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UNIQUENESS OF SOLUTION OF A NONLINEAR EVOLUTION DAM PROBLEM IN A HETEROGENEOUS POROUS MEDIUM

Messaouda Ben Attia, Elmehdi Zaouche, and Mahmoud Bousselsal

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Abstract. By choosing convenient test functions and using the method of doubling variables, we prove the uniqueness of the solution to a nonlinear evolution dam problem in an arbitrary heterogeneous porous medium of $\mathbb{R}^n$ ($n \in \{2,3\}$) with an impermeable horizontal bottom.

Keywords: test function, method of doubling variables, nonlinear evolution dam problem, heterogeneous porous medium, uniqueness.

Mathematics Subject Classification: 35A02, 76S05.

1. INTRODUCTION

Without loss of generality, we can assume that $n=2$. Let Ω be a bounded domain in $\mathbb{R}^2$ with horizontal bottom and locally Lipschitz boundary $\partial\Omega := \Gamma$ which represents a porous medium and let $x=(x_1,x_2)$ be the generic point of Ω. Let A, B and D be real numbers such that $B>A$. The boundary Γ is divided into two parts such that one part $\Gamma_1=[A,B]\times\{D\}$ is the impervious part and the other Γ_2 is the pervious part which is a nonempty relatively open subset of Γ (see Figure 1). For a positive real number T, let $Q=\Omega\times(0,T)$ be the space-time cylinder. Let $a:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying for some positive constants α,β and $p>1$,

$$\forall r\in\mathbb{R}: \quad \alpha|r|^p \le a(r)r, \tag{1.1}$$

$$\forall r\in\mathbb{R}: \quad |a(r)| \le \beta|r|^{p-1}, \tag{1.2}$$

$$\forall r_1,r_2\in\mathbb{R},\ r_1\neq r_2: \quad (a(r_1)-a(r_2))(r_1-r_2)>0 \tag{1.3}$$

and let $h:(A,B)\to\mathbb{R}$ be a Lipschitz continuous function of the variable x_1 such that for two positive constants $\underline{h}$ and $\overline{h}$,

$$\forall x_1\in(A,B): \quad \underline{h}\le h(x_1)\le\overline{h}. \tag{1.4}$$

Moreover, let $g_0 : \Omega \to \mathbb{R}$ be a measurable function satisfying

$$0 \leq g_0 \leq 1 \quad \text{a.e. in } \Omega. \tag{1.5}$$

We are concerned with a flow of an incompressible fluid through Ω in a time interval $[0, T]$ in which the fluid is governed by the following nonlinear version

$$\Omega \times \mathbb{R}^2 \mapsto \mathbb{R},\ (x, (r, s)) \mapsto h(x_1)a(s) \tag{1.6}$$

of Darcy's law. The pressure $\mathbf{p}$ and velocity v of the fluid are related by

$$v = -h(x_1)a((\mathbf{p} + x_2)_{x_2}).$$

Notice that if $k : (A, B) \to \mathbb{R}$ is a function such that $x_1 \mapsto (k(x_1))^{p-1}$ fulfills the conditions imposed on h,

$$M(x) = \begin{pmatrix} 0 & 0 \\ 0 & k(x_1) \end{pmatrix}$$

is the matrix permeability of the porous medium and $a(r) = |r|^{p-2}r$, we obtain the following form of nonlinear Darcy's law which corresponds to the nonhomogeneous p-Laplacian:

$$|M(x)\nabla(\mathbf{p} + x_2)|^{p-2}M(x)\nabla(\mathbf{p} + x_2) = \underbrace{(k(x_1))^{p-1}}_{h(x_1)}\underbrace{|(\mathbf{p} + x_2)_{x_2}|^{p-2}(\mathbf{p} + x_2)_{x_2}}_{a((\mathbf{p}+x_2)_{x_2})}.$$

Let $\varphi \in C_x^{0,1} \cap C_t^1$ be a nonnegative function defined in $\overline{Q}$ which represents the assigned pressure on $\Gamma_2 \times (0, T)$ and let Σ be the parabolic boundary defined by $\Sigma_1 \cup \Sigma_2$, where $\Sigma_1 = \Gamma_1 \times (0, T)$ and $\Sigma_2 = \Gamma_2 \times (0, T) = \Sigma_3 \cup \Sigma_4$ with $\Sigma_3 = (\Gamma_2 \times (0, T)) \cap \{\varphi > 0\}$ and $\Sigma_4 = (\Gamma_2 \times (0, T)) \cap \{\varphi = 0\}$.

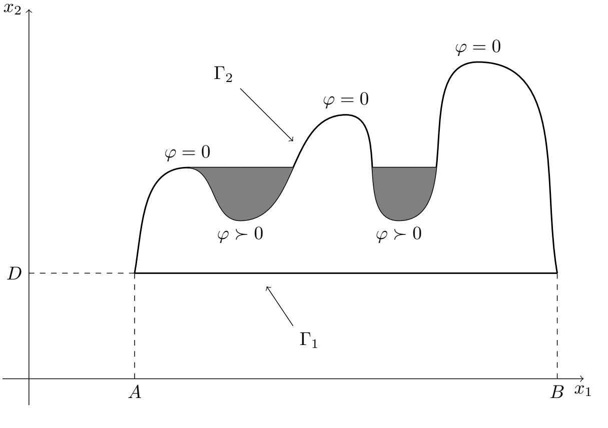

Fig. 1. A dam

Under above mentioned assumptions, the fluid flow is governed by following equations:

$$\begin{cases} u \geq x_2,\ 0 \leq g \leq 1,\ g(u - x_2) = 0 & \text{in } Q, \\ h(x_1)(a(u_{x_2}) - ga(1))_{x_2} + g_t = 0 & \text{in } Q, \\ u = \phi & \text{on } \Sigma_2, \\ g(\cdot, 0) = g_0 & \text{in } \Omega, \\ h(x_1)(a(u_{x_2}) - ga(1)) \cdot \nu = 0 & \text{on } \Sigma_1, \\ h(x_1)(a(u_{x_2}) - ga(1)) \cdot \nu \leq 0 & \text{on } \Sigma_4 \end{cases} \tag{1.7}$$

starting from the mass conservation law, where ν is the outward unit normal to the boundary $\partial\Omega$, $\phi = \varphi + x_2$, $u = \mathbf{p} + x_2$, g is the saturation of the fluid and (u, g) is the solution in search. The strong formulation (1.7) leads to the following weak formulation of the nonlinear heterogeneous evolution dam problem associated with the initial data g_0:

$$\begin{cases} \text{Find } (u, g) \in L^p(0, T; W^{1,p}(\Omega)) \times L^\infty(Q) \text{ such that:} \\ u \geq x_2,\ 0 \leq g \leq 1,\ g(u - x_2) = 0 \text{ in } Q, \\ u = \phi \quad \text{on } \Sigma_2, \\ \displaystyle\int_Q \left[h(x_1)(a(u_{x_2}) - ga(1))\xi_{x_2} + g\xi_t\right] dxdt \\ \displaystyle + \int_\Omega g_0(x)\xi(x, 0)\, dx \leq 0, \quad \forall \xi \in W^{1,p}(Q),\ \xi = 0 \text{ on } \Sigma_3, \\ \xi \geq 0 \text{ on } \Sigma_4,\ \xi(x, T) = 0 \text{ for a.e. } x \in \Omega. \end{cases} \tag{1.8}$$

In [16], the author established the existence of a solution by means of regularization for the evolution dam problem related to an incompressible fluid flow governed by a generalized nonlinear Darcy's law with Dirichlet boundary conditions on some part of the boundary using the Tychonoff fixed point theorem. He proved in [17] the continuity of solutions in t for this problem. Also, an existence of a solution was obtained in [7] by an approximation resulting from the compressible case.

For the homogeneous dam problem, the uniqueness of the solution has been obtained in [8] and [16] by the method of doubling variables, respectively, for linear and generalized nonlinear Darcy's laws. For a heterogeneous rectangular dam wet at the bottom and dry near to the top, the uniqueness for a linear evolution dam problem has been proved in [18], in both incompressible and compressible flows, by an idea from [11] in the homogeneous case. When $a(r) = r$, the uniqueness of the problem (1.8) in a rectangular porous medium has been obtained in [23] and [24] by the method of doubling variables, respectively, for incompressible and compressible flows. The technique of doubling variables is inspired by S.N. Kružhkov in [11] in order to obtain a L^1-contraction property for entropy solutions of hyperbolic problems. See [5, 6, 10, 12, 13, 15, 19–22] for some uses of this technique. For the applications of the Kružhkov method to stationary and non-stationary free boundary problems, we refer to [2–4, 8, 9, 16, 23].

In this paper, we choose convenient test functions and use the method of doubling variables to prove the uniqueness of the solution in a heterogeneous porous medium for the evolution dam problem (1.8) which is associated with an incompressible fluid governed by the nonlinear version (1.6) of Darcy's law. Our techniques are based on the uniqueness of solutions obtained in [16] and [23]. It should be noted that our uniqueness result is new in the context of a nonlinear evolution dam problem in an arbitrary heterogeneous bounded domain of $\mathbb{R}^n$ with $n \in \{2,3\}$. In Section 2, we give some properties of the solutions of (1.8) and in Section 3, we state and prove our uniqueness theorem that the solution of the problem (1.8) associated with the initial data g_0 is unique.

2. SOME PROPERTIES OF THE SOLUTIONS

In this section, we will give some properties of the solutions which are useful in proving our main result.

Lemma 2.1 ([16]). *Let $v \in W^{1,p}(Q)$ and $F \in W^{1,\infty}_{loc}(\mathbb{R}^2)$ be functions satisfying*

$$\begin{aligned}&F(u-x_2,v) \in L^p(0,T;W^{1,p}(\Omega)), \quad F(\phi-x_2,v)\in W^{1,p}(Q),\\&F(z_1,z_2)\geq 0 \ \text{ a.e. } (z_1,z_2)\in\mathbb{R}^2 \ \text{ and either } \frac{\partial F}{\partial z_1}(z_1,z_2)\geq 0 \ \text{ a.e. } (z_1,z_2)\in\mathbb{R}^2\\&\quad \text{or } \frac{\partial F}{\partial z_1}(z_1,z_2)\leq 0 \ \text{ a.e. } (z_1,z_2)\in\mathbb{R}^2.\end{aligned}$$

Then, if (u,g) is a solution of (1.8) and $\xi \in \mathcal{D}(\overline{\Omega}\times(0,T))$, we have

$$\begin{aligned}&\int_Q h(x_1)(a(u_{x_2})-ga(1))(F(u-x_2,v)\xi)_{x_2}+g(F(0,v)\xi)_t\,dxdt\\&\quad=\int_Q h(x_1)(a(u_{x_2})-ga(1))(F(\phi-x_2,v)\xi)_{x_2}+g(F(\phi-x_2,v)\xi)_t\,dxdt.\end{aligned}$$

In particular, if $F(\phi-x_2,v)\xi=0$ on Σ_2,

$$\int_Q h(x_1)(a(u_{x_2})-ga(1))(F(u-x_2,v)\xi)_{x_2}+g(F(0,v)\xi)_t\,dxdt.$$

The following corollary is an immediate consequence of Lemma 2.1.

Corollary 2.2. *Let $\epsilon>0$ and $k\geq 0$ be real numbers and let $\xi\in\mathcal{D}(\mathbb{R}^2\times(0,T))$ such that $\xi\geq 0$ and $\xi=0$ on Σ_3. If (u,g) is a solution of (1.8), we have*

$$\int_Q h(x_1)a(u_{x_2})\Big(\min\Big(\frac{(u-x_2-k)^+}{\epsilon},1\Big)\xi\Big)_{x_2}\,dxdt=0.$$

Let us set

$$\sigma_1 = \overline{\Sigma}_2 \cap \Sigma_1 = (\overline{\Gamma}_2 \cap \Gamma_1) \times (0,T),$$
$$\sigma_2 = \overline{\Sigma}_3 \cap \Sigma_4 = (\Gamma_2 \times (0,T)) \cap \overline{\{\varphi > 0\}} \cap \{\varphi = 0\}$$

and let us assume throughout the rest of the paper that σ_1 and σ_2 are $(1,q)$ polar sets of $\overline{Q}$ (see [1]), where q is the conjugate exponent of p. Since the empty set is the only $(1,q)$ polar set of $\overline{Q}$ in the case $p > 3$, then we can consider that $p \leq 3$.

We use a regularization by convolution with respect to the variables x_2 and t to prove the following proposition.

Proposition 2.3. *Let $\lambda \in [0,1]$ and let $\xi \in \mathcal{D}(\mathbb{R}^2 \times (0,T))$ such that $\xi \geq 0$ and $\xi = 0$ on $\Sigma_1 \cup \Sigma_3$. If (u,g) is a solution of (1.8), we have*

$$\int_Q \Big\{ h(x_1)(a(u_{x_2}) - a(1))\xi_{x_2} + (\lambda - g)^+ (h(x_1)a(1)\xi_{x_2} - \xi_t) \Big\}\, dxdt \leq 0. \tag{2.1}$$

Proof. We apply Corollary 2.2 for $k = 0$ to get

$$\int_Q h(x_1)a(u_{x_2}) \Big(\min\Big(\frac{u - x_2}{\epsilon}, 1\Big)\xi \Big)_{x_2} dxdt = 0. \tag{2.2}$$

On the other hand, we have

$$\int_Q h(x_1)a(1) \Big(\min\Big(\frac{u - x_2}{\epsilon}, 1\Big)\xi \Big)_{x_2} dxdt = 0 \tag{2.3}$$

since $\min(\frac{u-x_2}{\epsilon}, 1)\xi = 0$ on Σ and $(h(x_1))_{x_2} = 0$ a.e. in Q. Subtracting (2.3) from and (2.2), we get

$$\int_Q h(x_1)(a(u_{x_2}) - a(1)) \Big(\min\Big(\frac{u - x_2}{\epsilon}, 1\Big)\xi \Big)_{x_2} dxdt = 0,$$

which can be written as

$$\begin{aligned} &\frac{1}{\epsilon} \int_{Q \cap \{u - x_2 < \epsilon\}} \xi h(x_1)(a(u_{x_2}) - a(1))(u_{x_2} - 1)\, dxdt \\ &+ \int_Q \min\Big(\frac{u - x_2}{\epsilon}, 1\Big) h(x_1)(a(u_{x_2}) - a(1))\xi_{x_2}\, dxdt = 0. \end{aligned} \tag{2.4}$$

By (1.3) and the fact that $\xi h(x_1) \geq 0$ a.e. in Q, the first integral of (2.4) is nonnegative, then

$$\int_Q \min\Big(\frac{u - x_2}{\epsilon}, 1\Big) h(x_1)(a(u_{x_2}) - a(1))\xi_{x_2}\, dxdt \leq 0. \tag{2.5}$$

Letting $\epsilon \to 0$ in (2.5), we obtain

$$\int_Q h(x_1)(a(u_{x_2}) - a(1))\xi_{x_2}\, dxdt \leq 0, \tag{2.6}$$

and then (2.1) holds for $\lambda = 0$. Also, the inequality (2.1) holds for $\lambda = 1$ since $0 \leq g \leq 1$ a.e. in Q and ξ is a test function for (1.8),

$$\int_Q \big\{h(x_1)(a(u_{x_2}) - ga(1))\xi_{x_2} + g\xi_t\big\}\, dxdt \leq 0. \tag{2.7}$$

Now, we will prove (2.1) for $\lambda \in (0,1)$. Without loss of generality, we can assume that

$$d(\mathrm{supp}(\xi), \Sigma_1 \cup \Sigma_3) := \varepsilon_0 > 0.$$

Let us set

$$\mathcal{A} = ((\mathbb{R}^2 \times (0,T))\backslash\Sigma_1) \cup \Sigma_3 \cup \sigma_2$$

and

$$\mathcal{A}_{\varepsilon_0} = \Big\{(x,t) \in \mathbb{R}^2 \times (0,T)/d((x,t), \Sigma_1 \cup \Sigma_3 \cup \sigma_2) > \frac{\varepsilon_0}{2}\Big\}.$$

We extend u (resp. g) on $\mathcal{A}\backslash Q$ by x_2 (resp. 1) and still denote by u (resp. g) this function. Also, the function h can be extended to a Lipschitz function on $\mathbb{R}$, still denote by h. We use a regularization by convolution for $a(u_{x_2})$ and g with respect to the variables x_2 and t, $(a(u_{x_2}))_\varepsilon = \rho_\varepsilon * a(u_{x_2})$, $g_\varepsilon = \rho_\varepsilon * g$ where $\varepsilon \in (0, \frac{\varepsilon_0}{2})$, $\rho_\varepsilon \in \mathcal{D}(\mathbb{R} \times (0,T))$, $\mathrm{supp}(\rho_\varepsilon) \subset B(0,\varepsilon)$ is a regularizing sequence. We can use Fubini's theorem to write

$$\begin{aligned}
&\int_{\mathcal{A}_{\varepsilon_0}} \big\{h(x_1)((a(u_{x_2}))_\varepsilon - g_\varepsilon a(1))\xi_{x_2} + g_\varepsilon \xi_t\big\}\, dxdt \\
&= \int_{\mathcal{A}_{\varepsilon_0}} \Big\{ \int_{\mathbb{R}\times(0,T)} (a(u_{x_2}) - ga(1))(x_1, x_2 - y, t - s)\rho_\varepsilon(y,s)\, dyds\Big\} h(x_1)\xi_{x_2}\, dxdt \\
&\quad + \int_{\mathcal{A}_{\varepsilon_0}} \Big\{ \int_{\mathbb{R}\times(0,T)} g(x_1, x_2 - y, t - s)\rho_\varepsilon(y,s)\, dyds\Big\}\xi_t\, dxdt \\
&= \int_{\mathbb{R}\times(0,T)} \rho_\varepsilon(y,s)\Big\{\int_{\mathcal{A}_{\varepsilon_0}} h(x_1)(a(u_{x_2}) - ga(1))(x_1, x_2 - y, t - s)\xi_{x_2}\, dxdt\Big\}\, dyds \\
&\quad + \int_{\mathbb{R}\times(0,T)} \rho_\varepsilon(y,s)\Big\{\int_{\mathcal{A}_{\varepsilon_0}} g(x_1, x_2 - y, t - s)\xi_t\, dxdt\Big\}\, dyds.
\end{aligned}$$

Then, if we make the change of variables $z = x_2 - y$ and $\tau = t - s$, we get

$$\begin{aligned}
&\int\limits_{\mathcal{A}_{\varepsilon_0}} \left\{h(x_1)((a(u_{x_2}))_\varepsilon - g_\varepsilon a(1))\xi_{x_2} + g_\varepsilon \xi_t\right\} dxdt \\
&= \int\limits_{B(0,\varepsilon)} \rho_\varepsilon(y,s)\Big\{ \int\limits_{\mathcal{A}_{\varepsilon_0}} h(x_1)(a(u_z) - ga(1))(x_1,z,t) \\
&\qquad\qquad \times (\xi(x_1, z+y, \tau+s))_z \, dx_1 dz d\tau \Big\} dyds \\
&\quad + \int\limits_{B(0,\varepsilon)} \rho_\varepsilon(y,s)\Big\{ \int\limits_{\mathcal{A}_{\varepsilon_0}} g(x_1,z,t)(\xi(x_1,z+y,\tau+s))_\tau \, dx_1 dz d\tau \Big\} dyds \\
&= \int\limits_{B(0,\varepsilon)} \rho_\varepsilon(y,s)\Big\{ \int\limits_{Q} h(x_1)(a(u_z) - ga(1))(x_1,z,t) \\
&\qquad\qquad \times (\xi(x_1, z+y, \tau+s))_z \, dx_1 dz d\tau \Big\} dyds \\
&\quad + \int\limits_{B(0,\varepsilon)} \rho_\varepsilon(y,s)\Big\{ \int\limits_{Q} g(x_1,z,t)(\xi(x_1,z+y,\tau+s))_\tau \, dx_1 dz d\tau \Big\} dyds.
\end{aligned}$$

Observe that $(x_1, z, \tau) \mapsto \xi(x_1, z+y, \tau+s)$ is a nonnegative function in $\mathcal{D}(\mathbb{R}^2 \times (0,T))$ and vanishes on $\Sigma_1 \cup \Sigma_3$ for all $(y,s) \in B(0,\varepsilon)$. Therefore, since $\rho_\varepsilon \geq 0$, we deduce from (2.7) that

$$\int\limits_{\mathcal{A}_{\varepsilon_0}} \left\{h(x_1)((a(u_{x_2}))_\varepsilon - g_\varepsilon a(1))\xi_{x_2} + g_\varepsilon \xi_t\right\} dxdt \leq 0,$$

which can be written as

$$\int\limits_{\mathcal{A}_{\varepsilon_0}} \left\{h(x_1)((a(u_{x_2}))_\varepsilon - a(1))\xi_{x_2} + (\lambda - g_\varepsilon)(h(x_1)a(1)\xi_{x_2} - \xi_t)\right\} dxdt \leq 0$$

since

$$\int\limits_{\mathcal{A}_{\varepsilon_0}} h(x_1)a(1)\xi_{x_2}\, dxdt = \int\limits_{\mathcal{A}_{\varepsilon_0}} \lambda \xi_t \, dxdt = 0.$$

Similarly, using (2.6), we arrive at

$$\int\limits_{\mathcal{A}_{\varepsilon_0}} h(x_1)((a(u_{x_2}))_\varepsilon - a(1))\xi_{x_2}\, dxdt \leq 0.$$

Now, for any positive real number δ, we set

$$K_\delta = \min\Big(\frac{(\lambda - g_\varepsilon)^+}{\delta}, 1\Big)$$

which satisfies $K_\delta \in L^p_{\text{loc}}(\mathcal{A}_{\varepsilon_0})$, $K_{\delta x_2}, K_{\delta t} \in L^p_{\text{loc}}(\mathcal{A}_{\varepsilon_0})$. By using the integration by parts formula, we obtain

$$\begin{aligned}
&\int\limits_{\mathcal{A}_{\varepsilon_0}} \left\{h(x_1)((a(u_{x_2}))_\varepsilon - a(1))\xi_{x_2} + K_\delta(\lambda - g_\varepsilon)(h(x_1)a(1)\xi_{x_2} - \xi_t)\right\} dxdt \\
&- \frac{\delta}{2} \int\limits_{\mathcal{A}_{\varepsilon_0}} K_\delta^2 (h(x_1)a(1)\xi_{x_2} - \xi_t)\, dxdt \\
&= \int\limits_{\mathcal{A}_{\varepsilon_0}} \big\{h(x_1)((a(u_{x_2}))_\varepsilon - a(1))(K_\delta\xi)_{x_2} \\
&\qquad + (\lambda - g_\varepsilon)(h(x_1)a(1)(K_\delta\xi)_{x_2} - (K_\delta\xi)_t)\big\}\, dxdt \\
&\quad + \int\limits_{\mathcal{A}_{\varepsilon_0}} h(x_1)((a(u_{x_2}))_\varepsilon - a(1))((1-K_\delta)\xi)_{x_2}\, dxdt
\end{aligned}$$

and since (2.6) and (2.7) remain valid, respectively, for $K_\delta\xi$ and $(1-K_\delta)\xi$, it follows that

$$\begin{aligned}
&\int\limits_{\mathcal{A}_{\varepsilon_0}} \left\{h(x_1)((a(u_{x_2}))_\varepsilon - a(1))\xi_{x_2} + K_\delta(\lambda - g_\varepsilon)(h(x_1)a(1)\xi_{x_2} - \xi_t)\right\} dxdt \\
&- \frac{\delta}{2} \int\limits_{\mathcal{A}_{\varepsilon_0}} K_\delta^2 (h(x_1)a(1)\xi_{x_2} - \xi_t)\, dxdt \le 0.
\end{aligned} \tag{2.8}$$

Finally, we pass successively to the limit in (2.8) as $\delta \to 0$ and then as $\varepsilon \to 0$ and using Lebesgue's dominated convergence theorem, we obtain

$$\int\limits_{\mathcal{A}_{\varepsilon_0}} \left\{h(x_1)(a(u_{x_2}) - a(1))\xi_{x_2} + (\lambda - g)^+(h(x_1)a(1)\xi_{x_2} - \xi_t)\right\} dxdt \le 0,$$

and then (2.1) holds since $u = x_2$, $g = 1$ a.e. in $\mathcal{A}\backslash Q$ and ε_0 is arbitrary. □

We use Corollary 2.2 and Proposition 2.3 and argue as in the proof of [16, Lemma 5.2] to prove the following lemma:

Lemma 2.4. *Let χ be a function of $L^\infty(Q)$ satisfying*

$$0 \le \chi \le 1 \quad \text{and} \quad h(x_1)a(1)\chi_{x_2} - \chi_t = 0 \quad \text{in } \mathcal{D}'(Q).$$

Let $\xi, \xi_1, \xi_2 \in \mathcal{D}(\mathbb{R}^2 \times (0,T))$ such that $\xi, \xi_1 \ge 0$, $\xi = \xi_1 = 0$ on $\Sigma_1 \cup \Sigma_3$, $\xi_2 = 0$ on ∂Q and let k, λ, ϵ be nonnegative real numbers such that $\epsilon > 0$ and $\lambda \in 1 - H(k)$ with H denotes the maximal monotone graph associated to the Heaviside function. Then, if (u, g) is a solution of (1.8), we have

$$\int_Q \Big\{ h(x_1)a(u_{x_2})\Big(\min\Big(\frac{(u - x_2 - k)^+}{\epsilon}, 1\Big)\xi\Big)_{x_2} + (\lambda - g)^+(h(x_1)a(1)\xi_{1x_2} - \xi_{1t}) + (\lambda - \chi)^+(h(x_1)a(1)\xi_{2x_2} - \xi_{2t})\Big\}\, dxdt \le C(u, k, \xi_1),$$

where

$$\begin{aligned} C(u, 0, \xi_1) &= -\int_Q h(x_1)(a(u_{x_2}) - a(1))\xi_{1x_2}\, dxdt \\ &= \lim_{\epsilon \to 0} \int_Q h(x_1)(a(u_{x_2}) - a(1))\Big(\min\Big(\frac{u - x_2}{\epsilon}, 1\Big)\Big)_{x_2} \xi_1\, dxdt, \qquad (2.9) \\ C(u, k, \xi_1) &= 0 \quad \text{for } k > 0. \end{aligned}$$

We use Lemma 2.1 and employ the regularization by convolution with respect to the variables x_2 and t as in the proof of Proposition 2.3. We obtain by an argument similar to that in the proof of [16, Lemma 5.3] the following result:

Lemma 2.5. *Let us assume that $(0, h(x_1)a(1)).\nu \le 0$ on Γ_1. Let Ψ be a function of $C^\infty(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$ such that $\Psi(0) = 0$, $\Psi' \ge 0$, $\Psi \le 1$ and let k, λ, ϵ be the nonnegative real numbers defined in Lemma 2.4. Then, if (u, g) is a solution of (1.8) and $\xi \in \mathcal{D}(\mathbb{R}^2 \times (0,T))$, $\xi \ge 0$, $(1 - \Psi(u - x_2))\xi = 0$ on Σ_2, we have*

$$\int_Q \Big\{ h(x_1)(a(u_{x_2}) - \lambda a(1))\Big(\min\Big(\frac{(k - (u - x_2))^+}{\epsilon}, 1\Big)(1 - \Psi(u - x_2))\xi\Big)_{x_2} - (g - \lambda)^+(h(x_1)a(1)\xi_{x_2} - \xi_t)\Big\}\, dxdt \ge 0.$$

3. UNIQUENESS OF SOLUTION

In this section we state and prove our uniqueness theorem.

Theorem 3.1. *Assume that* (1.1)–(1.5) *and* $(0, h(x_1)a(1)).\nu \leq 0$ *on* Γ_1 *hold. Then, the solution of the problem* (1.8) *associated with the initial data* g_0 *is unique.*

We seek to obtain a comparison result for solutions which allows us to prove the uniqueness of the solution of the problem (1.8). First, we begin with the following two comparison lemmas of solutions.

Lemma 3.2. *Let* $\mathbf{B}$ *be a bounded open subset of* $\mathbb{R}^2$ *such that either* $\mathbf{B} \cap \Gamma = \emptyset$ *or* $\mathbf{B} \cap \Gamma$ *is a Lipschitz graph. For two solutions* (u_1, g_1) *and* (u_2, g_2) *to* (1.8), *we set* $u_m = \min(u_1, u_2)$ *and* $g_M = \max(g_1, g_2)$. *Then, for all* $\xi \in \mathcal{D}(\mathbf{B} \times (0, T))$, $\xi \geq 0$, *supp*$(\xi) \cap (\Sigma_1 \cup \Sigma_3) = \emptyset$ *and for* $i = 1, 2$, *we have*

$$\int_Q \big\{h(x_1)(a(u_{ix_2}) - a(u_{mx_2}) - (g_i - g_M)a(1))\xi_{x_2} + (g_i - g_M)\xi_t\big\}\, dxdt \leq 0. \tag{3.1}$$

Proof. Let (u_1, g_1) and (u_2, g_2) be two solutions of (1.8) and let ξ be the function defined in Lemma 3.2. We define

$$\forall (x, t, y, s) \in \overline{Q \times Q} :$$
$$\zeta(x, t, y, s) = \xi\Big(\frac{x + y}{2}, \frac{t + s}{2}\Big)\rho_{1,\delta_1}\Big(\frac{t - s}{2}\Big)\rho_{2,\delta_1}\Big(\frac{x_1 - y_1}{2}\Big)\rho_{3,\delta_2}\Big(\frac{x_2 - y_2}{2}\Big),$$

where δ_1, δ_2 are positive real numbers, $\rho_{1,\delta_1}, \rho_{2,\delta_1}, \rho_{3,\delta_2} \in \mathcal{D}(\mathbb{R})$, $\rho_{1,\delta_1},\ \rho_{2,\delta_1},\ \rho_{3,\delta_2} \geq 0$ in $\mathbb{R}$,

$$\int_{\mathbb{R}} \rho_{1,\delta_1}(t)\, dt = \int_{\mathbb{R}} \rho_{2,\delta_1}(t)\, dt = \int_{\mathbb{R}} \rho_{3,\delta_2}(t)\, dt = 1,$$

$$\text{supp}(\rho_{1,\delta_1}),\ \text{supp}(\rho_{2,\delta_1}) \subset (-\delta_1, \delta_1), \quad \text{supp}(\rho_{3,\delta_2}) \subset (-\delta_2, \delta_2)$$

and

$$\forall (x, y) \in (\mathbf{B} \cap \Omega) \times (\mathbf{B} \backslash \Omega), \quad \rho_{2,\delta_1}\Big(\frac{x_1 - y_1}{2}\Big)\rho_{3,\delta_2}\Big(\frac{x_2 - y_2}{2}\Big) = 0.$$

Notice that, by choosing δ_1 and δ_2 small enough, $\zeta \in \mathcal{D}(\mathbf{B} \times (0, T) \times \mathbf{B} \times (0, T))$ and

$$\zeta = 0 \quad \text{on } ((\Sigma_1 \cup \Sigma_3) \times Q) \cup (Q \times \Sigma). \tag{3.2}$$

So, applying Lemma 2.4 to (u_1, g_1) with

$$k = u_2(y, s) - y_2, \quad \lambda = g_2(y, s), \quad \xi(x, t) = \xi_1(x, t) = \zeta(x, t, y, s)$$

and $\xi_2(x, t) = 0$, we obtain for a.e. $(y, s) \in Q$,

$$\int_Q \Big\{h(x_1)a(u_{1x_2})\Big(\min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big)\xi\Big)_{x_2}$$
$$+ (g_2 - g_1)^+(h(x_1)a(1)\zeta_{x_2} - \zeta_t)\Big\}\, dxdt \leq C(u_1, u_2 - y_2, \zeta)$$

and integrating over Q, we get

$$\begin{aligned}\int\limits_{Q\times Q} &\Big\{h(x_1)a(u_{1x_2})\Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)\xi\Big)_{x_2}\\ &+(g_2-g_1)^+(h(x_1)a(1)\zeta_{x_2}-\zeta_t)\Big\}\,dxdtdyds \le \int\limits_Q C(u_1,u_2-y_2,\zeta)\,dyds.\end{aligned} \tag{3.3}$$

On the other hand, applying Lemma 2.5 to (u_2, g_2) with

$$k = u_1(x,t) - x_2, \quad \lambda = g_1(x,t), \quad \xi(y,s) = \zeta(x,t,y,s)$$

and $\Psi = 0$, we have for a.e. $(x,t) \in Q$,

$$\begin{aligned}\int\limits_Q &\Big\{h(y_1)(a(u_{2y_2})-g_1a(1))\Big(\min\Big(\frac{(u_1-x_2-u_2+y_2)^+}{\epsilon},1\Big)\zeta\Big)_{y_2}\\ &-(g_2-g_1)^+(h(y_1)a(1)\zeta_{y_2}-\zeta_t)\Big\}\,dyds \ge 0.\end{aligned} \tag{3.4}$$

Using (3.2) and the fact that the function $(y,s) \mapsto h(y_1)g_1a(1)$ does not depend on y_2, we find

$$\int\limits_Q h(y_1)g_1a(1)\Big(\min\Big(\frac{(u_1-x_2-u_2+y_2)^+}{\epsilon},1\Big)\zeta\Big)_{y_2}\,dyds = 0,$$

therefore, (3.4) can be written as

$$\begin{aligned}\int\limits_Q &\Big\{h(y_1)a(u_{2y_2})\Big(\min\Big(\frac{(u_1-x_2-u_2+y_2)^+}{\epsilon},1\Big)\zeta\Big)_{y_2}\\ &-(g_2-g_1)^+(h(y_1)a(1)\zeta_{y_2}-\zeta_t)\Big\}\,dyds \ge 0.\end{aligned}$$

By integrating over Q, we obtain

$$\int_{Q\times Q} \Big\{h(y_1)a(u_{2y_2})\Big(\min\Big(\frac{(u_1-x_2-u_2+y_2)^+}{\epsilon},1\Big)\zeta\Big)_{y_2} - (g_2-g_1)^+(h(y_1)a(1)\zeta_{y_2}-\zeta_t)\Big\}\,dxdtdyds \geq 0. \tag{3.5}$$

Since

$$\min\Big(\frac{(u_1-x_2-u_2+y_2)^+}{\epsilon},1\Big)\zeta = 0 \quad \text{on } (\Sigma\times Q)\cup(Q\times\Sigma)$$

and the functions $h(x_1)$ and u_1 (resp. $h(y_1)$ and u_2) do not depend on y_2 (resp. x_2), we have

$$\int_{Q\times Q} h(x_1)a(u_{1x_2})\Big(\min\Big(\frac{(u_1-x_2-u_2+y_2)^+}{\epsilon},1\Big)\zeta\Big)_{y_2}\,dxdtdyds = 0, \tag{3.6}$$

$$\int_{Q\times Q} h(y_1)a(u_{2y_2})\Big(\min\Big(\frac{(u_1-x_2-u_2+y_2)^+}{\epsilon},1\Big)\zeta\Big)_{x_2}\,dxdtdyds = 0, \tag{3.7}$$

$$u_{2x_2} = u_{1y_2} = 0 \quad \text{a.e. in } Q. \tag{3.8}$$

Subtracting (3.5) from and (3.3) and using (3.6)–(3.8), we get

$$\begin{aligned}\int_{Q\times Q} &\Big\{\big[h(x_1)a((\partial_{x_2}+\partial_{y_2})u_1)-h(y_1)a((\partial_{x_2}+\partial_{y_2})u_2)\big] \\ &\times(\partial_{x_2}+\partial_{y_2})\Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)\zeta\Big) \\ &+(g_2-g_1)^+\big[(h(x_1)\zeta_{x_2}+h(y_1)\zeta_{y_2})a(1)-\zeta_t-\zeta_s\big]\Big\}\,dxdtdyds \\ \leq \int_Q &C(u_1,u_2-y_2,\zeta)\,dyds,\end{aligned}$$

which leads to

$$\begin{aligned}
&\int\limits_{Q\times Q} (h(x_1)-h(y_1))a((\partial_{x_2}+\partial_{y_2})u_2)(\partial_{x_2}+\partial_{y_2})\zeta \\
&\qquad \times \min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)\,dxdtdyds \\
&+\int\limits_{Q\times Q} \Big\{h(x_1)\Big(a((\partial_{x_2}+\partial_{y_2})u_1)-a((\partial_{x_2}+\partial_{y_2})u_2)\Big)(\partial_{x_2}+\partial_{y_2})\zeta \\
&\qquad \times \min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big) \\
&\qquad + (g_2-g_1)^+\big[h(y_1)a(1)(\zeta_{x_2}+\zeta_{y_2})-\zeta_t-\zeta_s\big]\Big\}\,dxdtdyds \\
&+\int\limits_{Q\times Q} (g_2-g_1)^+(h(x_1)-h(y_1))a(1)\zeta_{x_2}\,dxdtdyds \\
&+\int\limits_{Q\times Q} (h(x_1)-h(y_1))a((\partial_{x_2}+\partial_{y_2})u_2)\zeta \\
&\qquad \times (\partial_{x_2}+\partial_{y_2})\Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)\Big)\,dxdtdyds \\
&+\int\limits_{Q\times(Q\cap\{u_2=y_2\})} h(x_1)(a(u_{1x_2})-a(1))\zeta\Big(\min\Big(\frac{u_1-x_2}{\epsilon},1\Big)\Big)_{x_2}\,dxdtdyds \\
&\le \int\limits_Q C(u_1,u_2-y_2,\zeta)\,dyds
\end{aligned} \tag{3.9}$$

taking into account the conditions (1.3) and (1.4). Let us consider the following change of variables:

$$z=\frac{x+y}{2},\quad \tau=\frac{t+s}{2},\quad \sigma=\frac{x-y}{2},\quad \theta=\frac{t-s}{2}. \tag{3.10}$$

Let J_1 and J_2 denote, respectively, the domains of the variables $z_2=\frac{x_2+y_2}{2}$ and $\sigma_2=\frac{x_2-y_2}{2}$ and let I denote the domain of the variables x_2 and y_2. Set

$$\begin{aligned}
&Q_1=(A,B)\times J_1\times(0,T),\quad Q_2=\Big(\frac{A-B}{2},\frac{B-A}{2}\Big)\times J_2\times\Big(-\frac{T}{2},\frac{T}{2}\Big),\\
&Q_3=(A,B)\times I\times(0,T),\quad Q_4=\Big(\frac{A-B}{2},\frac{B-A}{2}\Big)\times I\times\Big(-\frac{T}{2},\frac{T}{2}\Big).
\end{aligned}$$

Insert (3.10) in (3.9) yields

$$\begin{aligned} &D_{\epsilon,\delta_2,\delta_1} + E_{\epsilon,\delta_2,\delta_1} + F_{\delta_2,\delta_1} + G_{\epsilon,\delta_2,\delta_1} \\ &+ \int\limits_{Q\times(Q\cap\{u_2=y_2\})} h(x_1)(a(u_{1x_2}) - a(1))\zeta\Big(\min\Big(\frac{u_1 - x_2}{\epsilon}, 1\Big)\Big)_{x_2} dxdtdyds \\ &\le \int\limits_Q C(u_1, u_2 - y_2, \zeta)\, dyds, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} D_{\epsilon,\delta_2,\delta_1} = \int\limits_{Q_1\times Q_2} & (h(z_1+\sigma_1) - h(z_1-\sigma_1))a(\hat{u}_{2z_2})\hat{\zeta}_{z_2} \\ & \times \min\Big(\frac{(\hat{u}_1 - \hat{u}_2 - 2\sigma_2)^+}{\epsilon}, 1\Big)\, dzd\tau d\sigma d\theta, \end{aligned}$$

$$\begin{aligned} E_{\epsilon,\delta_2,\delta_1} = \int\limits_{Q_1\times Q_2} & \Big\{h(z_1+\sigma_1)(a(\hat{u}_{1z_2}) - a(\hat{u}_{2z_2}))\hat{\zeta}_{z_2} \\ & \times \min\Big(\frac{(\hat{u}_1 - \hat{u}_2 - 2\sigma_2)^+}{\epsilon}, 1\Big) \\ & + (\hat{g}_2 - \hat{g}_1)^+\Big(h(z_1-\sigma_1)a(1)\hat{\zeta}_{z_2} - \hat{\zeta}_\tau\Big)\Big\}\, dzd\tau d\sigma d\theta, \end{aligned}$$

$$\begin{aligned} F_{\delta_2,\delta_1} = \int\limits_{Q_3\times Q_4} & (\bar{g}_2 - \bar{g}_1)^+(h(z_1+\sigma_1) - h(z_1-\sigma_1)) \\ & \times a(1)\bar{\zeta}_{x_2}\, dz_1dx_2d\tau d\sigma_1dy_2d\theta, \end{aligned}$$

$$\begin{aligned} G_{\epsilon,\delta_2,\delta_1} = \int\limits_{Q_1\times Q_2} & (h(z_1+\sigma_1) - h(z_1-\sigma_1))a(\hat{u}_{2z_2})\hat{\zeta} \\ & \times \Big(\min\Big(\frac{(\hat{u}_1 - \hat{u}_2 - 2\sigma_2)^+}{\epsilon}, 1\Big)\Big)_{z_2} dzd\tau d\sigma d\theta \end{aligned}$$

with

$$\begin{aligned} &\hat{u}_1 = u_1(z+\sigma, \tau+\theta), \quad \hat{u}_2 = u_2(z-\sigma, \tau-\theta), \\ &\hat{\zeta} = \xi(z,\tau)\rho_{1,\delta_1}(\theta)\rho_{2,\delta_1}(\sigma_1)\rho_{3,\delta_2}(\sigma_2), \quad \hat{g}_1 = g_1(z+\sigma, \tau+\theta), \\ &\hat{g}_2 = g_2(z-\sigma, \tau-\theta), \quad \overline{u}_1 = u_1(z_1+\sigma_1, x_2, \tau+\theta), \\ &\bar{g}_1 = g_1(z_1+\sigma_1, x_2, \tau+\theta), \quad \bar{g}_2 = g_2(z_1-\sigma_1, y_2, \tau-\theta), \\ &\bar{\zeta} = \xi\Big(z_1, \frac{x_2+y_2}{2}, \tau\Big)\rho_{1,\delta_1}(\theta)\rho_{2,\delta_1}(\sigma_1)\rho_{3,\delta_2}\Big(\frac{x_2-y_2}{2}\Big). \end{aligned}$$

Since h is a Lipschitz continuous function and $\text{supp}(\rho_{2,\delta_1}) \subset (-\delta_1, \delta_1)$, there exists a constant K such that

$$\begin{aligned}
|F_{\delta_2,\delta_1}| &\leq K \int_{Q_3\times Q_4} |\sigma_1|(\bar{g}_2-\bar{g}_1)^+ a(1)|\bar{\zeta}_{x_2}|\, dz_1dx_2d\tau d\sigma_1 dy_2 d\theta \\
&\leq K\delta_1 \int_{Q_3\times Q_4} (\bar{g}_2-\bar{g}_1)^+ a(1)|\bar{\zeta}_{x_2}|\, dz_1dx_2d\tau d\sigma_1 dy_2 d\theta \\
&=: \delta_1 W^1_{\delta_2,\delta_1}, \qquad (3.12)\\
|G_{\epsilon,\delta_2,\delta_1}| &\leq K\delta_1 \int_{Q_1\times Q_2} \Big| a(\hat{u}_{2z_2})\Big(\min\Big(\frac{(\hat{u}_1-\hat{u}_2-2\sigma_2)^+}{\epsilon},1\Big)\Big)_{z_2}\Big|\hat{\zeta}\, dzd\tau d\sigma d\theta \\
&=: \delta_1 W^2_{\epsilon,\delta_2,\delta_1}, \qquad (3.13)\\
|D_{\epsilon,\delta_2,\delta_1}| &\leq K\delta_1 \int_{Q_1\times Q_2} \Big| a(\hat{u}_{2z_2})\min\Big(\frac{(\hat{u}_1-\hat{u}_2-2\sigma_2)^+}{\epsilon},1\Big)\hat{\zeta}_{z_2}\Big|\, dzd\tau d\sigma d\theta \\
&=: \delta_1 W^3_{\epsilon,\delta_2,\delta_1}. \qquad (3.14)
\end{aligned}$$

Notice that, $(W^1_{\delta_2,\delta_1})_{\delta_1>0}$, $(W^2_{\epsilon,\delta_2,\delta_1})_{\delta_1>0}$ and $(W^3_{\epsilon,\delta_2,\delta_1})_{\delta_1>0}$ are bounded, then, passing to the limit in (3.12)–(3.14) as $\delta_1 \to 0$, we obtain

$$\lim_{\delta_1\to 0}(F_{\delta_2,\delta_1}) = \lim_{\delta_1\to 0}(G_{\epsilon,\delta_2,\delta_1}) = \lim_{\delta_1\to 0}(D_{\epsilon,\delta_2,\delta_1}) = 0. \qquad (3.15)$$

On the other hand,

$$\begin{aligned}
\lim_{\delta_2\to 0}\big(\lim_{\delta_1\to 0}(E_{\epsilon,\delta_2,\delta_1})\big) = \int_Q \Big\{ & h(z_1)(a(u_{1z_2})-a(u_{2z_2}))\xi_{z_2}\min\Big(\frac{(u_1-u_2)^+}{\epsilon},1\Big) \\
&+ (g_2-g_1)^+(h(z_1)a(1)\xi_{z_2}-\xi_\tau)\Big\}\, dzd\tau,
\end{aligned} \qquad (3.16)$$

where $u_1 = u_1(z,\tau)$, $u_2 = u_2(z,\tau)$, $g_1 = g_1(z,\tau)$, $g_2 = g_2(z,\tau)$ and $\xi = \xi(z,\tau)$. Now, since, by (2.9) and the Lebesgue theorem, we have

$$\begin{aligned}
&\int_Q C(u_1, u_2-y_2, \zeta)\, dyds \\
&= \lim_{\epsilon\to 0}\int_{Q\cap\{u_2=y_2\}} \Big\{\int_Q h(x_1)(a(u_{1x_2})-a(1)) \\
&\qquad \times \zeta\Big(\min\Big(\frac{u_1-x_2}{\epsilon},1\Big)\Big)_{x_2} dxdt\Big\}\, dyds,
\end{aligned}$$

we obtain by letting successively $\delta_1 \to 0$, $\delta_2 \to 0$, $\epsilon \to 0$ in (3.11) and using (3.15)–(3.16),

$$\int_Q \big\{\chi_{\{u_1-u_2\geq 0\}} h(z_1)(a(u_{1z_2}) - a(u_{2z_2}))\xi_{z_2} + (g_2-g_1)^+(h(z_1)a(1)\xi_{z_2} - \xi_\tau)\big\}\, dzd\tau \leq 0,$$

where $\chi_{\{u_1-u_2\geq 0\}}$ denotes the characteristic function of the set $\{u_1 - u_2 \geq 0\}$. This leads to (3.1) for $i = 1$. If one exchanges the roles of (u_1, g_1) and (u_2, g_2), one also obtains (3.1) for $i = 2$. $\square$

Lemma 3.3. *Let* $\mathbf{B}$ *be a bounded open subset of* $\mathbb{R}^2$ *such that either* $\mathbf{B} \cap \Gamma = \emptyset$ *or* $\mathbf{B} \cap \Gamma$ *is a Lipschitz graph. Let* (u_1, g_1) *and* (u_2, g_2) *be two solutions of* (1.8) *and let* $\overline{g}$ *be a function of* $L^\infty(Q)$ *such that*

$$0 \leq \overline{g} \leq g_1, g_2 \text{ a.e. in } Q, \quad h(x_1)a(1)\overline{g}_{x_2} - \overline{g}_t = 0 \text{ in } \mathcal{D}'(Q). \tag{3.17}$$

Then, for all $\xi \in \mathcal{D}(\mathbf{B} \times (0,T))$, $\xi \geq 0$, *supp*$(\xi) \cap (\sigma_1 \cup \Sigma_4) = 0$ *and for* $i, j = 1, 2$, $i \neq j$, *we have*

$$\int_Q \big\{h(x_1)(a(u_{ix_2}) - a(u_{mx_2}) - (g_j - \overline{g})^+ a(1))\xi_{x_2} + (g_j - \overline{g})^+ \xi_t\big\}\, dxdt \leq 0. \tag{3.18}$$

Proof. Let (u_1, g_1) and (u_2, g_2) be two solutions of (1.8) and let ξ be the function defined in Lemma 3.3. We define

$$\forall (x,t,y,s) \in \overline{Q \times Q}:$$
$$\zeta(x,t,y,s) = \xi\Big(\frac{x+y}{2}, \frac{t+s}{2}\Big)\rho_{1,\delta_1}\Big(\frac{t-s}{2}\Big)\rho_{2,\delta_2}\Big(\frac{x_1-y_1}{2}\Big)\rho_{3,\delta_3}\Big(\frac{x_2-y_2}{2}\Big),$$

where δ_1, δ_2, δ_3 are positive real numbers, ρ_{1,δ_1}, ρ_{2,δ_2}, $\rho_{3,\delta_3} \in \mathcal{D}(\mathbb{R})$, ρ_{1,δ_1}, ρ_{2,δ_2}, $\rho_{3,\delta_3} \geq 0$ in $\mathbb{R}$,

$$\int_{\mathbb{R}} \rho_{1,\delta_1}(t)\, dt = \int_{\mathbb{R}} \rho_{2,\delta_2}(t)\, dt = \int_{\mathbb{R}} \rho_{3,\delta_3}(t)\, dt = 1,$$

$$\text{supp}(\rho_{1,\delta_1}) \subset (-\delta_1, \delta_1), \quad \text{supp}(\rho_{2,\delta_2}) \subset (-\delta_2, \delta_2), \quad \text{supp}(\rho_{3,\delta_3}) \subset (-\delta_3, \delta_3)$$

and

$$\forall (x,y) \in (\mathbf{B} \cap \Omega) \times (\mathbf{B} \backslash \Omega): \quad \rho_{2,\delta_2}\Big(\frac{x_1-y_1}{2}\Big)\rho_{3,\delta_3}\Big(\frac{x_2-y_2}{2}\Big) = 0.$$

For δ_1, δ_2 and δ_3 small enough, we have $\zeta \in \mathcal{D}(\mathbf{B} \times (0,T) \times \mathbf{B} \times (0,T))$ and

$$\zeta = 0 \quad \text{on } (\Sigma \times Q) \cup (Q \times (\sigma_1 \cup \Sigma_4)).$$

On the other hand, since $\text{supp}(\xi) \cap (\sigma_1 \cap \Sigma_4) = \emptyset$ and if we suppose that $\text{supp}(\xi) \cap \Sigma_3 \neq \emptyset$, we can find $r_0 \in \Big(0, \min\limits_{\text{supp}(\xi)\cap\Sigma_3} \varphi\Big)$ and $\Psi \in C^\infty(\mathbb{R}) \cap C^{0,1}(\mathbb{R})$ such that $\Psi' \geq 0$, $\Psi(r) = 0$

if $r \leq 0$ and $\Psi(r) = 1$ if $r \geq r_0$, and this function Ψ satisfies for δ_1, δ_2 and δ_3 small enough,

$$(1 - \Psi(u_2 - y_2))\zeta = 0 \quad \text{on } (\Sigma \times Q) \cup (Q \times \Sigma_2). \tag{3.19}$$

If $\text{supp}(\xi) \cap \Sigma_3 = \emptyset$, we choose $\Psi = 0$. Now, applying Lemma 2.4 to (u_1, g_1) with $k = u_2(y, s) - y_2$, $\lambda = g_2(y, s)$, $\xi(x, t) = \xi_2(x, t) = \zeta(x, t, y, s)$, $\xi_1(x, t) = 0$ and $\chi = \overline{g}$, we obtain for a.e. $(y, s) \in Q$,

$$\int_Q \Big\{ h(x_1) a(u_{1x_2}) \Big(\min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big)\zeta\Big)_{x_2} + (g_2 - \overline{g})^+ (h(x_1) a(1) \zeta_{x_2} - \zeta_t) \Big\} \, dxdt \leq 0$$

and integrating over Q, we get

$$\int_{Q \times Q} \Big\{ h(x_1) a(u_{1x_2}) \Big(\min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big)\zeta\Big)_{x_2} + (g_2 - \overline{g})^+ (h(x_1) a(1) \zeta_{x_2} - \zeta_t) \Big\} \, dxdtdyds \leq 0. \tag{3.20}$$

Similarly, for a.e. $(x, t) \in Q$, we apply Lemma 2.5 to (u_2, g_2) with $k = u_1(x, t) - x_2$, $\lambda = \overline{g}(x, t)$, $\xi(y, s) = \zeta(x, t, y, s)$, then we integrate over Q to obtain

$$- \int_{Q \times Q} \Big\{ h(y_1) \big[a(u_{2y_2}) - \overline{g} a(1) \big] \times \Big(\min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big)(1 - \Psi(u_2 - y_2))\zeta\Big)_{y_2} - (g_2 - \overline{g})^+ (h(y_1) a(1) \zeta_{y_2} - \zeta_s) \Big\} \, dxdtdyds \leq 0. \tag{3.21}$$

On the other hand, by Corollary 2.2, we have

$$\int_{Q \times Q} h(x_1) a(u_{1x_2}) \Big(\min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big)\zeta\Big)_{x_2} \, dxdtdyds = 0 \tag{3.22}$$

and the use of (3.17) and (3.19) leads to

$$\int_{Q\times Q} \overline{g}h(y_1)a(1) \times \Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)(1-\Psi(u_2-y_2))\zeta\Big)_{x_2}\,dxdtdyds = 0, \quad (3.23)$$

$$\int_{Q\times Q} h(y_1)a(u_{2y_2}) \times \Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)(1-\Psi(u_2-y_2))\zeta\Big)_{x_2}\,dxdtdyds = 0. \quad (3.24)$$

Addition of (3.20), (3.21), (3.22), (3.23) and (3.24) yields

$$K_{\epsilon,\delta_1,\delta_3,\delta_2} + L_{\delta_1,\delta_3,\delta_2} + M_{\epsilon,\delta_1,\delta_3,\delta_2} \leq 0, \quad (3.25)$$

where

$$K_{\epsilon,\delta_1,\delta_3,\delta_2} = \int_{Q\times Q} \Big\{\big[h(x_1)a(u_{1x_2}) - h(y_1)a(u_{2y_2})\big] \times (\partial_{x_2}+\partial_{y_2})\Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)\zeta\Big) + (g_2-\overline{g})^+\Big((\zeta_{x_2}+\zeta_{y_2})h(y_1)a(1) - \zeta_t - \zeta_s\Big)\Big\}\,dxdtdyds,$$

$$L_{\delta_1,\delta_3,\delta_2} = \int_{Q\times Q} (g_2-\overline{g})^+(h(x_1)-h(y_1))a(1)\zeta_{x_2}\,dxdtdyds,$$

$$\begin{aligned}M_{\epsilon,\delta_1,\delta_3,\delta_2} = &\int_{Q\times Q} h(y_1)\big[a((\partial_{x_2}+\partial_{y_2})u_2) - \overline{g}a(1)\big] \times (\partial_{x_2}+\partial_{y_2})\Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big) \times \Psi(u_2-y_2)\zeta\Big)\,dxdtdyds\\ &- \int_{Q\times Q} h(x_1)\big[a((\partial_{x_2}+\partial_{y_2})u_1) - \overline{g}a(1)\big] \times (\partial_{x_2}+\partial_{y_2})\Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)\zeta\Big)\,dxdtdyds\\ &- \int_{Q\times Q} \overline{g}a(1)(h(x_1)-h(y_1)) \times (\partial_{x_2}+\partial_{y_2})\Big(\min\Big(\frac{(u_1-x_2+y_2-u_2)^+}{\epsilon},1\Big)\zeta\Big)\,dxdtdyds.\end{aligned}$$

Passing to the limit in $L_{\delta_1,\delta_3,\delta_2}$ as $\delta_2 \to 0$, we arrive at

$$\lim_{\delta_2\to 0}(L_{\delta_1,\delta_3,\delta_2}) = 0. \tag{3.26}$$

On the other hand,

$$\begin{aligned}
&\lim_{\delta_3\to 0}\big(\lim_{\delta_2\to 0}(M_{\epsilon,\delta_1,\delta_3,\delta_2})\big)\\
&= \int_0^T\int_Q h(x_1)\big[a(u_{2x_2}) - \overline{g}a(1)\big]\Big(\min\Big(\frac{(u_1-u_2)^+}{\epsilon},1\Big)\Psi(u_2-x_2)\xi\Big)_{x_2}\rho_{1,\delta_1}\,dxdtds\\
&\quad - \int_0^T\int_Q h(x_1)\big[a(u_{1x_2}) - \overline{g}a(1)\big]\Big(\min\Big(\frac{(u_1-u_2)^+}{\epsilon},1\Big)\xi\Big)_{x_2}\rho_{1,\delta_1}\,dxdtds =: S_{\epsilon,\delta_1},
\end{aligned} \tag{3.27}$$

where

$$u_1 = u_1(x,t), \quad u_2 = u_2(x,s), \quad \overline{g} = \overline{g}(x,t), \quad \xi = \xi\Big(x,\frac{t+s}{2}\Big)$$

and $\rho_{1,\delta_1} = \rho_{1,\delta_1}(\frac{t-s}{2})$. Applying Lemma 2.1 to

$$F(z_1,z_2) = \min\Big(\frac{(z_1-z_2)^+}{\epsilon},1\Big)(1-\Psi(z_2))$$

with $v = u_2 - x_2$ and taking into account

$$(1-\Psi(u_2(x,s)-x_2))\xi\Big(x,\frac{t+s}{2}\Big)\rho_{1,\delta_1}\Big(\frac{t-s}{2}\Big) = 0$$

for all $(x,t,s) \in \Sigma_2 \times (0,T)$, we get

$$\begin{aligned}
&\int_0^T\int_Q h(x_1)\big[a(u_{1x_2}) - g_1 a(1)\big]\\
&\quad\times\Big(\min\Big(\frac{(u_1-u_2)^+}{\epsilon},1\Big)(1-\Psi(u_2-x_2))\xi\Big)_{x_2}\rho_{1,\delta_1}\,dxdtds = 0.
\end{aligned} \tag{3.28}$$

Using (3.28) and the fact that $\overline{g} \leq g_1$ and $\Psi(0) = 0$, we obtain from (3.27),

$$\begin{aligned} S_{\epsilon,\delta_1} = &\int_0^T \int_Q h(x_1)\big[a(u_{2x_2}) - g_2 a(1)\big] \\ &\qquad \times \Big(\min\Big(\frac{(u_1-u_2)^+}{\epsilon},1\Big)\Psi(u_2-x_2)\xi\Big)_{x_2}\rho_{1,\delta_1}\,dxdtds \\ &- \int_0^T \int_Q h(x_1)\big[a(u_{1x_2}) - g_1 a(1)\big] \\ &\qquad \times \Big(\min\Big(\frac{(u_1-u_2)^+}{\epsilon},1\Big)\Psi(u_2-x_2)\xi\Big)_{x_2}\rho_{1,\delta_1}\,dxdtds. \end{aligned} \tag{3.29}$$

Notice that

$$\pm\Big(\min\Big(\frac{(u_1-\phi(x,s))^+}{\epsilon},1\Big) - \min\Big(\frac{(\phi(x,t)-\phi(x,s))^+}{\epsilon},1\Big)\Big)\Psi(\phi(x,s)-x_2)\xi\rho_{1,\delta_1}$$

are test functions for (1.8) corresponding to (u_2, g_2). In addition, applying Lemma 2.1 to u_2 with $v = u_1 - x_2$,

$$F(z_1,z_2) = \min\Big(\frac{(z_2-z_1)^+}{\epsilon},1\Big)(1-\Psi(z_1))$$

and

$$F(z_1,z_2) = \min\Big(\frac{(z_2-z_1)^+}{\epsilon},1\Big),$$

subtracting one equation from the other and taking into account $\Psi(0) = 0$, $g_2(u_2 - x_2) = 0$ a.e. in Q, we deduce that

$$\begin{aligned} &\int_0^T \int_Q h(x_1)\big[a(u_{2x_2}) - g_2 a(1)\big]\Big(\min\Big(\frac{(u_1-u_2)^+}{\epsilon},1\Big)\Psi(u_2-x_2)\xi\Big)_{x_2}\rho_{1,\delta_1}\,dxdtds \\ &= \int_0^T \int_Q h(x_1)\Big\{\big[a(u_{2x_2}) - g_2 a(1)\big] \\ &\qquad \times \Big(\min\Big(\frac{(\phi(x,t)-\phi(x,s))^+}{\epsilon},1\Big)\Psi(\phi(x,s)-x_2)\xi\Big)_{x_2}\rho_{1,\delta_1} \\ &\qquad + g_2\Big(\min\Big(\frac{(\phi(x,t)-\phi(x,s))^+}{\epsilon},1\Big)\Psi(\phi(x,s)-x_2)\xi\rho_{1,\delta_1}\Big)_s\Big\}\,dxdtds. \end{aligned} \tag{3.30}$$

Similarly, if we apply Lemma 2.1 to u_1 with $v = u_2 - x_2$ and

$$F(z_1, z_2) = \min\Big(\frac{(z_1 - z_2)^+}{\epsilon}, 1\Big)\Psi(z_2),$$

we get

$$\int_0^T \int_Q h(x_1)\big[a(u_{1x_2}) - g_1 a(1)\big]$$

$$\times\Big(\min\Big(\frac{(u_1 - u_2)^+}{\epsilon}, 1\Big)\Psi(u_2 - x_2)\xi\Big)_{x_2}\rho_{1,\delta_1}\, dxdtds$$

$$= \int_0^T \int_Q h(x_1)\Big\{\big[a(u_{1x_2}) - g_1 a(1)\big] \tag{3.31}$$

$$\times\Big(\min\Big(\frac{(\phi(x,t) - \phi(x,s))^+}{\epsilon}, 1\Big)\Psi(\phi(x,s) - x_2)\xi\Big)_{x_2}\rho_{1,\delta_1}$$

$$+ g_1\Big(\min\Big(\frac{(\phi(x,t) - \phi(x,s))^+}{\epsilon}, 1\Big)\Psi(\phi(x,s) - x_2)\xi\rho_{1,\delta_1}\Big)_t\Big\}\, dxdtds.$$

Since g_1 (resp. g_2) does not depend on s (resp. t), we obtain from (3.29), (3.30) and (3.31),

$$S_{\epsilon,\delta_1} = \int_0^T \int_0^T \int_Q \Big\{h(x_1)\big[a(u_{2x_2}) - a(u_{1x_2}) + (g_1 - g_2)a(1)\big]$$

$$\times\Big(\min\Big(\frac{(\phi(x,t) - \phi(x,s))^+}{\epsilon}, 1\Big)\Psi(\phi(x,s) - x_2)\xi\Big)_{x_2} \tag{3.32}$$

$$- (g_1 - g_2)(\partial_t + \partial_s)\Big(\min\Big(\frac{(\phi(x,t) - \phi(x,s))^+}{\epsilon}, 1\Big)$$

$$\times \Psi(\phi(x,s) - x_2)\xi\Big)\Big\}\rho_{1,\delta_1} dxdtds.$$

We may then pass to the limit in (3.32) as $\delta_1 \to 0$ to deduce

$$\lim_{\delta_1\to 0}(\lim_{\delta_3\to 0}(\lim_{\delta_2\to 0}(M_{\epsilon,\delta_1,\delta_3,\delta_2})) = \lim_{\delta_1\to 0}(S_{\epsilon,\delta_1}) = 0. \tag{3.33}$$

Thus, for $K_{\epsilon,\delta_1,\delta_3,\delta_2}$, we use (1.3) and (1.4) to write

$$
\begin{aligned}
K_{\epsilon,\delta_1,\delta_3,\delta_2} = \int_{Q\times Q} & \Big\{ h(x_1)(a(u_{1x_2}) - a(u_{2y_2}))(\partial_{x_2} + \partial_{y_2})\zeta \\
& \times \min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big) \\
& + (g_2 - \overline{g})^+ (\zeta_{x_2} + \zeta_{y_2}) h(y_1) a(1) - \zeta_t - \zeta_s \Big) \Big\} \, dxdtdyds \\
+ \int_{Q\times Q} & \Big\{ h(x_1)(a(u_{1x_2}) - a(u_{2y_2}))\zeta \\
& \times (\partial_{x_2} + \partial_{y_2}) \min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big) \Big\} \, dxdtdyds \\
+ \int_{Q\times Q} & \Big\{ (h(x_1) - h(y_1)) a(u_{2y_2}) \\
& \times (\partial_{x_2} + \partial_{y_2}) \Big(\min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big)\zeta \Big) \Big\} \, dxdtdyds \\
\geq \int_{Q\times Q} & \Big\{ h(x_1)(a(u_{1x_2}) - a(u_{2y_2}))(\partial_{x_2} + \partial_{y_2})\zeta \\
& \times \min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big) \\
& + (g_2 - \overline{g})^+ (\zeta_{x_2} + \zeta_{y_2}) h(y_1) a(1) - \zeta_t - \zeta_s \Big) \Big\} \, dxdtdyds \\
+ \int_{Q\times Q} & \Big\{ (h(x_1) - h(y_1)) a(u_{2y_2}) \\
& \times (\partial_{x_2} + \partial_{y_2}) \Big(\min\Big(\frac{(u_1 - x_2 + y_2 - u_2)^+}{\epsilon}, 1\Big)\zeta \Big) \Big\} \, dxdtdyds.
\end{aligned}
$$

Now, we pass successively to the limit as $\delta_2 \to 0$, $\delta_3 \to 0$, $\delta_1 \to 0$, $\epsilon \to 0$ to get

$$
\begin{aligned}
& \int_{Q\times Q} \Big\{ \chi_{\{u_1 - u_2 \geq 0\}} h(x_1) \big[a(u_{1x_2}) - a(u_{2x_2}) \big] \zeta_{x_2} + (g_2 - \overline{g})^+ (h(x_1) a(1) \zeta_{x_2} - \zeta_t) \Big\} \, dxdt \\
& \leq \liminf_{\epsilon \to 0} \big(\lim_{\delta_1 \to 0} \big(\lim_{\delta_3 \to 0} \big(\lim_{\delta_2 \to 0} (K_{\epsilon,\delta_1,\delta_3,\delta_2}) \big) \big) \big).
\end{aligned}
\tag{3.34}
$$

Now, by letting successively $\delta_2 \to 0$, $\delta_3 \to 0$, $\delta_1 \to 0$, $\epsilon \to 0$ in (3.25) and using (3.26), (3.33) and (3.34), we obtain (3.18) for $i = 1$ and $j = 2$. Since we can exchange the roles of (u_1, g_1) and (u_2, g_2), we can also obtain (3.18) for $i = 2$ and $j = 1$. □

Proof of Theorem 3.1. Using Lemmas 3.2, 3.3 and applying arguments similar to [16, Lemmas 5.6, 5.7 and Theorem 5.8], we arrive at the following comparison result of solutions,

$$\int_Q \left\{h(x_1)(a(u_{ix_2}) - a(u_{mx_2}) - (g_i - g_M)a(1))\xi_{x_2} + (g_i - g_M)\xi_t\right\} dxdt \leq 0, \tag{3.35}$$
$$i = 1, 2,\ \forall \xi \in \mathcal{D}(\mathbf{B} \times (0,T)),\ \xi \geq 0,\ \xi(x,0) = \xi(x,T) = 0 \text{ a.e. in } \Omega.$$

If we choose $\xi \in \mathcal{D}(0,T)$, $\xi \geq 0$ in (3.35), we get

$$\int_Q (g_i - g_M)\xi_t \, dxdt \leq 0,$$

which can be written as

$$\frac{d}{dt}\int_\Omega (g_M - g_i)\, dx \leq 0 \quad \text{ in } \mathcal{D}'(0,T).$$

Since $g_i \in C^0([0,T]; L^1(\Omega))$ (see [17]) and $g_1(x,0) = g_2(x,0) = g_0(x)$ a.e. $x \in \Omega$, we obtain

$$\int_\Omega (g_M - g_i)\, dx = 0 \quad \text{ in } [0,T],$$

which leads to

$$g_1 = g_2 = g_M \quad \text{ a.e. in } Q. \tag{3.36}$$

Insert (3.36) in (3.35) yields

$$\forall \xi \in \mathcal{D}(\overline{\Omega} \times (0,T)),\ \xi \geq 0: \quad \int_Q h(x_1)(a(u_{ix_2}) - a(u_{mx_2}))\xi_{x_2}\, dxdt \leq 0. \tag{3.37}$$

From (1.3)–(1.4) and the fact that (3.37) remains true for $\xi = u_i - u_m$, we deduce that $(u_i - u_m)_{x_2} = 0$ a.e. in Q. Since $u_i - u_m = 0$ on Σ_2, we can extend $u_i - u_m$ to $\mathbb{R} \times (D, +\infty) \times (0,T)$ by 0 and still denote by $u_i - u_m$. Thus, for a.e. $(x_1, t) \in \mathbb{R} \times (0,T)$, there exists $w \in C^0([D, +\infty))$ such that $w(x_2) = (u_i - u_m)(x_1, x_2, t)$ a.e. $x_2 \in (D, +\infty)$ and

$$\forall z_1, z_2 \in [D, +\infty): \quad w(z_1) - w(z_2) = \int_{z_2}^{z_1} (u_i - u_m)_{x_2}(x_1, z, t)\, dz = 0,$$

which means that $w = c$ in $[D, +\infty)$ for some constant $c \geq 0$. Due to $w(x_2) = 0$ for x_2 large enough, it follows that $w = 0$ in $[D, +\infty)$, and hence $u_i = u_m$ a.e. in Q for $i = 1, 2$. Thus, the proof is complete. □

Remark 3.4. If E, F are real numbers such that $F > E$, $n = 3$ and $\Gamma_1 = [A, B] \times [E, F]$, the obtained uniqueness result remains true if we replace x_1 by $y' = (y_1, y_2) \in [A, B] \times [E, F]$ and x_2 by y_3, where $y = (y', y_3) = (y_1, y_2, y_3)$ is a generic point of $\Omega \subset \mathbb{R}^3$.

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Messaouda Ben Attia
benattiamessaouda1402@gmail.com

University of Kasdi Merbah, Ouargla
Laboratory of Applied Mathematics
B.P. 511, Ouargla 30000, Algeria

Elmehdi Zaouche (corresponding author)
elmehdi-zaouche@univ-eloued.dz
https://orcid.org/0000-0001-7777-9284

University of El Oued
Department of Mathematics
Labo. Oper. Theo. and Part. Diff. Eq.: Foundations and Applications
B.P. 789, El Oued 39000, Algeria

Mahmoud Bousselsal
bousselsal55@gmail.com

Ecole Normale Supérieure
Department of Mathematics
Laboratoire des équations aux dérivées partielles et H.M.
16050, Vieux-Kouba, Algiers, Algeria

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