Opuscula Math. 42, no. 1 (2022), 65–73
https://doi.org/10.7494/OpMath.2022.42.1.65 Opuscula Mathematica

EDGE HOMOGENEOUS COLORINGS

Tomáš Madaras, Alfréd Onderko, and Thomas Schweser

Communicated by Adam Paweł Wojda

Abstract. We explore four kinds of edge colorings defined by the requirement of equal number of colors appearing, in particular ways, around each vertex or each edge. We obtain the characterization of graphs colorable in such a way that the ends of each edge see (not regarding the edge color itself) q colors (resp. one end sees q colors and the color sets for both ends are the same), and a sufficient condition for 2-coloring a graph in a way that the ends of each edge see (with the omission of that edge color) altogether q colors. The relations of these colorings to M_q-colorings and role colorings are also discussed; we prove an interpolation theorem for the numbers of colors in edge coloring where all edges around each vertex have q colors.

Keywords: homogeneous coloring, M_q-coloring, line graph, role coloring.

Mathematics Subject Classification: 05C15.

1. INTRODUCTION

Edge colorings of various types are traditional and widely studied topic in graph theory. Many of them can be described using the following framework: for a graph G and a collection of its subgraphs $\{H_1, \ldots, H_t\}$, specify the properties of an edge coloring $\varphi : E(G) \to \{1, \ldots, k\}$ by stating size-related conditions on the number $|\varphi(H_j)| = |\varphi(e) : e \in E(H_j)|$ of colors used on edges of H_j for every $j = 1, \ldots, t$. For example, if H_j is the maximal star of G with the central vertex $v_j \in V(G)$, then the condition $|\varphi(H_j)| = |E(H_j)|$ describes the usual regular edge coloring, and if H_j is a facial cycle of a 2-connected plane graph G, then this condition describes the cyclic edge coloring (see the survey [5]). For $H_{j,k}$ being the maximal double star with the central edge v_jv_k, we obtain, in this way, the strong edge coloring (introduced in [9] and further investigated in [8]). If the collection of subgraphs H_j consists of all cycles and $|\varphi(H_j)| \geq 2$ is required, the color classes of the corresponding edge coloring are forests; the associated coloring invariant is the arboricity (see [15,16]). For H_j being all paths and all cycles on three or four edges, the condition $|\varphi(H_j)| \geq 3$ defines the star edge coloring (first considered in [14] and recently surveyed in [13]). Further examples are so called M_i- and m_i-colorings introduced by Czap [2]: the collection of subgraphs H_j consists again of all maximal stars, and it is required that $|\varphi(H_j)| \leq i$, respectively $|\varphi(H_j)| \geq i$ for a fixed positive integer i; the corresponding graph invariants $\mathcal{K}_i(G)$ and $k_i(G)$ (the maximum and the minimum number of colors that may be used in an M_i-, respectively m_i-coloring of G) were further studied in [1,3,4,6,11].

Here, we explore some further kinds of edge colorings, which are defined in a similar manner as M_i- or m_i-colorings. Let q be a fixed positive integer, G be a graph, and φ be an edge coloring of G. We use the following notation: if H is a maximal star of G with central vertex v, we put $\underline{\varphi}(v) := \varphi(H)$. Similarly, if H is a maximal double star with central edge $e = uv$, we put $\underline{\varphi}(e) := \varphi(H)$. Finally, we use the notation $\underline{\varphi}_u(e)$ for the set of colors used on the maximal star centered at u except of the color of e (that is, $\underline{\varphi}_u(e) = \{\varphi(uw) : \ w \neq v\}$).

We say that φ is:

- an $M_{=q}$-*coloring*, if for every vertex $v \in V(G)$, we have
$$|\underline{\varphi}(v)| = q,$$
- an $L_{=q}$-*coloring*, if for every edge $e \in E(G)$, $e = uv$ we have
$$|\underline{\varphi}_u(e) \cup \underline{\varphi}_v(e)| = q,$$
- an $E_{=q}$-*coloring*, if for every edge $e \in E(G), e = uv$ we have
$$|\underline{\varphi}_u(e)| = q \quad \text{and} \quad |\underline{\varphi}_v(e)| = q,$$
- an $E_{\approx q}$-*coloring*, if for every edge $e \in E(G), e = uv$ we have
$$|\underline{\varphi}_u(e)| = q \quad \text{and} \quad \underline{\varphi}_u(e) = \underline{\varphi}_v(e).$$

Note that every $M_{=q}$-coloring is both an M_q- and an m_q-coloring. The second coloring, $L_{=q}$-coloring, corresponds to q-homogeneous vertex coloring (considered in [12] and further in [20]; the defining property is the same number of colours used on neighbours of every vertex) of the line graph of G, but without the requirement to be proper. It is easy to see that every $E_{\approx q}$-coloring is also an $E_{=q}$-coloring and an $L_{=q}$-coloring. The converses are not true: for example, take a graph of 3-gonal prism D_3, color all edges in both triangular faces by 1 and 2, respectively, and the remaining edges by 3; we obtain an $L_{=2}$-coloring which is neither $E_{=2}$-coloring nor $E_{\approx 2}$-coloring. Similarly, the 4-coloring of D_3 in which both triangular faces are rainbow and all quadrangular faces bichromatic is an $E_{=2}$-coloring, but not $E_{\approx 2}$-coloring. Finally, the 2-coloring of D_3 obtained by coloring the edges of a perfect matching by 1 and all other edges by 2 is an $M_{=2}$-coloring, but not a coloring of any of three remaining types.

Regarding $M_{=q}$-coloring, we obtain the following interpolation coloring theorem.

Theorem 1.1. *Let $q \geq 2$ be an integer and let G be a graph with $\delta(G) \geq q$. Then, for each integer k, $q + 1 \leq k \leq \mathcal{K}_q(G)$, there exists an $M_{=q}$-coloring of G which uses k colors.*

For the other coloring concepts, we prove several existence results:

Theorem 1.2. *Let G be a graph with $\delta(G) \geq 4$. Then G is $L_{=2}$-colorable using two colors.*

Theorem 1.3. *Let $q \geq 2$ be an integer and let G be a connected graph. Then G is $E_{\approx q}$-colorable if and only if G is either a $(q+1)$-regular graph of Class 1 or $\delta(G) \geq 2q$; in addition, the number of colors used is $q+1$ in the former and q in the latter case.*

Theorem 1.4. *Let $q \geq 2$ be an integer and let G be a connected graph. Then G is $E_{=q}$-colorable if and only if G is either a $(q+1)$-regular graph or $\delta(G) \geq 2q$.*

2. PROOFS

The proof of Theorem 1.1 is implied by two lemmas which treat the existence of an $M_{=q}$-coloring for small and large number of used colors, respectively. The following fact is easy to see:

Observation 2.1. *If G is a graph and k is an integer, $\chi'(G) \leq k \leq |E(G)|$, then there is a proper edge coloring of G which uses k colors.*

Lemma 2.2. *Let q and k be integers, $q \geq 2$ and $q+1 \leq k \leq 2q-1$. Let G be a graph such that there are at least two non-pendant vertices of G and $\deg_G(v) \geq q$ for each such vertex v. Then there is an edge coloring φ of G which uses k colors and $|\underline{\varphi}(v)| = q$ for each vertex v of degree at least q.*

Proof. Let k and q be fixed throughout this proof. For the sake of simplicity we denote following two statements for an arbitrary graph G:

A(G): "There are at least two non-pendant vertices of the graph G and $\deg_G(v) \geq q$ for each such vertex v."
C(G): "There is an edge coloring φ of G which uses k colors and $|\underline{\varphi}(v)| = q$ for each vertex v of degree at least q."

The goal of the proof is to show that A(G) implies C(G).

Clearly, if A(G) is true then G has at least $2q-1$ edges. Denote by $V_{\geq q}(G)$ the set of vertices of degree at least q in G. Let

$$S(G) = \sum_{v \in V(G)} \max\{0, \deg_G(v) - q\}.$$

First suppose that $S(G) = 0$. In this case, the degree of each non-pendant vertex is precisely q. Thus, maximum degree of G is q and from Vizing's theorem we get $\chi'(G) \in \{q, q+1\}$. Therefore, $\chi'(G) \leq q+1 \leq k \leq 2q-1 \leq |E(G)|$ and from Observation 2.1 the existence of φ with required properties follows.

Suppose now that $S(G) > 0$ and for every graph G' with $S(G') < S(G)$ the statement A(G') implies C(G'). Since $S(G) > 0$, there is a vertex x of G of degree at least $q+1$. There are three major cases to consider.

Case 1. Every neighbor of x is pendant. Let y be a neighbor of x in G. Clearly, $V_{\geq q}(G-y) = V_{\geq q}(G)$ and therefore, $\mathrm{A}(G-y)$ is true. Since $S(G-y)$ is clearly smaller than $S(G)$, there is an edge coloring ψ of $G-y$ which uses k colors and $|\underline{\psi}(v)| = q$ for each $v \in V_{\geq q}(G-y)$. Set $\varphi(e) = \psi(e)$ for every edge $e \in E(G-y)$ and $\varphi(xy) = c$, where c is a color from $\underline{\psi}(x)$. It is easy to see that $\varphi(G) = \psi(G-y)$ as well as $\underline{\varphi}(v) = \underline{\psi}(v)$ for each vertex $v \neq y$ of G. Thus, φ is the coloring of G with required properties.

Case 2. The vertex x has a neighbor y of degree at least $q+1$ in G. By removing the edge xy we obtain the graph $G-xy$ such that $S(G-xy) < S(G)$ and $\mathrm{A}(G-xy)$ is true. Therefore, like in the previous case, there is an edge coloring ψ of $G-xy$ such that $|\psi(G-xy)| = k$ and $|\underline{\psi}(v)| = q$ for each $v \in V_{\geq q}(G-xy)$. Since both x and y are vertices of degree at least q in $G-xy$ and $k \leq 2q-1$, there is a color c from $\underline{\psi}(x) \cap \underline{\psi}(y)$. By assigning the color c to xy, we can extend ψ to all edges of G, thereby obtaining the coloring of G with required properties.

Case 3. The vertex x has a neighbor y of degree q. In addition we may suppose that none of the neighbors of x in G is of degree at least $q+1$, otherwise one could use the method described in the Case 2. Let G' be a graph obtained from $G-xy$ by adding a new vertex z and an edge yz. Like in the previous cases, it is easy to see that $S(G') < S(G)$ and $\mathrm{A}(G')$ is true. Let ψ be an edge coloring of G' which uses k colors and $|\underline{\psi}(v)| = q$ for each $v \in V_{\geq q}(G')$. Since ψ uses at most $2q-1$ colors, there is a color c in the set $\underline{\psi}(x) \cap \underline{\psi}(y)$. If $c = \psi(yz)$, one can define an edge coloring φ of G such that $\varphi(e) = \psi(e)$ for each edge $e \in E(G) \cap E(G')$ and $\varphi(xy) = \psi(yz)$. It is easy to see that in this case φ has the required properties, more precisely φ uses k colors and $\underline{\varphi}(v) = \underline{\psi}(v)$ for each vertex v of G.

Suppose now that the color c is different from $\psi(yz)$.If $\psi(yz)$ is used also on another edge in G', then there is a color $c' \in \underline{\psi}(x) \setminus \underline{\psi}(y)$ (since $\psi(yz) \notin \underline{\psi}(x)$). If we define an edge coloring φ of G by setting $\varphi(e) = \psi(e)$ for each $e \in E(G) \cap E(G')$ and $\varphi(xy) = c'$, then φ has desired properties. Hence, assume that the color $\psi(yz)$ is used only once. Consider the graph $G'[\psi^{-1}(c)]$. If there is a monochromatic path of color c between x and y in G', then it has length at least 2, thus it passes through a neighbor v of x. Since $|\underline{\psi}(v)| = q$ and c is used on at least two edges incident with v, $\deg_G(v) = \deg_{G'}(v) \geq q+1$, and we obtain the situation as in Case 2. Therefore, assume that x and y belong to different components of $G'[\psi^{-1}(c)]$. Let H be a component of $G[\psi^{-1}(c)]$ which contains x. Note that, for each $w \in V(H)$, $c \in \underline{\psi}(w)$. After recoloring the edges of H with color $\psi(yz)$ we can continue like in the first part of the Case 3. □

It is easy to see that, for an M_q-coloring of a graph G, the recoloring of all edges of the same color class with another color yields again an M_q-coloring; therefore, M_q-coloring of G exists for every number of colors whis does not exceed $\mathcal{K}_q(G)$.

Lemma 2.3. *Let $q \geq 2$ and $k \geq 2q$ be positive integers and let G be a graph with $\delta(G) \geq q$. If there is an M_q-edge coloring of G which uses k colors, then there is also an $M_{=q}$-edge coloring of G using the same number of colors.*

Proof. Suppose that for some graph G and integers q, k, the statement does not hold. Let Φ be the set of all M_q-edge colorings of G which use k colors. Let φ be the edge coloring from Φ such that

$$S(\varphi) = \sum_{v \in V(G)} \big(q - |\underline{\varphi}(v)|\big)$$

is the smallest possible. Clearly, $S(\varphi) > 0$, otherwise φ is an $M_{=q}$-coloring.

Let v_0 be a vertex such that $|\underline{\varphi}(v_0)| < q$. Denote by c a color used at least twice on edges incident with v_0 in φ, and let v_1 be a neighbor of v_0 such that $\varphi(v_0v_1) = c$. There are several cases to be considered.

Case 1. Suppose that $|\underline{\varphi}(v_1)| = q$.

Subcase 1.1. Let c be used only once on edges incident with v_1. Since

$$|\underline{\varphi}(v_0) \cup \underline{\varphi}(v_1)| \leq (q-1) + q = 2q - 1 < k,$$

there is a color

$$c' \in \varphi(G) \smallsetminus (\underline{\varphi}(v_0) \cup \underline{\varphi}(v_1).$$

By recoloring the edge v_0v_1 with c', we obtain an edge coloring $\varphi^* \in \Phi$ such that $S(\varphi^*) < S(\varphi)$, a contradiction.

Subcase 1.2. Let c be used at least twice on edges incident with v_1. Since

$$|\underline{\varphi}(v_1)| = q > |\underline{\varphi}(v_0)|,$$

there is a color

$$c' \in \underline{\varphi}(v_1) \smallsetminus \underline{\varphi}(v_0).$$

By recoloring the edge v_0v_1 with c', we obtain an edge coloring $\varphi^* \in \Phi$ with $S(\varphi^*) < S(\varphi)$, a contradiction.

Case 2. Suppose now that $|\underline{\varphi}(v_1)| < q$. Then there exists a color

$$c' \in \varphi(G) \setminus (\underline{\varphi}(v_0) \cup \underline{\varphi}(v_1))$$

(because $|\underline{\varphi}(v_0) \cup \underline{\varphi}(v_1)| < 2q \leq k$). By recoloring the edge v_0v_1 with c', we obtain an edge coloring $\varphi^* \in \Phi$ such that $|\underline{\varphi^*}(v_0)| > |\underline{\varphi}(v_0)|$ and $|\underline{\varphi^*}(v_1)| \geq |\underline{\varphi}(v_1)|$; hence $S(\varphi^*) < S(\varphi)$, a contradiction. □

Let us note that, for $E_{=q}$-coloring, an analogue of Theorem 1.1 does not hold. To demonstrate this, consider an $E_{=2}$-coloring of a 4-regular graph G. It is not hard to see that if an edge uv has a color c, then c appears also on two other edges incident with u and v, respectively. This implies that the color classes of $E_{=2}$-coloring of G are the collections of edge-disjoint cycles. Now, take the graph of the icosidodecahedron (which is plane and 4-regular, and consists of 20 triangular and 12 pentagonal faces in such a way that none two faces of the same size share an edge). For this graph, there exists an $E_{=2}$-coloring using 12 and 20 colors, as seen from the collection of all its triangular resp. pentagonal faces. If there exists such a coloring which uses 19 colors, then the edge set shall be partitioned into 19 cycles. Since the icosidodecahedron graph contains no 4-cycles, such a partition necessarily contains 18 3-cycles and a single 6-cycle which is formed by adjacent triangular and pentagonal face. This 6-cycle, however, prevents five 3-cycles to be part of the considered partition, a contradiction.

Note that an $E_{\approx q}$-coloring is a special case of an $E_{=q}$-coloring. Therefore, if a graph G admits an $E_{\approx q}$-coloring, then it also admits an $E_{=q}$-coloring, for given $q \geq 2$. On the other hand, if G does not admit an $E_{=q}$-coloring, then it does not admit an $E_{\approx q}$-coloring. Thus, the following lemmas prove both Theorem 1.3 and Theorem 1.4; Theorem 1.2 then follows from Theorem 1.3.

Lemma 2.4. *Let φ be an edge coloring of a $(q+1)$-regular graph G. Then, φ is an $E_{=q}$-coloring of G if and only if φ is a proper coloring of G.*

Proof. First, let φ be an $E_{=q}$-coloring of G. Suppose to the contrary that φ is not a proper coloring of G. Let e be the edge of G with endvertices u and v, such that $\varphi(e)$ is used at least twice on edges incident with v. Denote by $e_1, \ldots, e_{q+1}$ the edges incident with v in G, $e_1 = e$ and $\varphi(e_2) = \varphi(e_1)$. Since, φ is an $E_{=q}$-coloring, $|\{\varphi(e_2), \ldots, \varphi(e_{q+1}\}| = q$. Therefore, $\varphi(e_3)$ is different from $\varphi(e_2)$. Evidently, $\underline{\varphi}_v(e_3) = \underline{\varphi}_v(e) \setminus \{\varphi(e_3)\}$. Thus, $|\underline{\varphi}_v(e_3)| = q-1$, a contradiction; so, φ is a proper coloring of G.

On the other hand, it is easy to see that each proper coloring of G is also an $E_{=q}$-coloring of G, which completes the proof. □

It is easy to see that a proper coloring of a $(q+1)$-regular graph is an $E_{\approx q}$-coloring if and only if G is of class 1.

Lemma 2.5. *If G is an $E_{=q}$-colorable graph for some $q \geq 2$, then G does not contain a vertex v such that $q+1 < \deg_G(v) < 2q$.*

Proof. Suppose the lemma were false. Then we could find an $E_{=q}$-coloring φ of a graph G and a vertex $v \in V(G)$ such that $q+1 < \deg_G(v) < 2q$. Since $q+1 < \deg_G(v)$, there are two edges incident to v, denoted by e_1 and e_2, which are colored with the same color. On the other hand, $\deg_G(v) < 2q$ implies that there is an edge e_3 incident to v which has unique color, i.e., the color used on e_3 is used only once on edges incident to v. Therefore, $\underline{\varphi}_v(e_1) = \underline{\varphi}(v)$ and $\underline{\varphi}_v(e_3) = \underline{\varphi}(v) \setminus \{\varphi(e_3)\}$. Thus, sets $\underline{\varphi}_v(e_1)$ and $\underline{\varphi}_v(e_3)$ have clearly different cardinalities, a contradiction. □

Lemma 2.6. *Let $q \geq 2$ be an integer and let φ be an $E_{=q}$-coloring of a graph G. If $|\underline{\varphi}(v)| = q+1$ then $\deg_G(v) = q+1$.*

Proof. Suppose that φ be an $E_{=q}$-coloring of a graph G and there is a vertex $v \in V(G)$ such that $|\underline{\varphi}(v)| = q+1$ and $\deg_G(v) \geq q+2$. Then, there is a color c which is used on at least two edges incident to v. Let e be the edge of color c incident with v. Then $\underline{\varphi}_v(e) = \underline{\varphi}(v)$ and therefore, $|\underline{\varphi}_v(e)| = q+1$, a contradiction. □

Lemma 2.7. *Let $q \geq 2$ be an integer and let G be a graph with $\delta(G) \geq 2q$. Then G is $E_{\approx q}$-colorable using q colors.*

Proof. Let G be a graph with $\delta(G) \geq 2q$ which is not $E_{\approx q}$-colorable using q colors, and

$$S(G) = \sum_{v \in V(G)} (\deg_G(v) - 2q)$$

be minimal.

Suppose that there are two adjacent vertices u and v of degree at least $2q+1$ in G. Clearly, $S(G-uv) < S(G)$, thus, $G-uv$ is $E_{\approx q}$-colorable. We may expand $E_{\approx q}$-coloring of $G-uv$ to $E_{\approx q}$-coloring of G, which uses q colors, by simply coloring the edge uv with any color.

Now, suppose that G does not contain a pair of adjacent vertices of degree at least $2q+1$. First, we discuss the case, when there are at least two vertices, denoted by u_1 and u_2, of degree at least $2q+1$ in G. Let v_1 and v_2 be two vertices adjacent to u_1 and u_2, respectively. Let H be the graph which is obtained by removing an edge from K_{2q+1}, and let x_1 and x_2 be two vertices of H with $\deg_H(x_1) = \deg_H(x_2) = 2q-1$. We now construct a graph G' from the disjoint union of $G - u_1v_1 - u_2v_2$ and H, by adding edges x_1v_1 and x_2v_2. Clearly, $\delta(G') \geq 2q$ and $S(G') < S(G)$. Thus, there is an $E_{\approx q}$-coloring φ' of G' with q colors. Let φ be an edge coloring of G such that $\varphi(e) = \varphi'(e')$ for each $e \in E(G) \cap E(G')$, $\varphi(u_1v_1) = \varphi'(x_1v_1)$ and $\varphi(u_2v_2) = \varphi'(x_2v_2)$. Evidently, φ is an $E_{\approx q}$-coloring of G.

Finally, we discuss the case, when there is only one vertex of G, denoted by u, of degree at least $2q+1$. Since the sum of degrees of all vertices of a graph is even, the degree of u in G is at least $2q+2$. Let v_1 and v_2 be two neighbors of u, both of degree $2q$ in G. Similarly, as in previous case, let H be a graph obtained from K_{2q+1} by removing an edge. Denote by x_1 and x_2 vertices of degree $2q-1$ in H. Let G' be the graph obtained from the disjoint union of $G - uv_1 - uv_2$ and H, by adding edges x_1v_1 and x_2v_2. Analogously to previous case, $S(G') < S(G)$ implies the existence of $E_{\approx q}$-coloring of G', which can be easily transformed into $E_{\approx q}$-coloring of G.

Therefore, G does not contain a vertex of degree at least $2q+1$. Thus, G is $2q$-regular graph and by Petersen's 2-factor theorem, it is 2-factorable. By coloring q edge disjoint 2-factors with one color each, we obtain an $E_{\approx q}$-coloring, a contradiction. □

3. CONCLUDING REMARKS

Theorem 1.2 yields that line graph of each graph of minimum degree at least 4 is vertex 2-colorable in a way that each vertex contains both colors in its neighborhood. This coloring is known as the R6-role coloring (see [7]) and its existence for general graphs was shown to be NP-complete in [18]; it remains NP-complete even for split graphs, see [10]. For some graph classes, like chordal graphs ([19]) or cographs ([17]), the existence of a general 2-role coloring (that is, not restricted to R6-type) can be decided in polynomial time; however, the role and homogeneous colorability (in the sense that each vertex sees the same number of colors) of line- and related graphs as well as of selected graph operations seems to be still open.

Note also that there are graphs of minimum degree 2 which do not have an $L_{=2}$-coloring with just two colors – an easy example is theta-graph formed from three paths of length 5 (however, it is $L_{=2}$ colorable using three colors). Therefore, it would be interesting to find a graph of minimum degree 2 or 3 which is not $L_{=2}$-colorable (as we believe that it might exist).

Acknowledgements
This work was supported by the Slovak Research and Development Agency under the contract No. APVV-19-0153, by the Science Grant Agency - project VEGA 1/0574/21, and by UPJŠ internal grant VVGS-PF-2020-1430. Research partially supported by DAAD, Germany (as part of BMBF) and by the Ministry of Education, Science, Research and Sport of the Slovak Republic within the project 57447800.

REFERENCES

[1] K. Budajová, J. Czap, *m_2-edge coloring and maximum matching of graphs*, Int. J. Pure Appl. Math. **88** (2013), no. 2, 161–167.

[2] J. Czap, *M_i-edge colorings of graphs*, Appl. Math. Sci. **5** (2011), no. 49, 2437–2442.

[3] J. Czap, *A note on M_2-edge colorings of graphs*, Opuscula Math. **35** (2015), no. 3, 287–291.

[4] J. Czap, P. Šugerek, *M_i-edge colorings of complete graphs*, Appl. Math. Sci. **9** (2015), no. 77, 3835–3842.

[5] J. Czap, M. Horňák, S. Jendrol', *A survey on the cyclic coloring and its relaxations*, Discuss. Math. Graph Theory **41** (2021), 5–38.

[6] J. Czap, P. Šugerek, J. Ivančo, *M_2-edge colorings of cacti and graph joins*, Discuss. Math. Graph Theory **36** (2016), 59–69.

[7] M.G. Everett, S. Borgatti, *Role coloring a graph*, Math. Social Sci. **21** (1991), no. 2, 183–188.

[8] R.J. Faudree, A. Gyárfás, R.H. Schelp, Z. Tuza, *The strong chromatic index of graphs*, Ars Combin. **29B** (1990), 205–211.

[9] J.-L. Fouquet, J.-L. Jolivet, *Strong edge-colorings of graphs and applications to multi-k-gons*, Ars Combin. **16** (1983), 141–150.

[10] P. van 't Hof, D. Paulusma, J.M.M. van Rooij, *Computing role assignments of chordal graphs*, Theoret. Comput. Sci. **411** (2010), 3601–3613.

[11] J. Ivančo, *M_2-edge colorings of dense graphs*, Opuscula Math. **36** (2016), no. 5, 603–612.

[12] M. Janicová, T. Madaras, R. Soták, B. Lužar, *From NMNR-coloring of hypergraphs to homogenous coloring of graphs*, Ars Math. Contemp. **12** (2017), no. 2, 351–360.

[13] H. Lei, Y. Shi, *A survey on star edge-coloring of graphs*, arXiv:2009.08017, 2020.

[14] X.-S. Liu, K. Deng, *An upper bound on the star chromatic index of graphs with* $\delta \geq 7$, J. Lanzhou Univ. Nat. Sci. **44** (2008), 98–100.

[15] C.St.J.A. Nash-Williams, *Edge-disjoint spanning trees of finite graphs*, J. Lond. Math. Soc. **36** (1961), 445–450.

[16] C.St.J.A. Nash-Williams, *Decomposition of finite graphs into forests*, J. Lond. Math. Soc. **39** (1964), 12.

[17] C. Purcell, P. Rombach, *On the complexity of role colouring planar graphs, trees and cographs*, J. Discrete Algorithms **35** (2015), 1–8.

[18] F.S. Roberts, L. Sheng, *How hard is to determine if a graph has a 2-role assignment?*, Networks **37** (2001), 67–73.

[19] L. Sheng, *2-role assignments on triangulated graphs*, Theoret. Comput. Sci. **304** (2003), 201–214.

[20] M. Šurimová, T. Madaras, *Homogeneous colourings of graphs*, (2021), submitted.

Tomáš Madaras (corresponding author)
tomas.madaras@upjs.sk
https://orcid.org/0000-0003-4565-043X

P.J. Šafárik University in Košice
Institute of Mathematics, Faculty of Science
Jesenná 5, 04001 Košice, Slovakia

Alfréd Onderko
alfred.onderko@student.upjs.sk
https://orcid.org/0000-0003-4811-3955

P.J. Šafárik University in Košice
Institute of Mathematics, Faculty of Science
Jesenná 5, 04001 Košice, Slovakia

Thomas Schweser
thomas.schweser@tu-ilmenau.de
https://orcid.org/0000-0002-4969-7115

Technische Universitat Ilmenau
Institute of Mathematics
PF 100565, D-98684 Ilmenau, Germany

Received: July 28, 2021.
Revised: September 24, 2021.
Accepted: September 30, 2021.