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SOLUTION OF THE BOUNDARY VALUE PROBLEM OF HEAT CONDUCTION IN A CONE

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Abstract. In the paper we consider the boundary value problem of heat conduction in a non-cylindrical domain, which is an inverted cone, i.e. in the domain degenerating into a point at the initial moment of time. In this case, the boundary conditions contain a derivative with respect to the time variable; in practice, problems of this kind arise in the presence of the condition of the concentrated heat capacity. We prove a theorem on the solvability of a boundary value problem in weighted spaces of essentially bounded functions. The issues of solvability of the singular Volterra integral equation of the second kind, to which the original problem is reduced, are studied. We use the Carleman–Vekua method of equivalent regularization to solve the obtained singular Volterra integral equation.

Keywords: noncylindrical domain, cone, boundary value problem of heat conduction, singular Volterra integral equation, Carleman–Vekua regularization method.

Mathematics Subject Classification: 35K05, 45D99.

1. INTRODUCTION

Practice often requires to study the processes of heat transfer in domains of various shapes, the boundaries of which change with time. Problems of this kind arise, for example, in the theoretical study of energy or mass transfer processes associated with a change of the aggregate state of matter. Also, due to the constant increase in the use of contact technology, the problems of the optimal choice of the parameters of contact materials and their modes of operation are actual. Therefore, the study of thermophysical processes occurring in electrical contacts is very relevant in automation, instrumentation, welding technology, electrical equipment and in various devices, where contact elements serve as one of the main links [14, 20, 25].

A domain is usually called non-cylindrical in the literature if at least one of the parts of its boundary moves with time. If the boundaries of the domain do not change their shape with time, then the domain is called cylindrical. The theory of boundary value problems of heat conduction is well developed for such domains.

In most papers the domain in which the solution of the boundary value problem is sought does not degenerate into a point at the initial moment of time. In the several works [5, 6, 15–18] for solving such problems there is a technique which consists in reducing a non-cylindrical domain to a cylindrical one. There are a number of works [4, 7, 24] devoted to numerical methods for solving such problems.

Of particular interest are the boundary value problems of heat conduction in domains that degenerate into a point at the initial moment of time. For example, in the study of thermophysical processes in an electric arc of high-current disconnecting devices, the effect of contracting the axial section of the arc into a contact spot in the cathode field is observed. In this case, the diameter of the contact spot is several orders of magnitude smaller than the diameter of the section of the developed arc column, therefore it can be considered as a mathematical point [13, 23]. The solution domain changes over time according to the law determined by the conditions for opening the contacts. At the initial moment of time, the contacts are in a closed state and there is no domain for solving the problem. From a mathematical point of view, the peculiarity of the problem consists precisely in the presence of a moving boundary and the degeneration of the solution domain at the initial moment of time to a point.

The methods of separation of variables and integral transformations are generally not applicable to this type of problems, because although the problem remains within the framework of the classical methods of mathematical physics, it is impossible to coordinate the solution of the heat equation with the motion of the boundary of the heat transfer domain. Therefore, a research about boundary value problems in a domain degenerating at the initial moment of time is relevant. One-dimensional spatial variable boundary value problems in degenerate domains were studied in [1–3, 10–12]. By the method of heat potentials, such boundary value problems of heat conduction are reduced to the solution of singular Volterra type integral equations of the second kind. A singular Volterra type equation is understood as an equation whose kernel has the following property: the integral of the kernel of the equation does not tend to zero as the upper limit tends to the lower one. Such integral equations cannot be solved by the method of successive approximations, and in most cases the corresponding homogeneous integral equations have nonzero solutions.

In this paper, we consider a two-dimensional boundary value problem of heat conduction in a cone with boundary conditions containing the time derivative.

2. STATEMENT OF THE BOUNDARY VALUE PROBLEM

We consider in the domain $G = \{(r, t) :\ 0 < r < t,\ t > 0\}$ the following boundary value problem:

$$\frac{\partial u}{\partial t} - a^2 \cdot \frac{1}{r^{2\nu-1}} \frac{\partial}{\partial r} \left(r^{2\nu-1} \frac{\partial u}{\partial r} \right) = 0, \tag{2.1}$$

$$\left. \left(2 \cdot \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} \right) \right|_{r=t} = g(t), \tag{2.2}$$

$$r^{2\nu-1}\frac{\partial u}{\partial r}\bigg|_{r=0} = q(t), \tag{2.3}$$

where $0 < \nu < 1$.

Remark 2.1. Solution of the problem (2.1)–(2.3) for $g(t) = 0,\ q(t) = 0$, i.e. solution of a complete homogeneous problem, can be only a constant.

3. MAIN RESULT

For the problem (2.1)–(2.3), we proved the following theorem.

Theorem 3.1. *If the conditions $t^{\nu-\frac{1}{2}}g(t) \in L_\infty(0,\infty)$, $t^{1-\nu}q(t) \in L_\infty(0,\infty)$ are satisfied, then the boundary value problem* (2.1)–(2.3) *has a solution $u(r,t) = \tilde{u}(r,t) + C$, $\tilde{u}(r,t) \in L_\infty(G)$, $C = const$.*

The subsequent content of the paper is devoted to the proof of Theorem 3.1.

4. BOUNDARY VALUE PROBLEM TRANSFORMATION

To become free of the time derivative in (2.2), we conduct some transformations. To do this, we introduce a new unknown function:

$$\omega(r,t) = r^{2\nu-1}\frac{\partial u}{\partial r}. \tag{4.1}$$

Then, according to formal transformations, taking into account (4.1), problem (2.1)–(2.3) is reduced to the following one:

$$\frac{\partial \omega}{\partial t} = a^2\frac{\partial^2 \omega}{\partial r^2} - a^2\frac{2\nu-1}{r}\frac{\partial \omega}{\partial r}, \tag{4.2}$$

$$\frac{a^2}{r^{2\nu-1}}\left(\frac{\partial \omega}{\partial r} + \frac{2}{a^2}\omega\right)\bigg|_{r=t} = g(t), \tag{4.3}$$

$$\omega(r,t)\big|_{r=0} = q(t). \tag{4.4}$$

5. INTEGRAL REPRESENTATION OF THE SOLUTION OF THE PROBLEM (4.2)–(4.4) USING HEAT POTENTIALS

Note that the function

$$G\left(r,\xi,t-\tau\right) = \frac{1}{2a^2}\cdot\frac{r^\nu\cdot\xi^{1-\nu}}{t-\tau}\cdot\exp\left[-\frac{r^2+\xi^2}{4a^2(t-\tau)}\right]\cdot I_\nu\left(\frac{r\xi}{2a^2(t-\tau)}\right)$$

is the fundamental solution of the equation (4.2), ξ is a parameter. Hereinafter, $I_\nu(z)$ is the modified Bessel function of order ν.

We seek the solution of problem (4.2)–(4.4) as a sum of the single and double layer heat potentials:

$$\omega(r,t)=\int_0^t G\left(r,\xi,t-\tau\right)\big|_{\xi=\tau}\varphi(\tau)d\tau+\int_0^t \left.\frac{\partial G\left(r,\xi,t-\tau\right)}{\partial\xi}\right|_{\xi=0}\psi(\tau)d\tau. \tag{5.1}$$

Function (5.1) satisfies our equation (4.2) for any potential densities $\varphi(\tau)$ and $\psi(\tau)$. To determine them, we will use the boundary conditions (4.3)–(4.4). In view of this, we give another representation of the equality (5.1):

$$\begin{aligned}\omega(r,t)&=\int_0^t \frac{1}{2a^2}\cdot\frac{r^\nu\cdot\tau^{1-\nu}}{t-\tau}\exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right]I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right)\varphi(\tau)d\tau\\&+\int_0^t\frac{1}{(2a^2)^{\nu+1}}\cdot\frac{1}{2^\nu}\cdot\frac{r^{2\nu}}{(t-\tau)^{\nu+1}}\cdot\frac{1}{\nu\Gamma(\nu)}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\psi(\tau)d\tau,\end{aligned} \tag{5.2}$$

where

$$t^{\frac{1}{2}-\nu}e^{\frac{t}{4a^2}}\varphi(t)\in L_\infty(0,\infty).$$

6. REDUCTION OF BOUNDARY VALUE PROBLEM (4.2)–(4.4) TO A SINGULAR VOLTERRA TYPE INTEGRAL EQUATION

We require for the function $\omega(r,t)$, defined by equality (5.2), to satisfy the boundary conditions (4.3)–(4.4), which will allow us to define the functions $\varphi(t)$ and $\psi(t)$.

$$\begin{aligned}\lim_{r\to0}\omega(r,t)&=\frac{1}{(2a^2)^{\nu+1}2^\nu\nu\Gamma(\nu)}\lim_{r\to0}\int_0^t\frac{r^{2\nu}}{(t-\tau)^{\nu+1}}\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\psi(\tau)d\tau\\&=\left\|\frac{r^2}{4a^2(t-\tau)}=z\right\|=\frac{1}{2a^2\nu\Gamma(\nu)}\lim_{r\to0}\int_{\frac{r^2}{4a^2t}}^\infty z^{\nu-1}e^{-z}\psi\left(t-\frac{r^2}{4a^2z}\right)dz\\&=\frac{\psi(t)}{2a^2\nu\Gamma(\nu)}\int_0^\infty z^{\nu-1}e^{-z}dz=\frac{\psi(t)}{2a^2\nu}=q(t).\end{aligned}$$

From here, one of the sought-for densities $\psi(t)$ is directly determined:

$$\psi(t)=2a^2\nu q(t).$$

Thus, for the required solution of the problem (4.2)–(4.4), we have the following representation:

$$\omega(r,t)=\int\limits_0^t \frac{1}{2a^2}\cdot\frac{r^\nu\cdot\tau^{1-\nu}}{t-\tau}\exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right]I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right)\varphi(\tau)d\tau \\ +\tilde{q}(r,t), \tag{6.1}$$

where

$$\tilde{q}(r,t)=\frac{1}{(2a^2)^\nu}\cdot\frac{1}{2^\nu}\cdot\frac{1}{\Gamma(\nu)}\int\limits_0^t \frac{r^{2\nu}}{(t-\tau)^{\nu+1}}\cdot\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]q(\tau)d\tau.$$

Using the value of the derivative:

$$\begin{aligned}\frac{\partial\omega(r,t)}{\partial r}=&-\int\limits_0^t \frac{r^\nu\tau^{1-\nu}(r-\tau)}{4a^4(t-\tau)^2}\exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right]I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right)\varphi(\tau)d\tau\\ &+\int\limits_0^t \frac{r^\nu\tau^{2-\nu}}{4a^4(t-\tau)^2}\exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right]I_{\nu-1,\nu}\left(\frac{r\tau}{2a^2(t-\tau)}\right)\varphi(\tau)d\tau\\ &+\frac{\partial\tilde{q}(r,t)}{\partial r},\end{aligned}$$

where the notation $I_{\nu-1,\nu}(z)=I_{\nu-1}(z)-I_\nu(z)$ is used, we satisfy the boundary condition (4.3):

$$\begin{aligned}&\lim_{r\to t-0}\left(\frac{\partial\omega}{\partial r}+\frac{2}{a^2}\omega\right)\\ &=\int\limits_0^t \frac{3t^\nu\tau^{1-\nu}}{4a^4(t-\tau)}\exp\left[-\frac{t^2+\tau^2}{4a^2(t-\tau)}\right]I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right)\varphi(\tau)d\tau\\ &\quad+\int\limits_0^t \frac{t^\nu\tau^{2-\nu}}{4a^4(t-\tau)^2}\exp\left[-\frac{t^2+\tau^2}{4a^2(t-\tau)}\right]I_{\nu-1,\nu}\left(\frac{t\tau}{2a^2(t-\tau)}\right)\varphi(\tau)d\tau\\ &\quad+\tilde{q}\,(t,t)+\left.\frac{\partial\tilde{q}(r,t)}{\partial r}\right|_{r=t}\\ &\quad-\lim_{r\to t-0}\int\limits_0^t \frac{r^\nu\tau^{1-\nu}(r-\tau)}{4a^4(t-\tau)^2}\exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right]I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right)\varphi(\tau)d\tau.\end{aligned}$$

We denote the last limit by $N\left(\varphi e^{\frac{\tau}{4a^2}}\right)$ and find it.

$$\begin{aligned}
&N\left(\varphi e^{\frac{\tau}{4a^2}}\right)\\
&= -e^{-\frac{t}{4a^2}} \lim_{r\to t-0} \int_0^t \frac{r^\nu \tau^{1-\nu}(r-\tau)}{4a^4(t-\tau)^2} \exp\left[-\frac{(r-t)^2}{4a^2(t-\tau)}\right]\\
&\quad\times \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right) e^{\frac{\tau}{4a^2}}\varphi(\tau)d\tau = \left\| z = \frac{(r-t)^2}{4a^2(t-\tau)} \right\|\\
&= \frac{t\varphi(t)}{a^2} \lim_{r\to t-0} \int_{\frac{(r-t)^2}{4a^2 t}}^{\infty} \frac{e^{-z}}{t-r} \exp\left[-\frac{2t^2}{(r-t)^2}z - \frac{t}{2a^2}\right] I_\nu\left(\frac{2t^2}{(r-t)^2}z + \frac{t}{2a^2}\right) dz\\
&= \left\| e^{-z} I_\nu(z) \sim \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{z}},\ \text{for } z \gg 1 \right\| = \frac{\varphi(t)}{2a^2}.
\end{aligned}$$

Hence, we obtain an integral equation for the required density $\varphi(t)$:

$$\begin{aligned}
&\varphi(t) + \int_0^t \frac{t^\nu \tau^{2-\nu}}{2a^2(t-\tau)^2} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\nu-1,\nu}\left(\frac{t\tau}{2a^2(t-\tau)}\right) e^{-\frac{t-\tau}{4a^2}}\varphi(\tau)d\tau\\
&+\int_0^t \frac{3t^\nu \tau^{1-\nu}}{2a^2(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right) e^{-\frac{t-\tau}{4a^2}}\varphi(\tau)d\tau = F(t),
\end{aligned} \tag{6.2}$$

where

$$F(t) = 2t^{2\nu-1}g(t) - 2a^2\tilde{q}(t,t) - 2a^2 \left.\frac{\partial\tilde{q}(r,t)}{\partial r}\right|_{r=t}.$$

We write equation (6.2) in the following form:

$$\begin{aligned}
&\varphi(t) + \int_0^t \left[\frac{\tau^{2-\nu}}{t^{2-\nu}}\cdot e^{-\frac{t-\tau}{4a^2}}\right] N_1(t,\tau)\varphi(\tau)d\tau\\
&+\int_0^t \left[\frac{\tau^{2-\nu}}{t^{2-\nu}}\cdot e^{-\frac{t-\tau}{4a^2}}\right] N_2(t,\tau)\varphi(\tau)d\tau = F(t),
\end{aligned} \tag{6.3}$$

where

$$N_1(t,\tau) = \frac{1}{2a^2}\frac{t^2}{(t-\tau)^2} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_{\nu-1,\nu}\left(\frac{t\tau}{2a^2(t-\tau)}\right), \tag{6.4}$$

$$N_2(t,\tau) = \frac{3}{2a^2}\frac{t^2}{\tau(t-\tau)} \exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right). \tag{6.5}$$

Remark 6.1 ([21, p. 215])**.** Let the solution of the integral equation

$$y(t) + \int_a^t K(t,\tau)y(\tau)d\tau = f(t)$$

have the form

$$y(t) = f(t) + \int_a^t R(t,\tau)f(\tau)d\tau.$$

Then the solution of the more complicated integral equation

$$y(t) + \int_a^t K(t,\tau)\frac{g(\tau)}{g(t)}y(\tau)d\tau = f(t)$$

has the form

$$y(t) = f(t) + \int_a^t R(t,\tau)\frac{g(\tau)}{g(t)}f(\tau)d\tau.$$

According to this remark, we will seek a solution of the following equation

$$\varphi(t) + \int_0^t N_1(t,\tau)\varphi(\tau)d\tau + \int_0^t N_2(t,\tau)\varphi(\tau)d\tau = F(t). \tag{6.6}$$

Note the following property of the kernel $N(t,\tau) = N_1(t,\tau) + N_2(t,\tau)$, from which it follows that the integral equation (6.6), and together with it equation (6.3) are singular and to them the method of successive approximations cannot be applied.

Remark 6.2. For any value ν,$0 < \nu < 1$,

$$\lim_{t\to 0}\int_0^t N_1(t,\tau)d\tau = 1 \quad \text{and} \quad \lim_{t\to 0}\int_0^t N_2(t,\tau)d\tau = 0,$$

moreover

$$\int_0^t N_1(t,\tau)d\tau = 1, \quad \int_0^t N_2(t,\tau)d\tau = \frac{3}{2a^2}\cdot\frac{\Gamma(\nu)}{\Gamma(1+\nu)}\cdot t, \quad \forall t > 0.$$

Indeed,

$$\int_0^t \frac{1}{2a^2}\frac{t^2}{(t-\tau)^2}\exp\left[-\frac{t\tau}{2a^2(t-\tau)}\right]\left\{I_{\nu-1}\left(\frac{t\tau}{2a^2(t-\tau)}\right) - I_\nu\left(\frac{t\tau}{2a^2(t-\tau)}\right)\right\}d\tau$$

$$= \left\|\frac{t\tau}{2a^2(t-\tau)} = \eta\right\| = \int_0^\infty e^{-\eta}\left\{I_{\nu-1}(\eta) - I_\nu(\eta)\right\}d\eta = 1.$$

The second equality is proved similarly.

7. CHARACTERISTIC INTEGRAL EQUATION

We will seek a solution of the following “truncated” integral equation:

$$\varphi_1(t) + \int_0^t N_1(t,\tau)\varphi_1(\tau)d\tau = \Phi(t), \tag{7.1}$$

which, by Remark 6.2, is characteristic for the equation (6.6).

Remark 7.1. If a solution of equation (7.1) is found, then the solution of equation (6.6) will be obtained by the Carleman–Vekua regularization method.

We change the variables t, τ and introduce new functions:

$$t = \frac{1}{y}, \quad \tau = \frac{1}{x}; \quad \varphi(t) = \varphi\left(\frac{1}{y}\right) = \varphi_1(y), \quad \Phi(t) = \Phi\left(\frac{1}{y}\right) = \Phi_1(y),$$

then equation (7.1) reduces to the following integral equation with a difference kernel with respect to the unknown function $\varphi_1(y)$:

$$\varphi_1(y) + \int_y^\infty M_-\left(y-x\right)\varphi_1(x)dx = \Phi_1(y), \tag{7.2}$$

where

$$M_-\left(y-x\right) = \frac{1}{2a^2}\frac{1}{\left(x-y\right)^2}\exp\left[-\frac{1}{2a^2\left(x-y\right)}\right] I_{\nu-1,\nu}\left(\frac{1}{2a^2\left(x-y\right)}\right).$$

We apply the Laplace transform to both sides of equation (7.2). Then the solution of equation (7.2) can be written as follows [21, p. 561]:

$$\varphi_1(y) = \Phi_1(y) - \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\widehat{M_-}\left(-p\right)}{1+\widehat{M_-}\left(-p\right)}\widehat{\Phi_1}\left(p\right)e^{py}dp, \quad \mathrm{Re}p<0,$$

where

$$\widehat{M_-}\left(-p\right) = \int_0^\infty M_-(z)e^{pz}dz, \quad \mathrm{Re}p<0.$$

In order to find the image of the function $\widehat{M_-}\left(-p\right)$ we use:

1) the formula (29.169) from [8, p. 350],
2) the property: let $f(t) \risingdotseq \widehat{f}\,(p)$, then $\frac{1}{t}f(t) \risingdotseq \int_p^\infty \widehat{f}\,(p)\,dp$ [19, p. 506].

Thus, we have

$$\widehat{M_-}\left(-p\right) = \frac{1}{a^2}\int_{-\infty}^{p}\left[K_{\nu-1}\left(\frac{\sqrt{-q}}{a}\right)I_{\nu-1}\left(\frac{\sqrt{-q}}{a}\right) - K_\nu\left(\frac{\sqrt{-q}}{a}\right)I_\nu\left(\frac{\sqrt{-q}}{a}\right)\right]dq.$$

To calculate this integral, we use the formula (1.12.4.3) from [22, p. 44]:

$$\widehat{M_-}(-p) = \left\| \frac{\sqrt{-q}}{a} = z \right\| = 2 \int\limits_{\frac{\sqrt{-p}}{a}}^{\infty} z \left[K_{\nu-1}(z) I_{\nu-1}(z) - K_\nu(z) I_\nu(z) \right] dz$$

$$= z^2 \left[\left(1 + \frac{(\nu-1)^2}{z^2} \right) I_{\nu-1}(z) K_{\nu-1}(z) - \left\{ I_\nu(z) + \frac{\nu-1}{z} I_{\nu-1}(z) \right\} \right.$$

$$\left. \times \left\{ -K_\nu(z) + \frac{\nu-1}{z} K_{\nu-1}(z) \right\} \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} - z^2 \left[\left(1 + \frac{\nu^2}{z^2} \right) I_\nu(z) K_\nu(z) \right.$$

$$\left. - \left\{ I_{\nu-1}(z) - \frac{\nu}{z} I_\nu(z) \right\} \left\{ -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z) \right\} \right] \Big|_{\frac{\sqrt{-p}}{a}}^{\infty}$$

$$= 2 z I_\nu(z) K_{\nu-1}(z) \Big|_{\frac{\sqrt{-p}}{a}}^{\infty} = 1 - 2 \frac{\sqrt{-p}}{a} I_\nu \left(\frac{\sqrt{-p}}{a} \right) K_{\nu-1} \left(\frac{\sqrt{-p}}{a} \right).$$

Therefore, we get

$$\widehat{\varphi_1}(p) = \widehat{\Phi_1}(p) - \widehat{R_-^*}(-p) \cdot \widehat{\Phi_1}(p),$$

where

$$\widehat{R_-^*}(-p) = \frac{1 - 2 \frac{\sqrt{-p}}{a} I_{\nu-1} \left(\frac{\sqrt{-p}}{a} \right) K_\nu \left(\frac{\sqrt{-p}}{a} \right)}{2 \frac{\sqrt{-p}}{a} I_{\nu-1} \left(\frac{\sqrt{-p}}{a} \right) K_\nu \left(\frac{\sqrt{-p}}{a} \right)}$$

is the image of the resolvent.

7.1. FINDING THE ORIGINAL OF THE RESOLVENT

Let us find the original of expression $\widehat{R_-^*}(-p)$. For the convenience of calculations, we introduce the notation $\frac{\sqrt{-p}}{a} = z$ and find the original of the expression

$$R^*(z) = \frac{1 - 2 z I_{\nu-1}(z) K_\nu(z)}{2 z I_{\nu-1}(z) K_\nu(z)}.$$

Then we use the properties:

1) if $\varphi(t) \risingdotseq \widehat{\varphi}(p)$, then $\varphi(\alpha t) \risingdotseq \frac{1}{\alpha} \widehat{\varphi}\left(\frac{p}{\alpha}\right)$, $\alpha > 0$,
2) If $\widehat{\varphi}(p) \risingdotseq \varphi(t)$, then $\widehat{\varphi}\left(\sqrt{p}\right) \risingdotseq \frac{1}{2\sqrt{\pi}} \cdot \frac{1}{t^{\frac{3}{2}}} \int_0^\infty \tau e^{-\frac{\tau^2}{4t}} \varphi(\tau) d\tau$.

Let z_k be zeros of a function $I_{\nu-1}(z)$, then [19, p. 519]:

$$R^*(z) \risingdotseq R_-(y) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{2 \left[z I_{\nu-1}(z) K_\nu(z) \right]' \big|_{z=z_k}} \cdot e^{z_k y},$$

Since

$$\left[z I_{\nu-1}(z) K_\nu(z) \right]' \big|_{z=z_k} = z_k I_\nu(z_k) K_\nu(z_k),$$

then

$$R_{-}(y)=\sum_{k\in\mathbb{Z}\setminus\{0\}}A_{\nu,k}\cdot e^{z_k y},\quad A_{\nu,k}=\frac{1}{2z_k I_\nu\left(z_k\right)K_\nu\left(z_k\right)}. \tag{7.3}$$

Remark 7.2. To determine the zeros of the function $I_\beta(z)$, we use the equality $I_\beta(z)=e^{-\frac{\pi}{2}\beta i}J_\beta\left(iz\right)$, where $J_\beta(z)$ is the Bessel function – cylinder function of the first kind. For arbitrary real β the function $J_\beta(z)$ has infinitely many real zeros; for $\beta>-1$, all its zeros are real and equal $iz_k=\alpha_k,\ z_k=-i\alpha_k,\ \alpha_k\in\mathbb{R},\ k\in\mathbb{Z}\setminus\{0\}$ [9, p. 941].

Then from the equality (7.3) and the properties of the image 1) and 2) we obtain:

$$\widehat{R_{-}}\left(\frac{\sqrt{-p}}{a}\right)\risingdotseq R_{-}(y)=\frac{a^2}{2\sqrt{\pi}}\cdot\frac{1}{y^{\frac{3}{2}}}\sum_{k\in\mathbb{Z}\setminus\{0\}}A_{\nu,k}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}}\cdot e^{-i\alpha_k a^2\xi}d\xi. \tag{7.4}$$

7.2. ESTIMATION OF THE RESOLVENT $R_{-}(y)$

Let us prove the following lemma.

Lemma 7.3. *The resolvent $R_{-}(y)$ (7.4) satisfies the estimate*

$$R_{-}(y)\leqslant C\cdot\frac{1}{\sqrt{y}},\quad y>0.$$

Proof.

$$\begin{aligned}
R_{-}(y)&\leqslant\left|\frac{a^2}{2\sqrt{\pi}}\cdot\frac{1}{y^{\frac{3}{2}}}\sum_{k\in\mathbb{Z}\setminus\{0\}}A_{\nu,k}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}-i\alpha_k a^2\xi}d\xi\right|\\
&=\left|\frac{a^2}{2\sqrt{\pi}y^{\frac{3}{2}}}\left\{\sum_{k=-\infty}^{-1}A_{\nu,k}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}-i\alpha_k a^2\xi}d\xi+\sum_{k=1}^{\infty}A_{\nu,k}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}-i\alpha_k a^2\xi}d\xi\right\}\right|\\
&=\left|\frac{a^2}{2\sqrt{\pi}y^{\frac{3}{2}}}\left\{\sum_{n=1}^{\infty}A_{\nu,-n}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}-i\alpha_{-n}a^2\xi}d\xi+\sum_{n=1}^{\infty}A_{\nu,n}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}-i\alpha_n a^2\xi}d\xi\right\}\right|\\
&=\left\|z_n=-i\alpha_n,\quad z_{-n}=i\alpha_n\right\|\\
&\le\frac{a^2}{2\sqrt{\pi}y^{\frac{3}{2}}}\sum_{n=1}^{\infty}\left\{\left|A_{\nu,-n}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}+i\alpha_n a^2\xi}d\xi\right|+\left|A_{\nu,n}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}-i\alpha_n a^2\xi}d\xi\right|\right\}\\
&\le\frac{a^2}{2\sqrt{\pi}y^{\frac{3}{2}}}\int\limits_0^\infty \xi e^{-\frac{\xi^2}{4y}}d\xi\sum_{n=1}^{\infty}\left\{|A_{\nu,-n}|+|A_{\nu,n}|\right\}=\frac{a^2}{2\sqrt{y}}\sum_{n=1}^{\infty}\left\{|A_{\nu,-n}|+|A_{\nu,n}|\right\}.
\end{aligned}$$

Let us find the sum $S=\sum\limits_{n=1}^{\infty}\left\{|A_{\nu,-n}|+|A_{\nu,n}|\right\}$:

$$
\begin{aligned}
S &= \sum_{n=1}^{\infty} \left\{ \left| \frac{1}{2z_{-n} I_\nu (z_{-n}) K_\nu (z_{-n})} \right| + \left| \frac{1}{2z_n I_\nu (z_n) K_\nu (z_n)} \right| \right\} \\
&= \| z_n = -i\alpha_n, \quad z_{-n} = i\alpha_n \| \\
&= \sum_{n=1}^{\infty} \left\{ \left| \frac{1}{2i\alpha_n I_\nu (i\alpha_n) K_\nu (i\alpha_n)} \right| + \left| \frac{1}{-2i\alpha_n I_\nu (-i\alpha_n) K_\nu (-i\alpha_n)} \right| \right\} \\
&= \left\| \begin{matrix} K_\nu(z) = \dfrac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_\nu^{(1)} (iz), & I_\nu(z) = e^{-\frac{\pi}{2}\nu i} J_\nu (iz) \\ J_\nu (-z) = e^{\nu\pi i} J_\nu(z), & H_\nu^{(1)} (-z) = -e^{-\nu\pi i} H_\nu^{(2)}(z) \\ H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z), & H_\nu^{(2)}(z) = J_\nu(z) - iN_\nu(z) \end{matrix} \right\| \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{\alpha_n |J_\nu (\alpha_n)|} \left\{ \frac{1}{|J_\nu (\alpha_n) - iN_\nu (\alpha_n)|} \frac{1}{|J_\nu (\alpha_n) + iN_\nu (\alpha_n)|} \right\} \\
&= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{\alpha_n |J_\nu (\alpha_n)|} \cdot \frac{2}{\sqrt{J_\nu^2 (\alpha_n) + N_\nu^2 (\alpha_n)}} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{\alpha_n |J_\nu (\alpha_n)|} \cdot \frac{2}{\sqrt{J_\nu^2 (\alpha_n)}} \\
&= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{\alpha_n J_\nu^2 (\alpha_n)} \le \frac{2}{\pi} \int_{\alpha_1}^{\infty} \frac{d(\alpha_n)}{\alpha_n J_\nu^2 (\alpha_n)} = \|(1.8.4.1) \text{ from } [22, \text{ p. } 39]\| \\
&= \left. \frac{N_\nu (\alpha_n)}{J_\nu (\alpha_n)} \right|_1^{\infty} \le C(\alpha_1),
\end{aligned}
$$

where $N_\nu(z)$ is a cylinder function of the second kind (the Neumann function). Then we get

$$
R_-(y) \leqslant \frac{a^2}{2\sqrt{y}} \sum_{n=1}^{\infty} \{ |A_{\nu,-n}| + |A_{\nu,n}| \} \le \frac{C(\alpha_1) a^2}{2} \cdot \frac{1}{\sqrt{y}}.
$$

The lemma is proved. □

7.3. SOLUTION OF THE “CHARACTERISTIC” EQUATION

We found a solution of equation (7.2), which has the form

$$
\varphi_1(y) = \Phi_1(y) - \int_y^{\infty} R_- (x - y) \Phi_1(x) dx.
$$

We make the reverse replacements

$$
t = \frac{1}{y}, \quad \tau = \frac{1}{x}
$$

and write the solution of the characteristic equation (7.1) as follows

$$\varphi(t) = \Phi(t) - \int_0^t \tilde{R}(t,\tau)\Phi(\tau)d\tau,$$

where

$$\tilde{R}(t,\tau) \leq C\frac{\sqrt{t}}{\tau^{\frac{3}{2}}\sqrt{t-\tau}}. \tag{7.5}$$

The last inequality follows from the Lemma 7.3.

8. SOLUTION OF THE "COMPLETE" INTEGRAL EQUATION. THE CARLEMAN–VEKUA REGULARIZATION METHOD

Theorem 8.1. *The original integral equation* (6.6) *for any function* $t^{\frac{1}{2}-\nu}e^{\frac{t}{4a^2}} \cdot F(t) \in L_\infty(0,\infty)$ *has a unique solution in the class of functions*

$$t^{\frac{1}{2}-\nu}e^{\frac{t}{4a^2}} \cdot \varphi(t) \in L_\infty(0,\infty), \tag{8.1}$$

which can be found by the method of successive approximations.

Proof. To solve the original "complete" integral equation (6.6), we represent it as

$$\varphi(t) + \int_0^t N_1(t,\tau)\varphi(\tau)d\tau = F(t) - \int_0^t N_2(t,\tau)\varphi(\tau)d\tau$$

and apply the Carleman–Vekua regularization method. Assuming the right-hand side of equation (6.6) to be temporarily known, we write its solution

$$\varphi(t) = F(t) - \int_0^t N_2(t,\tau)\varphi(\tau)d\tau - \int_0^t \tilde{R}(t,\tau)\left\{F(\tau) - \int_0^\tau N_2\left(\tau,\xi\right)\varphi\left(\xi\right)d\xi\right\}d\tau. \tag{8.2}$$

We change the order of integration in the iterated integral and, then, change the roles of the variables τ and ξ, hence equation (8.2) takes the form

$$\varphi(t) + \int_0^t M(t,\tau)\varphi(\tau)d\tau = F(t) - \int_0^t \tilde{R}(t,\tau)F(\tau)d\tau, \tag{8.3}$$

where

$$M(t,\tau) = N_2(t,\tau) - \int_\tau^t \tilde{R}(t,\tau)N_2\left(\xi,\tau\right)d\xi.$$

By using (7.5), we obtain that the kernel $M(t,\tau)$ of the integral equation (6.6) has a weak singularity, since it satisfies the estimate

$$M(t,\tau) \leq D_1 \cdot \frac{\sqrt{t}}{\sqrt{\tau}\sqrt{t-\tau}} + D_2 \leq \begin{cases} D_1 \cdot \frac{\sqrt{2}}{\sqrt{\tau}} + D_2, & 0 < \tau \leq \frac{t}{2}, \\ D_1 \cdot \frac{\sqrt{2}}{\sqrt{t-\tau}} + D_2, & \frac{t}{2} \leq \tau < t. \end{cases}$$

This means that the solution of the integral equation (6.6) can be found by the method of successive approximations. The theorem is proved. □

9. SOLUTION OF THE BOUNDARY VALUE PROBLEM (2.1)–(2.3). PROOF OF THEOREM 3.1

From the integral representation for the solution (6.1) of the boundary value problem (4.2)–(4.4), we get

$$\omega(r,t) = \sum_{k=1}^{2} \omega_k(r,t),$$

where

$$\omega_1(r,t) = \int_0^t \frac{r^\nu \cdot \tau^{1-\nu}}{2a^2(t-\tau)} \exp\left[-\frac{r^2+\tau^2}{4a^2(t-\tau)}\right] I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right) \varphi(\tau)d\tau;$$

$$\omega_2(r,t) = \tilde{q}(r,t) = \frac{1}{(2a^2)^\nu 2^\nu \Gamma(\nu)} \int_0^t \frac{r}{(t-\tau)^{\nu+1}} \cdot \exp\left[-\frac{r^2}{4a^2(t-\tau)}\right] q(\tau)d\tau.$$

We estimate the first term, taking into account that $t^{\frac{1}{2}-\nu} e^{\frac{t}{4a^2}} \varphi(t) \in L_\infty(0,\infty)$:

$$\omega_1(r,t) = \sqrt{t} \exp\left[-\frac{t}{4a^2}\right] \int_0^t \frac{1}{2a^2} \cdot \frac{r^\nu \sqrt{\tau}}{(t-\tau)\sqrt{t}} \exp\left[-\frac{(r-\tau)(r+t-2\tau)}{4a^2(t-\tau)}\right]$$
$$\times \left\{\exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right)\right\} \cdot \left\{\tau^{\frac{1}{2}-\nu} \exp\left[\frac{\tau}{4a^2}\right] \varphi(\tau)\right\} d\tau.$$

Then we have

$$|\omega_1(r,t)| \leqslant C_1 \sqrt{t} \exp\left[-\frac{t}{4a^2}\right] \int_0^t \frac{1}{2a^2} \cdot \frac{r^\nu}{t-\tau} \exp\left[-\frac{r\tau}{2a^2(t-\tau)}\right] I_\nu\left(\frac{r\tau}{2a^2(t-\tau)}\right) d\tau$$
$$= \left\| \frac{r\tau}{2a^2(t-\tau)} = z \right\| = \frac{C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right] r^\nu}{2a^2} \int_0^\infty \frac{1}{\frac{r}{2a^2}+z} e^{-z} I_\nu(z) dz$$
$$= \|(2.15.3.3) \text{ from [22, p. 272]}\| = \frac{C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right] r^\nu}{2a^2} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\nu)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1+\nu)}$$
$$= \frac{C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right] r^\nu}{2a^2\nu}.$$

Therefore,

$$|\omega_1(r,t)| \leqslant \frac{C_1\sqrt{t}\exp\left[-\frac{t}{4a^2}\right] r^\nu}{2a^2\nu}$$

or, taking into consideration (4.1):

$$\frac{\partial u_1(r,t)}{\partial r} \le \frac{C_1\sqrt{t}r^{1-\nu}}{2a^2\nu}\exp\left[-\frac{t}{4a^2}\right],$$

$$u_1(r,t) \le C_1\int\limits_0^r \frac{\sqrt{t}r^{1-\nu}}{2a^2\nu}\exp\left[-\frac{t}{4a^2}\right]dr = \frac{C_1\sqrt{t}}{2a^2\nu}\cdot\frac{r^{2-\nu}}{2-\nu}\exp\left[-\frac{t}{4a^2}\right].$$

Now we estimate the second term, taking into account that $t^{1-\nu}q(t) \in L_\infty(0,\infty)$. By (4.1), we have

$$\begin{aligned}
|u_2(r,t)| &= \left|\frac{1}{(2a^2)^\nu\, 2^\nu\Gamma(\nu)}\int\limits_0^t \frac{q(\tau)}{(t-\tau)^{\nu+1}}d\tau\int\limits_0^r r_1\exp\left[-\frac{r_1^2}{4a^2(t-\tau)}\right]dr_1\right| \\
&= \left\|\frac{r_1^2}{4a^2(t-\tau)} = z\right\| \\
&= \frac{1}{(2a^2)^{\nu-1}\, 2^\nu\Gamma(\nu)}\int\limits_0^t \frac{|q(\tau)|}{(t-\tau)^\nu}\left|1-\exp\left[-\frac{r^2}{4a^2(t-\tau)}\right]\right|d\tau \\
&\le \frac{C_2}{(2a^2)^{\nu-1}\, 2^\nu\Gamma(\nu)}\int\limits_0^t \frac{d\tau}{\tau^{1-\nu}(t-\tau)^\nu} = \|\tau = tx\| \\
&= \frac{C_2}{(2a^2)^{\nu-1}\, 2^\nu\Gamma(\nu)}\int\limits_0^1 x^{\nu-1}(1-x)^{(1-\nu)-1}dx = \frac{C_2 B(\nu,1-\nu)}{(2a^2)^{\nu-1}\, 2^\nu\Gamma(\nu)}.
\end{aligned}$$

This implies the validity of the main result – Theorem 3.1.

10. CONCLUSION

The boundary value problem of heat conduction in a non-cylindrical domain degenerating into a point – a cone, by the method of heat potentials is reduced to a singular Volterra type integral equation of the second kind. We constructed for it a characteristic integral equation and found its explicit solution. Using the estimate for the resolvent of the characteristic equation, we found a solution of the original integral equation by the Carleman–Vekua regularization method. We proved a theorem on the unique solvability of the original boundary value problem (2.1)–(2.3) in weighted classes of essentially bounded functions.

The obtained results can be used to solve a similar boundary value problem when the boundary of the domain moves according to an arbitrary law $r = \alpha(t)$, $\alpha(0) = 0$.

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