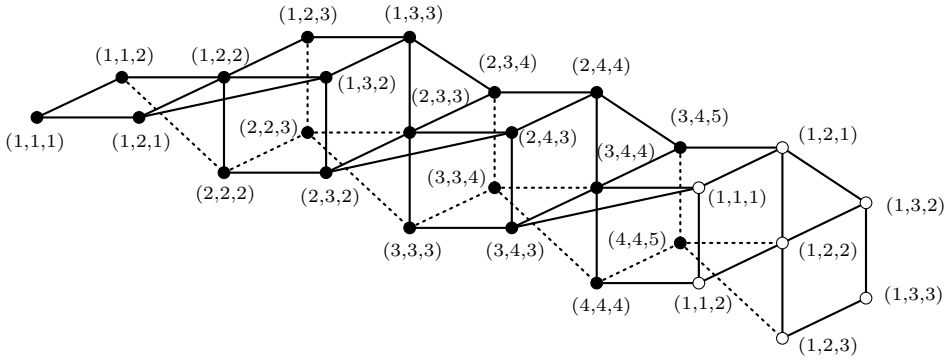


Fig. 13. The loopbox of size 2

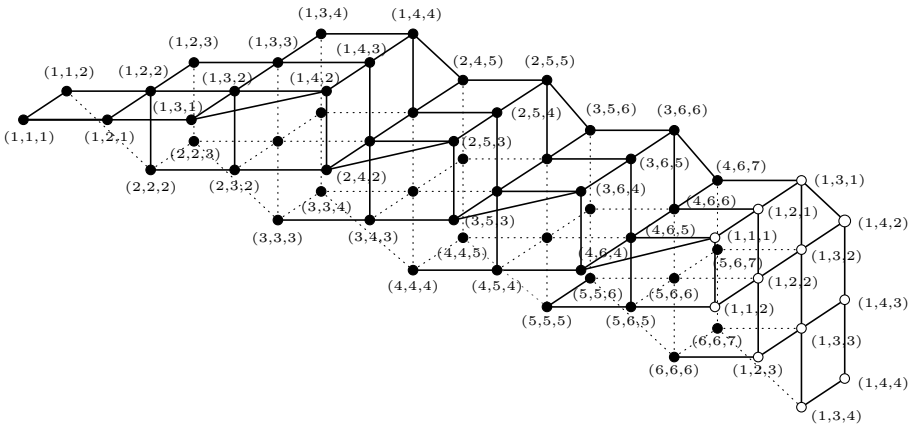


Fig. 14. The loopbox of size 3

Theorem 3.7. *Let $k \geq 1$ be an integer. Then $PD_\gamma(C_{4k+1}) \cong LB_k$.*

Proof. Figure 1 shows that

$$PD_\gamma(C_5) \cong K_5 \cong LB_1.$$

For $k = 2$, we have

$$PD_\gamma(C_9) \cong LB_2$$

(see Figure 13), where

$$\begin{aligned} (1, 1, 1) &= \{v_0, v_1, v_2, v_3, v_5, v_6\}, \\ (1, 2, 1) &= \{v_0, v_1, v_2, v_3, v_6, v_7\}, & (1, 1, 2) &= \{v_0, v_1, v_3, v_4, v_5, v_6\}, \\ (1, 2, 2) &= \{v_0, v_1, v_3, v_4, v_6, v_7\}, & (1, 2, 3) &= \{v_0, v_1, v_3, v_4, v_7, v_8\}, \\ (2, 2, 2) &= \{v_0, v_1, v_4, v_5, v_6, v_7\}, & (2, 2, 3) &= \{v_0, v_1, v_4, v_5, v_7, v_8\}, \\ (1, 3, 2) &= \{v_1, v_2, v_3, v_4, v_6, v_7\}, & (1, 3, 3) &= \{v_1, v_2, v_3, v_4, v_7, v_8\}, \\ (2, 3, 2) &= \{v_1, v_2, v_4, v_5, v_6, v_7\}, & (2, 3, 3) &= \{v_1, v_2, v_4, v_5, v_7, v_8\}, \\ (3, 3, 3) &= \{v_1, v_2, v_4, v_5, v_8, v_0\}, & (2, 4, 3) &= \{v_1, v_2, v_5, v_6, v_7, v_8\}, \\ (3, 4, 3) &= \{v_1, v_2, v_5, v_6, v_8, v_0\}, & (2, 3, 4) &= \{v_2, v_3, v_4, v_5, v_7, v_8\}, \\ (3, 3, 4) &= \{v_2, v_3, v_4, v_5, v_8, v_0\}, & (2, 4, 4) &= \{v_2, v_3, v_5, v_6, v_7, v_8\}, \\ (3, 4, 4) &= \{v_2, v_3, v_5, v_6, v_8, v_0\}, & (3, 4, 5) &= \{v_2, v_3, v_6, v_7, v_8, v_0\}, \\ (4, 4, 4) &= \{v_3, v_4, v_5, v_6, v_8, v_0\}, & (4, 4, 5) &= \{v_3, v_4, v_6, v_7, v_8, v_0\}. \end{aligned}$$

Let $k \geq 3$. Since each $\gamma_{pr}(C_{4k+1})$ -set must dominate the vertex v_0 , it contains either the pair $\{v_{4k-1}, v_{4k}\}$, $\{v_{4k}, v_0\}$, $\{v_0, v_1\}$, or $\{v_1, v_2\}$. We first find all $\gamma_{pr}(C_{4k+1})$ -sets containing the pair $\{v_{4k-1}, v_{4k}\}$. By Lemma 3.6(1), such a $\gamma_{pr}(C_{4k+1})$ -set must satisfy one of the following:

- (i) it contains the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0 ,
- (ii) it contains the pairs $\{v_{4k-3}, v_{4k-2}\}$ and $\{v_{4k-1}, v_{4k}\}$,
- (iii) it contains the pairs $\{v_{4k-1}, v_{4k}\}$ and $\{v_0, v_1\}$.

Note that each $\gamma_{pr}(C_{4k+1})$ -set containing the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0 is a union of a $\gamma_{pr}(P_{4k-3}(v_1 : v_{4k-3}))$ -set and $\{v_{4k-1}, v_{4k}\}$. By Theorem 2.9, we have

$$PD_\gamma(P_{4k-3}(v_1 : v_{4k-3})) \cong SG_{k,k,k-1}.$$

For all $x, y \in \{1, 2, \dots, k\}$ and $z \in \{1, 2, \dots, k-1\}$ with $x-y \leq 0$, $x-z \leq 1$, $y-z \geq 0$, let $B_{x,y,z}^{(1)}$ be the $\gamma_{pr}(P_{4k-3}(v_1 : v_{4k-3}))$ -set at the position (x, y, z) in $SG_{k,k,k-1}$, and let

$$D_{x,y,z}^{(1)} = B_{x,y,z}^{(1)} \cup \{v_{4k-1}, v_{4k}\}.$$

Thus $D_{x,y,z}^{(1)}$'s are the only $\gamma_{pr}(C_{4k+1})$ -sets containing the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0 , and they also form a stepgrid $SG_{k,k,k-1}$ in $PD_\gamma(C_{4k+1})$. By Lemma 2.4, without loss of generality, we may assume that $B_{x,k,z}^{(1)}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$.

By Corollary 2.10 (B1.1), the set $B_{x,k,k-1}^{(1)}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-4}, v_{4k-3}\}$ for all $x \in \{1, 2, \dots, k-1\}$, and $B_{k,k,k-1}^{(1)}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-4}, v_{4k-3}\}$. For each $x \in \{1, 2, \dots, k\}$, let

$$D_{x,k,k}^{(1)} = (D_{x,k,k-1}^{(1)} \setminus \{v_{4k-4}\}) \cup \{v_{4k-2}\}.$$

By Lemma 3.6(2), these $D_{x,k,k}^{(1)}$'s are the only $\gamma_{pr}(C_{4k+1})$ -sets containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$. By Corollary 2.10 (B2.1), the set $B_{1,1,1}^{(1)}$ contains the pairs $\{v_1, v_2\}$, $\{v_3, v_4\}$, and $B_{1,y,1}^{(1)}$ contains the pairs $\{v_1, v_2\}$, $\{v_4, v_5\}$ for all $y \in \{2, 3, \dots, k\}$. For each $y \in \{1, 2, \dots, k\}$, let

$$D_{1,y,0}^{(1)} = (D_{1,y,1}^{(1)} \setminus \{v_2\}) \cup \{v_0\}.$$

By Lemma 3.6(2), these $D_{1,y,0}^{(1)}$'s are the only $\gamma_{pr}(C_{4k+1})$ -sets containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$. Therefore, all $D_{x,y,z}^{(1)}$'s form the graph, named $D^{(1)}$, in $PD_\gamma(C_{4k+1})$ as shown in Figure 15.

Similarly, we can construct all $\gamma_{pr}(C_{4k+1})$ -sets as follows (the subscripts of all vertices are modulo $4k+1$): for all $x, y \in \{1, 2, \dots, k\}$ and $z \in \{1, 2, \dots, k-1\}$ with $x-y \leq 0$, $x-z \leq 1$, $y-z \geq 0$, and for each $i \in \{1, 2, 3, 4\}$,

$$D_{x,y,z}^{(i)} = B_{x,y,z}^{(i)} \cup \{v_{4k-2+i}, v_{4k-1+i}\},$$

where $B_{x,y,z}^{(i)}$ is a $\gamma_{pr}(P_{4k-3}(v_i : v_{4k-4+i}))$ -set,

$$\begin{aligned} D_{x,k,k}^{(i)} &= (D_{x,k,k-1}^{(i)} \setminus \{v_{4k-5+i}\}) \cup \{v_{4k-3+i}\}, \\ D_{1,y,0}^{(i)} &= (D_{1,y,1}^{(i)} \setminus \{v_{i+1}\}) \cup \{v_{i-1}\}. \end{aligned}$$

We get that these $D_{x,y,z}^{(i)}$'s are the only $\gamma_{pr}(C_{4k+1})$ -sets containing the pair $\{v_{4k-2+i}, v_{4k-1+i}\}$, and they form the graph $D^{(i)}$ (see Figure 15) in $PD_\gamma(C_{4k+1})$. By Lemma 2.4, without loss of generality, we may assume that $B_{x,k,z}^{(i)}$ contains the pair $\{v_{4k-5+i}, v_{4k-4+i}\}$, and then we get the following properties.

(B'1) Let $x \in \{1, 2, \dots, k\}$ and $z \in \{0, 1, \dots, k-1\}$. If $y = k$, then $D_{x,y,z}^{(i)}$ contains the pairs $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$; otherwise, it contains the pairs $\{v_{4k-6+i}, v_{4k-5+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B'1.1) $D_{x,k,k-1}^{(i)}$ contains the pairs $\{v_{4k-8+i}, v_{4k-7+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$ for all $x \in \{1, 2, \dots, k-1\}$, and $D_{k,k,k-1}^{(i)}$ contains the pairs $\{v_{4k-7+i}, v_{4k-6+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B'1.2) if $z \neq k-1$, then $D_{x,k,z}^{(i)}$ contains the pairs $\{v_{4k-9+i}, v_{4k-8+i}\}$, $\{v_{4k-5+i}, v_{4k-4+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.

(B'2) Let $y \in \{1, 2, \dots, k\}$ and $z \in \{1, 2, \dots, k\}$. If $x = 1$, then $D_{x,y,z}^{(i)}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$; otherwise, it contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_{i+1}, v_{i+2}\}$.

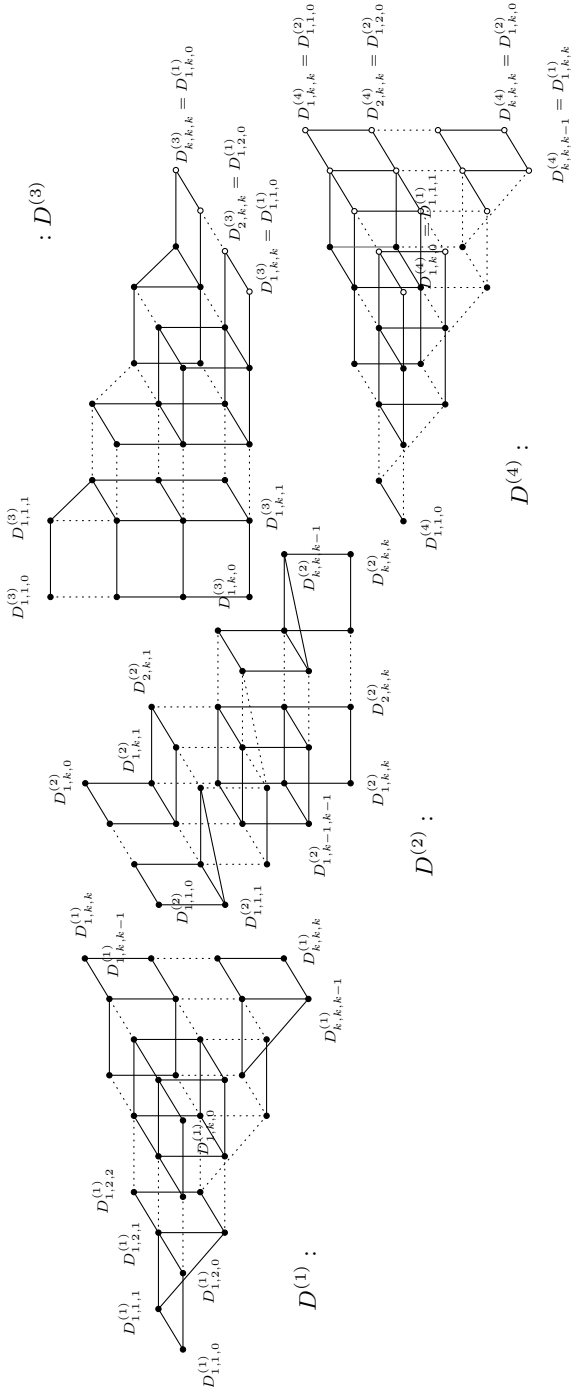


Fig. 15. The graphs $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, and $D^{(4)}$ in $PD_{\gamma}(C_{4k+1})$

- (B'2.1) $D_{1,1,1}^{(i)}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+2}, v_{i+3}\}$,
 $D_{1,y,1}^{(i)}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$, $\{v_{i+3}, v_{i+4}\}$ for
all $y \in \{2, 3, \dots, k\}$.
- (B'2.2) If $z \neq 1$, then $D_{1,y,z}^{(i)}$ contains the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_i, v_{i+1}\}$,
 $\{v_{i+4}, v_{i+5}\}$.
- (B'3) $D_{1,k,k}^{(i)}, D_{2,k,k}^{(i)}, \dots, D_{k,k,k}^{(i)}$ are the only $\gamma_{pr}(C_{4k+1})$ -sets in $D^{(i)}$ containing the
pairs $\{v_{4k-4+i}, v_{4k-3+i}\}$, $\{v_{4k-2+i}, v_{4k-1+i}\}$.
- (B'3.1) $D_{1,k,k}^{(i)}$ contains the pair $\{v_i, v_{i+1}\}$, and the others contain the pair
 $\{v_{i+1}, v_{i+2}\}$.
- (B'3.2) $D_{k,k,k}^{(i)}$ contains the pair $\{v_{4k-7+i}, v_{4k-6+i}\}$, and the others contain
the pair $\{v_{4k-8+i}, v_{4k-7+i}\}$.
- (B'4) $D_{1,1,0}^{(i)}, D_{1,2,0}^{(i)}, \dots, D_{1,k,0}^{(i)}$ are the only $\gamma_{pr}(C_{4k+1})$ -sets in $D^{(i)}$ containing
the pairs $\{v_{4k-2+i}, v_{4k-1+i}\}$, $\{v_{i-1}, v_i\}$.
- (B'4.1) $D_{1,1,0}^{(i)}$ contains the pair $\{v_{i+2}, v_{i+3}\}$, and the others contain the pair
 $\{v_{i+3}, v_{i+4}\}$.
- (B'4.2) $D_{1,k,0}^{(i)}$ contains the pair $\{v_{4k-5+i}, v_{4k-4+i}\}$, and the others contain the
pair $\{v_{4k-6+i}, v_{4k-5+i}\}$.

Note that $D^{(1)}$ and $D^{(2)}$ cannot have any common vertices in $PD_\gamma(C_{4k+1})$ since
otherwise there is a $\gamma_{pr}(C_{4k+1})$ -set containing the pairs $\{v_{4k-1}, v_{4k}\}$ and $\{v_{4k}, v_0\}$,
which is impossible. Similarly, $D^{(i)}$ and $D^{(i+1)}$ do not share any vertices in $PD_\gamma(C_{4k+1})$
for all $i \in \{2, 3\}$.

Then we consider all $\gamma_{pr}(C_{4k+1})$ -sets that are in both $D^{(1)}$ and $D^{(3)}$. Then
these sets must contain the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$. By (B'4) and (B'4.2),
 $D_{1,1,0}^{(1)}, D_{1,2,0}^{(1)}, \dots, D_{1,k,0}^{(1)}$ are the only $\gamma_{pr}(C_{4k+1})$ -sets in $D^{(1)}$ containing the pairs
 $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, and $D_{1,k,0}^{(1)}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$. By (B'3) and (B'3.2),
 $D_{1,k,k}^{(3)}, D_{2,k,k}^{(3)}, \dots, D_{k,k,k}^{(3)}$ are the only $\gamma_{pr}(C_{4k+1})$ -sets in $D^{(3)}$ containing the pairs
 $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, and $D_{k,k,k}^{(3)}$ contains the pair $\{v_{4k-4}, v_{4k-3}\}$. By Lemma 3.6(2),
for each $y \in \{1, 2, \dots, k\}$, we have

$$D_{1,y,0}^{(1)} = T_{1,y,0}^{(1)} \cup \{v_{4k-1}, v_{4k}, v_0, v_1\},$$

where $T_{1,y,0}^{(1)}$ is a $\gamma_{pr}(P_{4k-5}(v_3 : v_{4k-3}))$ -set, and

$$D_{y,k,k}^{(3)} = T_{y,k,k}^{(3)} \cup \{v_{4k-1}, v_{4k}, v_0, v_1\},$$

where $T_{y,k,k}^{(3)}$ is a $\gamma_{pr}(P_{4k-5}(v_3 : v_{4k-3}))$ -set. Since $D_{1,k,0}^{(1)}$ and $D_{k,k,k}^{(3)}$ contain the pair
 $\{v_{4k-4}, v_{4k-3}\}$, so do $T_{1,k,0}^{(1)}$ and $T_{k,k,k}^{(3)}$. By Lemma 2.2, $T_{1,k,0}^{(1)} = T_{k,k,k}^{(3)}$. By Theorem 2.6,
for each $y \in \{1, 2, \dots, k\}$, we have $T_{1,y,0}^{(1)} = T_{y,k,k}^{(3)}$, and hence $D_{1,y,0}^{(1)} = D_{y,k,k}^{(3)}$. Similarly,
we get $D_{1,y,0}^{(2)} = D_{y,k,k}^{(4)}$ for all $y \in \{1, 2, \dots, k\}$.

We next consider all $\gamma_{pr}(C_{4k+1})$ -sets, which are in both $D^{(1)}$ and $D^{(4)}$. Then these
sets must contain the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$. By (B'2), for all $y, z \in \{1, 2, \dots, k\}$,

all $D_{1,y,z}^{(1)}$'s are the only $\gamma_{pr}(C_{4k+1})$ -sets in $D^{(1)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, and they form the left graph in Figure 16. By (B'1), for all $x \in \{1, 2, \dots, k\}$ and $z \in \{0, 1, \dots, k-1\}$, all $D_{x,k,z}^{(4)}$'s are the only $\gamma_{pr}(C_{4k+1})$ -sets in $D^{(4)}$ containing the pairs $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, and they form the right graph in Figure 16. To show that $D_{1,y,z}^{(1)} = D_{z,k,y-1}^{(4)}$ for all $y, z \in \{1, 2, \dots, k\}$ with $y - z \geq 0$, it suffices to show that $D_{1,k,k}^{(1)} = D_{k,k,k-1}^{(4)}$. By (B'3.1), $D_{1,k,k}^{(1)}$ contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$. By (B'1.1), $D_{k,k,k-1}^{(4)}$ contains these three pairs as well. By Lemma 3.6(2), we have

$$D_{1,k,k}^{(1)} = T_{1,k,k}^{(1)} \cup \{v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}\}$$

and

$$D_{k,k,k-1}^{(4)} = T_{k,k,k-1}^{(4)} \cup \{v_{4k-3}, v_{4k-2}, v_{4k-1}, v_{4k}\},$$

where $T_{1,k,k}^{(1)}$ and $T_{k,k,k-1}^{(4)}$ are $\gamma_{pr}(P_{4k-5}(v_1 : v_{4k-5}))$ -sets containing the pair $\{v_1, v_2\}$. By Lemma 2.2, we get $T_{1,k,k}^{(1)} = T_{k,k,k-1}^{(4)}$, and thus $D_{1,k,k}^{(1)} = D_{k,k,k-1}^{(4)}$.

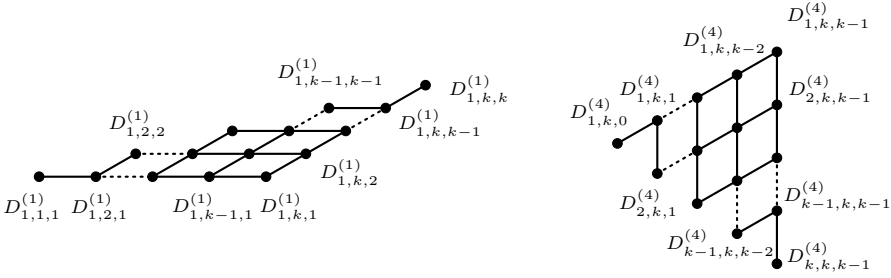


Fig. 16. The $\gamma_{pr}(C_{4k+1})$ -sets containing the pairs $\{v_{4k-1}, v_{4k}\}$ and $\{v_1, v_2\}$

Next, we consider all edges between a set in $D^{(1)}$ and a set in $D^{(2)}$. We first find all neighbors of $D_{1,y,0}^{(1)}$ in $D^{(2)}$ for each $y \in \{1, 2, \dots, k\}$. We show that $D_{1,1,0}^{(1)}$ is adjacent to $D_{k,k,k}^{(2)}$, and $D_{1,k,0}^{(1)}$ is adjacent to $D_{1,1,0}^{(2)}$. By (B'4), (B'4.1), (B'4.2), $D_{1,1,0}^{(1)}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_3, v_4\}$, the set $D_{1,y,0}^{(1)}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$ for each $y \in \{2, 3, \dots, k-1\}$, and $D_{1,k,0}^{(1)}$ contains the pairs $\{v_{4k-4}, v_{4k-3}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_0, v_1\}$, $\{v_4, v_5\}$. Since each set in $D^{(2)}$ contains the pair $\{v_{4k}, v_0\}$, the set $D_{1,y,0}^{(1)}$ is adjacent to some set in $D^{(2)}$ if and only if $(D_{1,y,0}^{(1)} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ or $(D_{1,y,0}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is a $\gamma_{pr}(C_{4k+1})$ -set. We have $(D_{1,1,0}^{(1)} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ is a $\gamma_{pr}(C_{4k+1})$ -set, but $(D_{1,y,0}^{(1)} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ is not if $y \neq 1$. Note that $(D_{1,1,0}^{(1)} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_0\}$. By (B'3.2), $D_{k,k,k}^{(2)}$ also contains these three pairs. By Lemma 2.2 and 3.6(2), we get

$(D_{1,1,0}^{(1)} \setminus \{v_1\}) \cup \{v_{4k-2}\}$ and $D_{k,k,k}^{(2)}$ are unions of a unique $\gamma_{pr}(P_{4k-5}(v_2 : v_{4k-4}))$ -set containing the pair $\{v_{4k-5}, v_{4k-4}\}$ and $\{v_{4k-2}, v_{4k-1}, v_{4k}, v_0\}$. Hence,

$$(D_{1,1,0}^{(1)} \setminus \{v_1\}) \cup \{v_{4k-2}\} = D_{k,k,k}^{(2)},$$

that is, $D_{1,1,0}^{(1)}$ is adjacent to $D_{k,k,k}^{(2)}$. Moreover, we see that $(D_{1,k,0}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is a $\gamma_{pr}(C_{4k+1})$ -set, but $(D_{1,y,0}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ is not if $y \neq k$. Note that $(D_{1,k,0}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_2\}$ contains the pairs $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, $\{v_4, v_5\}$. By (B'4), $D_{1,1,0}^{(2)}$ also contains these three pairs. By Lemmas 3.6(2) and 2.2, we have $(D_{1,k,0}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_2\} = D_{1,1,0}^{(2)}$, that is, $D_{1,k,0}^{(1)}$ is adjacent to $D_{1,1,0}^{(2)}$.

We next find all neighbors of $D_{x,k,k}^{(1)}$ in $D^{(2)}$ for each $x \in \{1, 2, \dots, k\}$. We claim that $D_{x,k,k}^{(1)}$ is adjacent to $D_{1,k,x-1}^{(2)}$ for each $x \in \{1, 2, \dots, k\}$, and $D_{k,k,k}^{(1)}$ is adjacent to $D_{1,k,k}^{(2)}$. By (B'3), (B'3.1), (B'3.2), $D_{1,k,k}^{(1)}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_1, v_2\}$, the set $D_{x,k,k}^{(1)}$ contains the pairs $\{v_{4k-7}, v_{4k-6}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$ for each $x \in \{2, 3, \dots, k-1\}$, and $D_{k,k,k}^{(1)}$ contains the pairs $\{v_{4k-6}, v_{4k-5}\}$, $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k-1}, v_{4k}\}$, $\{v_2, v_3\}$. Note that $D_{x,k,k}^{(1)}$ is adjacent to some set in $D^{(2)}$ if and only if $(D_{x,k,k}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ or $(D_{x,k,k}^{(1)} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$ -set. We have $(D_{x,k,k}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$ -set for each $x \in \{1, 2, \dots, k\}$, and then we let $N_x = (D_{x,k,k}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_0\}$. Note that N_1 contains the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and N_2, N_3, \dots, N_k contain the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$, and they form a path with k vertices in $D^{(2)}$. By (B'4.2), we have $D_{1,k,0}^{(2)}$ is the only $\gamma_{pr}(C_{4k+1})$ -set in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_1, v_2\}$, and by (B'1) and (B'2), we have that $D_{1,k,1}^{(2)}, D_{1,k,2}^{(2)}, \dots, D_{1,k,k-1}^{(2)}$ are the only $\gamma_{pr}(C_{4k+1})$ -sets in $D^{(2)}$ containing the pairs $\{v_{4k-3}, v_{4k-2}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$, and they form a path with k vertices in $D^{(2)}$. Then we can conclude that for each $x \in \{1, 2, \dots, k\}$, $N_x = D_{1,k,x-1}^{(2)}$, which means $D_{x,k,k}^{(1)}$ is adjacent to $D_{1,k,x-1}^{(2)}$. Moreover, $(D_{k,k,k}^{(1)} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$ -set, but $(D_{x,k,k}^{(1)} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ is not if $x \neq k$. Note that $(D_{k,k,k}^{(1)} \setminus \{v_{4k-3}\}) \cup \{v_0\}$ contains the pairs $\{v_{4k-2}, v_{4k-1}\}$, $\{v_{4k}, v_0\}$, $\{v_2, v_3\}$. By (B'3.1), $D_{1,k,k}^{(2)}$ also contains these three pairs. By Lemmas 3.6(2) and 2.2, we get $(D_{k,k,k}^{(1)} \setminus \{v_{4k-3}\}) \cup \{v_0\} = D_{1,k,k}^{(2)}$, that is, $D_{k,k,k}^{(1)}$ is adjacent to $D_{1,k,k}^{(2)}$.

Last but not least, we find all neighbors in $D^{(2)}$ of the other $\gamma_{pr}(C_{4k+1})$ -sets in $D^{(1)}$. We prove that $D_{x,k,z}^{(1)}$ is adjacent to $D_{1,z,x-1}^{(2)}$ for all $x \in \{1, 2, \dots, k\}$, $z \in \{1, 2, \dots, k-1\}$. Recall that for all $x, y \in \{1, 2, \dots, k\}$, $z \in \{1, 2, \dots, k-1\}$, $D_{x,y,z}^{(1)}$ contains the pair $\{v_{4k-1}, v_{4k}\}$ but not v_{4k-2}, v_0 . Then $D_{x,y,z}^{(1)}$ is adjacent to some set in $D^{(2)}$ if and only if $(D_{x,y,z}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is a $\gamma_{pr}(C_{4k+1})$ -set. By (B'1), $D_{x,y,z}^{(1)}$ contains the pairs $\{v_{4k-5}, v_{4k-4}\}$, $\{v_{4k-1}, v_{4k}\}$ for all $y \neq k$, so $(D_{x,y,z}^{(1)} \setminus \{v_{4k-1}\}) \cup \{v_0\}$ is not a dominating set. By (B'1) and (B'2), for all $z \in \{1, 2, \dots, k-1\}$, we have

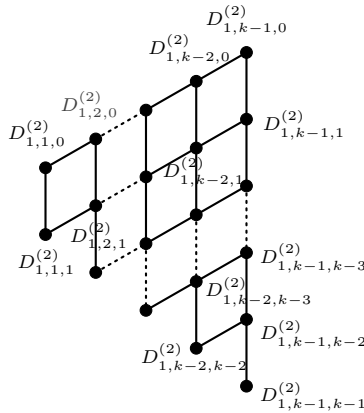


Fig. 18. The graph in $D^{(2)}$ induced by $D_{1,y,z}^{(2)}$'s

Thus, there is no edge between a set in $\widehat{D}^{(1)}$ and a set in $\widehat{D}^{(3)}$. Similarly, there is no edge between a set in $D^{(2)} - \{D_{1,y,0}^{(2)} : 1 \leq y \leq k\}$ and a set in $D^{(4)} - \{D_{y,k,k}^{(4)} : 1 \leq y \leq k\}$. Recall that $D_{1,y,z}^{(1)} = D_{z,k,y-1}^{(4)}$ for all $y, z \in \{1, 2, \dots, k\}$. Also, for all $y \in \{1, 2, \dots, k\}$, $D_{1,y,0}^{(1)} = D_{y,k,k}^{(3)}$, which has a neighbor in $D^{(4)}$. Hence, we consider all edges between a set in

$$\widetilde{D}^{(1)} = D^{(1)} - \{D_{1,y,z}^{(1)}, D_{1,y,0}^{(1)} : 1 \leq y, z \leq k\}$$

and a set in

$$\widetilde{D}^{(4)} = D^{(4)} - \{D_{z,k,y-1}^{(4)} : 1 \leq y, z \leq k\}.$$

Note that a set in $\widetilde{D}^{(1)}$ contains the pairs $\{v_{4k-1}, v_{4k}\}, \{v_2, v_3\}$ while a set in $\widetilde{D}^{(4)}$ contains the pair $\{v_1, v_2\}$ but not $\{v_{4k-1}, v_{4k}\}$. Thus, there is no edge between a set in $\widetilde{D}^{(1)}$ and a set in $\widetilde{D}^{(4)}$. This completes the proof. \square

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