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KNESER-TYPE OSCILLATION CRITERIA FOR SECOND-ORDER HALF-LINEAR ADVANCED DIFFERENCE EQUATIONS

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Abstract. The authors present Kneser-type oscillation criteria for a class of advanced type second-order difference equations. The results obtained are new and they improve and complement known results in the literature. Two examples are provided to illustrate the importance of the main results.

Keywords: second-order difference equations, advanced argument, half-linear, oscillation.

Mathematics Subject Classification: 39A10.

1. INTRODUCTION

Here we investigate the oscillatory behavior of solutions of the second-order quasi-linear difference equation with an advanced argument

$$\Delta(\mu(n)(\Delta u(n))^{\alpha}) + f(n)u^{\alpha}(\sigma(n)) = 0,\ n \geq n_0, \tag{1.1}$$

where we assume that α is a quotient of odd positive integers, $\{\mu(n)\}$ and $\{f(n)\}$ are positive real sequences, and $\{\sigma(n)\}$ is a sequence of integers with $\sigma(n) \geq n+1$ for all $n \geq n_0$. Throughout this paper, and without further mention, we assume that

$$\Pi(n_0) = \sum_{n=n_0}^{\infty} \frac{1}{\mu^{1/\alpha}(n)} < \infty. \tag{1.2}$$

By a *solution* of equation (1.1), we mean a nontrivial sequence $\{u(n)\}$ that satisfies (1.1) for all $n \geq n_0$. A solution $\{u(n)\}$ of (1.1) is called *oscillatory* if it is neither eventually negative nor eventually positive, and it is said to be *nonoscillatory* otherwise. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Difference equations with an advanced argument can be applied to a variety of real world problems in which the evolution rate depends on the future as well as on the present state. Thus, it would be possible to include an advanced argument in an equation in an effort to model events that depend on future actions. Such phenomena occur, for example, in areas such as population dynamics, economics, and control theory; see the monograph by Bellman and Cooke [4, Sections 3.4 and 5.6].

The oscillatory and asymptotic behavior of solutions of delay difference equations has received a great deal of attention in the last three decades; see, for example, [1, 2, 8, 10, 11, 13, 19] and the papers cited therein for recent results of this type. However, far few results are reported on the oscillation of advanced type difference equations (see [3, 5–7, 9, 12, 16–18, 20, 21]).

Our aim for this paper is to contribute to the under developed oscillation theory of second-order noncanonical difference equations with advanced arguments. We use only one condition to obtain the oscillation of equation (E). Furthermore, the results presented in this paper improve and complement those in [3, 7, 12, 18, 21]. For related results concerning oscillation of advanced type differential equations, we refer the reader to [9, 14, 15].

2. MAIN RESULTS

In this section, we present the main results in the paper. Define

$$R_* = \liminf_{n\to\infty} \mu^{1/\alpha}(n)\Pi^\alpha(\sigma(n))\Pi(n+1)f(n) \tag{2.1}$$

and set

$$\beta_0 = \frac{R_*}{\alpha}. \tag{2.2}$$

Notice that (2.1) implies that for any $\beta \in (0,1)$ there is an integer N_β such that

$$\mu^{1/\alpha}(n)\Pi^\alpha(\sigma(n))\Pi(n+1)f(n) \geq \alpha\beta \tag{2.3}$$

for $n \geq N_\beta$.

The following lemma provides important information about nonoscillatory solutions of (1.1) and will be used to prove our main results. We will also ask that

$$\sum_{s=n_0}^{\infty} \frac{1}{\mu^{1/\alpha}(s)\Pi^\alpha(s)} = \infty. \tag{2.4}$$

Lemma 2.1. *Let* $\beta_0 > 0$ *and condition* (2.4) *hold. If* (1.1) *has an eventually positive solution* $\{u(n)\}$*, then:*

(i) $\{u(n)\}$ *and* $\{\mu(n)(\Delta u(n))^\alpha\}$ *are decreasing;*
(ii) $\lim_{n\to\infty} u(n) = 0$*;*
(iii) $\{\frac{u(n)}{\Pi(n)}\}$ *is eventually nondecreasing;*
(iv) *the function*

$$z(n) = u(n) + \Pi(n)\mu^{1/\alpha}(n)\Delta u(n) \tag{2.5}$$

is nonnegative, and

$$\Delta z(n) \le \frac{-\beta u^\alpha(n+1)\Delta u(n)}{\mu(n+1)\Pi^\alpha(n+1)(\Delta u(n+1))^\alpha} \tag{2.6}$$

eventually.

Proof. Let $\{u(n)\}$ be a positive solution of (1.1) and choose $\beta \in (0,1)$ and an integer $N_\beta \ge n_0$ so that (2.3) holds for all $n \ge n_1$ for some $n_1 \ge N_\beta$. Then

$$\Delta(\mu(n)(\Delta u(n))^\alpha) < 0,$$

$\mu(n)(\Delta u(n))^\alpha$ is nonincreasing, and either $\Delta u(n) > 0$ or $\Delta u(n) < 0$ eventually.

(i) Assume, for the sake of a contradiction, that $\Delta u(n) > 0$ for all $n \ge n_2 \ge n_1$. Then $u(\sigma(n)) \ge u(\sigma(n_1)) = M > 0$, and so from (1.1) and (2.3), we have

$$-\Delta(\mu(n)(\Delta u(n))^\alpha) = f(n)u^\alpha(\sigma(n)) \ge \frac{\beta\alpha M^\alpha}{\mu^{1/\alpha}(n)\Pi^\alpha(\sigma(n))\Pi(n+1)}$$

for all $n \ge n_3$ for some $n_3 \ge n_2$. In view of (1.2), the sequence $\{\Pi(n)\}$ is decreasing, and so the above inequality becomes

$$-\Delta(\mu(n)(\Delta u(n))^\alpha) \ge \frac{\beta\alpha M^\alpha}{\mu^{1/\alpha}(n)\Pi^{\alpha+1}(n+1)}. \tag{2.7}$$

Summing (2.7) from n_3 to $n-1$, we obtain

$$\begin{aligned} -\mu(n)(\Delta u(n))^\alpha + \mu(n_3)(\Delta u(n_3))^\alpha &\ge \alpha\beta M^\alpha \sum_{s=n_3}^{n-1} \frac{1}{\mu^{1/\alpha}(s)\Pi^{\alpha+1}(s+1)} \\ &\ge \alpha\beta M^\alpha \sum_{s=n_3}^{n-1} \int_{\Pi(s+1)}^{\Pi(s)} \frac{dV}{V^{\alpha+1}} \\ &= \beta M^\alpha \left(\frac{1}{\Pi^\alpha(n)} - \frac{1}{\Pi^\alpha(n_3)}\right) \end{aligned} \tag{2.8}$$

or

$$\mu(n)(\Delta u(n))^\alpha \le u(n_3)(\Delta u(n_3))^\alpha - \beta M^\alpha \left(\frac{1}{\Pi^\alpha(n)} - \frac{1}{\Pi^\alpha(n_3)}\right) \to -\infty$$

as $n \to \infty$, which is a contradiction. This proves (i).

(ii) Since $\{u(n)\}$ is positive and decreasing, $\lim_{n\to\infty} u(n) = M \geq 0$. If $M > 0$, then using (2.3), we arrive at (see (2.8))

$$-\mu(n)(\Delta u(n))^\alpha \geq \beta M^\alpha \left(\frac{1}{\Pi^\alpha(n)} - \frac{1}{\Pi^\alpha(n_4)}\right). \tag{2.9}$$

for $n \geq n_4$ for some $n_4 \geq n_3$. Since $\frac{1}{\Pi(n)} \to \infty$ as $n \to \infty$, for any $\ell \in (0,1)$, there is an integer $n_5 \geq n_4$ such that

$$\frac{1}{\Pi^\alpha(n)} - \frac{1}{\Pi^\alpha(n_4)} \geq \frac{\ell}{\Pi^\alpha(n)}, \quad n \geq n_5. \tag{2.10}$$

Using (2.10) in (2.9), we obtain

$$-\Delta u(n) \geq \frac{(\beta M^\alpha \ell)^{1/\alpha}}{\mu^{1/\alpha}(n)\Pi(n)} = M_1 \frac{1}{\mu^{1/\alpha}(n)\Pi(n)},$$

where $M_1 = (\beta M^\alpha \ell)^{1/\alpha}$. Summing and using (2.4) shows $u(n) \to -\infty$ as $n \to \infty$, which is a contradiction to the positivity of $u(n)$. Hence, $M = 0$, which proves (ii).

(iii) Using the fact that $\{\mu^{1/\alpha}(n)\Delta u(n)\}$ is nonincreasing gives

$$u(n) \geq -\sum_{s=n}^{\infty} \frac{1}{\mu^{1/\alpha}(s)} \mu^{1/\alpha}(s)\Delta u(s) \geq -\Pi(n)\mu^{1/\alpha}(n)\Delta u(n), \tag{2.11}$$

that is,

$$\Delta\left(\frac{u(n)}{\Pi(n)}\right) = \frac{\mu^{1/\alpha}(n)\Pi(n)\Delta u(n) + u(n)}{\mu^{1/\alpha}(n)\Pi(n)\Pi(n+1)} \geq 0,$$

and so (iii) holds.

(iv) From (2.11), it is clear that $z(n) \geq 0$. Now from the definition of $z(n)$, we see that

$$\Delta z(n) = \Pi(n+1)\Delta(\mu^{1/\alpha}(n)\Delta u(n)). \tag{2.12}$$

By the Mean-Value Theorem,

$$\Delta((\mu(n)(\Delta u(n))^\alpha)^{1/\alpha}) = \frac{1}{\alpha} t^{1/\alpha - 1}\Delta(\mu(n)(\Delta u(n))^\alpha), \tag{2.13}$$

where

$$\mu(n+1)(\Delta u(n+1))^\alpha < t < \mu(n)(\Delta u(n))^\alpha. \tag{2.14}$$

From (2.12), (2.13), (2.3), and equation (1.1), we obtain

$$\Delta z(n) = -\frac{1}{\alpha}\frac{t^{1/\alpha}}{t}\Pi(n+1)f(n)u^\alpha(\sigma(n)) \leq \frac{-\beta t^{1/\alpha}u^\alpha(\sigma(n))}{\mu^{1/\alpha}(n)\Pi^\alpha(\sigma(n))t}. \tag{2.15}$$

From (2.14) and the fact that t is negative,

$$\Delta z(n) \leq \frac{-\beta\mu^{1/\alpha}(n)\Delta u(n)u^\alpha(\sigma(n))}{\mu^{1/\alpha}(n)\mu(n+1)(\Delta u(n+1))^\alpha\Pi^\alpha(\sigma(n))}.$$

Finally, applying (iii), we obtain

$$\Delta z(n) \leq \frac{-\beta u^{\alpha}(n+1)\Delta u(n)}{\mu(n+1)\Pi^{\alpha}(n+1)(\Delta u(n+1))^{\alpha}}.$$

This shows (iv) holds and completes the proof of the lemma. □

Lemma 2.2. *Let $\beta_0 > 0$ and condition (2.4) hold. If $\{u(n)\}$ is an eventually positive solution of (1.1), then for any fixed $\beta \in (0,1)$, $\{u(n)/\Pi^{1-\beta}(n)\}$ is nondecreasing for sufficiently large n.*

Proof. Fix $\beta \in (0,1)$ and let $\{u(n)\}$ be a positive solution of (1.1). Choose $n_1 \geq n_0$ so that $u(n) > 0$, (2.3) holds, and parts (i)–(iv) of Lemma 2.1 hold for all $n \geq n_1$. Using (2.11) in (2.6), we obtain

$$\Delta z(n) \leq \beta \Delta u(n).$$

Summing the last inequality from n to ∞ and using part (ii) of Lemma 2.1, we have

$$z(n) \geq z(\infty) - \beta u(\infty) + \beta u(n) \geq \beta u(n),$$

which in view of the definition of $z(n)$ gives

$$(1-\beta)u(n) \geq -\mu^{1/\alpha}(n)\Pi(n)\Delta u(n). \tag{2.16}$$

Now

$$\Delta\left(\frac{u(n)}{\Pi^{1-\beta}(n)}\right) = \frac{\Pi^{1-\beta}(n)\Delta u(n) - u(n)\Delta\Pi^{1-\beta}(n)}{\Pi^{1-\beta}(n)\Pi^{1-\beta}(n+1)}. \tag{2.17}$$

Since $1-\beta < 1$, by the Mean-Value theorem, we have

$$-\Delta\Pi^{1-\beta}(n) \geq \frac{(1-\beta)\Pi^{-\beta}(n)}{\mu^{1/\alpha}(n)}.$$

From this, (2.16), and (2.17),

$$\Delta\left(\frac{u(n)}{\Pi^{1-\beta}(n)}\right) = \frac{\Pi(n)\mu^{1/\alpha}(n)\Delta u(n) + (1-\beta)u(n)}{\mu^{1/\alpha}(n)\Pi(n)\Pi^{1-\beta}(n+1)} \geq 0.$$

This proves the lemma. □

Theorem 2.3. *In addition to (2.4), assume that*

$$\lim_{n\to\infty} \frac{\Pi(n+1)}{\Pi(\sigma(n))} = \infty. \tag{2.18}$$

If $R_* > 0$, then equation (1.1) is oscillatory.

Proof. Choose $\beta \in (0,1)$, let $\{u(n)\}$ be a positive solution of (1.1), and choose $n_1 \geq n_0$ so that $u(n) > 0$, (2.3) holds, and Lemmas 2.1 and 2.2 hold for $n \geq n_1$. Using (2.3) and the fact that $\{\frac{u(n)}{\Pi^{1-\beta}(n)}\}$ is nondecreasing, from (1.1) we see that

$$\begin{aligned} -\Delta(\mu(n)(\Delta u(n))^{\alpha}) = f(n)u^{\alpha}(\sigma(n)) &\geq \frac{\alpha\beta}{\mu^{1/\alpha}(n)\Pi^{\alpha}(\sigma(n))\Pi(n+1)}u^{\alpha}(\sigma(n)) \\ &= \frac{\alpha\beta}{\mu^{1/\alpha}(n)\Pi^{\alpha\beta}(\sigma(n))\Pi(n+1)\Pi^{\alpha(1-\beta)}(\sigma(n))}u^{\alpha}(\sigma(n)) \\ &\geq \frac{\alpha\beta}{\mu^{1/\alpha}(n)\Pi^{\alpha+1}(n+1)}u^{\alpha}(\sigma(n+1))\left(\frac{\Pi(n+1)}{\Pi(\sigma(n))}\right)^{\alpha\beta} \\ &\geq B^{\alpha}\beta\frac{\alpha u^{\alpha}(n+1)}{\mu^{1/\alpha}(n)\Pi^{\alpha+1}(n+1)}, \end{aligned} \tag{2.19}$$

where we have used the fact that (2.18) implies $\frac{\Pi(n+1)}{\Pi(\sigma(n))} \geq B^{1/\beta}$ for some $B > 0$ and $n \geq n_2$ for some $n_2 \geq n_1$. Summing (2.19) from n_2 to $n-1$, we obtain

$$\begin{aligned} -\mu(n)(\Delta u(n))^{\alpha} &\geq B^{\alpha}\beta\sum_{s=n_2}^{n-1}\frac{\alpha u^{\alpha}(s+1)}{\mu^{1/\alpha}(s)\Pi^{\alpha+1}(s+1)} \\ &\geq B^{\alpha}\beta u^{\alpha}(n)\sum_{s=n_2}^{n-1}\frac{\alpha(\Pi(s)-\Pi(s+1))}{\Pi^{\alpha+1}(s+1)} \\ &\geq B^{\alpha}\beta u^{\alpha}(n)\sum_{s=n_2}^{n-1}\alpha\int_{\Pi(s+1)}^{\Pi(s)}\frac{dV}{V^{\alpha+1}} \\ &= B^{\alpha}\beta u^{\alpha}(n)\left(\frac{1}{\Pi^{\alpha}(n)}-\frac{1}{\Pi^{\alpha}(n_2)}\right) \\ &\geq B^{\alpha}\beta\ell_1\frac{u^{\alpha}(n)}{\Pi^{\alpha}(n)} \end{aligned}$$

for $n \geq n_3$ for some $\ell_1 \in (0,1)$ and $n_3 \geq n_2$.

Since $\ell_1 \in (0,1)$, $\beta \in (0,1)$, and $B > 0$ is arbitrary, we can choose B such that $B > \frac{1}{(\ell\beta)^{1/\alpha}}$. Then (2.20) yields

$$-\Pi(n)\mu^{1/\alpha}(n)\Delta u(n) > B(\ell\beta)^{1/\alpha}u(n) > u(n),$$

which contradicts (2.11). This completes the proof of the theorem. □

To prove our next result, we assume that there is a constant $\lambda > 0$ such that

$$\frac{\Pi(n+1)}{\Pi(\sigma(n))} \geq \lambda. \tag{2.20}$$

Theorem 2.4. *Let $\beta_0 > 0$ and conditions (2.4) and (2.20) hold. If*

$$\limsup_{n\to\infty} \Pi^\alpha(n) \sum_{s=n_1}^{n-1} f(s) \left(\frac{\Pi(\sigma(s))}{\Pi(s+1)}\right)^\alpha > \frac{(1-\beta)^\alpha}{\lambda^{\alpha\beta}} \tag{2.21}$$

for any $n_1 \geq n_0$, then (1.1) is oscillatory.

Proof. Choose $\beta \in (0,1)$, let $\{u(n)\}$ be a positive solution of (1.1), and choose $n_1 \geq n_0$ so that $u(n) > 0$, (2.3) holds, and Lemmas 2.1 and 2.2 hold for $n \geq n_1$. Summing (1.1) from n_1 to $n-1$ gives

$$\begin{aligned} -\mu(n)(\Delta u(n))^\alpha &= -\mu(n_1)(\Delta u(n_1)) + \sum_{s=n_1}^{n-1} f(s) u^\alpha(\sigma(s)) \\ &\geq \sum_{s=n_1}^{n-1} f(s) \Pi^{(1-\beta)\alpha}(\sigma(s)) \left(\frac{u(\sigma(s))}{\Pi^{(1-\beta)}(\sigma(s))}\right)^\alpha \\ &\geq \sum_{s=n_1}^{n-1} f(s) \Pi^{(1-\beta)\alpha}(\sigma(s)) \left(\frac{u(s+1)}{\Pi^{(1-\beta)}(s+1)}\right)^\alpha \end{aligned}$$

since $\frac{u(n)}{\Pi(n)^{1-\beta}}$ is nondecreasing. Then, from (2.20),

$$-\mu(n)(\Delta u(n))^\alpha \geq u^\alpha(n) \sum_{s=n_1}^{n-1} f(s) \left(\frac{\Pi(\sigma(s))}{\Pi(s+1)}\right)^\alpha \lambda^{\alpha\beta}. \tag{2.22}$$

Now using (2.16) in (2.22), we obtain

$$1 \geq \frac{\Pi^\alpha(n)\lambda^{\alpha\beta}}{(1-\beta)^\alpha} \sum_{s=n_1}^{n-1} f(s) \left(\frac{\Pi(\sigma(s))}{\Pi(s+1)}\right)^\alpha,$$

which contradicts (2.21). This completes the proof. □

Remark 2.5. Clearly our Theorem 2.4 improves Corollary 8 of [12].

3. EXAMPLES

In this section, we provide two examples to illustrate the main results.

Example 3.1. Consider the second-order noncanonical advanced difference equation

$$\Delta(n^3(n+1)^3(\Delta u(n))^3) + (n+1)^6 u^3((n+1)^2) = 0,\ n \geq 1. \tag{3.1}$$

Here $\mu(n) = n^3(n+1)^3$, $f(n) = (n+1)^6$, $\sigma(n) = (n+1)^2$, and $\alpha = 3$. Also, $\Pi(n) = \frac{1}{n}$ and

$$\lim_{n\to\infty} \frac{\Pi(n+1)}{\Pi(\sigma(n))} = \lim_{n\to\infty}(n+1) = \infty,$$

so (2.18) is satisfied. Notice that (2.3) holds for $\beta = \frac{1}{3}$, and it is clear that (2.4) also holds. Now

$$R_* = \liminf_{n\to\infty} n(n+1)\frac{1}{(n+1)^6}\frac{1}{(n+1)}(n+1)^6 = \liminf_{n\to\infty} n > 0.$$

Therefore, by Theorem 2.3, equation (3.1) is oscillatory.

Example 3.2. Consider the second-order noncanonical advanced difference equation

$$\Delta(n(n+1)\Delta u(n)) + \gamma u(2n) = 0,\ n \geq 1. \tag{3.2}$$

Here $\mu(n) = n(n+1)$, $f(n) = \gamma \geq 1$, $\sigma(n) = 2n$, and $\alpha = 1$. Now $\Pi(n) = \frac{1}{n}$ and condition (2.20) becomes

$$\liminf_{n\to\infty} \frac{\Pi(n+1)}{\Pi(\sigma(n))} = \liminf_{n\to\infty} \frac{2n}{(n+1)} = 2 = \lambda.$$

We see that

$$R_* = \liminf_{n\to\infty} n(n+1).\frac{1}{2n}.\frac{1}{(n+1)}\gamma = \gamma/2 > 0$$

which implies $\beta_0 > 0$. By taking $\beta = \frac{1}{2}$, we see that (2.3) is satisfied. Clearly, (2.4) holds as well. Condition (2.21) becomes

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{s=1}^{n-1}\gamma\left(\frac{s+1}{2s}\right) > \limsup_{n\to\infty} \frac{1}{n}\sum_{s=1}^{n-1}\gamma\left(\frac{s}{2s}\right) = \limsup_{n\to\infty} \frac{1}{n}\left(\frac{n-1}{2}\right)\gamma = \gamma/2.$$

That is, $\gamma/2 > \frac{1}{2\sqrt{2}}$, or $\gamma > \frac{1}{\sqrt{2}}$, and so (2.21) holds. Therefore, by Theorem 2.4, equation (3.2) is oscillatory.

Remark 3.3. Note that the Example 3.2 was considered in [12] where it was shown that equation (3.2) is oscillatory if $\gamma > 2$; in [16] it was shown that the equation (3.2) is oscillatory if $\gamma = 2$. Therefore, our Theorem 2.4 improves Theorem 4 in [12] and Theorem 3.3 in [16].

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REFERENCES

[1] R.P. Agarwal, *Difference Equations and Inequalities,* Dekker, New York, 2000.

[2] R.P. Agarwal, M. Bohner, S.R. Grace, D. O'Regan, *Discrete Oscillation Theory,* Hindawi, New York, 2005.

[3] R.P. Agarwal, C. Zhang, T. Li, *New Kamenev type oscillation criteria for second order nonlinear advanced dynamic equations*, Appl. Math. Comput. **225** (2013), 822–823.

[4] R. Bellman, K.L. Cooke, *Differential-Difference Equations*, Math. Sci. Eng., vol. 6, Academic Press, 1963.

[5] G.E. Chatzarakis, I.P. Stavroulakis, *Oscillation of advanced difference equations with variable arguments*, Electron. J. Qual. Theory Differ. Equ. **2012** (2012), no. 79, 1–16.

[6] G.E. Chatzarakis, E. Thandapani, *New oscillation criterion of first order difference equations with advanced argument*, Adv. Math. Sci. Journal **10** (2021), 971–979.

[7] E. Chandrasekar, G.E. Chatzarakis, G. Palani, E. Thandapani, *Oscillation criteria for advanced difference equations of second order*, Appl. Math. Comput. **372** (2020), 124963.

[8] G.E. Chatzarakis, P. Dinaker, S. Selvarangam, E. Thandapani, *Oscillation properties of second order quasilinear difference equations with unbounded delay and advanced neutral terms*, Math. Bohem. (2021), to appear.

[9] G.E. Chatzarakis, J. Džurina, I. Jadlovská, *New oscillation criteria for second order half linear advanced differential equations*, Appl. Math. Comput. **347** (2019), 404–416.

[10] G. Chatzarakis, S.R. Grace, I. Jadlovská, *Oscillation theorems for certain second order nonlinear retarded difference equations*, Math. Slovaca (2021), to appear.

[11] C. Dharuman, J.R. Graef, E. Thandapani, K.S. Vidhyaa, *Oscillation of second order difference equation with a sublinear neutral term*, J. Math. Appl. **40** (2017), 59–67.

[12] P. Dinakar, S. Selvarangam, E. Thandapani, *New oscillation condition for second order halflinear advanced difference equations*, Int. J. Math. Eng. Manag. Sci. **4** (2019), 1459–1470.

[13] S.R. Grace, J. Alzabut, *Oscillation results for nonlinear second order difference equations with mixed neutral terms*, Adv. Difference Equ. **2020** (2020), Article no. 8.

[14] J.R. Graef, *Oscillation of higher order functional differential equations with an advanced argument*, Appl. Math. Lett., to appear.

[15] I. Jadlovská, *Oscillation criteria of Kneser-type for second-order half-linear advanced differential equations*, Appl. Math. Lett. **106** (2020), 106354.

[16] R. Kanagasabapathi, S. Selvarangam, J.R. Graef, E. Thandapani, *Oscillation results for nonlinear second order difference equations with advanced arguments*, Indian J. Math. (2021), to appear.

[17] O. Ocalan, O. Akhi, *Oscillatory properties for advanced difference equations*, Novi Sad J. Math. **37** (2007), 39–47.

[18] B. Ping, M. Han, *Oscillation of second order difference equations with advanced arguments*, Conference Publication, Amer. Inst. Math. Sciences (2003), 108–112.

[19] H. Wu, L. Erbe, A. Peterson, *Oscillation of solutions to second order half-linear delay dynamic equation on time scales*, Electron. J. Differential Equations **2016** (2016), no. 71, 1–15.

[20] B. Zhang, S.S. Cheng, *Comparison and oscillation theorems for an advanced type difference equations*, Ann. Differ. Equ. **4** (1995), 485–494.

[21] Z. Zhang, T. Li, *Oscillation theorems for second order advanced functional difference equations*, Comput. Math. Appl. **36** (1998), 11–18.

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