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DOUBLE PHASE PROBLEMS: A SURVEY OF SOME RECENT RESULTS

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Abstract. We review some recent results on double phase problems. We focus on the relevant function space framework, which is provided by the generalized Orlicz spaces. We also describe the basic tools and methods used to deal with double phase problems, given that there is no global regularity theory for these problems.

Keywords: double phase integrand, generalized Orlicz spaces, regularity theory, maximum principle, Nehari manifold.

Mathematics Subject Classification: 35J20, 35J60.

1. INTRODUCTION

In this paper we will review some recent results and advances on elliptic (stationary) equations driven by a differential operator of the form $\Delta_p^a u + \Delta_q u$, where $\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du)$ with $a \in L^\infty(\Omega)$, $a(z) \geqslant 0$ for a.a. $z \in \Omega$, $1 < q < p < \infty$. We do not assume that the weight function $a(\cdot)$ is bounded away from zero and so the integral (energy) functional corresponding to this operator has an integrand $\vartheta(z,x) = a(z)|x|^p + |x|^q$ which exhibits unbalanced growth, namely, we have

$$|x|^q \leqslant \vartheta(z,x) \leqslant a(z)|x|^p + |x|^q \text{ for a.a. } x \in \Omega, \text{ all } x \in \mathbb{R}.$$

Such integral functionals were first investigated by Zhikov [40,41] and Marcellini [22,23], in the context of problems of elasticity theory and of the calculus of variations. In elasticity theory, the weight function $a(z)$ appears as the modulating coefficient and it dictates the geometry of the composite made of two different materials with distinct power hardening exponents p and q. Such functionals are known in the literature as "double phase functionals". They are a special case of problems with nonstandard growth and nonuniform ellipticity. Double phase functionals are also related to non-Newtonian fluids (see [19,25]). Recently, the interest for such problems was revived with the works

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of Mingione and coworkers (see [2,5,11]) and of Marcellini and coworkers (see [9,10,24]). We also mention the works of De Filippis [8] (problems with (p,q)-growth and a manifold constraint) and of Ragusa–Tachikawa [37] (double phase problems with variable exponents). Finally, there are the important review papers of Mingione–Rădulescu [25] and Rădulescu [36], which focus on the regularity theory for such problems. For double phase problems there are only local regularity results and a global regularity theory (up to the boundary, similar to that of Lieberman [18]), remains so far elusive. This eliminates some important tools available to problems with standard growth (see for example, Papageorgiou–Rădulescu–Zhang [30]). For this reason, we need to come up with new techniques to deal with such equations. The unbalanced growth of the relevant integrand, leads to a functional framework which goes beyond the standard Lebesgue and Sobolev spaces and uses generalized Orlicz and generalized Orlicz–Sobolev spaces (also known as Musielak–Orlicz spaces). A comprehensive treatment of such spaces can be found in the books of Harjulehto–Hästö [16] and of Musielak [26]. In the next section we review some basic aspects of the theory of these spaces.

2. GENERALIZED ORLICZ SPACES

The starting point of the theory of these spaces are the notions of a Φ-function and of a "generalized Φ-function".

Definition 2.1. A function $\vartheta : \mathbb{R}_+ = [0,+\infty) \to \mathbb{R}_+$ is called an "Φ-function" (denoted by $\vartheta \in \Phi$), if $\vartheta(\cdot)$ is nondecreasing, continuous, convex, $0 < \vartheta(t)$ for all $t > 0$, $\vartheta(0) = 0$ and $\lim_{x\to+\infty} \vartheta(x) = +\infty$. A function $\vartheta : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ ($\Omega \subseteq \mathbb{R}^N$ a bounded domain) is called a "generalized Φ-function" (denoted by $\vartheta \in \Phi(\Omega)$) if for all $x \in \mathbb{R}_+$, $z \to \vartheta(z,x)$ is measurable and for a.a. $z \in \Omega$, $\vartheta(z,\cdot) \in \Phi$.

We say that $\vartheta \in \Phi(\Omega)$ is "locally integrable", if for all $x \geqslant 0$, $\vartheta(\cdot,x) \in L^1(\Omega)$.

Remark 2.2. A generalized Φ-function is a Carathéodory function and therefore jointly measurable. Evidently $\vartheta(z,x) = a(z)x^p + x^q$, $z \in \Omega$, $x \geqslant 0$ is a generalized Φ-function.

By $M(\Omega)$ we denote the linear space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual we identify two such functions which differ only on a Lebesgue-null set. Given $\vartheta \in \Phi(\Omega)$, the generalized Orlicz space $L^\vartheta(\Omega)$ is defined by

$$L^\vartheta(\Omega) = \{u \in M(\Omega) : \text{ there exists } \lambda > 0 \text{ such that } \rho_\vartheta(\lambda u) < \infty\}$$

with $\rho_\vartheta(\cdot)$ being the modular function defined by

$$\rho_\vartheta(u) = \int_\Omega \vartheta(z, |u(z)|)\mathrm{d}z.$$

We equip the generalized Orlicz space $L^\vartheta(\Omega)$ with the so-called "Luxemburg norm" defined by

$$\|u\|_\vartheta = \inf\left\{\lambda > 0 : \int_\Omega \vartheta\left(z, \frac{|u|}{\lambda}\right)\mathrm{d}z \leqslant 1\right\}.$$

Proposition 2.3. *If $\vartheta \in \Phi(\Omega)$, then $L^{\vartheta}(\Omega)$ is a Banach space.*

Definition 2.4. Let ϑ, $\xi \in \Phi(\Omega)$,

(a) We say that ϑ is "weaker" than ξ, denoted by

$$\vartheta \preceq \xi,$$

if there exist constants c_1, $c_2 > 0$ and a function $h \in L^1(\Omega)$ such that

$$\vartheta(z,x) \leqslant c_1 \xi(z, c_2 x) + h(z) \text{ for a.a. } z \in \Omega, \text{all } x \geqslant 0.$$

(b) We say that ϑ satisfies the "(Δ_2)-condition", if there exist a constant $c > 0$ and a function $h \in L^1(\Omega)$ such that

$$\vartheta(z, 2x) \leqslant c\vartheta(z,x) + h(z) \text{ for a. a. } z \in \Omega, \text{all } x \geqslant 0.$$

Proposition 2.5. *If ϑ, $\xi \in \Phi(\Omega)$ and $\vartheta \preceq \xi$, then*

$$L^{\xi}(\Omega) \hookrightarrow L^{\vartheta}(\Omega) \text{ continuously.}$$

If $\vartheta \in \Phi(\Omega)$ satisfies the (Δ_2)-condition, then we can have a direct definition of $L^{\vartheta}(\Omega)$ using the modular function $\rho_{\vartheta}(\cdot)$.

Proposition 2.6. *If $\vartheta \in \Phi(\Omega)$ satisfies the (Δ_2)-condition, then*

$$L^{\vartheta}(\Omega) = \{u \in M(\Omega) : \rho_{\vartheta}(u) < \infty\}.$$

Remark 2.7. Note that the double phase integrand satisfies the (Δ_2)-condition.

Proposition 2.8. *If $\vartheta \in \Phi(\Omega)$, then*

(a) $u_n(z) \to u(z)$ *a.e. in* $\Omega \Rightarrow \|u\|_{\vartheta} \leqslant \liminf_{n\to\infty} \|u_n\|_{\vartheta}$;
(b) *if* $\{u_n\}_{n\in\mathbb{N}} \subseteq L^{\vartheta}(\Omega)$ *and* $|u_n(z)| \uparrow |u(z)|$ *a.e. in* Ω, *then* $u \in L^{\vartheta}(\Omega)$ *and* $\|u_n\|_{\vartheta} \uparrow \|u\|_{\vartheta}$.

If $\vartheta \in \Phi(\Omega)$, then the "conjugate" in the sense of convex analysis of $\vartheta(z,\cdot)$ is defined by

$$\vartheta^*(z,y) = \sup\{yx - \vartheta(z,x) : x \geqslant 0\}$$

Note that $\vartheta^*(z,\cdot)$ is convex and $\vartheta^* \in \Phi(\Omega)$. If $\vartheta(z,\cdot) \leqslant \xi(z,\cdot)$ for a.a. $z \in \Omega$, then $\xi^*(z,\cdot) \leqslant \vartheta^*(z,\cdot)$ for a.a. $z \in \Omega$. Moreover, $\vartheta^{**} = (\vartheta^*)^*$ is the greatest convex minorant of $\vartheta(z,\cdot)$. Therefore since $\vartheta(z,\cdot)$ is convex, then $\vartheta^{**} = \vartheta$.

Using the notion of conjugate function, we can state the following Hölder's-type inequality for the generalized Orlicz spaces.

Proposition 2.9. *If $\vartheta \in \Phi(\Omega)$ but instead of convex $\vartheta(z,\cdot)$ there exists $\gamma \geqslant 1$ such that*

$$\frac{\vartheta(z,x)}{x} \leqslant \gamma \frac{\vartheta(z,y)}{y} \tag{2.1}$$

for a.a. $z \in \Omega$, all $0 < x < y$, then we have

$$\int_{\Omega} |uv| \mathrm{d}z \leqslant 2\|u\|_{\vartheta}\|v\|_{\vartheta^*} \quad \textit{for all } u \in L^{\vartheta}(\Omega), v \in L^{\vartheta^*}(\Omega).$$

Remark 2.10. If $\vartheta(z,\cdot)$ satisfies (2.1), then we say that it is "almost increasing". If $\vartheta(z,\cdot)$ is convex, then for $0<x<y$ we have

$$\vartheta(z,x)=\vartheta\left(z,\frac{x}{y}y+0\right)\leqslant\frac{x}{y}\vartheta(z,y)+\left(1-\frac{x}{y}\right)\vartheta(z,0)=\frac{x}{y}\vartheta(x,y)$$
$$\Rightarrow x\to\frac{\vartheta(z,x)}{x}\text{ is nondecreasing.}$$

Therefore, Hölder's inequality holds in this case. The constant 2 in the inequality is optimal. Finally, we have

$$\|u\|_{\vartheta}=\sup\left[\int_{\Omega}|uv|\mathrm{d}z:\|v\|_{\vartheta^*}\leqslant 1\right]\quad\text{for all } u\in L^{\vartheta}(\Omega).$$

Definition 2.11. Let ϑ, $\xi\in\Phi(\Omega)$. We say that ϑ, ξ are "equivalent", denoted by $\vartheta\backsim\xi$, if there exists $k\geqslant 1$ such that

$$\vartheta\left(z,\frac{x}{k}\right)\leqslant\vartheta(z,x)\leqslant\vartheta(z,kx)\quad\text{for a.a. } z\in\Omega,\text{all } x\geqslant 0.$$

Proposition 2.12. *If ϑ, $\xi\in\Phi(\Omega)$, satisfy* (2.1) *and $\vartheta\backsim\xi$, then $L^{\vartheta}(\Omega)=L^{\xi}(\Omega)$ and the corresponding Luxemburg norms are equivalent.*

Introducing additional properties on $\vartheta(z,\cdot)$ we can improve the structure of the Banach space $L^{\vartheta}(\Omega)$.

Definition 2.13. We say that $\vartheta\in\Phi$ is an "N-function" (denoted by $\vartheta\in N$), if $\vartheta(\cdot)$ satisfies

$$\lim_{x\to 0^+}\frac{\vartheta(x)}{x}=0,\quad\text{(sublinear near } 0^+),$$
$$\lim_{x\to+\infty}\frac{\vartheta(x)}{x}=+\infty,\quad\text{(superlinear near } +\infty).$$

We say that $\vartheta\in\Phi(\Omega)$ is a "generalized N-function" (denoted by $\vartheta\in N(\Omega)$), if for all $x\geqslant 0$, $z\to\vartheta(z,x)$ is measurable and for a.a. $z\in\Omega$, $\vartheta(z,\cdot)\in N$.

Remark 2.14. $\vartheta\in N(\Omega)\Rightarrow\vartheta^*\in N(\Omega)$.

For N-functions, we can establish a close relation between the Luxemburg norm and the modular function $\rho_{\vartheta}(\cdot)$.

Proposition 2.15. *If $\vartheta\in N(\Omega)$ and satisfies the (Δ_2)-condition and $\{u_n,u\}_{n\in\mathbb{N}}\subseteq L^{\vartheta}(\Omega)$, then*

(a) $\rho_{\vartheta}(u)<1$ (*resp.* $=1$, >1) $\Leftrightarrow\|u\|_{\vartheta}<1$ (*resp.* $=1$, >1),
(b) $\rho_{\vartheta}(u_n)\to 0\Leftrightarrow\|u_n\|_{\vartheta}\to 0$,
(c) $\rho_{\vartheta}(u_n)\to+\infty\Leftrightarrow\|u_n\|_{\vartheta}\to+\infty$,
(d) $u_n\to u$ *in* $L^{\vartheta}(\Omega)\Leftrightarrow\rho_{\vartheta}(u_n)\to\rho_{\vartheta}(u)$.

Let $k(z,x)$ denote the right derivative of $\vartheta(z,\cdot)$ at x. Then

$$\vartheta(z,x) = \int_0^x k(z,s)\mathrm{d}s \quad \text{for all } (z,x) \in \Omega \times \mathbb{R}_+.$$

We have:

Proposition 2.16. *If $\vartheta \in N(\Omega)$ and satisfies the (Δ_2)-condition, then for every $u \in L^{\vartheta}(\Omega)$, we have $k(\cdot,|u(\cdot)|) \in L^{\vartheta^*}(\Omega)$.*

Proposition 2.17. *If $\vartheta \in N(\Omega)$ and it is locally integrable, then $L^{\vartheta}(\Omega)$ is a separable Banach space.*

Remark 2.18. We point out that when $\vartheta(z,x)$ is the double phase integrand all the above results hold.

Now we introduce the corresponding generalized Orlicz–Sobolev spaces. So, let $\vartheta \in \Phi(\Omega)$. The generalized Orlicz–Sobolev space $W^{1,\vartheta}(\Omega)$ is defined by

$$W^{1,\vartheta}(\Omega) = \left\{u \in M(\Omega) : |Du| \in L^{\vartheta}(\Omega)\right\}$$

with Du being the weak gradient of u. We equip this space with the following norm

$$\|u\|_{1,\vartheta} = \|u\|_{\vartheta} + \|Du\|_{\vartheta} \quad (\|Du\|_{\vartheta} = \||Du|\|_{\vartheta})\,.$$

Moreover, if $\vartheta \in N(\Omega)$ and it is locally integrable, then we define

$$W_0^{1,\vartheta}(\Omega) = \overline{C_c^{\infty}(\Omega)}^{\|\cdot\|_{1,\vartheta}}.$$

Proposition 2.19. *If $\vartheta \in N(\Omega)$ it is locally integrable and*

$$\operatorname*{ess\,inf}_{\Omega} \vartheta(\cdot,1) > 0,$$

then $W^{1,\vartheta}(\Omega)$ and $W_0^{1,\vartheta}(\Omega)$ are separable Banach spaces. Moreover, they are reflexive provided $L^{\vartheta}(\Omega)$ is reflexive.

Concerning the reflexivity of $L^{\vartheta}(\Omega)$, we have the following result (Riesz representation theorem).

Theorem 2.20. *If $\vartheta \in N(\Omega)$, both ϑ, ϑ^* are locally integrable and*

$$\frac{\vartheta(z,x)}{x}, \frac{\vartheta^*(z,y)}{y} \geqslant c > 0 \quad \textit{for a.a} \;\; z \in \Omega, \;\; \textit{all } x,y \geqslant u_0,$$

then $L^{\vartheta}(\Omega)^* = L^{\vartheta^*}(\Omega)$ (so $L^{\vartheta}(\Omega)$ is reflexive).

We can say more if $\vartheta(z,\cdot)$ has additional properties.

Definition 2.21. We say that $\vartheta \in N(\Omega)$ is "uniformly convex", if given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\vartheta\left(z, \frac{x+u}{2}\right) \leqslant (1-\delta)\left(\frac{\vartheta(z,x)+\vartheta(z,u)}{2}\right)$$

for a.a. $z \in \Omega$, all $x,u \in \mathbb{R}_+$ with $|x-u| > \varepsilon \max\{x,u\}$.

Remark 2.22. The double phase integrand $\vartheta(z,x) = a(z)x^p + x^q$ is uniformly convex.

It is natural to ask whether the uniform convexity of the function $\vartheta(z,\cdot)$ implies the uniform convexity of the generalized Orlicz space $L^{\vartheta}(\Omega)$. It turns out that this is true provided an additional condition holds.

Proposition 2.23. *If $\vartheta \in N(\Omega)$ is uniformly convex and there exist $r > 1$ and $\gamma \geqslant 1$ such that*

$$\frac{\vartheta(z,x)}{x^r} \leqslant \gamma \frac{\vartheta(z,y)}{y^r}$$

for a.a. $z \in \Omega$, all $0 < y < x$, then $L^{\vartheta}(\Omega)$ is uniformly convex (thus reflexive by the Milman-Pettis theorem).

Remark 2.24. If $\vartheta(z,x)$ is the double phase integrand, then we can take $r \geqslant p$ and $\gamma \geqslant 1$.

Corollary 2.25. *If $\vartheta \in N(\Omega)$ is as in Proposition 2.23, then $W^{1,\vartheta}(\Omega)$ and $W_0^{1,\vartheta}(\Omega)$ are uniformly convex Banach spaces.*

We can state the following Sobolev embedding theorem for these spaces when $\vartheta(z,x)$ is the double phase integrand.

Theorem 2.26. *If $\vartheta(z,x) = a(z)x^p + x^q$ for all $(z,x) \in \Omega \times \mathbb{R}_+$ with $a \in L^{\infty}(\Omega)$, $a(z) \geqslant 0$ for a.a. $z \in \Omega$, then*

(a) *$L^{\vartheta}(\Omega) \hookrightarrow L^s(\Omega)$, $W_0^{1,\vartheta}(\Omega) \hookrightarrow W_0^{1,s}(\Omega)$ continuously for all $s \in [1,q]$,*
(b) *if $q \neq N$, then $W_0^{1,\vartheta}(\Omega) \hookrightarrow L^s(\Omega)$ continuously for all $s \in [1,q^*]$, if $q = N$, then $W_0^{1,\theta}(\Omega) \hookrightarrow L^s(\Omega)$ continuously for all $s \in [1,\infty)$,*
(c) *if $q \leqslant N$, then $W_0^{1,\vartheta}(\Omega) \hookrightarrow L^s(\Omega)$ compactly for all $s \in [1,q^*)$, if $q > N$, then $W_0^{1,\vartheta}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ compactly,*
(d) *$L^p(\Omega) \hookrightarrow L^{\vartheta}(\Omega)$ continuously.*

Remark 2.27. The above embeddings remain true if $W_0^{1,p}(\Omega)$ is replaced by $W^{1,p}(\Omega)$.

Suppose that $p < q^*$ (this is true, if for example $\frac{p}{q} < 1 + \frac{1}{N}$). Then we have that the Poincaré inequality holds on $W_0^{1,\vartheta}(\Omega)$ with $\vartheta(z,x)$ being the double phase integrand.

Proposition 2.28. *If $\vartheta(z,x) = a(z)x^p + x^q$ for all $(z,x) \in \Omega \times \mathbb{R}_+$ with $a \in L^{\infty}(\Omega)$, $a(z) \geqslant 0$ for a.a. $z \in \Omega$, $1 < q < p < \min\{q^*, N\}$, then there exists $c > 0$ such that*

$$\|u\|_{\vartheta} \leqslant c\|Du\|_{\theta} \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Remark 2.29. This proposition implies that on $W_0^{1,\vartheta}(\Omega)$ we can consider the equivalent norm

$$\|u\| = \|Du\|_{\vartheta} \quad \text{for all } u \in W_0^{1,\vartheta}(\Omega).$$

Proposition 2.28 remains true if we assume that $\vartheta \in N(\Omega)$ is locally integrable and the embedding $W^{1,\vartheta}(\Omega) \hookrightarrow L^{\vartheta}(\Omega)$ is compact.

Definition 2.30. A weight $a \in L^\infty(\Omega)$ with $0 < a(z)$ for a.a. $z \in \Omega$ belongs in the Muckenhoupt class $\tilde{A}_p$ $(p > 1)$ if

$$\sup_Q \left[\fint_Q a(z)\mathrm{d}z \right] \left[\fint_Q a(z)^{1-p'}\mathrm{d}z \right]^{p-1} < \infty$$

the supremum taken over all cubes Q with sides parallel to the coordinate axes.

Remark 2.31. Recall that if $u \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\fint_Q |u|\mathrm{d}z = \frac{1}{|Q|_N} \int_Q |u|\mathrm{d}z$$

with $|\cdot|_N$ being the Lebesgue measure on $\mathbb{R}^N$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We know that $a \in \tilde{A}_p$ if and only if the averaging operator

$$L^p_a(\Omega) \ni u \to a_Q(u) = \fint_Q a(z)\mathrm{d}z\chi_Q(z)$$

is uniformly bounded for all Q (see Cruz Uribe–Fiorenza [7, p. 152]).

By $C^{0,1}(\bar{\Omega})$ we denote the space of Lipschitz continuous functions on $\bar{\Omega}$. We impose the following conditions on the weight function $a(\cdot)$ and the exponents p, q

H_0: $a \in C^{0,1}(\bar{\Omega}) \cap \tilde{A}_p$, $1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$.

Remark 2.32. As we already mentioned earlier, the last inequality in H_0 implies that $p < q^*$.

Let $\vartheta_0(z,x) = a(z)x^p$ for all $(z,x) \in \Omega \times \mathbb{R}_+$. On account of Proposition 2.3, $L^{\vartheta_0}(\Omega)$ and $W_0^{1,\vartheta_0}(\Omega)$ are Banach spaces. Moreover, since $\vartheta_0(z,\cdot)$ is uniformly convex, on account of Corollary 2.25, both spaces are uniformly convex, thus reflexive.

The next embedding theorem, is due to Papageorgiou–Rădulescu–Zhang [30].

Proposition 2.33. *If hypotheses H_0 hold with $a(z) > 0$ for all $z \in \Omega$, then $W_0^{1,\vartheta_0}(\Omega) \hookrightarrow L^{\vartheta_0}(\Omega)$ compactly.*

Remark 2.34. As a consequence of Propositions 2.28 and 2.33, we infer that on $W_0^{1,\vartheta}(\Omega)$ the Poincaré inequality holds.

This embedding result allows one to study the spectral properties of the operator Δ_p^a and then consider double phase problems with "sublinear" reaction (see Papageorgiou–Pudelko–Rădulescu [27]).

3. REGULARITY AND MAXIMUM PRINCIPLE

In this section we present some regularity results for double phase equations and a maximum principle.

So, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary. We consider the following Dirichlet double phase equation

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{3.1}$$

A function $u \in W_0^{1,\vartheta}(\Omega)$ ($\vartheta(z,x) = a(z)x^p + x^q$), is said to be a weak solution of problem (3.1) if

$$\int_\Omega \big(a(z)|Du|^{p-2}Du + |Du|^{q-2}Du, Dh\big)_{\mathbb{R}^N}\, \mathrm{d}z = \int_\Omega f(z,u)h\mathrm{d}z.$$

for all $h \in W_0^{1,\vartheta}(\Omega)$.

We impose the following conditions on the data of problem (3.1).

H_1: $a \in C^{0,1}(\bar{\Omega})$, $a(z) \geqslant 0$ for all $z \in \bar{\Omega}$, $1 < q < p < N$ and $\frac{p}{q} < 1 + \frac{1}{N}$.
H_2: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(z,x)| \leqslant \hat{a}(z)\left[1 + |x|^{r-1}\right] \text{ for a.a. } z \in \Omega, \text{all } x \in \mathbb{R}$$

with $\hat{a} \in L^\infty(\Omega)$ and $1 < r < q^*$.

Using these hypotheses and employing the Moser iteration approach, Gasinski–Winkert [14, Theorem 3.1], proved the following boundedness result for the weak solutions of (3.1).

Proposition 3.1. *If hypotheses H_1, H_2 hold and $u \in W_0^{1,\vartheta}(\Omega)$ is a weak solution of* (3.1)*, then $u \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$.*

What can be said if we generalize the integrability property of the source term? So, let $h \in L^r(\Omega)$, $r > 1$ and consider the following double phase Dirichlet problem

$$-\Delta_p^a u(z) - \Delta_q u(z) = h(z) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{3.2}$$

The weak solutions of (3.2) satisfy the same estimates as the weak solutions of the p-Laplacian equation (see Guedda–Veron [15, Proposition 1.3]).

Proposition 3.2. *If hypotheses H_0 hold and $u \in W_0^{1,\vartheta}(\Omega)$ is a weak solution of* (3.2)*, then we have*

$$\|u\|_s \leqslant c\|h\|_r^{\frac{1}{q-1}}$$

with

$$s = \begin{cases} \frac{N(q-1)r}{N-qr}, & \text{if } 1 < r < \frac{N}{p}, \\ +\infty, & \text{if } \frac{N}{p} < r \end{cases}$$

and $c = c(N, \Omega, q) > 0$.

This result also can be proved using a standard Moser iteration procedure.

Consider the double phase functional

$$I(u;D) = \int_D [a(z)|Du|^p + |Du|^q]\mathrm{d}z \quad \text{with } D \subset\subset \Omega.$$

Definition 3.3. A function $u \in W^{1,1}(\Omega)$ is said to be a "local minimizer" of the functional $I(\cdot,\cdot)$, if $a(\cdot)|Du(\cdot)|^p + |Du(\cdot)|^q \in L^1(\Omega)$ and

$$I(u;\mathrm{supp}\varphi) \leqslant I(u+\varphi;\mathrm{supp}\varphi)$$

for all $\varphi \in W^{1,q}(\Omega)$ with compact support in Ω.

Under hypotheses H_1, the local minimizers of the functional $I(\cdot,\cdot)$ exhibit local Hölder gradient continuity. More precisely, we have the following result due to Baroni–Colombo–Mingione [2, Theorem 1].

Proposition 3.4. *If hypotheses H_1 hold and $u \in W^{1,1}(\Omega)$ is local minimizer of the functional $I(\cdot,\cdot)$, then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for some $\alpha \in (0,1)$.*

This result can be extended to anisotropic double phase integrals. This was done by Ragusa-Tachikawa [37, Theorem 1.2]. So, we consider the integral functional

$$J(u;D) = \int_D \left[a(z)|Du|^{p(z)} + |Du|^{q(z)}\right]\mathrm{d}z, \quad D \subset\subset \Omega.$$

We assume the following on the variable exponents p, q.

H_3 : $p, q \in C^{0,\alpha}(\Omega)$,$1 < q_0 \leqslant q(z) \leqslant p(z)$ for all $z \in \Omega$, $\alpha \in (0,1]$, $a \in C^{0,\beta}(\Omega)$ with $\beta \in (0,1]$, $a(z) \geqslant 0$ for all $z \in \Omega$ and

$$\sup_{z\in\Omega} \frac{p(z)}{q(z)} < 1 + \frac{\gamma}{N}, \quad \gamma = \min\{\alpha,\beta\}.$$

Proposition 3.5. *If hypotheses H_3 hold and $u \in W^{1,1}(\Omega)$ is a local minimizer of the functional $J(\cdot;\cdot)$, then $u \in C^{1,\eta}_{\mathrm{loc}}(\Omega)$ for some $\eta \in (0,1)$.*

We see that the regularity results are local. There are no general global (up to the boundary) regularity results, similar to those for balanced problems (see Lieberman [18]). This fact eliminates from consideration many tools which are very helpful and effective when dealing with balanced problems. One such basic tool is the equivalence of Hölder and Sobolev local minimizers (see Papageogiou–Rădulescu-Zhang [29, Proposition A.3]). For those problems the nonlinear maximum principle (see Pucci–Serrin [35, pp. 111,120]), guarantees that the positive solutions of the problem belong to intC_+, with C_+ being the positive cone of $C^1_0(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. Then the aforementioned equivalence of local minimizers does the job. In the unbalanced case this approach is no longer available. Nevertheless, we have the following maximum principle, which in many occasions is helpful (see Papageprgiou–Vetro–Vetro [33, Proposition 2.4])

Proposition 3.6. *If hypotheses H_1 hold, $\xi \in L^\infty(\Omega)$, $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$, $\xi \not\equiv 0$, $u \in W_0^{1,\vartheta}(\Omega)\backslash\{0\}$, $u(z) \geqslant 0$ for a.a. $z \in \Omega$ and*

$$-\Delta_p^a u - \Delta_q u + \xi(z)u^{p-1} \geqslant 0 \quad \text{in } W_0^{1,\vartheta}(\Omega)^*,$$

then for every $K \subseteq \Omega$ compact, we have

$$u(z) \geqslant c_K > 0 \quad \text{for a.a.} z \in K.$$

More on the regularity theory of problems with nonstandard growth and nonuniform ellipticity (double phase problems are a special case), can be found in the recent survey paper of Mingione–Rădulescu [25], where the reader can also find a rich bibliography on the subject.

4. EIGENVALUE PROBLEMS

In this section we deal with some eigenvalue problems related to the double phase differential operator. Our presentation is based on the works of Colasuonno–Squassina [4] and Papageorgiou–Pudelko–Rădulescu [27].

We start with a fundamental property of the double phase differential operator. So, for $\vartheta(z,x) = a(z)|x|^p + |x|^q$, we consider the nonlinear operator

$$V : W_0^{1,\vartheta}(\Omega) \to W_0^{1,\vartheta}(\Omega)^* = W^{-1,\vartheta^*}(\Omega)$$

defined by

$$\langle V(u), h\rangle \int_\Omega \left(a(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} + |Du|^{q-2}(Du, Dh)_{\mathbb{R}^N}\right) \mathrm{d}z$$

for all u, $h \in W_0^{1,\vartheta}(\Omega)$.

This nonlinear operator has the following basic properties.

Proposition 4.1. *If hypotheses H_1 hold, then $V(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$, that is,*

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,\vartheta}(\Omega) \text{ and } \limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \leqslant 0 \Rightarrow u_n \to u \text{ in } W_0^{1,\vartheta}(\Omega).$$

Evidently, the $(S)_+$ property of $V(\cdot)$ is helpful in proving the compactness condition (PS or C-condition) for the energy functional of the problem. We point out that various techniques have been proposed in the literature in order to recover the compactness in several circumstances. We refer to Tang and Cheng [39] who proposed a new approach to restore the compactness of Palais-Smale sequences and to Tang and Chen [38] who introduced an original method to recover the compactness of minimizing sequences. A related approach has been developed by Chen and Tang [3] in the framework of Cerami sequences.

The result can be extended to anisotropic double phase operators. So,

$$\vartheta_a(z,x) = a(z)|x|^{p(z)} + |x|^{q(z)}$$

and let $V_a : W_0^{1,\vartheta_a}(\Omega) \to W_0^{1,\vartheta_a}(\Omega)^*$ be defined by

$$\langle V_a(u), h\rangle = \int_\Omega \left(a(z)|Du|^{p(z)-2}(Du, Dh)_{\mathbb{R}^N} + |Du|^{q(z)-2}(Du, Dh)_{\mathbb{R}^N}\right) \mathrm{d}z,$$

for all $u, h \in W_0^{1,p(z)}(\Omega)$.

We assume the following on the variable exponents p, q and the weight function $a(\cdot)$. In what follows given $\eta \in C(\bar{\Omega})$, we define

$$\eta_- = \min_{\bar{\Omega}} \eta, \quad \text{and} \quad \eta_+ = \max_{\bar{\Omega}} \eta.$$

H_4: $p, q, a \in C^{0,1}(\bar{\Omega})$, $1 < q_- \leqslant q_+ < p_- \leqslant p_+ < N$, $a(z) \geqslant 0$ for all $z \in \bar{\Omega}$ and $\frac{p_+}{q_-} < 1 + \frac{1}{N}$.

Proposition 4.2. *If hypotheses H_4 hold, then the operator $V_a : W_0^{1,\vartheta_a}(\Omega) \to W_0^{1,\vartheta_a}(\Omega)^*$ is bounded, continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (see Proposition 4.1).*

First we consider the eigenvalue problem for the double phase differential operator. So, for $u \in W_0^{1,\vartheta}(\Omega)$ $(\vartheta(z,x) := \alpha(z)|x|^p + |x|^q)$, we set

$$k(u) = \|u\|_\vartheta, \quad K(u) = \|Du\|_\vartheta, \quad M = \{u \in W_0^{1,\vartheta}(\Omega) : k(u) = 1\}.$$

As usual we consider the Rayleigh quotient

$$\frac{K(u)}{k(u)} = \frac{\|Du\|_\vartheta}{\|u\|_\vartheta} \quad \text{forall } u \in W_0^{1,\vartheta}(\Omega).$$

Then we define the first eigenvalue $\hat{\lambda}_1^\vartheta$, by

$$\hat{\lambda}_1^\vartheta = \inf\left\{\frac{\|Du\|_\vartheta}{\|u\|_\vartheta} : \ u \in W_0^{1,\vartheta}(\Omega), \ u \neq 0\right\} = \inf\{K(u) : \ u \in M\}.$$

Then we have the following result due to Colasuonno–Squassina [4, Theorem 1.1].

Proposition 4.3. *If hypotheses H_1 hold, then $\hat{\lambda}_1^\vartheta > 0$ and there exists $\hat{u}_1^\vartheta \in M \cap L^\infty(\Omega)$ such that $\hat{u}_1^\vartheta(z) \geqslant 0$ for a.a. $z \in \Omega$ and it solves*

$$-div\left(\frac{pa(z)}{\lambda^{p-1}}|Du|^{p-2}Du + \frac{q}{\lambda^{q-1}}|Du|^{q-2}Du\right) = \lambda S(u)(pa(z)|u|^{p-2}u + q|u|^{q-2}u) \ \mathit{in} \ \Omega.$$

with $\lambda = \hat{\lambda}_1^\vartheta$ and

$$S(u) = \frac{\frac{1}{\lambda^p}\int_\Omega (pa(z)|Du|^p + q|Du|^q)\,\mathrm{d}z}{\int_\Omega (pa(z)|u|^p + q|u|^q)\,\mathrm{d}z}.$$

Remark 4.4. Therefore $\hat{u}_1^\vartheta$ solves the eigenvalue problem

$$\begin{aligned} &-\operatorname{div}\left(\frac{pa(z)}{K(u)^{p-1}}|Du|^{p-2}Du + \frac{q}{K(u)^{q-1}}|Du|^{q-2}Du\right) \\ &= \lambda S(u)\left(\frac{pa(z)}{k(u)}|u|^{p-2}u + \frac{q}{k(u)}|u|^{q-2}u\right) \quad \text{in}\Omega, \end{aligned} \tag{4.1}$$

where

$$S(u) = \frac{k(u)^p \int_\Omega (pa(z)|Du|^p + q|Du|^q)\,\mathrm{d}z}{K(u)^p \int_\Omega (pa(z)|u|^p + q|u|^q)\,\mathrm{d}z}$$

We call $\hat{u}_1^\vartheta$ an eigenfunction for the eigenvalue problem (4.1). From Proposition 3.6, we know that for all $K \subseteq \Omega$ compact, we have $\hat{u}_1^\vartheta(z) \geqslant c_K > 0$ a.a. $z \in K$. In particular, then $\hat{u}_1^\vartheta(z) > 0$ for a.a. $z \in \Omega$.

As in the case of the p-Laplacian (see Gasinski–Papageorgiou [13]), we can define higher eigenvalues using the Ljusternik–Schnirelmann minimax scheme. So, let

$$\mathcal{D}_n = \{C \subseteq W_0^{1,\vartheta}(\Omega) : 0 \notin C,\ C : \text{compact, symmetric}, \gamma(C) \geqslant n\}.$$

with $\gamma(\cdot)$ being the Krasnoselski genus. We define

$$\hat{\lambda}_n^\vartheta = \inf_{C \in \mathcal{D}_n} \max_{u \in C} \|Du\|_\vartheta,\ n \in \mathbb{N}$$

Then the following result hold (see Colasuonno–Squassina [4, Theorem 1.2]).

Proposition 4.5. *The sequence* $\{\hat{\lambda}_n^\vartheta\}_{n\in\mathbb{N}}$ *is nondecreasing,* $\hat{\lambda}_n^\vartheta \to +\infty$ *as* $n \to \infty$ *and are all eigenvalues of* (4.1).

Also these eigenvalues depend continuously on the exponents p, q. More precisely, we have the following result (see Colasuonno–Squassina [4, Theorem 1.3]).

Proposition 4.6. *If* $p_m \to p$, $q_m \to q$ *in* $(1, \infty)$ *and*

$$\begin{aligned} &\vartheta_m(z,x) = a(z)|x|^{p_m} + |x|^{q_m},\ m \in \mathbb{N}, \\ &\vartheta(z,x) = a(z)|x|^p + |x|^q,\ \text{for all } (z,x) \in \Omega \times \mathbb{R}, \end{aligned}$$

then for all $n \in \mathbb{N}$, *we have*

$$\hat{\lambda}_n^{\vartheta_m} \to \hat{\lambda}_n^\vartheta.$$

Remark 4.7. We have also a kind of continuous dependence on the domain Ω, namely if $\{\Omega_m\}_{m\in\mathbb{N}}$ is an increasing sequence of open subsets of Ω such the $\Omega = \cup_{m\in\mathbb{N}}\Omega_m$, then $\hat{\lambda}_1^\vartheta(\Omega_m) \to \hat{\lambda}_1^\vartheta(\Omega)$ as $m \to \infty$.

Next we deal with the following eigenvalue problem

$$-\Delta_p^a u(z) = \hat{\lambda} a(z)|u|^{p-2}u \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{4.2}$$

In this case we make the following hypotheses on the data of (4.2).

H_5: $a \in C^{0,1}(\bar{\Omega}) \cap \tilde{A}_p$, $a(z) > 0$ for all $z \in \Omega$, $1 < q < p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$.

The study of problem (2.1) is important when investigating double phase Dirichlet boundary value problems with a $(p-1)$-sublinear, resonant source term. The spectral analysis of (4.2) can be found in the recent work of Papageorgiou–Pudelko–Rădulescu [27].

Recall that $\vartheta_0(z,x) = a(z)|x|^p$. From Proposition 2.33 we know that

$$W_0^{1,\vartheta_0}(\Omega) \hookrightarrow L^{\vartheta_0}(\Omega) \text{ compactly.}$$

Using this fact, we can show the following result.

Proposition 4.8. *If hypotheses H_5 hold, then problem* (4.2) *has a smallest eigenvalue $\hat{\lambda}_1^a > 0$ and every corresponding eigenfunction $\hat{u} \in W_0^{1,\vartheta_0}(\Omega)$ satisfies $\hat{u} \in L^\infty(\Omega)$ and $\hat{u}(z) > 0$ or $\hat{u}(z) < 0$ for a. a. $z \in \Omega$* (*that is, the corresponding eigenfunctions have fixed sign*)*. In fact $\hat{\lambda}_1^a$ is the only eigenvalue with eigenfunctions of fixed sign.*

Proposition 4.9. *If hypotheses H_5 hold and if $\hat{\lambda} \in (\hat{\lambda}_1^a, \mu)$ is an eigenvalue of* (4.2)*, then every eigenfunction $\hat{u} \in W_0^{1,\vartheta_0}(\Omega)$ corresponding to $\hat{\lambda}$ is in $L^\infty(\Omega)$ and is nodal* (*sign changing*)*. Moreover, if*

$$\Omega_+ = \{\hat{u} > 0\}, \quad \Omega_- = \{\hat{u} < 0\},$$

then $0 < c \leqslant |\Omega_\pm|_N$ for all $\hat{\lambda} \in (\hat{\lambda}_1^a, \mu)$.

As usual using the Rayleigh quotient we have

$$\hat{\lambda}_1^a = \inf\left[\frac{\rho_a(Du)}{\rho_a(u)} :\ u \in W_0^{1,\vartheta_0}(\Omega), u \neq 0\right], \tag{4.3}$$

where for all $v \in L^{\vartheta_0}(\Omega)$, we have

$$\rho_a(v) = \int_\Omega a(z)|v|^p \mathrm{d}z.$$

In (4.3) the infimum is realized on the corresponding eigenspace and we show that:

Proposition 4.10. *If hypotheses H_5 hold, then $\hat{\lambda}_1^a$ is isolated in the spectrum of* (4.2) *and simple* (*that is, if $\hat{u}_1$, $\hat{u}_2$ are two eigenfunctions corresponding to $\hat{\lambda}_1^a$, then $\hat{u}_1 = \mu \hat{u}_2$ for some $\mu \neq 0$*)*.*

Using the Ljusternik–Schnirelmann minimax scheme we can generate a whole nondecreasing divergent sequence $\{\hat{\lambda}_n^a\}_{n \in \mathbb{N}}$ of eigenvalues of (4.2). It is easily seen that the spectrum $\hat{\sigma}_a(p,q)$ of (4.2) is closed in $\mathbb{R}_+$. Then the second eigenvalue of (4.2) is defined by

$$\tilde{\lambda}_2^a = \inf\left[\hat{\lambda} \in \sigma_a(p,q) : \hat{\lambda} > \hat{\lambda}_1^a\right] \quad \text{(see Proposition 4.10).}$$

This eigenvalue coincides with the second Ljusternik–Schnirelmann eigenvalue, that is,

$$\tilde{\lambda}_1^a = \hat{\lambda}_1^a.$$

For this eigenvalue we have an alternative minimax characterization different from the one provided by the Ljusternik–Schnirelmann scheme. So, let

$$M = \{u \in W_0^{1,\vartheta}(\Omega) : \|u\|_{\vartheta_0} = 1\},$$

$\hat{u}_1 \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$ is the positive, $L^{\vartheta_0}(\Omega)$-normalized (that is, $\|\hat{u}_1\|_{\vartheta_0} = 1$) eigenfunction corresponding to $\hat{\lambda}_1^a > 0$,

$$\Gamma = \{\gamma \in C([-1,1], M) : \gamma(-1) = -\hat{u}_1, \gamma(1) = \hat{u}_1\}.$$

Recall that $\hat{u}_1(z) > 0$ for a.a. $z \in \Omega$.

Proposition 4.11. *If hypotheses H_5 hold, then*

$$\hat{\lambda}_2^a = \inf_{\gamma \in \Gamma} \max_{-1 \leqslant t \leqslant 1} \rho_a(D\gamma(t)).$$

The above results use the fact that the operator $A_p^a : W_0^{1,\vartheta}(\Omega) \to W_0^{1,\vartheta}(\Omega)^*$ defined by

$$\langle A_p^a(u), h\rangle = \int_\Omega a(z)|Du|^{p-2}(Du, Dh)_{\mathbb{R}^N} \mathrm{d}z$$

for all $u, h \in W_0^{1,\vartheta_0}(\Omega)$, has the same properties as the operator $V(\cdot)$.

Proposition 4.12. *If hypotheses H_5 hold, then the operator $A_p^a(\cdot)$ is bounded, continuous, strictly monotone* (*thus maximal monotone too*) *and of type* $(S)_+$ (*see Proposition 4.1*).

Applications of these results to resonant double phase problems can be found in Papageorgiou–Pudelko–Rădulescu [27].

5. PROBLEMS WITH COMPETING NONLINEARITIES

In this section we examine some parametric Dirichlet problems, in which the reaction (source) has nonlinearities of different structure (competing nonlinearities).

We start with the well-known "concave-convex" problem. So, with $\Omega \subseteq \mathbb{R}^N$ a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the following Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = \xi(z)u(z)^{\tau-1} + \lambda u(z)^{r-1} \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < \tau < q < p < r,\ p < N,\ \lambda > 0,\ u \geqslant 0. \end{cases} \tag{P_λ}$$

We see that in the reaction we have two different terms. One is the $(q-1)$-sublinear (concave) term $x \to \xi(z)x^{\tau-1}$, $x \geqslant 0$ (recall that $\tau < q$). The other is the parametric, $(p-1)$-superlinear (convex) term $x \to \lambda x^{r-1}$, $x \geqslant 0$ with $\lambda > 0$ being the parameter (recall that $p < r$). So, problem (P_λ) is the double phase version of the "concave-convex problem". We mention an essential difference from the usual concave-convex problems. In problem (P_λ) the parameter $\lambda > 0$ multiplies the convex term, while in the standard concave-convex problems, the parameter $\lambda > 0$ multiplies the concave term, see

Ambrosetti–Brezis–Cerami [1] (semilinear problems driven by the Laplacian), Garcia Azorero–Manfredi–Peral Alonso [12] (nonlinear problems driven by the p-Laplacian) and Leonardi–Papageorgiou [17] (nonlinear Robin problems with an indefinite potential). Then the structure of our problem is different and this combined with the lack of a global regularity theory (see the discussion before Proposition 3.6), leads to a different approach based on the Nehari method. An inspection of the tools used in the corresponding balanced problems (see [1, 12, 17]), reveals that they can not be used in the present setting due to the lack of a global regularity theory. In particular, we can not use the equivalence of Hölder and Sobolev local minimizers. This leads to a different approach which uses the Nehari manifold. We set

$$\psi_\vartheta(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q \quad \text{forall } u \in W_0^{1,\vartheta}(\Omega).$$

We see that $\psi_\vartheta \in C^1(W_0^{1,\vartheta}(\Omega))$ and

$$\langle \psi_\vartheta'(u), h\rangle = \langle V(u), h\rangle \quad \text{forall } u, h \in W_0^{1,\vartheta}(\Omega).$$

We introduce the following hypothese on the data of (P_λ)

H_6: $a \in C^{0,1}(\bar{\Omega})$, $a(z) > 0$ for all $z \in \Omega$, $1 < \tau < q < p < r < q^*$, $p < N$, $\frac{p}{q} < 1 + \frac{1}{N}$ and $\xi \in L^\infty(\Omega)\backslash\{0\}$, $\xi(z) \geqslant 0$ for a.a. $z \in \Omega$.

Then the energy (Euler) functional for problem (P_λ) is defined by

$$\varphi_\lambda(u) = \psi_\vartheta(u) - \frac{1}{\tau}\int_\Omega \xi(z)|u|^\tau \mathrm{d}z - \frac{\lambda}{r}\|u\|_r^r \text{ for all } u \in W_0^{1,\vartheta}(\Omega).$$

Clearly $\varphi_\lambda \in C^1\left(W_0^{1,\vartheta}(\Omega)\right)$.

The Nehari manifold for the functional $\varphi_\lambda(\cdot)$ is defined by

$$N_\lambda = \left\{u \in W_0^{1,\vartheta}(\Omega) : \langle \varphi_\lambda'(u), u\rangle = 0, u \neq 0\right\}.$$

Evidently the Nehari manifold contains the nontrivial solutions of (P_λ). The Nehari manifold N_λ is much smaller than $W_0^{1,\vartheta}(\Omega)$ and so properties of $\varphi_\lambda(\cdot)$ which fail globally, will be true for $\varphi_\lambda|_{N_\lambda}$. In our case the presence of the convex (superlinear) term makes $\varphi_\lambda(\cdot)$ unbounded below. However, we can show that

$$\varphi_\lambda|_{N_\lambda} \text{ is coercive (thus bounded below).} \tag{5.1}$$

The Nehari manifold is linked to the behavior of the so-called fibering function. For fixed $u \in W_0^{1,\vartheta}(\Omega)$, this function is defined by

$$\beta_\lambda^u(t) = \varphi_\lambda(tu) \quad \text{forall } t \geqslant 0.$$

Then $\beta_\lambda^u \in C^2(0,\infty)$. Note that

$$u \in N_\lambda \Leftrightarrow (\beta_\lambda^u)'(1) = 0.$$

More generally, we can say that

$$tu \in N_\lambda \Leftrightarrow (\beta_\lambda^u)'(t) = 0.$$

It follows that the elements of the Nehari manifold correspond to the stationary points of the corresponding fibering maps. Therefore, it is natural to subdivide N_λ into sets corresponding to local minima, local maxima and points of infection. Hence we define

$$\begin{aligned} N_\lambda^+ &= \{u \in N_\lambda : (\beta_\lambda^u)''(1) > 0\} \ \text{(local minima)} \\ N_\lambda^- &= \{u \in N_\lambda : (\beta_\lambda^u)''(1) < 0\} \ \text{(local maxima)} \\ N_\lambda^0 &= \{u \in N_\lambda : (\beta_\lambda^u)''(1) = 0\} \ \text{(points of inflection)} \end{aligned}$$

We have the following decomposition of the Nehari manifold

$$N_\lambda = N_\lambda^+ \cup N_\lambda^- \cup N_\lambda^0. \tag{5.2}$$

On account of (5.1) we have

$$m_\lambda^+ = \inf_{N_\lambda^+} \varphi_\lambda > -\infty \ \text{(provided } N_\lambda^+ \neq \emptyset).$$

Using the definition of $N_\lambda^+ \subseteq N_\lambda$, we show that

$$m_\lambda^+ < 0. \tag{5.3}$$

Moreover, via a contradiction argument, we show that

$$N_\lambda^0 = \emptyset \quad \text{for all } \lambda > 0 \text{ small}. \tag{5.4}$$

At the same time we have

$$N_\lambda^\pm \neq \emptyset \quad \text{for all } \lambda > 0 \text{ small}. \tag{5.5}$$

Then from (5.2), (5.4), (5.5), we infer that

$$N_\lambda = N_\lambda^+ \cup N_\lambda^- \quad \text{for all } \lambda > 0 \text{ small}. \tag{5.6}$$

Using (5.1), (5.6) and the fibering map, we show that for all $\lambda > 0$ small, there exists $u_0^\lambda \in N_\lambda^+$ such that

$$\varphi_\lambda(u_0^\lambda) = m_\lambda^+ < 0 = \varphi_\lambda(0) \ \text{(see (9))}.$$

Although u_0^λ is a constrained minimizer of $\varphi_\lambda(\cdot)$, using the Lagrange multiplier rule, we show that u_0^λ is a critical point of φ_λ, that is,

$$u_0^\lambda \in K_{\varphi_\lambda} = \left\{u \in W_0^{1,\vartheta}(\Omega) : \varphi_\lambda'(u) = 0\right\} \ \text{(critical set of } \varphi_\lambda(\cdot)).$$

We do a similar analysis for the component N_λ^- ($\lambda > 0$ small, see (5.6)). So, we set

$$m_\lambda^- = \inf_{N_\lambda^-} \varphi_\lambda.$$

Using the direct method of the calculus of variations (see (5.1)), we show the existence of a $\hat{u}_\lambda \in N_\lambda^-$ such that

$$m_\lambda^- = \varphi_\lambda(\hat{u}_\lambda).$$

Again we have that $\hat{u}_\lambda \in K_{\varphi_\lambda}$. Therefore both components N_λ^+ and N_λ^- are natural constraints for the function $\varphi_\lambda(\cdot)$ (see Papageorgiou–Rădulescu–Repovš [28, p. 425]).

So, we have

$$u_0^\lambda \in K_{\varphi_\lambda} \cap N_\lambda^+, \quad \hat{u}_\lambda \in K_{\varphi_\lambda} \cap N_\lambda^- \quad \text{for } \lambda > 0 \text{ small}$$

and so are nontrivial solutions of (P_λ). Since

$$\varphi_\lambda(u) = \varphi_\lambda(|u|) \quad \text{forall} u \in W_0^{1,\vartheta}(\Omega),$$

we may assume that $u_0^\lambda, \hat{u}_\lambda \geqslant 0$. Invoking Proposition 3.6, we have $u_0^\lambda(z), \hat{u}_\lambda(z) > 0$ for a. a. $z \in \Omega$. So, we can state the following multiplicity theorem for (P_λ) (see Liu–Papageorgiou [21]).

Theorem 5.1. *If hypotheses (H_6) hold, then for all $\lambda > 0$ small problem (P_λ) has at least two positive solutions $u_0^\lambda, \hat{u}_\lambda \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega)$ and $u_0^\lambda(z) > 0, \hat{u}_\lambda(z) > 0$ for a.a. $z \in \Omega$.*

This theorem reveals a discontinuity property for the "spectrum" of (P_λ). Indeed, if $\lambda = 0$, the problem has a unique solution, while for $\lambda > 0$ small problem (P_λ) has at least two positive solutions. This discontinuity is no longer true when the parameter $\lambda > 0$ multiplies the concave term.

Using a similar version of the Nehari method, we can also treat problems, where we have competition of singular and superlinear terms. So, the problem under consideration is the following:

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = \lambda u(z)^{-\mu} + \xi(z) u(z)^{r-1} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < q < p < r, 0 < \mu < 1, \lambda > 0, \ u \geqslant 0. \end{cases} \tag{P_λ'}$$

For this problem we have the following multiplicity theorem (see Liu–Popogeorgiou [20]).

Theorem 5.2. *If hypotheses H_6 hold, then for all $\lambda > 0$ small problem (P_λ') has at least two positive solutions*

$$\begin{aligned} &u_\lambda, \hat{u}_\lambda \in W_0^{1,\vartheta}(\Omega) \cap L^\infty(\Omega), \\ &0 < u_\lambda(z), \hat{u}_\lambda(z) \text{ a.e. in } \Omega, \\ &\varphi_\lambda(u_\lambda) < 0 < \varphi_\lambda(\hat{u}_\lambda). \end{aligned}$$

In this case the energy functional $\varphi_\lambda(\cdot)$ is given by

$$\varphi_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{1-\mu}\int_\Omega |u|^{1-p}\mathrm{d}z - \frac{1}{r}\int_\Omega \xi(z)|u|^r\mathrm{d}z$$

for all $u \in W_0^{1,\vartheta}(\Omega)$.

Note that on account of the singular term, $\varphi_\lambda(\cdot)$ is not C^1. This is a source of difficulties in the study of (P'_λ).

Additional results on singular double phase problem can be found in Crespo Blanco–Papageorgiou–Winkert [6] and Papageorgiou–Rădulescu–Zhang [31].

Finally, we examine the case where we have the competition of a parametric $(q-1)$-linear term and of a $(p-1)$-superlinear perturbation. So, the problem is the following

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = \lambda |u(z)|^{q-2}u(z) + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < q < p, \ \lambda \in \mathbb{R}. \end{cases} \tag{P''_λ}$$

The hypotheses on the data of this problem are the following.

H_7: $a \in L^\infty(\Omega)$, $a(z) \geqslant 0$ for a.a. $z \in \Omega$, $a \neq 0$, $1 < q < p$, $\frac{p}{q} < 1 + \frac{1}{N}$.

H_8: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and
- (a) $|f(z,x)| \leqslant \hat{a}(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a} \in L^\infty(\Omega)$ and $p < r < q^*$,
- (b) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to+\infty} \frac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and if $\varrho(z,x) = f(z,x)x - pF(z,x)$, then $0 < \hat{c} \leqslant \liminf_{x\to\pm\infty} \frac{\varrho(z,x)}{|x|^p}$ uniformly for a.a. $z \in \Omega$,
- (c) $\lim_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} = 0$ uniformly for a.a. $z \in \Omega$,
- (d) for a.a. $z \in \Omega$, the quotient function $x \to \frac{f(z,x)}{|x|^{p-1}}$ is increasing on $(0, +\infty)$ and on $(-\infty, 0)$.

Remark 5.3. Note that hypotheses H_8-(ii) implies that for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-superlinear. Hypotheses H_8-(iv) is weaker than the standard Nehari monotonicity condition which requires that the quotient function $x \to \frac{f(z,x)}{|x|^{p-1}}$ is strictly increasing on $(0, +\infty)$ and on $(-\infty, 0)$ (see Gasinski–Winkert [14] and Liu–Dai [19]; see also [32] for Neumann problems).

In this case the energy (Euler) functional is

$$\varphi_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{q}\|u\|_q^q - \int_\Omega F(z,u)\mathrm{d}z.$$

In order to produce constant sign solutions, we also introduce the positive and negative truncations of φ_λ, namely the functional

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda}{q}\|u^\pm\|_q^q - \int_\Omega F(z, \pm u^\pm)\mathrm{d}z.$$

Evidently, $\varphi_\lambda, \varphi_\lambda^\pm \in C^1(W_0^{1,\vartheta}(\Omega))$.

We introduce the following sets:

$$\begin{aligned}
N^\lambda &= \left\{u \in W_0^{1,\vartheta}(\Omega) : \langle \varphi_\lambda'(u), u\rangle = 0,\ u \neq 0\right\},\\
N_+^\lambda &= \left\{u \in W_0^{1,\vartheta}(\Omega) : \langle (\varphi_\lambda^+)'(u), u\rangle = 0,\ u \geqslant 0,\ u \neq 0\right\},\\
N_-^\lambda &= \left\{u \in W_0^{1,\vartheta}(\Omega) : \langle (\varphi_\lambda^-)'(u), u\rangle = 0,\ u \leqslant 0,\ u \neq 0\right\},\\
N_0^\lambda &= \left\{u \in W_0^{1,\vartheta}(\Omega) : \langle \varphi_\lambda'(u), u^\pm\rangle = 0,\ u^\pm \neq 0\right\}.
\end{aligned}$$

Note that N^λ is the Nehari manifold for the energy functional $\varphi_\lambda(\cdot)$ and $N_\pm^\lambda, N_0^\lambda \subseteq N^\lambda$. Also N_+^λ (resp. N_-^λ) contains the positive (resp. negative) solutions, while N_0^λ contains the nodal solutions of (P_λ'').

Let $\hat{\lambda}_1(q) > 0$ denote the principal eigenvalue of $(-\Delta_q, W_0^{1,q}(\Omega))$. Using a different version of the Nehari method we obtain the following multiplicity theorem (see Papageorgiou–Vetro–Vetro [34]).

Theorem 5.4. *If hypotheses H_7, H_8 hold and $\lambda < \hat{\lambda}_1(q)$, then problem (P_λ'') has at least three nontrivial solutions*

$$\begin{aligned}
&u_0 \in N_+^\lambda \cap L^\infty(\Omega) \text{ (positive solution)},\\
&v_0 \in N_-^\lambda \cap L^\infty(\Omega) \text{ (negative solution)},\\
&y_0 \in N_0^\lambda \cap L^\infty(\Omega) \text{ (nodal solution)}.
\end{aligned}$$

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